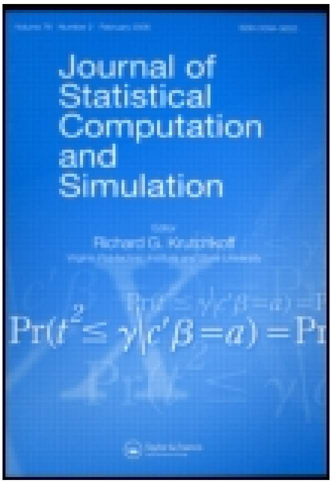

Journal of Statistical Computation and Simulation

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gscs20

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Published online: 25 Nov 2013.

Journal of Statistical Computation and Simulation, 2015
Vol. 85, No. 6, 1071–1089, http://dx.doi.org/10.1080/00949655.2013.861838

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(*Received 17 January 2013; accepted 30 October 2013*)

We consider the problem of making statistical inference on unknown parameters of a lognormal distribution under the assumption that samples are progressively censored. The maximum likelihood estimates (MLEs) are obtained by using the expectation-maximization algorithm. The observed and expected Fisher information matrices are provided as well. Approximate MLEs of unknown parameters are also obtained. Bayes and generalized estimates are derived under squared error loss function. We compute these estimates using Lindley's method as well as importance sampling method. Highest posterior density interval and asymptotic interval estimates are constructed for unknown parameters. A simulation study is conducted to compare proposed estimates. Further, a data set is analysed for illustrative purposes. Finally, optimal progressive censoring plans are discussed under different optimality criteria and results are presented.

Keywords: approximate maximum likelihood estimate; Bayes estimate; EM algorithm; Fisher information matrix; importance sampling; Lindley's method; maximum likelihood estimate; optimal censoring

1. Introduction

In general, reliability and life testing experiments are performed to study products failure time distributions. Such experiments often give rise to censored data. Type-I and type-II censoring have been the two most common schemes for generating censored samples. Many authors have studied various lifetime distributions under these schemes and one may refer to Balakrishnan and Cohen [1] for a review. These two censoring schemes do not allow for removal of units at time point other than the final time point of the experiment. Many studies of life and bio-assays are conducted where live units are removed in between from the experiment for further investigations. So, a more practical censoring scheme known as progressive censoring which allows intermediate removal of units can be used. A progressively censored sample can be obtained as follows. Let a total of n experimental units be placed on a life test. At the time of first failure, the R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. Similarly, when the second failure occurs, R_2 of the remaining $n-R_1-2$ surviving units are again randomly removed. Finally, when mth failure occurs, the remaining $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ units are removed from the experiment. This yields a progressively type-II censored sample of

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size m from a parent sample of size n and here $(R_1, R_2, \ldots, R_m)$ is referred as the corresponding progressive censoring scheme. The numbers m and R_i are prefixed before the start of the experiment. Progressive censoring has received considerable attentions of many authors, see for instance, the monograph by Balakrishnan and Aggarwala [2] and the review article by Balakrishnan [3] for various work done on progressive censoring scheme. One may also refer to the works of Balasooriya and Balakrishnan,[4] Balakrishnan et al.,[5] Balakrishnan and Asgharzadeh,[6] Asgharzadeh,[7] Pradhan and Kundu,[8] among others.

A random variable X following a two parameter lognormal $LN(\mu, \tau)$ distribution has the density function of the form

$$f(x; \mu, \tau) = \frac{1}{x\sqrt{\tau}} \phi\left(\frac{\ln x - \mu}{\sqrt{\tau}}\right), \quad x > 0, \ -\infty < \mu < \infty, \ \tau > 0,$$

where $\phi(\cdot)$ denotes the density function of a standard normal random variable. In this paper μ and τ are treated as unknown parameters. In fact, e^{μ} and e^{τ} can be treated as scale and shape parameters of $LN(\mu, \tau)$ distribution, respectively. The distribution can assume variety of shapes and this flexibility makes it useful for adequate fitting of data which mostly occur in socio-economic, bio-assays, industrial and agricultural experiments. One may refer to Nelson [9] for a discussion on these accounts and some related properties of lognormal distribution. We mention that Weibull distribution has been extensively used by many authors to analyse various failure time data, see for instance, Kundu [10] and Gupta and Kundu.[11] It is to be noted that hazard function of a $LN(\mu, \tau)$ distribution behaves in a manner similar to that of the Weibull distribution. Although not much work has been done on lognormal distribution, however, because of stated similarity and seeing the usefulness of the Weibull distribution it renders that many failure time data can also be adequately analysed using $LN(\mu, \tau)$ distribution as well, particularly when samples are censored. In this paper, the problem of making statistical inference on the unknown parameters of a lognormal distribution is considered under the assumption that samples are progressively censored.

