SIAM J. MATH. ANAL.
Vol. 47, No. 1, pp. 614–625

NO TOUCHDOWN AT ZERO POINTS OF THE PERMITTIVITY PROFILE FOR THE MEMS PROBLEM*

JONG-SHENQ GUO† AND PHILIPPE SOUPLET‡

Abstract. We study the quenching behavior for a semilinear heat equation arising in models of micro-electro-mechanical systems (MEMS). The problem involves a source term with a spatially dependent potential, given by the dielectric permittivity profile, and quenching corresponds to a touchdown phenomenon. It is well known that quenching does occur. We prove that touchdown cannot occur at zero points of the permittivity profile. In particular, we remove the assumption of compactness of the touchdown set, made in all previous work on the subject and whose validity is unknown in most typical cases. This answers affirmatively a conjecture made in [W. Guo, Z. Pan, and M. J. Ward, *SIAM J. Appl. Math.*, 66 (2005), pp. 309–338] on the basis of numerical evidence. The result crucially depends on a new type I estimate of the quenching rate, that we establish. In addition we obtain some sufficient conditions for compactness of the touchdown set, without a convexity assumption on the domain. These results may be of some qualitative importance in applications to MEMS optimal design, especially for devices such as microvalves.

Key words. micro-electro-mechanical systems, touchdown, quenching points, permittivity profile, type I estimate, compactness

AMS subject classifications. Primary, 35K55, 35B40, 35B44; Secondary, 74K15, 74F15

DOI. 10.1137/140970070

1. Introduction and main results.

In this paper, we consider the problem

$$u_t = \Delta u + f(x)(1-u)^{-p}, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(x,0) = 0, \quad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, of class $C^{2+\nu}$ for some $\nu > 0$,

$$p > 0 \text{ and } f \text{ is a nonnegative, Hölder continuous function on } \overline{\Omega}, \text{ with } f \not\equiv 0. \tag{1.4}$$

A typical case of interest is the following:

$$u_t = \Delta u + \lambda |x|^m (1-u)^{-2}, \quad x \in \Omega,\ t > 0, \tag{1.5}$$

$$u = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.6}$$

$$u(x,0) = 0, \quad x \in \Omega, \tag{1.7}$$

where m, λ are positive constants. This problem arises in the study of modeling the dynamic deflection of an elastic membrane inside a micro-electro-mechanical system (MEMS). The full model is

$$\epsilon u_{tt} + u_t = \Delta u + \frac{\lambda g(x)}{(1-u)^2\left(1 + \alpha \int_\Omega \frac{1}{1-u} dx\right)^2}, \quad x \in \Omega,\ t > 0, \tag{1.8}$$

*Received by the editors May 22, 2014; accepted for publication (in revised form) November 20, 2014; published electronically February 3, 2015.
http://www.siam.org/journals/sima/47-1/97007.html

†Department of Mathematics, Tamkang University, Tamsui, New Taipei City 25137, Taiwan (jsguo@mail.tku.edu.tw). The research of this author was partially supported by the National Science Council of Taiwan under the grant NSC 102-2115-M-032-003-MY3.

‡Sorbonne Paris Cité, Laboratoire Analyse, Géométrie et Applications, CNRS (UMR 7539), Université Paris 13, 93430 Villetaneuse, France (souplet@math.univ-paris13.fr).

where ϵ is the ratio of the interaction due to the inertial and damping terms, λ is proportional to the applied voltage, u is the deflection of the membrane (the natural physical dimension being thus $n = 2$). The function $g(x)$, called the permittivity profile, represents varying dielectric properties of the membrane. One of the physically suggested dielectric profiles is the power-law profile $g(x) = |x|^m$ with $m > 0$. The integral in (1.8) arises due to the fact that the device is embedded in an electrical circuit with a capacitor of fixed capacitance. The parameter α denotes the ratio of this fixed capacitance to a reference capacitance of the device. As for the initial condition (1.3), it means that the membrane has initially no deflection, the voltage being switched on at $t = 0$. For the details of background and derivation of this model, we refer the reader to [22, 23, 6].

The case when $\epsilon = 0$ is studied in [14, 15, 13] for $\alpha > 0$ and f a constant. We shall here concentrate on the case when $\epsilon = \alpha = 0$ (so that there is no capacitor in the circuit) and f is nonconstant. It has been studied extensively in past years; see, e.g., the works [11, 18, 10, 16, 17, 20, 26]. For the study of stationary solutions, we refer to [21, 12, 1, 9, 2, 3, 19, 20, 26].

By the standard parabolic theory, there exists a unique classical solution of (1.1)–(1.3) in a short time interval. Also, by the strong maximum principle, we have $u > 0$ in Ω for $t > 0$. Moreover, the solution u of (1.1)–(1.3) can be continued as long as $\max_{x\in\overline{\Omega}} u(x,t) < 1$. We shall let $[0, T)$ be the maximal existence time interval of u, where $T \le \infty$. If $T < \infty$, then quenching occurs in finite time, i.e.,

$$\limsup_{t\to T^-} \left\{\max_{x\in\overline{\Omega}} u(x,t)\right\} = 1.$$

It is well known (see, e.g., [4, 20] and the references therein) that the solution of (1.1)–(1.3) quenches in finite time when λ is sufficiently large. A point $x = x_0$ is a *quenching point* if there exists a sequence $\{(x_n, t_n)\}$ in $\overline{\Omega} \times (0, T)$ such that

$$x_n \to x_0,\ t_n \uparrow T, \text{ and } u(x_n, t_n) \to 1 \text{ as } n \to \infty.$$

The set of all quenching points is called the quenching set, denoted by $\mathcal{Q}$. In the context of MEMS, quenching corresponds to a touchdown phenomenon.

