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Optimal lot-sizing policy for a manufacturer with defective items in a supply chain with up-stream and down-stream trade credits ☆

Kuo-Ren Lou *, Lu Wang

Department of Management Sciences, Tamkang University, Tamsui, Taipei 251, Taiwan, ROC

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ABSTRACT

In this paper, we establish an economic production quantity model for a manufacturer (or wholesaler) with defective items when its supplier offers an up-stream trade credit M while it in turn provides its buyers (or retailers) a down-stream trade credit N. The proposed model is in a general framework that includes numerous previous models as special cases. In contrast to the traditional differential calculus approach, we use a simple-to-understand and easy-to-apply arithmetic–geometric inequality method to find the optimal solution. Furthermore, we provide some theoretical results to characterize the optimal solution. Finally, several numerical examples are presented to illustrate the proposed model and the optimal solution.

1. Introduction

In the classical inventory economic order quantity (EOQ) model, it is implicitly assumed that a buyer must pay for the purchased items immediately upon receiving the items. However, in practice, a seller frequently offers his/her buyers a delay of payment for settling the amount owed to him/her. Usually, there is no interest charge if the outstanding amount is paid within the permissible delay period. However, if the payment is not paid in full by the end of the permissible delay period, then interest is charged on the outstanding amount. The permissible delay in payment produces two benefits to the seller: (1) it attracts new buyers who may consider it to be a type of price reduction, and (2) it may be applied as an alternative to price discount because it does not provoke competitors to reduce their prices and thus introduce lasting price reductions. On the other hand, the policy of granting credit terms adds an additional dimension of default risk to the seller because the longer the permissible delay, the higher the default risk.

During the past two decades, many researchers have studied various inventory models with trade credit financing. Goyal (1985) was the first proponent for developing an economic order quantity (EOQ) model under the conditions of permissible delay in payments. Aggarwal and Jaggi (1995) extended Goyal's model to allow for deteriorating items. Then Jamal, Sarker, and Wang (1997) further generalized Aggarwal and Jaggi's model to allow for shortages. Teng (2002) amended Goyal's model by incorporating the fact that unit price is significantly higher than unit cost. Huang (2003) extended Goyal's model to a supply chain in which the supplier offers the wholesaler the permissible delay period M (i.e., the upstream trade credit), and the wholesaler in turn provides the trade credit period N (with $N < M$) to its retailers (i.e., the downstream trade credit). Teng and Goyal (2007) amended Huang's model by complementing his shortcomings. Liao (2008) extended Huang's model to analyze the impact of the two-level trade credit financing on an economic production quantity (EPQ) model for deteriorating items. Soni and Shah (2008) presented an inventory model with a stock-dependent demand under progressive payment scheme. Teng (2009a) established optimal ordering policies for a retailer who offers distinct trade credits to its good and bad credit customers. Teng and Chang (2009) developed optimal manufacturer's replenishment policies under two levels of trade credit financing. Kreng and Tan (2010) studied optimal replenishment decisions under two-level trade credit policy depending on the order quantity. Teng, Krommyda, Skouri, and Lou (2011) extended the model by Soni and Shah (2008) to allow for: a nonzero ending-inventory, a profit-maximization objective, a limited warehouse's capacity and deteriorating items. Many related articles can be found in Chang, Teng, and Goyal (2008), Chang, Teng, and Chern (2010), Goyal, Teng, and Chang (2007), Huang (2004, 2007), Huang and Hsu (2008), Ouyang, Chang, and Shum (2012), Shinn and Hwang (2003), Skouri, Konstantaras, Papachristos, and Teng (2011), Yang, Ouyang, Wu, and Yen (2011), Yang, Pan, Ouyang, and Teng (2012), and their references.

Recently, Kreng and Tan (2011) proposed the optimal replenishment decisions to the manufacturer (or wholesaler) with finite replenishment rate and imperfect product quality in a supply

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* Corresponding author.
E-mail address: 109880@mail.tku.edu.tw (K.-R. Lou).