We have organized rest of the paper as follows. In Section 2, maximum likelihood estimation is discussed. The expectation-maximization (EM) algorithm is proposed to compute the maximum likelihood estimates (MLEs) of unknown parameters μ and τ. Next, in Section 3, we provide the observed and expected Fisher information matrices. Asymptotic confidence intervals are constructed using the observed information matrix of the MLEs. Approximate maximum likelihood estimates (AMLEs) are obtained in Section 4. Bayes estimation is discussed in Section 5. Lindley's method and importance sampling method are used to derive approximate Bayes estimates for μ and τ. Highest posterior density (HPD) interval estimates are also constructed for unknown parameters. In Section 6, we compare the performance of all estimates using simulations and a data set is analysed for illustrative purposes. The optimum censoring plan is discussed in Section 7. Finally, some conclusions are made in Section 8.

2. Maximum likelihood estimation

Suppose that n independent units whose lifetime follow $LN(\mu, \tau)$ distribution are put on a life test and then a progressively type-II censored sample of size m is observed under the scheme $R = (R_1, R_2, \ldots, R_m)$. We denote the observed sample by $X = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$. Writing $X_{(j)}$ in place of $X_{j:m:n}, j = 1, 2, \ldots, m$ for convenience, we observe that the likelihood function of μ and τ is

$$l(\mu, \tau | x) \propto \prod_{i=1}^{m} \frac{1}{x_{(i)}\sqrt{\tau}} \phi(y_{(i)})[1 - \Phi(y_{(i)})]^{R_i},$$

where $y_{(i)} = ((\ln x_{(i)} - \mu)/\sqrt{\tau})$, $x = (x_{(1)}, x_{(2)}, \ldots, x_{(m)})$ and $\Phi(\cdot)$ denotes the distribution function of a standard normal distribution. The likelihood equations of μ and τ are obtained as

$$\sum_{i=1}^{m} y_{(i)} + \sum_{i=1}^{m} R_i \frac{\phi(y_{(i)})}{1 - \Phi(y_{(i)})} = 0, \tag{1}$$

$$-m + \sum_{i=1}^{m} y_{(i)}^2 + \sum_{i=1}^{m} R_i \frac{y_{(i)}\phi(y_{(i)})}{1 - \Phi(y_{(i)})} = 0. \tag{2}$$

The MLEs of unknown parameters can be computed by solving the above two nonlinear equations. One may refer to Balakrishnan and Mi [12] (see also [5]) regarding the existence of a unique root for Equations (1) and (2). These equations do not yield analytic solution for desired estimates. So, some numerical techniques such as Newton–Raphson should be employed to compute the MLEs. Here we apply EM algorithm suggested by Dempster et al.,[13] to compute the desired MLEs of μ and τ. We mention that this progressively censored lognormal problem for deriving MLEs can be viewed as an incomplete data problem, see for instance, Ng et al.[14] We write the observed data as $X = (X_{(1)}, X_{(2)}, \ldots, X_{(m)})$ and unobserved (censored) data as $Z = (Z_1, Z_2, \ldots, Z_m)$ with each of Z_j being a $1 \times R_j$ vector such that $Z_j = (Z_{j1}, Z_{j2}, \ldots, Z_{jR_j})$, $j = 1, 2, \ldots, m$. The complete data are given by $W = (X, Z)$. Now after ignoring the constant term, the log-likelihood function of the complete data $L_c(W; \mu, \tau)$ is given by

$$\ln L_c(W; \mu, \tau) = -\frac{n}{2} \ln \tau - \sum_{i=1}^{m} \ln x_{(i)} - \frac{1}{2\tau} \sum_{i=1}^{m} (\ln x_{(i)} - \mu)^2 - \sum_{j=1}^{m} \sum_{k=1}^{R_j} \ln z_{jk}$$
$$- \frac{1}{2\tau} \sum_{j=1}^{m} \sum_{k=1}^{R_j} (\ln z_{jk} - \mu)^2.$$

