Vehicle-Bridge Interaction Element for Dynamic Analysis

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Abstract: The objective of this study is to develop an element that is both accurate and efficient for modeling the vehicle-bridge interaction (VBI) in analysis of railway bridges carrying high-speed trains, which may consist of a number of cars in connection. In this study, a train is modeled as a series of sprung masses lumped at the bogie positions and a bridge with track irregularities by beam elements. Two sets of equations of motion that are coupled can be written, one for the bridge and the other for each of the sprung masses. To resolve the problem of coupling, the sprung mass equation is first discretized using Newmark's finite difference formulas and then condensed to that of the bridge element in contact. The element derived is referred to as the vehicle-bridge interaction element, which has the same number of degrees of freedom (DOF) as the parent element, while possessing the properties of symmetry and bandedness in element matrices. For this reason, conventional assembly procedures can be employed to forming the structure equations. The applicability of the VBI element is demonstrated in the numerical studies.

INTRODUCTION

Following the pioneer works of Stokes (1849) and Willis (1849) in the mid-19th century, the vibration of bridges caused by the passage of railroad cars has been investigated by a great number of researchers. Partly enhanced by the construction of high-speed rails worldwide, this subject is becoming a focus of increasing interest. In a survey paper by Diana and Cheli (1989), issues relating to the train-bridge interactions have been discussed, with 90 papers cited. By modeling a moving vehicle as a moving load, moving mass, or moving sprung mass considering the suspension effects, the dynamic response of bridges induced by moving vehicles has been studied by researchers from time to time (Timoshenko 1922; Jeffcott 1929; Lowan 1935; Ayre and Jacobsen 1950; Ayre et al. 1950; Biggs 1964; Frýba 1972; Chu et al. 1979; Stanišić 1985; Sadiku and Leipholz 1987; Chatterjee et al. 1994). More sophisticated models that consider the various dynamic characteristics of vehicles or railroad cars have also been implemented in the study of vehicle-bridge interactions (VBIs) (Veletsos and Huang 1970; Garg and Dukkipati 1984; Yang and Lin 1995). Recently, closed-form solution has been obtained by Yang et al. (1996) for the response of simple beams subjected to action of high-speed trains modeled as a sequence of moving loads with regular nonuniform intervals. In his study, the resonant and cancellation effects of waves generated by the motion of wheel loads on the bridge have been related to the ratio of the railroad car to bridge lengths.

In studying the dynamic response of a vehicle-bridge system, two sets of equations of motion can be written, each for the vehicle and the bridge. It is the interaction forces existing at the contact points of the two subsystems that make the two sets of equations coupled. Because the contact points are time dependent, so are the system matrices, which therefore must be updated and factorized at each time step. To solve these two sets of equations, procedures of an iterative nature are usually adopted (Hwang and Nowak 1991; Green and Cebon 1994; Yang and Fonder 1996). For instance, by first assuming the displacements for the contact points, one can solve the vehicle equations to obtain the interaction forces and then proceed to solve the bridge equations for improved values of displacements for the contact points. The advantage of such an approach is that the reponse of vehicles and bridges at any time step is made available. However, the convergence rate of iteration is likely to be low, when dealing with the more realistic case of a bridge carrying a large number of vehicles in motion, for there exists twice the number of contact points if each vehicle is modeled as two sprung mass systems.

Other approaches for solving the VBI problems include those based on the condensation method. Garg and Dukkipati (1984) used the Guyan reduction scheme to condense the vehicle degrees of freedom (DOF) to the associated bridge DOF. Recently, Yang and Lin (1995) used the dynamic condensation method to eliminate all the vehicle DOF on the element level. However, if the behavior of the vehicle is concerned, which serves as an indicator of the riding comfort, these two approaches cannot be considered acceptable, because of the approximations made in relating the vehicle to bridge DOF. Such a drawback will be overcome in this paper.

To resolve the dependency of system matrices on the wheel load positions, the condensation technique that eliminates the vehicle DOF on the element level will be adopted in this paper. First, two sets of equations of motion are written, one for the bridge and the other for each of the sprung masses that make up the train. The sprung mass equation is then discretized, using Newmark's finite difference formulas, and condensed to those of the bridge element in contact. The result is a VBI element that possesses the same number of DOF as the parent element. The applicability of the present VBI element will be illustrated in the numerical studies.

VBI ELEMENT

As shown in Fig. 1, a bridge is modeled as a beam-like structure and the train traveling over the bridge with constant speed v is idealized as a series of lumped masses supported by the suspension systems, as represented by the springs and dashpots, which in turn are acting on the bridge. In this study, an interaction element is defined such that it consists of a bridge (beam) element and the masses and suspension units of the car bodies directly acting on it, as shown in Fig. 2, where the rail irregularity $r(x)$ and ballast with stiffness k_B are also indicated. For the parts of the birdge that are not directly under the action of the vehicles, they are modeled by conventional bridge elements. However, for the remaining parts that are in direct contact with the vehicles, interaction elements considering the effects of suspension units have to be used instead. In this study, the notation [ ] is used for a square matrix, { } for a column vector, and ⟨ ⟩ for a row vector.

The model depicted in Fig. 2 will be grossly referred to the sprung moss model throughout this paper. Let the stiffness and

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Note. Associate Editor: James M. Nau. Discussion open until April 1, 1998. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on December 2, 1996. This paper is part of the ***Journal of Structural Engineering***, Vol. 123, No. 11, November, 1997. ©ASCE, ISSN 0733-9445/97/0011-1512–1518/$4.00 + $.50 per page. Paper No. 14646.