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chain, in which the manufacturer receives an up-stream trade credit M from its supplier while provides its retailers a down-stream trade credit N with $N < M$. They then developed four theoretical results. However, they ignored the fact that the manufacturer offers his/her retailers a permissible delay period N, and, hence, the manufacturer receives sales revenue from N to $T + N$, not from 0 to T as shown in their model. In this paper, we not only extend their EPQ model to complement the above mentioned shortcomings but also relax some dispensable assumptions of $N < M$ and others. In our view the permissible delay period N offered by the manufacturer is independent of the permissible delay period M offered by the supplier. The manufacturer must choose an appropriate value of N based on the prevalent market conditions. In many situations manufacturers may be forced to offer a permissible delay period to their retailers while receiving no permissible delay period ($M = 0$) from their suppliers. As a result, the proposed model here is in a general framework that includes numerous previous models such as Goyal (1985), Teng (2002), Huang (2003), Teng and Goyal (2007), Liao (2008), Chang, Teng, and Chern (2010), and Kreng and Tan (2011) as special cases.

The rest of this paper is organized as follows. In Section 2, we first define the assumptions and notation used throughout the entire paper, and then establish the manufacturer annual total profit in a supply chain with both up-stream and down-stream trade credits. To maximize the annual total profit for the manufacturer, we use a simple-to-understand and easy-to-apply arithmetic–geometric inequality method to obtain the optimal solution, instead of the traditional differential calculus approach in Section 3. Furthermore, some theoretical results are established to obtain the optimal solution. In Section 4, several numerical examples are provided to illustrate the theoretical results and managerial insights. Finally, the conclusions and suggestions for the future research are given in Section 5.

2. Mathematical formulation

For simplicity, we use the following notation and assumptions throughout the entire paper. Then we establish the mathematical model.

2.1. Notation

D	the demand rate per year
P	the production rate per year, $P \geqslant D$
A	the ordering (or set-up) cost per order (lot)
ρ	$1 - \frac{D}{P} \geqslant 0$, the fraction of no production
c	the unit purchasing price
d	the screening cost per unit
p	the percentage of defective items (which consists of imperfect items and scrap items) in a lot
q	the percentage of scrap items in defective items
T	the replenishment cycle time in years
Q	the production lot size in units per cycle, which is $DT/(1-p)$ because $Q - pQ = DT$
s	the unit selling price of good items, $s \geqslant c$
v	the unit price of imperfect items, $v < c$
c_s	the unit disposal cost for scrap items
h	the unit stock holding cost per item per year excluding interest charges
I_e	the interest earned per dollar per year
I_k	the interest charged per dollar in stocks per year by the supplier
M	the manufacturer's trade credit period offered by a supplier in years
N	the customer's trade credit period offered by a manufacturer in years
$TP(T)$	the annual total profit, which is a function of T
T^*	the optimal replenishment cycle time of $TP(T)$
$TP(T^*)$	the optimal annual total profit.

2.2. Assumptions

1. The manufacturer's annual production rate P is higher than the annual demand rate D, which is known and constant. In order to satisfy the demand, it is necessary to assume that $(1-p)P > D$ (i.e., $p < 1 - D/P = \rho$).
2. In today's time-based competition, we may assume without loss of generality (WLOG) that shortages are not allowed.
3. During the credit period M, the manufacturer's sales revenue is deposited in an interest bearing account with the rate of I_e. At the end of the supplier's permissible delay M, the manufacturer keeps the profit from sales revenue, pays the rest to the supplier, and starts paying for the interest charges on the unpaid balance to the supplier with the rate of I_k.
4. Under modern automatic screening machines and electronic control systems, we may assume WLOG that a 100% screening process is sufficiently quick to inspect all items such that items are inspected faster than produced. In short, the production period and the screening process are expected to end simultaneously.
5. Each production lot Q has defective rate of p. Those pQ defective items in each cycle comprise $(1-q)\,pQ$ imperfect (or reworkable) items and $q\,pQ$ scrap (or unworkable) items. The scrap items must be removed from inventory at the end of the screening process at a disposal cost c_s per unit. Re-workable items are sold in a single batch at a discount price v per unit at the end of the cycle.
6. Time horizon is infinite.