We need to compute the pseudo log-likelihood function $L_s(\mu, \tau)$ in the E-step which is obtained from the function L_c and it is seen that

$$L_s(\mu, \tau) = -\frac{n}{2} \ln \tau - \sum_{i=1}^{m} \ln x_{(i)} - \frac{1}{2\tau} \sum_{i=1}^{m} (\ln x_{(i)} - \mu)^2 - \sum_{j=1}^{m} \sum_{k=1}^{R_j} E(\ln Z_{jk} | Z_{jk} > x_{(j)})$$
$$- \frac{1}{2\tau} \sum_{j=1}^{m} \sum_{k=1}^{R_j} E((\ln Z_{jk} - \mu)^2 | Z_{jk} > x_{(j)}).$$

After some simplifications this is rewritten as

$$L_s(\mu, \tau) = -\frac{n}{2} \ln \tau - \frac{n\mu^2}{2\tau} + \left(\frac{\mu}{\tau} - 1\right) \sum_{i=1}^{m} \ln x_{(i)} + \left(\frac{\mu}{\tau} - 1\right) \sum_{j=1}^{m} \sum_{k=1}^{R_j} E(\ln Z_{jk} | Z_{jk} > x_{(j)})$$
$$- \frac{1}{2\tau} \sum_{i=1}^{m} (\ln x_{(i)})^2 - \frac{1}{2\tau} \sum_{j=1}^{m} \sum_{k=1}^{R_j} E((\ln Z_{jk})^2 | Z_{jk} > x_{(j)}). \tag{3}$$

Also, the related expectations are defined and evaluated as

$$A(x_{(j)}; \mu_{(k)}, \tau_{(k)}) = E(\ln Z_{jk} | Z_{jk} > x_{(j)}) = \mu + Q_{(j)} \sqrt{\tau}$$

and

$$B(x_{(j)};\mu_{(k)},\tau_{(k)}) = E((\ln Z_{jk})^2|Z_{jk} > x_{(j)}) = \tau(1 + y_{(j)}Q_{(j)}) + 2\mu\sqrt{\tau}Q_{(j)} + \mu^2,$$

where $y_{(j)} = ((\ln x_{(j)} - \mu)/\sqrt{\tau})$, and $Q_{(j)} = \phi(y_{(j)})/(1 - \Phi(y_{(j)}))$.

Now the M-step involves the maximization the function $L_s(\mu, \tau)$ as defined in (3). If $(\mu_{(k)}, \tau_{(k)})$ be the kth stage estimate of (μ, τ) then the corresponding $(k+1)$th stage estimate $(\mu_{(k+1)}, \tau_{(k+1)})$ is obtained by maximizing the function

$$L_s(\mu,\tau) = -\frac{n}{2}\ln\tau - \frac{n\mu^2}{2\tau} + \left(\frac{\mu}{\tau} - 1\right)\sum_{i=1}^{m}\ln x_{(i)} - \frac{1}{2\tau}\sum_{i=1}^{m}(\ln x_{(i)})^2 + \left(\frac{\mu}{\tau} - 1\right)\sum_{j=1}^{m}R_jA(x_{(j)};\mu_{(k)},\tau_{(k)}) - \frac{1}{2\tau}\sum_{j=1}^{m}R_jB(x_{(j)};\mu_{(k)},\tau_{(k)}).$$

As a consequence, the desired updated estimates turn out to be

$$\hat{\mu}_{(k+1)} = \frac{1}{n}\left[\sum_{i=1}^{m}\ln x_{(i)} + \sum_{j=1}^{m}R_jA(x_{(j)};\mu_{(k)},\tau_{(k)})\right]$$

and

$$\hat{\tau}_{(k+1)} = \frac{1}{n}\left[\sum_{i=1}^{m}(\ln x_{(i)})^2 + \sum_{j=1}^{m}R_jB(x_{(j)};\mu_{(k)},\tau_{(k)})\right] - \hat{\mu}^2_{(k+1)}.$$

3. Fisher information matrix

This section deals with computing the observed and expected Fisher information matrices using the method of Louis.[15] Note that observed information matrix is helpful in constructing asymptotic confidence intervals for unknown parameters. The expected Fisher information matrix will be utilized in obtaining optimal censoring plans. In the next subsection, we obtain observed information matrix.