Note that in the typical case of (1.5), there is no source at $x = 0$ due to the spatially dependent coefficient $|x|^m$. A long-standing open problem, even in one space dimension, is to determine whether or not $x = 0$ is a quenching point. More generally, for problem (1.1)–(1.3), the question is whether a point x_0 such that $f(x_0) = 0$ can be a quenching point. In [10, 16], under the assumption that the quenching set is a compact subset of Ω, it is shown that x_0 is not a quenching point if $f(x_0) = 0$. On the other hand, the compactness assumption was proved in [16] by adapting a moving plane argument from [7, 11] when f is constant or, more generally, when f is nonincreasing as one approaches the boundary. However, for the typical problem (1.5)–(1.7) it is unknown whether the quenching set is compact. Actually, supported by numerical evidence provided in [18], the following conjecture was made (see [10, 16, 4]).

Conjecture. *The point $x = 0$ is not a quenching point for problem* (1.5)–(1.7).

In the present paper, we give an affirmative answer to this conjecture, as well as for the case of general f, in any space dimension. Our main result is the following.

THEOREM 1.1. *Assume* (1.4) *and let the solution u of problem* (1.1)–(1.3) *be such that $T < \infty$. If $x_0 \in \Omega$ is such that $f(x_0) = 0$, then x_0 is not a quenching point.*

In particular, as a special case, we have that 0 is not a quenching point for problem (1.5)–(1.7). Actually, we have been able to answer this question *independently* of the compactness issue of the quenching set. In fact, as a key step—of independent interest—to the proof of Theorem 1.1, we prove the following estimate, which in particular guarantees that the quenching rate is of *type I* on any compact subset of Ω. In what follows, we denote

$$\delta(x) = \mathrm{dist}(x, \partial\Omega), \quad x \in \overline{\Omega},$$

the function distance to the boundary.

THEOREM 1.2. *Assume* (1.4) *and let the solution* u *of problem* (1.1)–(1.3) *be such that* $T < \infty$. *Then there exists a constant* $\gamma > 0$ *(independent of* x, t*) such that*

$$1 - u(x,t) \geq \gamma\, \delta(x)\, (T-t)^{1/(p+1)}, \quad x \in \Omega,\ 0 < t < T. \tag{1.9}$$

Theorem 1.2 will be proved via a nontrivial modification of the Friedman–McLeod method ([7]; see also [11]). Once Theorem 1.2 is proved, Theorem 1.1 will be deduced by constructing a suitable local supersolution.

The compactness of the quenching set remains an open question. In particular, we do not know if Theorem 1.1 remains true if $f(x_0) = 0$ with $x_0 \in \partial\Omega$ (in other words, can a zero *boundary* point of the permittivity profile be a quenching point?). As mentioned above, this cannot occur if we assume in addition that f is nonincreasing as one approaches the boundary. Actually, as a consequence of Theorem 1.2 and of suitable comparison arguments, we have been able to obtain two further criteria for the quenching set to be compact. We note that, unlike in the aforementioned criterion, we do not require any convexity of the domain Ω.

THEOREM 1.3. *Assume* (1.4) *and let the solution* u *of problem* (1.1)–(1.3) *be such that* $T < \infty$. *Assume either*

$$0 < p < 1 \tag{1.10}$$

or

$$f(x) = o\big(\delta^{p+1}(x)\big) \quad \text{as } \delta(x) \to 0. \tag{1.11}$$

Then quenching does not occur near the boundary, i.e., $\mathcal{Q} \subset \Omega$.

Going back to MEMS modeling, it seems that Theorem 1.1 and the case of (1.11) in Theorem 1.3 may be of some importance in applications, at least from the qualitative point of view (see [18, 4] for more details). This is especially true for particular devices of MEMS type such as microvalves, where the touchdown behavior is explicitly exploited, since the touchdown or quenching set then corresponds to the lid or closing area of the valve. As a consequence of our results, we see that the latter has to be part of the positive set of the function f. The choice of f, through an appropriate repartition of the dielectric coating, can thus be used in the optimal design of the microvalve. In this respect, it would be desirable to gain further information about the structure of the quenching set, but this seems a difficult mathematical problem for nonconstant f, even in one space dimension.

Remark. We point out that Theorems 1.1–1.3 still hold if we replace as in [20] the zero initial data by a nonnegative C^2 function u_0 such that $u_0 < 1$ in $\overline{\Omega}$, $\Delta u_0 + f(x)(1-u_0)^{-p} \geq 0$ in Ω, and $u_0 = 0$ on $\partial\Omega$. Indeed, this assumption guarantees that $u_t > 0$ and the proofs can then be modified in a straightforward way.