Now, we are ready to establish the EPQ model with defective items under a supply chain with up-stream and down-stream trade credits.

2.3. The mathematical model

The manufacturer's annual total profit consists of the following elements:

1. Procurement cost per year = $\frac{A+cQ}{T} = \frac{A}{T} + \frac{cD}{1-p}$,
2. Screening cost per year = $\frac{dQ}{T} = \frac{dD}{1-p}$,
3. Disposal cost per year = $\frac{c_s qpQ}{T} = c_s qp\frac{D}{1-p}$,
4. Holding cost per year $= \frac{h}{T}\Big\{(P-D)\frac{Q^2}{2P^2} + \big[(P-D)\frac{Q}{P} - qpQ + (1-q)pQ\big]\big(T - \frac{Q}{P}\big)/2\Big\} = \frac{hD^2T}{2(1-p)^2}\left\{\frac{\rho}{P} + [\rho - pq + (1-q)p]\left(\frac{1-p}{D} - \frac{1}{P}\right)\right\} \equiv kDT$,
5. Revenue received from good items per year = sD, and
6. Revenue received from repaired items per year = $\frac{v(1-q)pQ}{T} = \frac{v(1-q)pD}{1-p}$.

Since $p < 1 - D/P = \rho$ from Assumption 1, we know that the constant $k = \frac{hD}{2(1-p)^2}\left\{\frac{\rho}{P} + [\rho - pq + (1-q)p]\left(\frac{1-p}{D} - \frac{1}{P}\right)\right\}$ is positive.

In addition, the manufacturer's interests payable and charged are derived as follows. According to the values of N and M, there are two possible cases: (1) $N < M$, and (2) $N \geqslant M$. Let us discuss the case in which $N < M$ first, and then the other case.

2.3.1. Case 1: $N < M$

The manufacturer buys all parts at time zero and must pay the purchasing cost at time M. Based on the values of M (i.e., the time at which the manufacturer must pay the supplier to avoid interest charge) and $T+N$ (i.e., the time at which the manufacturer receives the payment from the last customer), the manufacturer has only two possible sub-cases to compute the capital opportunity costs. If $T+N \geqslant M$, then the manufacturer cannot receive the last payment by M, and thus must finance all items sold after $M-N$ (i.e., the manufacturer receives those payments after M, and finances them at time M). On the other hand, if $T+N < M$, then the manufacturer can receive all payments by M, and thus there is no interest charged involved. However, Kreng and Tan (2011) ignored the fact that the manufacturer receives revenue from N to $T+N$, not from 0 to T. Consequently, they misclassified the interests payable and earned into the following four cases: (1) $T \leqslant N \leqslant M$, (2) $N \leqslant T \leqslant M$, (3) $N \leqslant M \leqslant T \leqslant PM(1-p)/D$, and (4) $N \leqslant M \leqslant PM(1-p)/D \leqslant T$. Now, let us use proper classification to discuss the detailed formulation in each sub-case.

2.3.2. Sub-case 1-1: $M \leqslant T+N$

In this sub-case, the manufacturer pays off all units sold by $M-N$ at time M, keeps the profits, and starts paying for the interest payable on the items sold after $M-N$. The graphical representation of this sub-case is shown in Fig. 1. However, the manufacturer cannot payoff the supplier by M because its supplier credit period M is shorter than its customer last payment time $T+N$. Hence, the manufacturer must finance all items sold after time $M-N$ at an interest charged I_k per dollar per year. The manufacturer makes DT good items and pQ defective items per cycle. Based on Assumption 5, we get the interest payable on pQ defective items from time M through T per cycle is

$$\begin{cases} cI_k(pQ)(T-M) = cI_k p\frac{DT}{1-p}(T-M) \text{ if } T \geq M, \\ 0 \text{ if } T < M. \end{cases} \tag{1}$$