3.1. *Observed Fisher information matrix*

The missing information principle of Louis [15] suggests that

$$\text{Observed information} = \text{Complete information} - \text{Missing information}. \quad (4)$$

Now define notations X = the observed data, W = the complete data, $\theta = (\mu, \tau)$, $I_X(\theta)$ = the observed information, $I_W(\theta)$ = the complete information, $I_{W|X}(\theta)$ = the missing information, then Equation (4) can be reexpressed as

$$I_X(\theta) = I_W(\theta) - I_{W|X}(\theta). \quad (5)$$

Note that the complete information $I_W(\theta)$ is given by

$$I_W(\theta) = -E\left[\frac{\partial^2 \ln L_c(W;\theta)}{\partial\theta^2}\right] = \begin{bmatrix} \frac{n}{\tau} & 0 \\ 0 & \frac{n}{2\tau^2} \end{bmatrix}. \quad (6)$$

The Fisher information in one observation, which is censored at the time of the jth failure time $x_{(j)}$ is given by

$$I^{(j)}_{W|X}(\theta) = -E_{Z_j|X_{(j)}}\left[\frac{\partial^2 \ln f_{Z_j}(z_j|x_{(j)},\theta)}{\partial\theta^2}\right] = E_{Z_j|X_{(j)}}\left[\frac{\partial \ln f_{Z_j}(z_j|x_{(j)},\theta)}{\partial\theta}\right]^2$$
$$= \begin{bmatrix} a_{11}(x_{(j)};\mu,\tau) & a_{12}(x_{(j)};\mu,\tau) \\ a_{21}(x_{(j)};\mu,\tau) & a_{22}(x_{(j)};\mu,\tau)\end{bmatrix}.$$

The elements of the above matrix are computed using the conditional distribution of $Z_{jk}|X_{(j)} = x_{(j)}$, $k = 1, 2, \ldots, R_j$. This conditional distribution is derived as (see [8,14])

$$f_{Z|X}(z_j|X_{(j)} = x_{(j)}) = \begin{cases} \dfrac{f(z_j;\theta)}{1-\Phi(y_{(j)})} & \text{if } z_j > x_{(j)}, \\ 0 & \text{otherwise.}\end{cases}$$

Consequently, we are able to observed that

$$a_{11}(x_{(j)};\mu,\tau) = \frac{1}{\tau}[1 + y_{(j)}Q_{(j)} - Q^2_{(j)}],$$
$$a_{21}(x_{(j)};\mu,\tau) = \frac{1}{2\tau^{1.5}}[Q_{(j)} + y_{(j)}Q_{(j)}(y_{(j)} - Q_{(j)})],$$
$$a_{22}(x_{(j)};\mu,\tau) = \frac{1}{4\tau^2}[2 + y_{(j)}Q_{(j)}(1 - y_{(j)}Q_{(j)} + y^2_{(j)})].$$

Thus, the total missing information is given by

$$I_{W|X}(\theta) = \sum_{j=1}^{m} R_j I^{(j)}_{W|X}(\theta). \quad (7)$$

Finally, using Equations (6) and (7) in Equation (5), we obtain the observed information matrix. It is to be noted again that the asymptotic variance covariance matrix of $(\hat{\mu}, \hat{\tau})$ is given by $[I_X(\theta)]^{-1}$.

Further, the two-sided $100(1-\xi)\%$ asymptotic confidence intervals for the parameters μ and τ can be obtained as $\hat{\mu} \pm Z_{\xi/2}\sqrt{\mathrm{Var}(\hat{\mu})}$ and $\hat{\tau} \pm Z_{\xi/2}\sqrt{\mathrm{Var}(\hat{\tau})}$, respectively, where $Z_{\xi/2}$ is the upper $\xi/2$th percentile of the standard normal distribution.