J. Struct. Eng., 1997, 123(11): 1512-1518

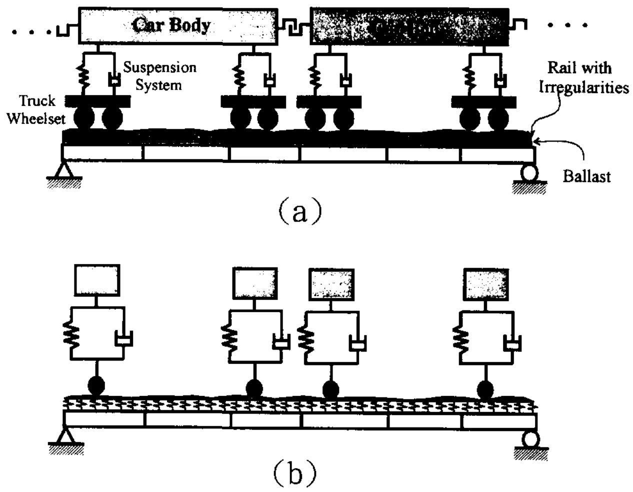

FIG. 1. Train-Bridge System: (a) General Model; (b) Sprung Mass Model

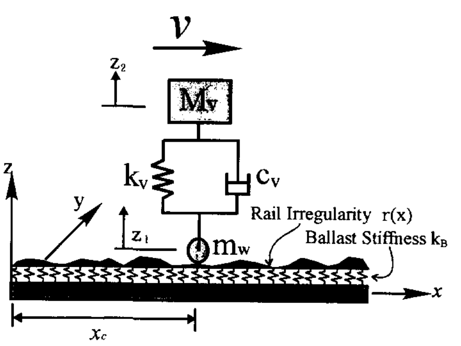

FIG. 2. Vehicle-Bridge Interaction Element

damping coefficients of the suspension system be denoted by k_v and c_v, respectively, the mass of the wheel assembly by m_w, and the lumped mass of the car body by M_v. Also, let the vertical displacements of the two nodes from the static equilibrium positions be denoted by the generalized coordinates $\{z\}^T = \langle z_1, z_2 \rangle$. Corresponding to the nodal displacements $\{z\}^T$ are the external forces $\{p_v\}^T = \langle p, 0 \rangle$, where $p = -(M_v + m_w)g$ and g = the acceleration of gravity. The equations of motion for the sprung mass model in Fig. 2 can be written as (Frýba 1972)

$$\begin{bmatrix} m_w & 0 \\ 0 & M_v \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} c_v & -c_v \\ -c_v & c_v \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} + \begin{bmatrix} k_v & -k_v \\ -k_v & k_v \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} n + f_c \\ 0 \end{Bmatrix} \quad (1)$$

where f_c = the interaction force existing between the wheel mass and the bridge element. Let x_c denote the acting position of the sprung mass (see Fig. 2) and $\{N_c\}$ a vector containing cubic Hermitian interpolation functions for the vertical displacement of the beam evaluated at the contact point x_c, i.e., $\{N_c\} = \{N_i(x_c)\}$. The interaction force can be expressed as

$$f_c = k_B(\langle N_c \rangle \{u_b\} + r_c - z_1) \geq 0 \quad (2)$$

where the condition of $f_c \geq 0$ is imposed to exclude the separation of the vehicle from the bridge; k_B = ballast stiffness; $\{u_b\}$ = nodal displacements of the beam; and r_c = rail irregularity at the contact point x_c, which is given for any instant. The equations of motion for the bridge element can be written as

$$[m_b]\{\ddot{u}_b\} + [c_b]\{\dot{u}_b\} + [k_b]\{u_b\} = \{p_b\} - \{N_c\}f_c \quad (3)$$

where $[m_b]$, $[c_b]$, and $[k_b]$ = mass, damping, and stiffness matrices, respectively, of the bridge element; and $\{p_b\}$ = external nodal loads. Regarding the bridge element as a three-dimensional solid beam element, one can assign six DOF to each node of the element, with three for translations and the other three for rotations. For damping of the Rayleigh type, the three matrices $[m_b]$, $[c_b]$, and $[k_b]$ are available elsewhere [see for instance Paz (1986)].

As can be seen from (1) and (3), the vehicle and the bridge interact with each other through the contact force f_c, which varies as a function of time and position. To ensure that the vehicle is in contact with the bridge, the reactive force exerted by the bridge on the sprung mass must be of sufficiently low amplitude. Whenever the contact force f_c is less than zero, the wheel mass jumps upward and the contact condition between the vehicle and the bridge is violated. Such a condition will be excluded from the present discussion. From the first line of (1), along with the second line, the contact force f_c also can be expressed as

$$f_c = -p + m_w\ddot{z}_1 + M_v\ddot{z}_2 \quad (4)$$

From (3) and (4), it is obvious that the dynamic response of the beam is affected not only by the moving loads, but also by the suspension systems.

The system equations as given in (1)–(3) are nonlinear in nature, which can only be solved by incremental methods, with iterations for removing the unbalanced forces. Consider a typical incremental step from time t to $t + \Delta t$. To this end, the system equations in (1)–(3) should be interpreted as those established for the deformed position at time $t + \Delta t$. Let $\{z\}_{t+\Delta t} = \{z\}_t + \{\Delta z\}$ = vehicle displacement increments occurring during the step considered. By the use of (2), the sprung mass equation in (1) can be written in an incremental form using the strategy presented by Yang et al. (1990) as follows:

$$\begin{bmatrix} m_w & 0 \\ 0 & M_v \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix}_{t+\Delta t} + \begin{bmatrix} c_v & -c_v \\ -c_v & c_v \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}_{t+\Delta t} + \begin{bmatrix} k_v + k_B & -k_v \\ -k_v & k_v \end{bmatrix} \begin{Bmatrix} \Delta z_1 \\ \Delta z_2 \end{Bmatrix} = \begin{Bmatrix} p + k_B(\langle N_c \rangle \{u_b\} + r_c) \\ 0 \end{Bmatrix}_{t+\Delta t} - \begin{Bmatrix} q_{s1} \\ q_{s2} \end{Bmatrix}_t \quad (5)$$

where $\{q_s\}_t$ = internal resistant forces of the vehicle suspension at time t, i.e.

$$\begin{Bmatrix} q_{s1} \\ q_{s2} \end{Bmatrix}_t = \begin{bmatrix} k_v + k_B & -k_v \\ -k_v & k_v \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}_t \quad (6)$$

Similarly, by the use of (2) and noting that $\{u_b\}_{t+\Delta t} = \{u_b\}_t + \{\Delta u_b\}$, the bridge equation in (3) can be written in incremental form as

$$[m_b]\{\ddot{u}_b\}_{t+t\Delta t} + [c_b]\{\dot{u}_b\}_{t+\Delta t} + [[k_b] + k_B\{N_c\}\langle N_c \rangle]\{\Delta u_b\} = \{p_b\}_{t+\Delta t} - \{N_c\}k_B(r_c - z_1)_{t+\Delta t} - [[k_b] + k_B\{N_c\}\langle N_c \rangle]\{u_b\}_t \quad (7)$$

As can be seen, the two system equations as presented in (5) and (7) for the sprung mass and the bridge, respectively, are coupled. In the following, the sprung mass equation in (5) will first be reduced to an equivalent stiffness equation using Newmark's single-step finite difference scheme. The sprung mass DOF can then be eliminated and condensed to those of the bridge element in contact.