The interest payable on DT good items per cycle is cI_k multiplied by the area of the triangle EBC as shown in Fig. 1 given by

$$cI_k\left[\frac{D(T+N-M)^2}{2}\right]. \tag{2}$$

Therefore, if $M \leqslant T+N$ then the interest payable per cycle is

$$\begin{cases} cI_k D\left[\frac{p}{1-p}T(T-M) + \frac{(T+N-M)^2}{2}\right] \text{ if } T \geqslant M, \\ cI_k D\frac{(T+N-M)^2}{2} \text{ if } T < M. \end{cases} \tag{3}$$

Notice that Kreng and Tan (2011) did not recognize that the last customer buys good items at time T, and pays the manufacturer at time $T+N$ due to its customer trade credit period N. Consequently, they obtained the inappropriate interest payable per cycle as

$$cI_k\left[\frac{D(T-M)^2}{2}\right],$$

and

$$cI_k\left\{(P-D)\frac{Q^2}{2P^2} + \left[(P-D)\frac{Q}{P} - qpQ + (1-q)pQ\right]\left(T - \frac{Q}{P}\right)\Big/2 - (P-D)M^2/2\right\}, \tag{4}$$

which are different from (3).

On the other hand, the manufacturer starts selling good items at time 0, but getting the money at time N. Consequently, the manufacturer accumulates revenue in an account that earns I_e per dollar per year starting from N through M. Therefore, the interest earned on good items per cycle is sI_e multiplied by the area of the triangle NME as shown in Fig. 1. In addition, if $T < M$, then imperfect $(1-q)pQ$ items is sold at T and earned interest from time T through M. Hence, the interest earned per cycle is given by

$$\begin{cases} sI_e D\frac{(M-N)^2}{2} \text{ if } T \geqslant M, \\ sI_e D\frac{(M-N)^2}{2} + vI_e(1-q)pQ(M-T) \text{ if } T < M. \end{cases} \tag{5}$$

Notice that Kreng and Tan (2011) ignored the fact that the manufacturer starts getting the revenue at time N, not at time 0 as shown in their model. As a result, they obtained the following inappropriate interest earned as

$$\begin{cases} \frac{sI_e D(M^2-N^2)}{2} \text{ if } T \geq M, \\ \frac{sI_e D(2MT-T^2-N^2)}{2} + vI_e(1-q)pQ(M-T) \text{ if } T < M. \end{cases} \tag{6}$$

Dividing (3) and (5) by T, and simplifying terms, we yield the annual total profit for the manufacturer as

$$TP_{1-1a}(T) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + cI_k\left(\frac{M}{1-p} - N\right)\right]D - \left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]DT - \frac{2A - (sI_e - cI_k)D(M-N)^2}{2T} \quad \text{if } T \geq M. \tag{7}$$

$$TP_{1-1b}(T) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + cI_k(M-N) + vI_e(1-q)M\frac{p}{1-p}\right]D - \left[k + \frac{cI_k}{2} + vI_e(1-q)\frac{p}{1-p}\right] \times DT - \frac{2A - (sI_e - cI_k)D(M-N)^2}{2T} \quad \text{if } T < M. \tag{8}$$

2.3.3. Sub-case 1-2: $M > T+N$

In this sub-case, the manufacturer receives the total revenue at time $T+N$, and is able to pay the supplier the total purchase cost at time M. Since the customer last payment time $T+N$ is shorter than the supplier credit period M, the manufacturer faces no interest charged. The interest earned on imperfect $(1-q)pQ$ items from time T through M is $vI_e(1-q)pQ(M-T)$ per cycle. However, the interest earned on DT good items per cycle is sI_e multiplied by the area of the trapezoid on the interval $[N,M]$ as shown in Fig. 2.. As a result, the interest earned per year is given by

$$\frac{vI_e}{T}(1-q)pQ(M-T) + \frac{sI_e}{T}\left[\frac{DT^2}{2} + DT(M-T-N)\right]. \tag{9}$$

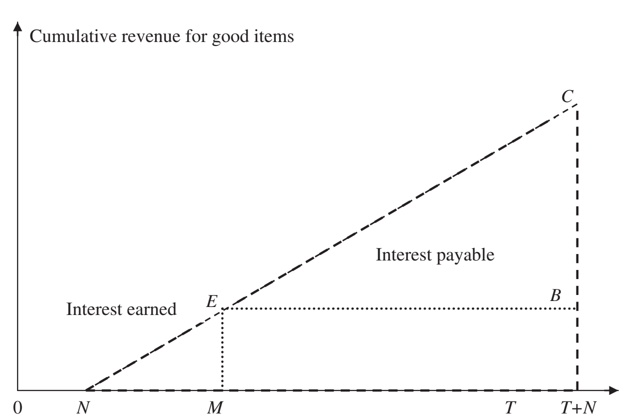

Fig. 1. $M \leqslant T+N$.