3.2. *Expected Fisher information matrix*

This subsection deals with finding the expected Fisher information matrix based on progressively censored sample X and the associated censoring scheme $R = (R_1, R_2, \ldots, R_m)$. Note that the expected Fisher information matrix is given by

$$E[I_X(\theta)] = I_W(\theta) - E[I_{W|X}(\theta)].$$

Since missing information matrix involves only function of $X_{(j)}$ so in order to evaluate above expression we need the probability density function of $X_{(j)}$, $j = 1, 2, \ldots, m$ which is given by (see [2])

$$f_{X_{(j)}}(x) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\frac{1}{x\sqrt{\tau}}\phi\left(\frac{\ln x-\mu}{\sqrt{\tau}}\right)\left\{1-\Phi\left(\frac{\ln x-\mu}{\sqrt{\tau}}\right)\right\}^{r_i-1}, \quad 0<x<\infty,$$

where $r_j = m-j+1+\sum_{i=j}^{m} R_i$, $c_{j-1}=\prod_{i=1}^{j} r_i$, $j=1,2,\ldots,m$, and $a_{1,1}=1$, $a_{i,j}=\prod_{k=1,k\neq i}^{j} 1/(r_k-r_i)$, $1\le i\le j\le m$. Thus, we have

$$E[I_{W|X}(\theta)] = \sum_{j=1}^{m} R_j E_{X_{(j)}}\begin{bmatrix} a_{11}(x_{(j)};\mu,\tau) & a_{12}(x_{(j)};\mu,\tau)\\ a_{21}(x_{(j)};\mu,\tau) & a_{22}(x_{(j)};\mu,\tau)\end{bmatrix},$$

where

$$E_{X_{(j)}}[a_{11}(x_{(j)};\mu,\tau)] = \frac{1}{\tau}[1+e_{1j}(\mu,\tau)-e_{2j}(\mu,\tau)],$$
$$E_{X_{(j)}}[a_{12}(x_{(j)};\mu,\tau)] = \frac{1}{2\tau^{1.5}}[e_{3j}(\mu,\tau)+e_{4j}(\mu,\tau)-e_{5j}(\mu,\tau)],$$
$$E_{X_{(j)}}[a_{22}(x_{(j)};\mu,\tau)] = \frac{1}{4\tau^{2}}[2+e_{1j}(\mu,\tau)-e_{6j}(\mu,\tau)+e_{7j}(\mu,\tau)],$$

and

$$e_{1j}(\mu,\tau) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\int_0^1 \Phi^{-1}(u)\phi(\Phi^{-1}(u))(1-u)^{r_i-2}\,\mathrm{d}u,$$

$$e_{2j}(\mu,\tau) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\int_0^1 (\phi(\Phi^{-1}(u)))^2(1-u)^{r_i-3}\,\mathrm{d}u,$$

$$e_{3j}(\mu,\tau) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\int_0^1 \phi(\Phi^{-1}(u))(1-u)^{r_i-2}\,\mathrm{d}u,$$

$$e_{4j}(\mu,\tau) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\int_0^1 (\Phi^{-1}(u))^2\phi(\Phi^{-1}(u))(1-u)^{r_i-2}\,\mathrm{d}u,$$

$$e_{5j}(\mu,\tau) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\int_0^1 \Phi^{-1}(u)(\phi(\Phi^{-1}(u)))^2(1-u)^{r_i-3}\,\mathrm{d}u,$$

$$e_{6j}(\mu,\tau) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\int_0^1 (\Phi^{-1}(u))^2(\phi(\Phi^{-1}(u)))^2(1-u)^{r_i-3}\,\mathrm{d}u,$$

$$e_{7j}(\mu,\tau) = c_{j-1}\sum_{i=1}^{j} a_{i,j}\int_0^1 (\Phi^{-1}(u))^3(\phi(\Phi^{-1}(u)))(1-u)^{r_i-2}\,\mathrm{d}u.$$

Finally, the expected asymptotic variance–covariance matrix of $(\hat{\mu},\hat{\tau})$ is obtained as

$$V(\theta) = \begin{bmatrix} V_{11}(n,R) & V_{12}(n,R)\\ V_{21}(n,R) & V_{22}(n,R)\end{bmatrix} = (E[I_X(\theta)])^{-1}.$$