Based on Newmark's β method with constant average acceleration, i.e., with $\beta = 0.25$ and $\gamma = 0.5$ (Clough and Penzien 1993)

$$\{\dot{z}\}_{t+\Delta t} = \{\dot{z}\}_t + [(1 - \gamma)\{\ddot{z}\}_t + \gamma\{\ddot{z}\}_{t+\Delta t}]\Delta t \quad (8a)$$

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$$\{z\}_{t+\Delta t} = \{z\}_t + \{\dot{z}\}_t\Delta t + [(0.5 - \beta)\{\ddot{z}\}_t + \beta\{\ddot{z}\}_{t+\Delta t}]\Delta t^2 \quad (8b)$$

from which

$$\{z\}_{t+\Delta t} = \{z\}_t + \{\Delta z\} \quad (9a)$$

$$\{\ddot{z}\}_{t+\Delta t} = a_0\{\Delta z\} - a_2\{\dot{z}\}_t - a_3\{\ddot{z}\}_t \quad (9b)$$

$$\{\dot{z}\}_{t+\Delta t} = \{\dot{z}\}_t + a_6\{\ddot{z}\}_t + a_7\{\ddot{z}\}_{t+\Delta t} \quad (9c)$$

where the coefficients and those to appear are defined as

$$a_0 = \frac{1}{\beta\Delta t^2};\quad a_1 = \frac{\gamma}{\beta\Delta t};\quad a_2 = \frac{1}{\beta\Delta t};\quad a_3 = \frac{1}{2\beta} - 1;\quad a_4 = \frac{\gamma}{\beta} - 1;$$

$$a_5 = \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right);\quad a_6 = \Delta t(1 - \gamma);\quad a_7 = \gamma\Delta t \quad (10)$$

With the relations given in (9), the sprung mass equation in (5) can be manipulated to yield the equivalent stiffness equation

$$\begin{bmatrix} k_v + k_B + a_0 m_w + a_1 c_v & -k_v - a_1 c_v \\ -k_v - a_1 c_v & k_v + a_0 M_v + a_1 c_v \end{bmatrix} \begin{Bmatrix} \Delta z_1 \\ \Delta z_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} p + k_B r_c + k_B \langle N_c \rangle \{u_b\} \\ 0 \end{Bmatrix}_{t+\Delta t} - \left(\begin{Bmatrix} q_{s1} \\ q_{s2} \end{Bmatrix}_t + \begin{Bmatrix} q_{e1} \\ q_{e2} \end{Bmatrix}_t \right) \quad (11)$$

where

$$q_{e1,t} = -m_w(a_2\dot{z}_1 + a_3\ddot{z}_1) - c_v[a_4(\dot{z}_1 - \dot{z}_2) + a_5(\ddot{z}_1 - \ddot{z}_2)] \quad (12a)$$

$$q_{e2,t} = -M_v(a_2\dot{z}_2 + a_3\ddot{z}_2) - c_v[a_4(\dot{z}_2 - \dot{z}_1) + a_5(\ddot{z}_2 - \ddot{z}_1)] \quad (12b)$$

As can be seen, the equivalent stiffness equation in (11) is a single-step finite difference equation, from which the displacement increments $\{\Delta z\}$ of the sprung mass can be solved and related to the bridge displacements $\{u_b\}_{t+\Delta t}$ as

$$\begin{Bmatrix} \Delta z_1 \\ \Delta z_2 \end{Bmatrix} = -\frac{1}{D} \begin{Bmatrix} k_v + a_0 M_v + a_1 c_v \\ k_v + a_1 c_v \end{Bmatrix} (p + k_B r_{c,t+\Delta t} + k_B \langle N_c \rangle \{u_b\}_{t+\Delta t})$$

$$- \frac{1}{D} \begin{Bmatrix} (q_{s1,t} + q_{e1,t})a_0 M_v + (\hat{q}_{s,t} + \hat{q}_{e,t})(k_v + a_1 c_v) \\ (q_{s1,t} + q_{e1,t})(a_0 m_w + k_B) + (\hat{q}_{s,t} + \hat{q}_{e,t})(k_v + a_1 c_v) \end{Bmatrix} \quad (13)$$

where $\hat{q}_{e,t} = (q_{e1} + q_{e2})_t$, $\hat{q}_{s,t} = (q_{s1} + q_{s2})_t = k_B z_{1,t}$, and

$$D = \begin{vmatrix} k_v + k_B + a_0 m_w + a_1 c_v & -k_v - a_1 c_v \\ -k_v - a_1 c_v & k_v + a_0 M_v + a_1 c_v \end{vmatrix} \quad (14)$$

The equation as given in (13) is exactly the master-slave relation for condensing the sprung mass DOF to the bridge DOF, of which the order of error is not greater than that implied by the finite difference formulas in (8). Compared with the approximate master-slave relation used by Yang and Lin (1995), the present relation in (13) has the advantage of being simple, accurate, and explicit. Because of this, the element to be derived in this paper can be used effectively in computation of both the vehicle and the bridge responses.