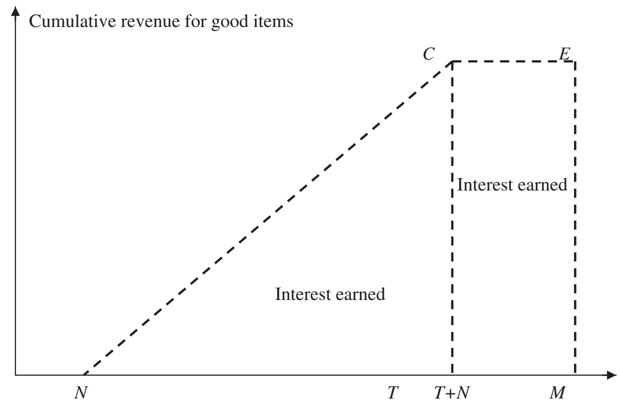

Fig. 2. $M > T + N$.

Using (9) and simplifying terms, we obtain the annual total profit for the manufacturer as

$$TP_{1-2}(T) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + sI_e(M-N) + vI_e(1-q)M\frac{p}{1-p}\right]D - \left[k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}\right]DT - \frac{A}{T}. \tag{10}$$

2.3.4. Case 2: $N \geqslant M$

In this case, the customer's trade credit period N is equal to or larger than the supplier credit period M. Consequently, there is no interest earned on DT good items. In addition, the manufacturer must finance all DT good items at time M at an interest charged I_k per dollar per year, and start to payoff the loan after time N. Hence, the interest payable on good items per cycle is cI_k multiplied by the area of the trapezoid on the interval $[M, T+N]$, as shown in Fig. 3. Therefore, the interest payable on DT good items per year is given by

$$\frac{cI_k}{T}\left[(N-M)DT + \frac{DT^2}{2}\right]. \tag{11}$$

The interest payable on pQ defective items per cycle is the same as in (1). Therefore, if $N \geqslant M$ then the interest payable per year is

$$\begin{cases} cI_k D\left[\frac{p}{1-p}(T-M) + (N-M) + \frac{T}{2}\right] & \text{if } T \geq M, \\ cI_k D(N - M + \frac{T}{2}) & \text{if } T < M. \end{cases} \tag{12}$$

However, if $T < M$, then imperfect $(1-q)pQ$ items is sold at T and starts to earn interest from time T through M. Hence, the interest earned on imperfect items per year is given by

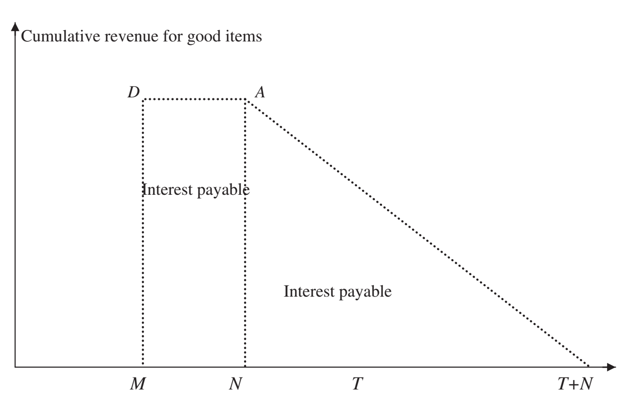

Fig. 3. $N \geqslant M$.

$$\begin{cases} 0 & \text{if } T \geq M, \\ vI_e(1-q)pQ(M-T)/T & \text{if } T < M. \end{cases} \tag{13}$$