4. Approximate maximum likelihood estimation

It is observed that the likelihood equations (1) and (2) do not yield explicit estimators for MLEs. In this section, we provide explicit estimators for μ and τ by obtaining approximate maximum likelihood estimators. We mention that the likelihood equations are nonlinear because of the term $\phi(y_{(i)})/(1-\Phi(y_{(i)}))$. Define

$$h(y_{(i)}) = \frac{\phi(y_{(i)})}{1-\Phi(y_{(i)})}.$$

We approximate the function $h(y_{(i)})$ by expanding it in a Taylor series around $E(Y_{(i)}) = \nu_{(i)}$. From Balakrishnan and Sandhu,[16] it is seen that if $U_{(i)}$ is the ith progressively type-II censored sample from the $U(0,1)$ distribution then,

$$Y_{(i)} =^d \Phi^{-1}(U_{(i)}).$$

Further, we have

$$\nu_{(i)} = E(Y_{(i)}) \approx \Phi^{-1}(\alpha_{(i)}),$$

where $\alpha_{(i)} = E(U_{(i)})$ and it is known from Balakrishnan and Aggarwala [2] that

$$\alpha_{(i)} = 1 - \prod_{j=m-i+1}^{m} \frac{j + R_{m-j+1} + \cdots + R_m}{j + 1 + R_{m-j+1} + \cdots + R_m}.$$

Now by expanding $h(y_{(i)})$ around $\nu_{(i)}$ and keeping only the first two terms we have the following approximation:

$$\begin{aligned} h(y_{(i)}) &\approx h(\nu_{(i)}) + (y_{(i)} - \nu_{(i)})h'(\nu_{(i)}) \\ &= \alpha_i + \beta_i y_{(i)}, \end{aligned} \tag{8}$$

where $\alpha_i = h(\nu_{(i)}) - \nu_{(i)}h'(\nu_{(i)})$ and $\beta_i = h'(\nu_{(i)})$ for $i = 1, 2, \ldots, m$. Next, we show that β_i is nonnegative for all i. We have

$$h'(y) = \frac{\phi(y)}{(1-\Phi(y))^2} h_1(y),$$

where $h_1(y) = \phi(y) - y(1-\Phi(y))$. We observed that $h_1(y)$ is decreasing in y. Moreover, $\lim_{y \to -\infty} h_1(y) = \infty$ and $\lim_{y\to\infty} h_1(y) = 0$. This means that $h_1(y)$ is nonnegative for all y, which in turn implies that $h'(y)$ is non negative for all y. Hence, it is proved that $\beta_i \geq 0$ for all i. Further, using the approximation (8) in Equations (1) and (2) and then simplifying, we see that AMLEs of μ and τ are of the forms, respectively,

$$\hat{\mu}_A = \frac{\sum_{i=1}^m (1 + R_i\beta_i) \ln x_{(i)} + \sqrt{\hat{\tau}_A} \sum_{i=1}^m R_i\alpha_i}{m + \sum_{i=1}^m R_i\beta_i},$$

and

$$\hat{\tau}_A = \left(\frac{A_1 + \sqrt{A_1^2 + 4mB_1}}{2m} \right)^2,$$

where $A_1 = \sum_{i=1}^m R_i\alpha_i(\ln x_{(i)} - K)$, $B_1 = \sum_{i=1}^m (\ln x_{(i)} - K)^2(1 + R_i\beta_i)$, and $K = (\sum_{i=1}^m (1 + R_i\beta_i) \ln x_{(i)})/(m + \sum_{i=1}^m R_i\beta_i)$.

Using approximate likelihood function the asymptotic variance–covariance matrix can be obtained as

$$I(\theta) = \begin{bmatrix} \dfrac{\partial^2 \ln l(\mu,\tau|x)}{\partial \mu^2} & \dfrac{\partial^2 \ln l(\mu,\tau|x)}{\partial \mu \partial \tau} \\ \dfrac{\partial^2 \ln l(\mu,\tau|x)}{\partial \tau \partial \mu} & \dfrac{\partial^2 \ln l(\mu,\tau|x)}{\partial \tau^2} \end{bmatrix}^{-1}_{\theta=\hat{\theta}_A}.$$

Further, the two-sided asymptotic confidence intervals using the normality property of AMLEs for the parameters μ and τ can be obtained.