By the fact that $z_{1,t+\Delta t} = z_{1,t} + \Delta z_1$ and using the first line of (13) for Δz_1, one can derive from (7) the condensed equations of the beam at time $t + \Delta t$ with the interaction effect of the sprung mass taken into account, that is

$$[m_b]\{\ddot{u}_b\}_{t+\Delta t} + [c_b]\{\dot{u}_b\}_{t+\Delta t} + [\hat{k}_b]\{\Delta u_b\} = (\{p_b\}_{t+\Delta t} + \{p_s\}_{t+\Delta t})$$

$$- (\{f_s\}_t + [\hat{k}_b]\{u_b\}_t) \quad (15)$$

where $[\hat{k}_b]$ = stiffness matrix for the condensed system; $\{p_s\}_{t+\Delta t}$ = external loads caused by the sprung mass; $\{f_s\}_t$ = resistant forces associated with the sprung mass; and $[\hat{k}_b]\{u_b\}_t$ = resistant forces exerted by the bridge element at time t

$$[\hat{k}_b] = [k_b] + k_B \frac{a_0}{D} [(M_v + m_w)(k_v + a_1 c_v) + a_0 M_v m_w]\{N_c\}\langle N_c \rangle \quad (16a)$$

$$\{p_s\}_{t+\Delta t} = -k_B \left[r_{c,t+\Delta t} - (p + k_B r_{c,t+\Delta t}) \frac{1}{D}(k_v + a_0 M_v + a_1 c_v) \right] \{N_c\} \quad (16b)$$

$$\{f_s\}_t = k_B \left[\frac{1}{D} \{(q_{s1,t} + q_{e1,t})a_0 M_v + (\hat{q}_{s,t} + \hat{q}_{e,t})(k_v + a_1 c_v)\} - z_{1,t} \right] \{N_c\} \quad (16c)$$

Clearly, the stiffness matrix $[\hat{k}_b]$ is step dependent, because it is a function of the loading position x_c, as implied by the shape function $\{N_c\}$. The equations as given in (15) represent exactly the equations of motion for the VBI element. With the use of the present element, the maximum element length must be less than the minimum spacing between axles. Such a restriction can be released if the foregoing procedure is generalized to include the effect of multiaxle loadings.

Consider the special case of moving loads. By letting $k_v = 0$, $m_w = 0$, and $c_v = 0$, one can obtain from (14) the determinant as $D = a_0 M_v k_B$, from (6) the resistant force $q_{s1,t} = k_B z_{1,t}$, and from (12a) the sprung mass force $q_{e1,t} = 0$. Further, from (16a)–(16c), the following can be obtained: $[\hat{k}_b] = [k_b]$, $\{p_s\}_{t+\Delta t} = p\{N_c\}$ and $\{f_s\}_t = \{0\}$. Consequently, the condensed beam equation in (15) reduces to

$$[m_b]\{\ddot{u}_b\}_{t+\Delta t} + [c_b]\{\dot{u}_b\}_{t+\Delta t} + [k_b]\{\Delta u_b\} = \{p_b\}_{t+\Delta t} + p\{N_c\} - [k_b]\{u_b\}_t \quad (17)$$

where the term $p\{N_c\}$ = action of the moving load, as is expected. On the other hand, by setting the damping coefficient c_v and the mass m_w of the wheel set assembly equal to zero and assigning a very large number to the stiffnesses k_v and k_B, one can arrive at the moving mass model.

Because the VBI element has exactly the same number of DOF as the parent beam element and because it possesses the property of symmetry in element matrices, conventional element assembly processes can be applied to constructing the equations of motion for the entire vehicle-bridge system based on the element equations in (15), that is

$$[M_b]\{\ddot{U}_b\}_{t+\Delta t} + [C_b]\{\dot{U}_b\}_{t+\Delta t} + [K_b]\{\Delta U_b\} = \{P_b\}_{t+\Delta t} - \{F_b\}_t \quad (18)$$

where $\{\Delta U_b\}$ = displacement increments of the bridge from time t to $t + \Delta t$

$$\{U_b\}_{t+\Delta t} = \{U_b\}_t + \{\Delta U_b\} \quad (19)$$

and each of the terms is obtained by assembling over all beam elements of the bridge, in particular

$$\{P_b\}_{t+\Delta t} = \sum_{elm=1}^{n.o.e.} [\{p_b\}_{t+\Delta t} + \{p_s\}_{t+\Delta t}] \quad (20a)$$

$$\{F_b\}_t = \sum_{elm=1}^{n.o.e.} [\{f_s\}_t + [\hat{k}_b]\{u_b\}_t] \quad (20b)$$

By the same Newmark procedure, the displacement increments $\{\Delta U_b\}$ at time $t + \Delta t$ can be solved from the system equations in (18). Correspondingly, the acceleration and velocity can be calculated from equations similar to (9) for the bridge element as

$$\{\ddot{U}_b\}_{t+\Delta t} = a_0\{\Delta U_b\} - a_2\{\dot{U}_b\}_t - a_3\{\ddot{U}_b\}_t \quad (21a)$$

$$\{\dot{U}_b\}_{t+\Delta t} = \{\dot{U}_b\}_t + a_6\{\ddot{U}_b\}_t + a_7\{\ddot{U}_b\}_{t+\Delta t} \quad (21b)$$

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Moreover, the vehicle displacement increments $\{\Delta z\}$ can be obtained from (13) and the total responses from the Newmark-type equations in (9). Here, it is noted that the vehicle acceleration $\{\ddot{z}\}$ serves as a measure of riding comfort for passengers.

As was stated, because of condensation of the VBIs on the element level, conventional element assembly procedures can be directly applied using the present approach. For this reason, the amount of effort required in programming and computation is minimal, compared with approaches that perform condensation on the structure level or with no condensation at all. Such an advantage becomes more obvious in the study of bridges subjected to a sequence of moving vehicular loads. In such cases, all one needs is a proper bookkeeping scheme to identify at each time step the acting position of each wheel load of the cars composing the train. It should be noted that the VBI element derived here is applicable not only for modeling railroad cars with regular intervals and constant sizes, but also for vehicles that constitute a random traffic flow through the highway bridges.