2. Proof of Theorem 1.2.

2.1. General strategy and basic computation. When the compactness of the quenching set is known, type I estimates can be proved by means of the maximum principle applied, in a strict subdomain of Ω, to the well-known auxiliary function (cf. [7, 11])

$$J(x,t) := u_t - \varepsilon(1-u)^{-p},$$

where ε is a small positive constant. In the present situation, the possible noncompactness of the quenching set prevents one from verifying that $J \geq 0$ on the boundary of any subdomain of Ω and the method is not directly applicable.

To overcome this, our basic idea is to consider a modified function J as follows:

$$J(x,t) = u_t - \varepsilon a(x) h(u), \tag{2.1}$$

where $a(x)$ is an auxiliary function such that $a = 0$ on $\partial\Omega$, hence also $J = 0$. The construction is delicate and requires specific properties for a, which will be given later. As for the function $h(u)$, it will be a perturbation of the nonlinearity, namely,

$$h(u) = (1-u)^{-p} + (1-u)^{-q}, \quad 0 < q < p. \tag{2.2}$$

Before specializing, we first present the basic computations.

LEMMA 2.1. *Let J, h be given by* (2.1)–(2.2), *where $a \in C^2(\Omega)$ is a nonnegative function. Then*

$$J_t - \Delta J - p f(x)(1-u)^{-p-1} J = \varepsilon R, \tag{2.3}$$

where

$$R = (p-q)a(x)f(x)(1-u)^{-p-q-1} + a h''(u)|\nabla u|^2 + 2h'(u)\nabla a \cdot \nabla u + h(u)\Delta a. \tag{2.4}$$

Moreover, $h'' > 0$ and, at any point $x \in \Omega$ such that $a(x) > 0$, we have

$$R \geq \underbrace{(p-q)a(x)f(x)(1-u)^{-p-q-1}}_{\mathcal{T}_1} + \underbrace{h(u)\Delta a}_{\mathcal{T}_2} - \underbrace{\frac{h'^2(u)|\nabla a|^2}{a h''(u)}}_{\mathcal{T}_3}. \tag{2.5}$$

Proof. We compute

$$\begin{aligned}
J_t &= u_{tt} - \varepsilon a(x) h'(u) u_t,\\
\nabla J &= \nabla u_t - \varepsilon\big(a(x)h'(u)\nabla u + h(u)\nabla a(x)\big),\\
\Delta J &= \Delta u_t - \varepsilon\big(a(x)h'(u)\Delta u + a(x)h''(u)|\nabla u|^2 + 2h'(u)\nabla a(x)\cdot\nabla u + h(u)\Delta a(x)\big).
\end{aligned}$$

Setting $g(u) = (1-u)^{-p}$ and omitting the variables x, u without risk of confusion, we get

$$\begin{aligned}
J_t - \Delta J &= (u_t - \Delta u)_t - \varepsilon a h'(u_t - \Delta u) + \varepsilon(a h''|\nabla u|^2 + 2h'\nabla a \cdot \nabla u + h\Delta a)\\
&= f(x) g' u_t - \varepsilon f(x) a h' g + \varepsilon(a h''|\nabla u|^2 + 2h'\nabla a\cdot\nabla u + h\Delta a).
\end{aligned}$$

Using $u_t = J + \varepsilon a h$, we have

$$J_t - \Delta J - f(x) g' J = \varepsilon R,$$

where

$$R = f(x)a(g'h - h'g) + ah''|\nabla u|^2 + 2h'\nabla a \cdot \nabla u + h\Delta a.$$

On the other hand, we have

$$\begin{aligned} g'h - h'g &= p(1-u)^{-p-1}\big((1-u)^{-p} + (1-u)^{-q}\big) \\ &\quad - (1-u)^{-p}\big(p(1-u)^{-p-1} + q(1-u)^{-q-1}\big) \\ &= (p-q)(1-u)^{-p-q-1}, \end{aligned}$$

hence (2.4). Finally, since $h'' > 0$, for all $x \in \Omega$ such that $a(x) > 0$, we may write

$$R = (p-q)af(x)(1-u)^{-p-q-1} + h\Delta a + ah''\left[|\nabla u|^2 + 2\frac{h'\nabla a \cdot \nabla u}{ah''}\right],$$

hence (2.5). ∎

2.2. Construction of the function $a(x)$. We see that, in order to guarantee $R \geq 0$, the (negative) term $\mathcal{T}_3$ on the right-hand side of (2.5) must be absorbed by a positive contribution coming either

- from the term $\mathcal{T}_1$ (generated by the perturbation in (2.2)), provided $f(x) > 0$, or
- from the term $\mathcal{T}_2$, provided $\Delta a(x) > 0$.

Since $a(x)$ is nonnegative and vanishes at the boundary, we cannot have $\Delta a > 0$ everywhere. Actually, we shall consider functions $a(x)$ which are positive in Ω and suitably convex everywhere except on a small ball B, where f is uniformly positive. Also, it will be necessary to split the parabolic cylinder $\Omega \times (T/2, T)$ into suitable subregions, taking into account the "large" and "small" parts of the function $u(x,t)$.

The following essential lemma gives the construction of the appropriate function $a(x)$.