5. Bayes estimation

Suppose that $X_{(1)}, X_{(2)}, \dots, X_{(m)}$ is progressively type-II censored sample from a $LN(\mu, \tau)$ distribution. Based on such sample, Bayes estimators of the unknown parameters μ and τ are derived under the squared error loss function. We propose to use a natural bivariate prior distribution for μ and τ given as

$$\pi(\mu, \tau) = \pi_2(\tau)\pi_1(\mu|\tau),$$

where $\pi_1(\mu|\tau)$ follows a normal $N(a_1, \tau/b_1)$ distribution and $\pi_2(\tau)$ follows an inverse gamma $IG(p_2, q_2/2)$ distribution with corresponding prior means being a_1 and $q_2/2(p_2 - 1)$, $p_2 > 1$, respectively. Here hyperparameters a_1, b_1, p_2, and q_2 are chosen to reflect the prior knowledge about the unknown parameters. We further refer to Crow and Shimizu [17] for a discussion on considering such prior distributions for parameters of a lognormal distribution. The corresponding posterior distribution is given by

$$\pi(\mu, \tau|x) = \frac{\pi_2(\tau) \times \pi_1(\mu|\tau) \times l(\mu, \tau|x)}{\int_{-\infty}^{\infty} \int_0^{\infty} \pi_2(\tau) \times \pi_1(\mu|\tau) \times l(\mu, \tau|x)\, \mathrm{d}\tau\, \mathrm{d}\mu}, \quad -\infty < \mu < \infty,\ 0 < \tau < \infty.$$

Therefore, if $g(\mu, \tau)$ is any function of μ and τ then its Bayes estimate, under the squared error loss function, has the form

$$\hat{g}_{B1}(x) = \frac{\int_{-\infty}^{\infty} \int_0^{\infty} g(\mu, \tau)\pi(\mu, \tau|x)\, \mathrm{d}\tau\, \mathrm{d}\mu}{\int_{-\infty}^{\infty} \int_0^{\infty} \pi(\mu, \tau|x)\, \mathrm{d}\tau\, \mathrm{d}\mu}. \tag{9}$$

If we take a noninformative prior $\pi^*(\mu, \tau) = 1/\tau$, $-\infty < \mu < \infty$, $0 < \tau < \infty$, one has the posterior distribution of (μ, τ)

$$\pi^*(\mu, \tau|x) \propto \frac{\pi^*(\mu, \tau) \times l(\mu, \tau|x)}{\int_{-\infty}^{\infty} \int_0^{\infty} \pi^*(\mu, \tau) \times l(\mu, \tau|x)\, \mathrm{d}\tau\, \mathrm{d}\mu}, \quad -\infty < \mu < \infty,\ 0 < \tau < \infty.$$

Further, the generalized Bayes estimate of $g(\mu, \tau)$ is given by

$$\hat{g}_{B2}(x) = \int_{-\infty}^{\infty} \int_0^{\infty} g(\mu, \tau)\pi^*(\mu, \tau|x)\, \mathrm{d}\tau\, \mathrm{d}\mu. \tag{10}$$

Since the estimators defined in Equations (9) and (10) cannot be simplified to closed forms, one may use some approximation methods to obtain the estimates. We consider Lindley's method and importance sampling method in this study.

5.1. *Lindley's method*

We use the method of Lindley [18] to obtain approximate explicit Bayes estimators of unknown parameters μ and τ. We have briefly outlined the method in Appendix 1. It can be shown that Bayes estimates of μ and τ using Lindley's method under squared error loss function are of the form

$$\hat{\mu}_{B1} = \hat{\mu} + 0.5[\sigma_{11}^2 l_{30} + 3\sigma_{11}\sigma_{21}l_{21} + \sigma_{11}\sigma_{22}l_{12} + 2\sigma_{12}\sigma_{21}l_{12} + \sigma_{12}\sigma_{22}l_{03}] + \rho_1\sigma_{11} + \rho_2\sigma_{12}, \tag{11}$$

and

$$\hat{\tau}_{B1} = \hat{\tau} + 0.5[\sigma_{11}\sigma_{21}l_{30} + 3\sigma_{12}\sigma_{22}l_{12} + \sigma_{11}\sigma_{22}l_{21} + 2\sigma_{12}\sigma_{21}l_{21} + \sigma_{22}^2 l_{03}] + \rho_1\sigma_{21} + \rho_2\sigma_{22}. \tag{12}$$

Expressions for terms appearing in the right-hand side of Equations (11) and (12) are given in Appendix 1.