INCREMENTAL DYNAMIC ANALYSIS WITH ITERATIONS

During the passage of vehicles over a bridge, on the one hand, the vehicles excite the bridge by their interaction forces; on the other hand, the bridge affects the behavior of the vehicles by its motion. Such a phenomenon is typical of a nonlinear interaction problem, which can only be solved by procedures of incremental and iterative nature. For the present purpose, consider the system equations of motion in (18). By the finite difference equations in (19) and (21), the equations of motion in (18) can be manipulated to yield the following equivalent stiffness equations:

$$[\bar{K}_b]_{t+\Delta t}\{\Delta U_b\} = \{P_b\}_{t+\Delta t} - \{\bar{F}_b\}_t \qquad (22)$$

where the effective stiffness matrix $[\bar{K}_b]_{t+\Delta t}$ is

$$[\bar{K}_b]_{t+\Delta t} = a_0[M_b] + a_1[C_b] + [K_b] \qquad (23)$$

and the effective resistant force vector $\{\bar{F}_b\}_t$ is

$$\{\bar{F}_b\}_t = \{F_b\}_t - [M_b](a_2\{\dot{U}_b\}_t + a_3\{\ddot{U}_b\}_t) - [C_b](a_4\{\dot{U}\}_t + a_5\{\ddot{U}_b\}_t) \qquad (24)$$

Here, it should be noted that both the effective stiffness matrix $[\bar{K}_b]_{t+\Delta t}$ and the load vector $\{P_b\}_{t+\Delta t}$ remain constant within each time step.

For the purpose of performing iterations, the equivalent stiffness equations for the bridge in (22), which has been presented in incremental form, should be modified to include the feature of iteration, that is

$$[\bar{K}_b]_{t+\Delta t}\{\Delta U_b\}^i = \{P_b\}_{t+\Delta t} - \{\bar{F}_b\}^{i-1}_{t+\Delta t} \qquad (25)$$

in which the right superscript i on each symbol indicates the number of iterations. The right hand of (25) should now be interpreted as the external load increments for the first iteration ($i = 1$) and as the system unbalanced forces for the following iterations ($i \geq 2$) (Yang and Kuo 1994). The philosophy for modifying an equation originally presented in incremental form into one in iterative form is that throughout the process of iteration, all the terms associated with time t be interpreted as those for the $(i-1)$st iterative step and all the terms associated with time $t + \Delta t$ as the ith iterative step. For instance, the resistant force vector in (24) should be interpreted as

$$\{\bar{F}_b\}^{i-1}_{t+\Delta t} = \{F_b\}^{i-1}_{t+\Delta t} - [M_b](a_2\{\dot{U}_b\}^{i-1}_{t+\Delta t} + a_3\{\ddot{U}_b\}^{i-1}_{t+\Delta t}) - [C_b](a_4\{\dot{U}_b\}^{i-1}_{t+\Delta t} + a_5\{\ddot{U}_b\}^{i-1}_{t+\Delta t}) \qquad (26)$$

Eq. (25) represents a typical nonlinear equation that can be encountered in the study of a great number of nonlinear problems. To solve problems of this sort, the (modified) Newton-Raphson method that performs iterations at constant loads can be employed (Yang and Kuo 1994). The initial conditions to the system equations in (25) are

$$\{\bar{F}_b\}^0_{t+\Delta t} = \{\bar{F}_b\}^l_t \quad \{U_b\}^0_{t+\Delta t} = \{U_b\}^l_t \qquad (27)$$

where the right superscript l = the last iteration of the previous time step. Accordingly, the system equations in (25) reduce to the following for the first iteration of each time step:

$$[\bar{K}]_{t+\Delta t}\{\Delta U_b\}^1 = \{P_b\}_{t+\Delta t} - \{\bar{F}_b\}^l_t \qquad (28)$$

Here, it should be noted that the effective stiffness matrix $[\bar{K}]_{t+\Delta t}$ remains constant within each incremental step, which need not be updated for each iteration.

For each iteration, the displacement increments $\{U_b\}^i$ can be solved from (25) and the total displacements of the bridge are

$$\{U_b\}^i_{t+\Delta t} = \{U_b\}^{i-1}_{t+\Delta t} + \{\Delta U_b\}^i \qquad (29)$$

The acceleration and velocity of the bridge can be computed from (21) with due account taken into the feature of iteration

$$\{\ddot{U}_b\}^i_{t+\Delta t} = a_0\{\Delta U_b\}^i - a_2\{\dot{U}_b\}^{i-1}_{t+\Delta t} - a_3\{\ddot{U}_b\}^{i-1}_{t+\Delta t} \qquad (30a)$$

$$\{\dot{U}_b\}^i_{t+\Delta t} = \{\dot{U}_b\}^{i-1}_{t+\Delta t} + a_6\{\ddot{U}_b\}^{i-1}_{t+\Delta t} + a_7\{\ddot{U}_b\}^i_{t+\Delta t} \qquad (30b)$$

Once the bridge displacements $\{U_b\}^i_{t+\Delta t}$ are made available, the element displacements $\{u_b\}^i_{t+\Delta t}$ can be computed thereby. It follows that the vehicle displacement increments $\{\Delta z\}^i$ can be computed from (13), by treating terms with subscript $t + \Delta t$ as those associated with the ith iterative step and terms with subscript t associated with the $(i-1)$st iterative step. The total responses for the sprung mass can be determined from (9) as

$$\{z\}^i_{t+\Delta t} = \{z\}^{i-1}_{t+\Delta t} + \{\Delta z\} \qquad (31a)$$

$$\{\ddot{z}\}^i_{t+\Delta t} = a_0\{\Delta z\}^i - a_2\{\dot{z}\}^{i-1}_{t+\Delta t} - a_3\{\ddot{z}\}^{i-1}_{t+\Delta t} \qquad (31b)$$

$$\{\dot{z}\}^i_{t+\Delta t} = \{\dot{z}\}^{i-1}_{t+\Delta t} + a_6\{\ddot{z}\}^{i-1}_{t+\Delta t} + a_7\{\ddot{z}\}^i_{t+\Delta t} \qquad (31c)$$

with the following initial condition: $\{z\}^0_{t+\Delta t} = \{z\}^l_t$.