LEMMA 2.2. *Let h be given by* (2.2). *Let $x_0 \in \Omega, \rho > 0$ with $\overline{B}(x_0,\rho) \subset \Omega$, and denote the open set*

$$\Omega_{x_0,\rho} = \Omega \setminus \overline{B}(x_0,\rho).$$

Then there exists a function $a \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ with the following properties:

$$hh''a\Delta a - {h'}^2|\nabla a|^2 \geq 0 \quad \textit{for all } x \in \overline{\Omega_{x_0,\rho}} \textit{ and all } 0 \leq u < 1, \tag{2.6}$$

$$C_1\delta^{p+1}(x) \leq a(x) \leq C_2\delta^{p+1}(x) \quad \textit{for all } x \in \overline{\Omega} \tag{2.7}$$

for some constants $C_1, C_2 > 0$.

Proof. For $0 < u < 1$, computing

$$h'(u) = p(1-u)^{-p-1} + q(1-u)^{-q-1}, \tag{2.8}$$

$$h''(u) = p(p+1)(1-u)^{-p-2} + q(q+1)(1-u)^{-q-2}, \tag{2.9}$$

it follows that

$$\begin{aligned} hh'' &= p(p+1)(1-u)^{-2p-2} + (p(p+1) + q(q+1))(1-u)^{-p-q-2} \\ &\quad + q(q+1)(1-u)^{-2q-2} \\ &\geq \frac{p+1}{p}\Big[p^2(1-u)^{-2p-2} + 2pq(1-u)^{-p-q-2} + q^2(1-u)^{-2q-2}\Big], \end{aligned}$$

where we used $p(p+1)+q(q+1) > 2(p+1)q$ due to $0 < q < p$. Therefore, we have

$$hh'' \geq \frac{p+1}{p}(h')^2 \quad \text{for } 0 \leq u < 1. \tag{2.10}$$

Next we introduce a suitable harmonic function ϕ, namely, the unique solution of the problem

$$\begin{aligned} \Delta\phi &= 0, \quad x \in \Omega_{x_0,\rho}, \\ \phi &= 0, \quad x \in \partial\Omega, \\ \phi &= 1, \quad x \in \partial B(x_0,\rho). \end{aligned}$$

The function ϕ is smooth and, by the strong maximum principle and the Hopf lemma, we have $0 < \phi < 1$ in $\Omega_{x_0,\rho}$ and

$$c_1\delta(x) \leq \phi(x) \leq c_2\delta(x), \quad x \in \overline{\Omega_{x_0,\rho}}, \tag{2.11}$$

for some positive constants c_1, c_2. Then we set

$$a(x) = \phi^{p+1}(x), \quad x \in \overline{\Omega_{x_0,\rho}}. \tag{2.12}$$

Since $a \in C^2(\Omega_{x_0,\rho})$, the boundary $\partial B(x_0,\rho)$ is smooth, and $a = 1$ on $\partial B(x_0,\rho)$, the function a can be extended in $B(x_0,\rho)$ in such a way that $a \in C^2(\Omega)$ and $a > 0$ in Ω.

On the other hand, on $\Omega_{x_0,\rho}$, we compute

$$\nabla a = (p+1)\phi^p\nabla\phi, \qquad \Delta a = (p+1)[\phi^p\Delta\phi + p\phi^{p-1}|\nabla\phi|^2] = p(p+1)\phi^{p-1}|\nabla\phi|^2,$$

hence

$$a\Delta a = \frac{p}{p+1}|\nabla a|^2 \quad \text{on } \Omega_{x_0,\rho}. \tag{2.13}$$

Combining (2.10) and (2.13), we get (2.6). Property (2.7) follows from (2.11), (2.12) and $a > 0$ in Ω. ∎

Before going further, let us recall the following useful lower bound on u_t.

Lemma 2.3. *There exists a constant $c_0 > 0$ such that*

$$u_t \geq c_0\delta(x) \quad \text{on } \Omega \times [T/2, T). \tag{2.14}$$

Proof. Although the proof of the lemma is standard (cf. [7, 11, 16]), we provide a proof here for completeness. Setting $v = u_t$, we see that v satisfies

$$\begin{aligned} v_t &= \Delta u + pf(x)(1-u)^{-p-1}v, \quad x \in \Omega,\ 0 < t < T, \\ v &= 0, \quad x \in \partial\Omega,\ 0 < t < T, \\ v(x,0) &= f(x), \quad x \in \Omega, \end{aligned}$$

so that $v = u_t \geq 0$ in $Q_T := \Omega \times (0,T)$ by the maximum principle.

Applying the maximum principle again, we deduce that $u_t \geq z$ in Q_T, where z is the solution of the heat equation in Q_T, with zero boundary condition and initial condition $z(\cdot,0) = f$. Since z satisfies the estimate (2.14) in virtue of the Hopf lemma and the strong maximum principle, so does u_t. ∎

2.3. Proof of Theorem 1.2.

Step 1. *Preparations.* Since $f \geq 0$ and $f \not\equiv 0$ is continuous, we may pick a point $x_0 \in \Omega$ and $\rho > 0$ such that $B(x_0, 2\rho) \subset \Omega$ and

$$\sigma_1 := \inf_{x \in \overline{B}(x_0,\rho)} f(x) > 0. \tag{2.15}$$

We then take $a \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ as given by Lemma 2.2, and define J by

$$J(x,t) = u_t - \varepsilon a(x) h(u)$$

with $\varepsilon > 0$ to be fixed later and

$$h(u) = (1-u)^{-p} + (1-u)^{-q}, \quad 0 \leq u < 1, \quad \text{where } q = p/2 \tag{2.16}$$