Therefore, we obtain the annual total profit for the manufacturer as

$$TP_{2a}(T) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + cI_k\left(\frac{M}{1-p} - N\right)\right]D - \left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]DT - \frac{A}{T} \quad \text{if } T \geq M. \tag{14}$$

$$TP_{2b}(T) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + cI_k(M-N) + vI_e(1-q)M\frac{p}{1-p}\right]D - \left[k + \frac{cI_k}{2} + vI_e(1-q)\frac{p}{1-p}\right]DT - \frac{A}{T} \quad \text{if } T < M. \tag{15}$$

3. Optimal solution

For simplicity, we apply an arithmetic–geometric inequality method to obtain the optimal solution, such as in Cárdenas-Barrón (2011) and Teng (2009b). As we know, the arithmetic mean is always greater than or equal to the geometric mean. In short, for any two real positive numbers, say a and b, we have

$$\frac{a+b}{2} \geq \sqrt{ab}. \tag{16}$$

The equation holds only if $a = b$. For Sub-case 1-1a: $M \leqslant T + N$ and $M \leqslant T$, to maximize $TP_{1-1a}(T)$ in (7) is equivalent to minimize $\left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]DT + \frac{2A - (sI_e - cI_k)D(M-N)^2}{2T}$.

If $2A - (sI_e - cI_p)D(M-N)^2 \leqslant 0$, then it is obvious that $T^*_{1-1a} = 0$ and $TP_{1-1a}(T^*_{1-1a}) = 0$. On the other hand, if $2A - (sI_e - cI_p)D(M-N)^2 > 0$, and $\left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]DT = \frac{2A - (sI_e - cI_k)D(M-N)^2}{2T} > 0$, then by applying the above mentioned arithmetic–geometric inequality in (16), we know that the optimal replenishment cycle time is

$$T^*_{1-1a} = \sqrt{\frac{2A - (sI_e - cI_k)D(M-N)^2}{2D\left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]}}, \tag{17}$$

and the optimal annual total profit is

$$TP_{1-1a}(T^*_{1-1a}) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + cI_k\left(\frac{M}{1-p} - N\right)\right]D - \sqrt{2D\left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]\left[2A - (sI_e - cI_k)D(M-N)^2\right]}. \tag{18}$$

For Sub-case 1-1b: $M \leqslant T + N$ and $M > T$, by using the similar argument, we obtain the optimal replenishment cycle time as follows. If $2A - (sI_e - cI_p)D(M-N)^2 \leqslant 0$, then $T^*_{1-1b} = 0$ and $TP_{1-1b}(T^*_{1-1b}) = 0$. If $2A - (sI_e - cI_p)D(M-N)^2 > 0$, then

$$T^*_{1-1b} = \sqrt{\frac{2A - (sI_e - cI_k)D(M-N)^2}{2D\left[k + \frac{cI_k}{2} + vI_e(1-q)\frac{p}{1-p}\right]}}, \tag{19}$$

and the optimal annual total profit as

$$TP_{1-1b}(T^*_{1-1b}) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + cI_k(M-N) + vI_e(1-q)M\frac{p}{1-p}\right]D - \sqrt{2D\left[k + \frac{cI_k}{2} + vI_e(1-q)\frac{p}{1-p}\right]\left[2A - (sI_e - cI_k)D(M-N)^2\right]}. \tag{20}$$

Likewise, for Sub-case 1-2: $M > T + N$, we get the optimal replenishment cycle time as

$$T^*_{1-2} = \sqrt{\frac{A}{k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}}}, \tag{21}$$

and the optimal annual total profit as

$$TP_{1-2}(T^*_{1-2}) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + sI_e(M-N) + vI_e(1-q)M\frac{p}{1-p}\right]D - 2\sqrt{AD\left[k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}\right]}. \tag{22}$$