PROCEDURE OF ITERATIVE ANALYSIS

The following is a summary of the procedure for performing the incremental-iterative analysis based on the modified Newton-Raphson algorithm:

1. Read in all the structure and vehicle data.
2. Start with time $t = 0$ and set up the following initial conditions: $\{F_b\}^l_0 = \{0\}$, $\{U_b\}^l_0 = \{\dot{U}_b\}^l_0 = \{\ddot{U}_b\}^l_0 = \{0\}$, and $\{z\}^l_0 = \{\dot{z}\}^l_0 = \{\ddot{z}\}^l_0 = \{0\}$. Calculate the mass matrices $[m_b]$ for all elements and assemble the structure mass matrix $[M_b]$. Select a proper time increment Δt for the Newmark integration scheme.
3. For the incremental step, let $t = t + \Delta t$ and $i = 1$. If t is larger than a specified value, stop the process. Otherwise, determine the acting position x_c of each wheel load, the rail irregularity r_c, and the shape vector $\{N_c\}$ for elements with wheel loads acting on them.
4. For elements carrying no wheel loads, calculate the element matrix $[k_b]$; and for elements in contact with wheel loads, calculate the modified element matrix $[\hat{k}_b]$ using (16*a*).
5. Assemble the bridge matrix $[K_b]$ and load vector $\{P_b\}_{t+\Delta t}$, using (20*a*) for the latter. The damping matrix

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$[C_b]$ is calculated by assuming it to be of the Rayleigh type. Compute the equivalent stiffness matrix $[\bar{K}_b]_{t+\Delta t}$ using (23). For the present problem, the system matrices $[\bar{K}_b]_{t+\Delta t}$ and $\{P_b\}_{t+\Delta t}$ are constant for each time step.

6. Determine the resistant force vector $\{\bar{F}_b\}^{i-1}_{t+\Delta t}$ using (26). For $i \geq 1$, check if the unbalanced forces $(\{P_b\}_{t+\Delta t} - \{\bar{F}_b\}^{i-1}_{t+\Delta t})$ are less than a given tolerance. If yes and if the contact condition $f_c \geq 0$ as given in (2) is satisfied, go to step 3 for the next increment.
7. Solve the displacement increments $\{\Delta U_b\}^i$ from the system equations in (25). Determine the vehicle displacement increments $\{\Delta z\}^i$ from (13), where terms with subscript $t + \Delta t$ should be interpreted as those for the ith iterative step and terms with subscript t for the $(i - 1)$st step.
8. Find the total displacements $\{U_b\}^i_{t+\Delta t}$ for the bridge from (29) and $\{z\}^i_{t+\Delta t}$ for the vehicle from (31*a*). Compute the displacement derivatives $\{\dot{U}_b\}^i_{t+\Delta t}$ and $\{\ddot{U}_b\}^i_{t+\Delta t}$ for the bridge using (30) and $\{\dot{z}\}^i_{t+\Delta t}$ and $\{\ddot{z}\}^i_{t+\Delta t}$ for each sprung mass using (31*b*) and (31*c*).
9. Let $i = i + 1$ and go to step 6 for the next iterative step.

Using the present procedure, the coupling effect between the bridge and the vehicle is considered through the condensed elements. Because of this, the number of cycles required for iteration is generally small, compared with approaches that do not rely on the condensation technique. The other advantage is that the vertical response of the sprung masses is obtained as part of the solution, which serves as a measure of the riding comfort of passengers, in addition to the response of the bridge.

DEFINITION FOR IMPACT FACTOR

In the design of bridge structures, the dynamic response resulting from the passage of moving vehicles has been considered indirectly by increasing the stresses caused by static live loads by an impact factor, which is defined as the ratio of the maximum dynamic response to the maximum static response of the bridge under the same load minus one. The impact factor adopted in this study is defined as follows:

$$I = \frac{R_d(x) - R_s(x)}{R_s(x)} \tag{32}$$

where $R_d(x)$ and $R_s(x)$ = maximum dynamic and static response, respectively, of the bridge at cross section x because of the passage of the moving load. Such a definition is more rational and computationally more convenient than the dynamic increment factor used by the American Association of State Highway and Transportation Officials (AASHTO), because both the maximum dynamic and the static responses are calculated for the same cross section. Such an advantage will become obvious when dealing with moving loads that appear as a series or as a random flow (Yang and Yau 1996).

NUMERICAL EXAMPLES

Three examples are prepared to verify the VBI element and the procedure of solution presented in this paper. First, the dynamic responses solved by the present method for a simple beam subjected to a moving sprung mass will be compared with those considering the contribution of the first mode of vibration. Second, by modeling a train as a sequence of moving lumped loads or lumped masses, the impact responses of bridges excited by the train will be investigated and compared with existing solutions. Finally, the dynamic responses of a cantilever subjected to different models of vehicles in motion will be studied. In each case, the beam is modeled as 10 elements.

Simple Beam Subjected to Moving Sprung Mass

As shown in Fig. 3, a simple beam of span length L = 25 m is subjected to a moving sprung mass. The following data are adopted: Young's modules E = 2.87 GPa, Poisson's ratio v = 0.2, moment of inertia I = 2.90 m^4, mass per unit length m = 2,303 kg/m, suspended mass M_v = 5,750 kg, suspension stiffness k_v = 1,595 kN/m, speed v = 100 km/h, frequency of the bridge ω_1 = 30.02 rad/s, frequency of the sprung mass system ω_v = 16.66 rad/s, and mass ratio M_v/mL = 0.1. By representing the deflection of the beam as $u_b = q_b(t)\sin(\pi x/L)$, the displacement of sprung mass as $q_v(t)$ and neglecting the effect of damping, the equations of motion for the vibration of the beam and the sprung mass moving at speed v can be given as (Biggs 1964)

$$\begin{Bmatrix} \ddot{q}_b \\ \ddot{q}_v \end{Bmatrix} + \begin{bmatrix} 2\omega_v^2 \frac{M_v}{mL} \sin^2 \frac{\pi vt}{L} + \omega_1^2 & -2\omega_v^2 \frac{M_v}{mL} \sin \frac{\pi vt}{L} \\ -\omega_v^2 \frac{M_v}{mL} \sin \frac{\pi vt}{L} & \omega_v^2 \end{bmatrix} \begin{Bmatrix} q_b \\ q_v \end{Bmatrix} = \begin{Bmatrix} -2 \frac{M_v g}{mL} \sin \frac{\pi vt}{L} \\ 0 \end{Bmatrix} \tag{33}$$

The dynamic responses of the midpoint displacement of the beam subjected to the moving load and the sprung mass have been plotted in Fig. 4. As can be seen, the response obtained by the present procedure using the VBI element and based on the sprung mass assumption agrees well with the single mode solution to (33).