(any choice of $q \in (0,p)$ would do). Note that

$$\sigma_2 := \inf_{x \in \overline{B}(x_0,\rho)} a(x) > 0. \tag{2.17}$$

Next, we split the cylinder $\Sigma := \Omega \times (T/2, T)$ into three subregions as follows:

$$\Sigma_1 = \big(\Omega \setminus \overline{B}(x_0,\rho)\big) \times (T/2,T),$$
$$\Sigma_2^\eta = \big\{(x,t) \in \overline{B}(x_0,\rho) \times (T/2,T);\ u(x,t) \geq 1-\eta\big\},$$

and

$$\Sigma_3^\eta = \big\{(x,t) \in \overline{B}(x_0,\rho) \times (T/2,T);\ u(x,t) < 1-\eta\big\},$$

where the number $\eta \in (0,1)$ will be specified later on.

Step 2. *Parabolic inequality for J in the subregions Σ_1 and Σ_2^η.* It follows from properties (2.5) in Lemma 2.1 and (2.6) in Lemma 2.2, along with $a > 0$, $f \geq 0$ in Ω, and $h'' > 0$, that

$$J_t - \Delta J - pf(x)(1-u)^{-p-1} J \geq 0 \quad \text{in } \Sigma_1. \tag{2.18}$$

Next, in view of (2.16) and (2.8), we have

$$|h\Delta a| \leq C_3 (1-u)^{-p}, \quad |h'\nabla a| \leq C_3(1-u)^{-p-1} \quad \text{in } \Sigma$$

for some positive constant C_3 independent of ε, η. Also, from (2.9) and (2.17) we get

$$ah'' \geq \sigma_2 p(p+1)(1-u)^{-p-2} \quad \text{in } \overline{B}(x_0,\rho) \times (0,T).$$

Consequently, recalling the definition of R in Lemma 2.1, it follows from (2.5), (2.15), and (2.17) that

$$\begin{aligned}(1-u)^{p+q+1} R &\geq (p-q) f(x) a + h\Delta a (1-u)^{p+q+1} - \frac{(h'|\nabla a|)^2}{ah''}(1-u)^{p+q+1} \\ &\geq (p-q)\sigma_1\sigma_2 - C_4(1-u)^{q+1} \geq (p-q)\sigma_1\sigma_2 - C_4\eta^{q+1} \quad \text{in } \Sigma_2^\eta\end{aligned}$$

for some positive constant C_4 independent of ε, η. Owing to (2.3), we may thus choose $\eta \in (0,1)$ small, independent of ε, such that

$$J_t - \Delta J - pf(x)(1-u)^{-p-1} J \geq 0 \quad \text{in } \Sigma_2^\eta. \tag{2.19}$$

Step 3. *Control of J on Σ_3^η and conclusion.* Now that η has been fixed, using (2.7), (2.14), and (2.16), we may choose $\varepsilon > 0$ small enough, so that

$$J \geq \delta(x)\Big[c_0 - 2C_2\varepsilon\delta^p(x)(1-u)^{-p}\Big] \geq \delta(x)\Big[c_0 - 2C_2\varepsilon\delta^p(x)\eta^{-p}\Big] \geq 0 \quad \text{in } \Sigma_3^\eta \tag{2.20}$$

and

$$J(x,T/2) \geq \delta(x)\Big[c_0 - 2C_2\varepsilon\delta^p(x)\big(1 - \|u(\cdot,T/2)\|_\infty\big)^{-p}\Big] \geq 0 \quad \text{in } \overline{\Omega}, \tag{2.21}$$

where c_0 is the constant in Lemma 2.3 and C_1, C_2 are the constants in (2.7). Observe that, as a consequence of (2.20) and $\Sigma = \Sigma_1 \cup \Sigma_2^\eta \cup \Sigma_3^\eta$, we have

$$\big\{(x,t) \in \Sigma;\ J(x,t) < 0\big\} \subset \Sigma_1 \cup \Sigma_2^\eta. \tag{2.22}$$

Also, since $a = 0$ on $\partial\Omega$, we have

$$J = 0 \quad \text{on } \partial\Omega \times (T/2, T). \tag{2.23}$$

On the other hand, by standard parabolic regularity, we observe that

$$J \in C^{2,1}(\Sigma) \cap C(\overline{\Omega} \times [T/2, T)).$$

It follows from (2.18)–(2.19), (2.21)–(2.23), and the maximum principle (see, e.g., [25, Proposition 52.4 and Remark 52.11(a)]) that

$$J \geq 0 \quad \text{in } \Sigma.$$

Then, for $T/2 < t < s < T$ and $x \in \Omega$, taking (2.7) into account, an integration in time gives

$$\begin{aligned}(1-u(x,t))^{p+1} &\geq (1-u(x,s))^{p+1} + C_1\varepsilon(p+1)\delta^{p+1}(x)(s-t)\\ &\geq C_1\varepsilon(p+1)\delta^{p+1}(x)(s-t).\end{aligned}$$

Letting $s \to T$, we finally deduce (1.9) in Σ, hence in $\Omega \times (0,T)$. Note that the constants C_1, ε are independent of x, t, and so is the constant γ in (1.9). □