For Case 2a: $N \geqslant M$ and $T \geqslant M$, we know that the optimal replenishment cycle time is

$$T^*_{2a} = \sqrt{\frac{A}{D\left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]}} \tag{23}$$

and the optimal annual total profit is

$$TP_{2a}(T^*_{2a}) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} + cI_k\left(\frac{M}{1-p} - N\right)\right]D - 2\sqrt{AD\left[k + cI_k\left(\frac{p}{1-p} + \frac{1}{2}\right)\right]}. \tag{24}$$

Similarly, for Case 2b: $N \geqslant M$ and $T < M$, the optimal replenishment cycle time is

$$T^*_{2b} = \sqrt{\frac{A}{D\left[k + \frac{cI_k}{2} + vI_e(1-q)\frac{p}{1-p}\right]}}, \tag{25}$$

and the optimal annual total profit is

$$TP_{2b}(T^*_{2b}) = \left[s + \frac{v(1-q)p - (c+d+c_s qp)}{1-p} - cI_k(N-M) + vI_e(1-q)M\frac{p}{1-p}\right]D - 2\sqrt{AD\left[k + \frac{cI_k}{2} + vI_e(1-q)\frac{p}{1-p}\right]}. \tag{26}$$

For simplicity, let

$$\Delta = A - \left[k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}\right]D(M-N)^2. \tag{27}$$

Thus, we have the following theoretical results.

Theorem 1. *For $N \leqslant M$, we obtain:*

1. *If $\Delta \geqslant 0$, then the optimal replenishment cycle time is either T^*_{1-1a} as in (17) or T^*_{1-1b} as in (19).*
2. *If $\Delta < 0$, then the optimal replenishment cycle time is either T^*_{1-1a} as in (17) or T^*_{1-2} as in (21).*

Proof. For $N \leqslant M$, there are 3 possible distinct cases: (Case 1-1*a*) $M \leqslant T + N$ and $M \leqslant T$, (Case 1-1*b*) $M \leqslant T + N$ and $M > T$, and (Case 1.2) $M > T + N$. Since $M \leqslant T^*_{1-1b} + N$ in Case 1-1*b*, we know from (19) that if and only if

$$T^*_{1-1b} = \sqrt{\frac{2A - (sI_e - cI_k)D(M-N)^2}{2D\left[k + \frac{cI_k}{2} + vI_e(1-q)\frac{p}{1-p}\right]}} \geqslant M - N,$$

then

$$\Delta = A - \left[k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}\right]D(M-N)^2 \geq 0. \tag{28}$$

Similarly, since $M > T^*_{1-2} + N$ in Case 1.2, we know from (21) that if and only if

$$T^*_{1-2} = \sqrt{\frac{A}{k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}}} < M - N$$

,then

$$\Delta = A - \left[k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}\right]D(M-N)^2 < 0. \tag{29}$$

The proof immediately follows by (28) and (29).

4. Numerical examples

To illustrate the above results, we provide the following numerical examples.

Example 1. Let D = 1000 unit/year, P = 2000 unit/year, A = \$100/order, c = \$20/unit, d = \$1/unit, v = \$10/unit, s = \$60/unit, c_s = \$5/unit, h = \$5/unit/year, I_k = 0.05/year, I_e = 0.01/year, p = 0.1, q = 0.5, M = 0.25 year, and N = 0.1 year. Since M = 0.25 > N = 0.1, we have three possible solutions. We first calculate $k = \frac{hD}{2(1-p)^2}\left\{\frac{\rho}{P} + [\rho - pq + (1-q)p]\left(\frac{1-p}{D} - \frac{1}{P}\right)\right\} \approx 1.39$, and then find T^*_{1-1a} in (17), T^*_{1-1b} in (19), and T^*_{1-2} in (21) separately as follows:

$T^*_{1-1a} \approx 0.2286$ satisfies $M \leqslant T^*_{1-1a} + N$, but violates $T^*_{1-1a} > M$,
$T^*_{1-1b} \approx 0.2349$ satisfies $M \leqslant T^*_{1-1b} + N$ and $T^*_{1-1b} < M$, and
$T^*_{1-2} \approx 7.6822$ violates $M > T^*_{1-2} + N$, respectively.