From the response of vertical acceleration for the midpoint of the beam shown in Fig. 5, one observes that inclusion of the higher modes can result in oscillation of the acceleration response, which was neglected in the solution of (33). The responses of the deflection and vertical acceleration of the sprung mass have been plotted in Figs. 6 and 7, respectively. The differences in these two figures between the present solution and that of (31) can be attributed mainly to the omission of higher modes in the latter. A comparison of Figs. 5 and 7

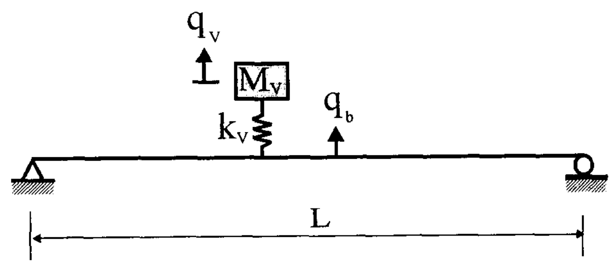

FIG. 3. Beam with Moving Sprung Mass

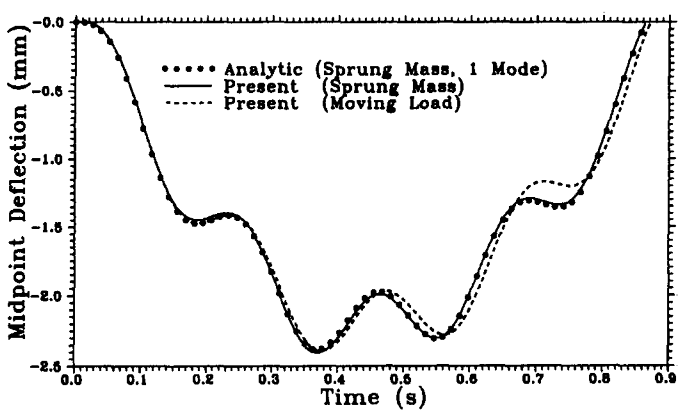

FIG. 4. Midpoint Vertical Deflection of Beam

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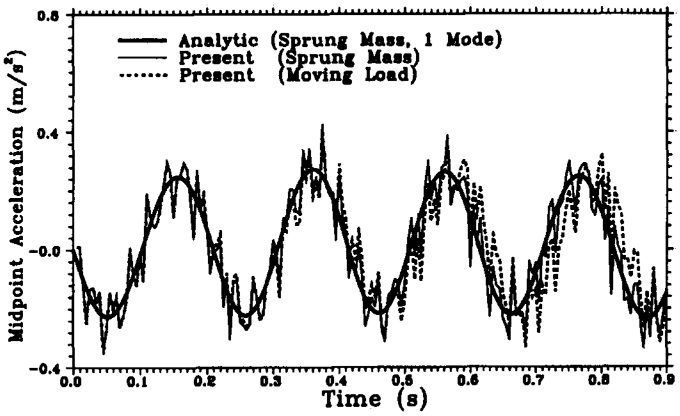

FIG. 5. Midpoint Vertical Acceleration of Beam

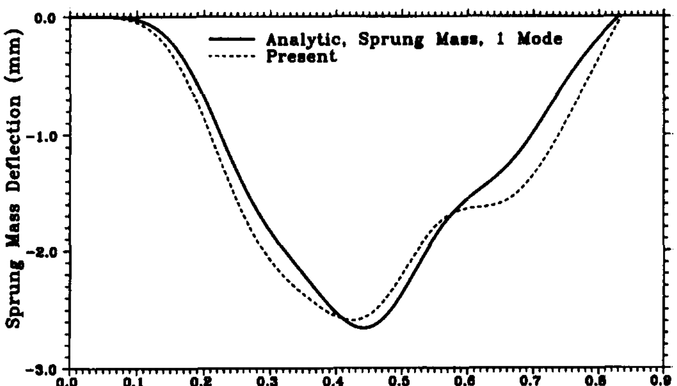

FIG. 6. Deflection of Sprung Mass

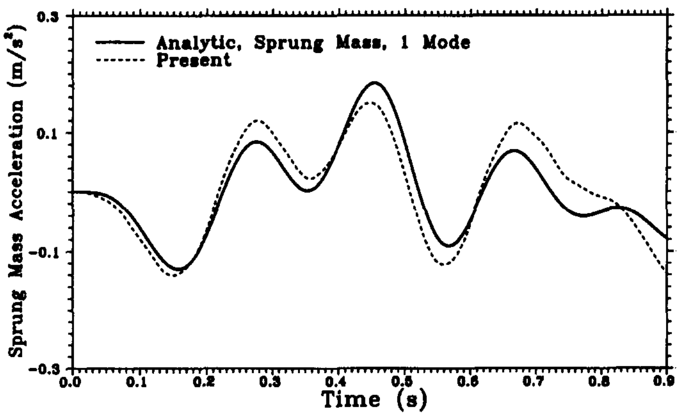

FIG. 7. Vertical Acceleration of Sprung Mass

indicates that the sprung mass is more sensitive than the beam to the omission of higher-order terms. It should be noted that the vertical acceleration of the sprung mass serves as a measure of the comfort level for passengers riding on the vehicle.