3. Proof of Theorem 1.1. With the type I estimate (1.9) of Theorem 1.2 at hand, the proof is done via a suitable local comparison function. Let $x_0 \in \Omega$ be such that $f(x_0) = 0$ and take $b_0 \in (0,1)$ small such that

$$\overline{B}(x_0, 2b_0) \subset \Omega. \tag{3.1}$$

We consider the following function

$$w(x,t) := 1 - A\big[\phi(x) + (T-t)\big]^{1/(p+1)} \quad \text{in } \overline{B}(x_0,b) \times [0,T),$$

where

$$\phi(x) := \kappa b^2\left(1 - \frac{|x-x_0|^2}{b^2}\right)^2.$$

Here, $\kappa \in (0,1)$ and $b \in (0,b_0)$ are constants to be chosen later and A is a fixed positive constant such that $A \leq \gamma b_0$ and $A \leq (1+T)^{-1/(p+1)}$ (where γ is the constant given in (1.9)). Note that

$$w(x,0) = 1 - A[\phi(x) + T]^{1/(p+1)} \geq 0 \quad \text{for } x \in \overline{B}(x_0, b)$$

and

$$\begin{aligned} w(x,t) &= 1 - A(T-t)^{1/(p+1)} \\ &\geq 1 - \gamma\delta(x)(T-t)^{1/(p+1)} \geq u(x,t) \quad \text{ for } (x,t) \in \partial B(x_0,b) \times (0,T), \end{aligned}$$

due to (3.1), $A \leq \gamma b_0$, and (1.9).

We compute, in $B(x_0,b) \times (0,T)$,

$$\begin{aligned} &w_t - \Delta w - f(x)(1-w)^{-p} \\ &\quad = \frac{A}{p+1}\big[\phi(x) + (T-t)\big]^{-1+\frac{1}{p+1}} + \frac{A}{p+1}\big[\phi(x) + (T-t)\big]^{-1+\frac{1}{p+1}}\Delta\phi \\ &\qquad - \frac{Ap}{(p+1)^2}\big[\phi(x) + (T-t)\big]^{-2+\frac{1}{p+1}}|\nabla\phi|^2 - f(x)A^{-p}\big[\phi(x) + (T-t)\big]^{-p/(p+1)} \\ &\quad = \frac{A}{p+1}\big[\phi(x) + (T-t)\big]^{-p/(p+1)} \\ &\qquad \times \left\{1 + \Delta\phi - \frac{p}{p+1}\big[\phi(x) + (T-t)\big]^{-1}|\nabla\phi|^2 - (p+1)A^{-p-1}f(x)\right\}, \end{aligned}$$

hence

$$\begin{aligned} w_t - \Delta w - f(x)(1-w)^{-p} & \\ \geq \frac{A}{p+1}\big[\phi(x) + (T-t)\big]^{-p/(p+1)} &\left\{1 + \Delta\phi - \frac{p}{p+1}\frac{|\nabla\phi|^2}{\phi} \right. \\ &\left. \quad - (p+1)A^{-p-1}f(x)\right\}. \end{aligned} \tag{3.2}$$

Moreover, in $B(x_0,b)$ we have

$$\begin{aligned} \nabla\phi(x) &= -4\kappa\left(1 - \frac{|x-x_0|^2}{b^2}\right)(x - x_0), \\ \Delta\phi(x) &= -4n\kappa\left(1 - \frac{|x-x_0|^2}{b^2}\right) + 8\kappa\left(\frac{|x-x_0|}{b}\right)^2 \geq -4n\kappa, \\ \frac{|\nabla\phi(x)|^2}{\phi(x)} &= 16\kappa\left(\frac{|x-x_0|}{b}\right)^2 \leq 16\kappa. \end{aligned}$$

Now, since $f(x_0) = 0$, we may choose $b > 0$ small enough so that

$$(p+1)A^{-p-1}\sup_{x\in B(x_0,b)} f(x) \leq \frac{1}{3}.$$

Then we can choose $\kappa = \kappa(n) \in (0,1)$ small enough such that

$$\Delta\phi(x) \geq -\frac{1}{3}, \quad \frac{|\nabla\phi(x)|^2}{\phi(x)} \leq \frac{1}{3} \quad \text{ for all } x \in B(x_0,b).$$

Therefore, $w_t - \Delta w - f(x)(1-w)^{-p} \geq 0$ in $B(x_0,b) \times (0,T)$, and it follows from the comparison principle that $u \leq w$ in $B(x_0,b) \times (0,T)$. Since $\min_{\overline{B}(x_0,b/2)} \phi > 0$, this implies that $x = x_0$ is not a quenching point and the theorem is proved. ∎