Consequently, the optimal replenishment time is $T^*_{1-1b} \approx 0.2349$, and the optimal annual total profit from (8), (20) is $TP_{1-1b}(T^*_{1-1b}) = 36,626.40$. To validate the correctness of Theorem 1, we check

$$\Delta = A - \left[k + \frac{sI_e}{2} + vI_e(1-q)\frac{p}{1-p}\right]D(M-N)^2 = 100 - 38.1250 = 61.8750 > 0,$$

which satisfies the condition in Part 1 of Theorem 1. The optimal solution is the same as that stated in Part 1 of Theorem 1. Hence, Theorem 1 is true in this example.

Example 2. For simplicity, we use the same data as in Example 1 except M = 0.2 years. Substituting $k \approx 1.39$ and the values of corresponding parameters into (17), (19), and (21), we have the following results:

$T^*_{1-1a} \approx 0.2258$ satisfies $M \leqslant T^*_{1-1a} + N$ and $T^*_{1-1a} > M$,
$T^*_{1-1b} \approx 0.2320$ satisfies $M \leqslant T^*_{1-1b} + N$, but violates $T^*_{1-1b} < M$, and
$T^*_{1-2} \approx 7.6822$ violates $M > T^*_{1-2} + N$, respectively.

Consequently, the optimal replenishment time is $T^*_{1-1a} \approx 0.2258$, and the optimal annual total profit from (7), (18) is $TP_{1-1b}(T^*_{1-1b}) = 36,591.97$.

Example 3. For convenience, we use the same data as in Example 1 except M = 0.1 year, and N = 0.2 year. In this example, N = 0.2 > M = 0.1. There are two possible cases as shown in (23) and (25). Substituting $k \approx 1.39$ and the corresponding values of the parameters into (23) and (25), we get the following results:

$T^*_{2a} \approx 0.2235$ satisfies $T^*_{2a} > M$, and
$T^*_{2b} \approx 0.2297$ violates $T^*_{2b} < M$, respectively.

Therefore, the optimal replenishment time is $T^*_{2a} \approx 0.2235$.

Table 1
Sensitivity analysis on parameters.

Parameter	T^*	$TP(T^*)$
$p = 0.1$	0.2349	36,626.40
$p = 0.2$	0.2244	34,039.00
$p = 0.3$	0.2128	30,713.50
$q = 0.5$	0.2349	36,626.40
$q = 0.4$	0.2333	36,790.40
$q = 0.3$	0.2317	36,954.50
$v = 10$	0.2349	36,626.40
$v = 14$	0.3654	36,848.90
$v = 18$	0.5216	37,071.40
$c_s = 5$	0.2349	36,626.40
$c_s = 7$	0.2349	36,515.20
$c_s = 9$	0.2349	36,404.10

Example 4. Using the same data as those in Example 1, we study the sensitivity analysis on the optimal solution with respect to p, q, v, and c_s in appropriate unit. The computational results are shown in Table 1.

The sensitivity analysis reveals that: (1) a higher value of p causes lower values of T^*, and $TP(T^*)$, (2) a higher value of q causes higher values of T^* while a lower value of $TP(T^*)$, (3) a higher value of v causes slightly lower values of T^*, but a higher value of $TP(T^*)$, and (4) a higher value of c_s causes a lower value of $TP(T^*)$, meanwhile T^* remains unchanged.

5. Conclusion

In this paper, we have proposed an EPQ model with up-stream and down-stream trade credits in a general framework that includes numerous previous models such as Goyal (1985), Teng (2002), Huang (2003), Teng and Goyal (2007), Liao (2008), Chang et al. (2010), and Kreng and Tan (2011) as special cases. In order to obtain the explicit closed-form solution without using traditional differential calculus, we have used a simple arithmetic–geometric inequality method to obtain the optimal solution for the manufacturer. In addition, we have established some theoretical results to characterize the optimal solution. Finally, we have provided several numerical examples to illustrate the proposed model and its optimal solution.

The research presented in this paper can be extended in several ways. For instance, we may generalize the constant demand rate to any non-decreasing demand rate. Also, we could extend the model to allow for shortages. Finally, we can consider the effect of inflation rates on the economic order quantity.

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