Simply Supported Beam Subjected to Moving Train

Consider a simply supported beam with the following properties: $L = 20$ m, $I = 3.81$ m^4, $E = 29.43$ GPa, $m = 34{,}088$ kg/m, and damping ratio = 2.5%. The train traveling over the bridge contains 10 bogies, which can be modeled as a sequence of moving lumped loads with regular nonuniform intervals as

$$|\leftarrow L_c \rightarrow|\leftarrow L_d \rightarrow|\cdots|\leftarrow L_d \rightarrow|\leftarrow L_c \rightarrow|\leftarrow L_d \rightarrow|\leftarrow L_c \rightarrow|$$

where $L_c = 18$ m; $L_d = 6$ m; and "|" represents the lumped load p. Assume the weight p of the lumped load to be $p = 215.6$ kN and the mass to be $M_v = 22{,}000$ kg and $m_w = 0$ kg. Two cases are considered herein. In the first case, by letting the suspension stiffness k_v, damping c_v, and ballast stiffness k_B equal zero, the moving load model can be obtained. In the second case, by letting the stiffnesses k_v and k_B equal a very large number, say, with $k_v = k_B = 9.0 \times 10^6$ kN/m, the moving mass model is obtained. The impact factors I calculated for the midpoint displacement of the beam subjected to the moving loads using the two models have been plotted in Fig. 8, against a nondimensional speed parameter S, defined as the ratio of the exciting frequency of the moving vehicle $\pi v/L$ to the fundamental frequency ω of the bridge. Also shown in the figure are the results based on the analytical work of Yang et al. (1996) by eigenfunction expansion. From this figure, it is obvious that the present solutions correlate very well with those of the analytical study. Besides, the moving mass model tends to reduce the frequency of vibration of the vehicle-bridge system, in the sense that the critical speed for the peak response to occur becomes smaller.

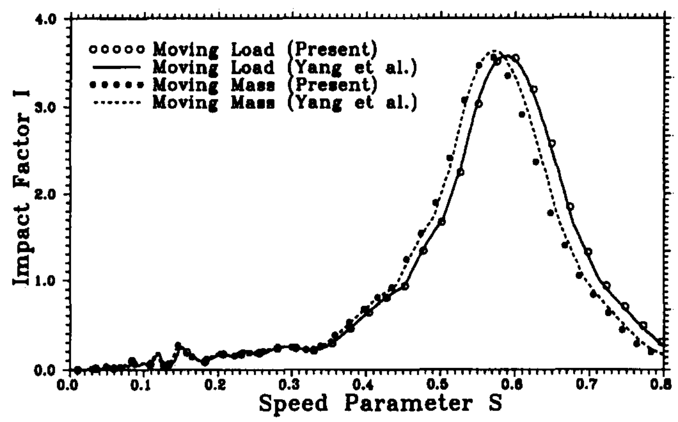

FIG. 8. Impact Response for Bridge Sustaining a Moving Train

Free-Fixed Beam with Various Models for Moving Vehicles

Fig. 9 shows a cantilever subjected to a moving lumped mass. The following data are assumed: length $L = 7.62$ m;

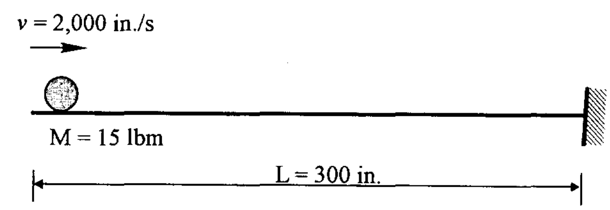

FIG. 9. Cantilever with Mass Moving at Constant Speed

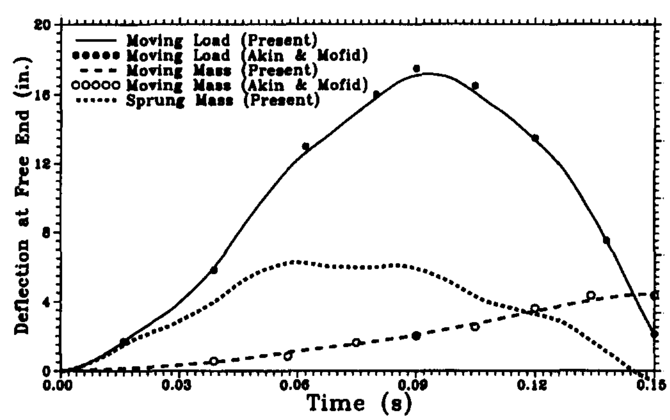

FIG. 10. Comparison of Results for Cantilever

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velocity $v = 50.8$ m/s; flexural rigidity $EI = 9.474 \times 10^6$ N/m^2; mass of the beam per unit length $m = 0.1192$ kg/m; lumped mass $M_v = 6.81$ kg; and moving load = 66.74 N. The results for the free-end deflection of the cantilever obtained by the moving mass and moving load models have been plotted in Fig. 10, along with those of Akin and Mofid (1989). As can be seen, good agreement has been achieved between the present solutions and those of Akin and Mofid (1989). In addition, by modeling the moving mass as a sprung mass with suspension stiffness $k_v = 10.43$ kN/m, assuming 0.1% damping for the cantilever, and letting the sprung mass be always in contact with the cantilever, the dynamic response solved for the free end has also been shown in Fig. 10. It can be seen that for most of the acting period of the vehicle on the cantilever, i.e., 0.15 s, the moving load model tends to produce the largest response, the sprung mass the second, and the moving mass the least.

CONCLUDING REMARKS

In this study, the equations of motion for the vehicle is first discretized using Newmark's finite difference formulas and then condensed to the bridge equations, considering the condition of contact between the vehicles and the bridge. The element thus derived is referred to as the VBI element, which has the same number of DOF as the parent element, while possessing the properties of symmetry and bandedness in element matrices. As such, conventional assembly process can be directly applied to forming the equations of motion for the entire vehicle-bridge system. Characterized by the fact that the VBI is duly taken into account, the derived element can be reliably used in computing the vehicle response, which serves as a measure of passengers' riding comfort, in addition to the bridge response. The applicability of the derived element has been demonstrated in the numerical examples. Besides, it is concluded that high modes of vibration of the bridge affect more significantly the response of the vehicle than the bridge. Compared with the moving load model, the use of the moving mass model tends to reduce the frequency of vibration of the entire vehicle-bridge system, as the critical speed at which the peak response occurs becomes smaller.

ACKNOWLEDGMENTS

The research reported here has been sponsored in part by the National Science Council of the Republic of China through Grants NSC85-2211-E002-002 and NSC86-2221-E002-074. Such financial aid is gratefully acknowledged.

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