4. Proof of Theorem 1.3.

(i) Case of (1.10). Assume without loss of generality that $0 \neq x_0 \in \partial\Omega$ with $B(0,|x_0|) \cap \Omega = \emptyset$. This (exterior ball condition) is possible due to the assumption that $\partial\Omega \in C^{2+\nu}$. We look for a supersolution of the form $z(x) = 1 - C(d-r)^\beta$, $r = |x|$, with $\beta > 1$, $C > 0$, $d > |x_0|$ to be chosen. For $0 < r < d$, we have:

$$z_r = \beta C(d-r)^{\beta-1},$$
$$z_{rr} = -\beta(\beta-1)C(d-r)^{\beta-2}.$$

Set $\omega := \Omega \cap \{x \in \mathbb{R}^n;\ |x_0| < |x| < d\}$. Choosing $\beta = 2/(p+1) > 1$, so that $\beta - 2 = -\beta p$, we compute, in ω,

$$\begin{aligned}
&-\Delta z - f(x)(1-z)^{-p}\\
&\quad = \beta(\beta-1)C(d-|x|)^{\beta-2} - \frac{\beta C(n-1)(d-|x|)^{\beta-1}}{|x|}\\
&\qquad - f(x)\big[C(d-|x|)^\beta\big]^{-p}\\
&\quad = C(d-|x|)^{-\beta p}\left[\beta(\beta-1) - \frac{\beta(n-1)(d-|x|)}{|x|} - C^{-p-1}f(x)\right].
\end{aligned}$$

Next taking $d = d(p,n,|x_0|) > |x_0|$ close to $|x_0|$ and $C = C(p, \|f\|_\infty) > 0$ large, we then have, in ω,

$$\begin{aligned}
&-\Delta z - f(x)(1-z)^{-p}\\
&\quad \geq C(d-|x|)^{-\beta p}\left[\beta(\beta-1) - \frac{\beta(n-1)(d-|x_0|)}{|x_0|} - C^{-p-1}\|f\|_\infty\right] \geq 0.
\end{aligned}$$

By taking d possibly closer to $|x_0|$, we have

$$z(x) \geq 1 - C(d-|x_0|)^\beta \geq 0 \quad \text{in } \overline{\omega},$$

hence also $z(x) \geq u(x,t) = 0$ for $x \in \partial\omega \cap \{|x| < d\} = \partial\omega \cap \partial\Omega$ and $t \in (0,T)$. Since, on the other hand, $z(x) = 1 > u(x,t)$ for $x \in \partial\omega \cap \{|x| = d\}$ and $0 < t < T$, we deduce from the comparison principle that $z \geq u$ on $\omega \times (0,T)$. Therefore x_0 is not a quenching point.

(ii) Case of (1.11). The proof relies on estimate (1.9) and is a modification of that of Theorem 1.1. Let $\Omega_\eta = \{x \in \Omega;\ \delta(x) < \eta\}$. There exists $\eta_0 > 0$ such that Ω_η is a smooth bounded domain for all $\eta \in (0, \eta_0)$, due to Ω being a smooth domain. We have $\partial\Omega_\eta = \partial\Omega \cup \Gamma_\eta$, where $\Gamma_\eta = \{x \in \Omega;\ \delta(x) = \eta\}$.

Let ψ_η be the unique solution of the problem

$$\begin{aligned}
\Delta\psi_\eta &= 0, \quad x \in \Omega_\eta,\\
\psi_\eta &= 1, \quad x \in \partial\Omega,\\
\psi_\eta &= 0, \quad x \in \Gamma_\eta.
\end{aligned}$$

The function ψ_η is smooth and, by the strong maximum principle, we have $0 < \psi_\eta < 1$ in Ω_η. Letting

$$\phi = k\psi_\eta^2,$$

we consider the function

$$w(x,t) := 1 - \gamma\eta\big[\phi(x) + (T-t)\big]^{1/(p+1)} \quad \text{in } \overline{\Omega}_\eta \times [0,T),$$

where $k \in (0,1)$ and $\eta \in (0,\eta_0)$ are constants to be chosen later and γ is the constant given in (1.9). First assuming $\eta \le \eta_1 := \min(\eta_0, \gamma^{-1}(1+T)^{-1/(p+1)})$, we have

$$w(x,0) = 1 - \gamma\eta[\phi(x) + T]^{1/(p+1)} \ge 0 \quad \text{in } \overline{\Omega}_\eta,$$

along with $w(x,t) \ge 0$ on $\partial\Omega \times (0,T)$ and, by (1.9),

$$\begin{aligned} w(x,t) &= 1 - \gamma\eta(T-t)^{1/(p+1)} = 1 - \gamma\delta(x)(T-t)^{1/(p+1)} \\ &\ge u(x,t) \quad \text{on } \Gamma_\eta \times (0,T). \end{aligned}$$

Formula (3.2) remains valid, with A replaced by $\gamma\eta$ and, moreover, we have

$$\Delta\phi = 2k|\nabla\psi_\eta|^2 \quad \text{and} \quad \frac{|\nabla\phi|^2}{\phi} = 4k|\nabla\psi_\eta|^2.$$

Therefore,

$$\begin{aligned} &w_t - \Delta w - f(x)(1-w)^{-p} \\ &\ge \frac{\gamma\eta}{p+1}\big[\phi(x) + (T-t)\big]^{-p/(p+1)} \left\{1 - \frac{4kp}{p+1}|\nabla\psi_\eta|^2 - (p+1)(\gamma\eta)^{-p-1} f(x)\right\} \end{aligned}$$

in $\Omega_\eta \times [0,T)$. Now, by assumption (1.11), we may choose $\eta \in (0,\eta_1)$ small enough so that

$$(p+1)(\gamma\eta)^{-p-1} \sup_{x\in\Omega_\eta} f(x) \le \frac{1}{2}.$$

Then we can choose $k = k(\eta) > 0$ small enough so that

$$4k|\nabla\psi_\eta|^2 \le \frac{1}{2} \quad \text{for all } x \in \Omega_\eta.$$

Therefore, $w_t - \Delta w - f(x)(1-w)^{-p} \ge 0$ in $\Omega_\eta \times (0,T)$, and it follows from the comparison principle that $u \le w$ in $\Omega_\eta \times (0,T)$. Since $\min_{\overline{\Omega}_{\eta/2}} \psi_\eta > 0$, this guarantees that no quenching occurs near the boundary. ∎

Remark. Although it is not clear if the case of (1.10) in Theorem 1.3 has a direct relation with this fact, it is interesting to recall that, when f is constant, quenching for problem (1.1)–(1.3) is incomplete (in the sense of existence of a suitable weak continuation after $t = T$) if and only if $0 < p < 1$ (cf. [24, 5, 8]).

Acknowlegment. Ph. S. is grateful for the hospitality of the Mathematics Department of the Tamkang University, where part of this work was done.

REFERENCES

[1] P. Esposito, N. Ghoussoub, and Y. Guo, *Compactness along the first branch of unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math., 60 (2007), pp. 1731–1768.

[2] P. Esposito, *Compactness of a nonlinear eigenvalue problem with a singular nonlinearity*, Commun. Contemp. Math., 10 (2008), pp. 17–45.

[3] P. Esposito and N. Ghoussoub, *Uniqueness of solutions for an elliptic equation modeling MEMS*, Methods Appl. Anal., 15 (2008), pp. 341–354.

[4] P. Esposito, N. Ghoussoub, and Y. Guo, *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, Courant Lect. Notes Math. 20, AMS, Providence, RI, 2010.

[5] M. Fila, H.A. Levine, and J.L. Vázquez, *Stabilization of solutions of weakly singular quenching problems*, Proc. Amer. Math. Soc., 119 (1993), pp. 555–559.

[6] G. Flores, G. Mercado, J.A. Pelesko, and N. Smyth, *Analysis of the dynamics and touchdown in a model of electrostatic MEMS*, SIAM J. Appl. Math., 67 (2007), pp. 434–446.

[7] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J., 34 (1985), pp. 425–447.

[8] V.A. Galaktionov and J.L. Vázquez, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math., 50 (1997), pp. 1–67.

[9] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices: Stationary case*, SIAM J. Math. Anal., 38 (2007), pp. 1423–1449.

[10] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices* II: *Dynamic case*, NoDEA Nonlinear Differential Equations Appl., 15 (2008), pp. 115–145.

[11] J.-S. Guo, *On the quenching behavior of the solution of a semilinear parabolic equation*, J. Math. Anal. Appl., 151 (1990), pp. 58–79.

[12] J.-S. Guo, *Quenching problem in nonhomogeneous media*, Differential Integral Equations, 10 (1997), pp. 1065–1074.

[13] J.-S. Guo, *Recent developments on a nonlocal problem arising in the micro-electro mechanical system*, Tamkang J. Math., 45 (2014), pp. 229–241.

[14] J.-S. Guo, B. Hu, and C.-J. Wang, *A nonlocal quenching problem arising in micro-electro mechanical systems*, Quart. Appl. Math., 67 (2009), pp. 725–734.

[15] J.-S. Guo and N.I. Kavallaris, *On a nonlocal parabolic problem arising in electrostatic MEMS control*, Discrete Contin. Dynam. Systems, 32 (2012), pp. 1723–1746.

[16] Y. Guo, *On the partial differential equations of electrostatic MEMS devices* III: *Refined touchdown behavior*, J. Differential Equations, 244 (2008), pp. 2277–2309.

[17] Y. Guo, *Global solutions of singular parabolic equations arising from electrostatic MEMS*, J. Differential Equations, 245 (2008), pp. 809–844.

[18] Y. Guo, Z. Pan, and M.J. Ward, *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties*, SIAM J. Appl. Math., 66 (2005), pp. 309–338.

[19] Z. Guo and J. Wei, *Asymptotic behavior of touchdown solutions and global bifurcations for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Anal., 7 (2008), pp. 765–786.

[20] N.I. Kavallaris, T. Miyasita, and T. Suzuki, *Touchdown and related problems in electrostatic MEMS device equation*, NoDEA Nonlinear Differential Equations Appl., 15 (2008), pp. 363–385.

[21] H.A. Levine, *Quenching, nonquenching, and beyond quenching for solution of some parabolic equations*, Ann. Mat. Pura Appl. (4), 155 (1989), pp. 243–260.

[22] J.A. Pelesko and A.A. Triolo, *Nonlocal problems in MEMS device control*, J. Engrg. Math., 41 (2001), pp. 345–366.

[23] J.A. Pelesko and D.H. Bernstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, Boca Raton, FL, 2002.

[24] D. Phillips, *Existence of solutions of quenching problems*, Appl. Anal., 24 (1987), pp. 253–264.

[25] P. Quittner and Ph. Souplet, *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*, Birkhäuser Adv. Texts, Birkhäuser, Basel, 2007.

[26] D. Ye and F. Zhou, *On a general family of nonautonomous elliptic and parabolic equations*, Calc. Var., 37 (2010), pp. 259–274.