We can also compute approximate Bayes estimates of μ and τ using AMLEs in a similar manner. Calculation can follow from Appendix 1. In such a case, we use approximate likelihood function. Explicit expressions for respective generalized Bayes estimates of μ and τ using MLEs and AMLEs can similarly be evaluated. Details are not presented here.

5.2. *Importance sampling method*

In this section, we provide some more approximate Bayes estimates using a Markov Chain Monte Carlo (MCMC) technique. We propose importance sampling method to generate samples from the posterior distribution of μ and τ and then using these samples, we compute desired Bayes estimates. Furthermore, HPD intervals for unknown parameters are also constructed using the method of Chen and Shao [19] (see Appendix 2 for details).

First observe that the posterior distribution μ and τ under the considered bivariate prior distribution is

$$\pi(\mu,\tau|x) \propto IG_\tau\left(\frac{m}{2}+p_2, \frac{1}{2}\left\{\sum_{i=1}^{m}(\ln x_{(i)})^2 + a_1^2 b_1 + q_2 - \frac{(\sum_{i=1}^m \ln x_{(i)} + a_1 b_1)^2}{m+b_1}\right\}\right)$$
$$N_{\mu|\tau}\left(\frac{\sum_{i=1}^m \ln x_{(i)} + a_1 b_1}{m+b_1}, \frac{\tau}{m+b_1}\right) h(\mu,\tau),$$

where $h(\mu,\tau) = \prod_{i=1}^m \{1-\Phi((\ln x_{(i)}-\mu)/\sqrt{\tau})\}^{R_i}$. We generate samples from $\pi(\mu,\tau|x)$ as follows.

Step 1. Generate $\tau_1 \sim IG_\tau(\cdot,\cdot)$ and $\mu_1 \sim N_{\mu|\tau_1}(\cdot,\cdot)$.

Step 2. Repeat Step 1 s times to obtain $(\mu_1,\tau_1), (\mu_2,\tau_2), \ldots, (\mu_s,\tau_s)$.

Bayes estimates of μ and τ under squared error loss is now computed as,

$$\hat{\mu}^* = \frac{\sum_{i=1}^s \mu_i h(\mu_i,\tau_i)}{\sum_{i=1}^s h(\mu_i,\tau_i)}$$

and

$$\hat{\tau}^* = \frac{\sum_{i=1}^s \tau_i h(\mu_i,\tau_i)}{\sum_{i=1}^s h(\mu_i,\tau_i)}.$$

In addition, we can compute approximate generalized Bayes estimates of μ and τ with respect to the noninformative prior $\pi(\mu, \tau) = 1/\tau$, $-\infty < \mu < \infty$, $0 < \tau < \infty$, in a similar manner as above using the importance sampling method.

6. Simulation and data analysis

6.1. *Simulation results*

In this subsection, a Monte Carlo simulation study is conducted to compare the performance of proposed estimators. We simulate progressively type-II censored samples of size m from a given sample of size n with given censoring scheme. We compute MLEs of μ and τ using the EM algorithm and respective AMLEs are taken as the initial guesses for implementing the algorithm. Two different approximate Bayes estimates using Lindley and MCMC samples are computed under squared error loss. For simulation purpose, the unknown parameters are assigned values as $\mu = 0$ and $\tau = 1$. Both the average estimates and mean-squared error (MSE) values are computed based on 5000 replications. We denote Prior 1 for noninformative Bayes estimates and Prior 2 for proper Bayes estimates in which case hyperparameters are given values as $a_1 = 0.01$, $b_1 = 1$, $p_2 = 3$, and $q_2 = 4$. We are able to suggest these values based on our discussion given in the beginning of Section 5. We further mention that, given the true values of parameters, prior means of μ and τ have been taken into considerations in specifying values for hyperparameters. For convenience, short notations are used to represent different censoring schemes, for instance, scheme $(5, 0, 0, 0, 0)$ is denoted by $(5, 0^{*4})$. In Table 1, various average estimates and corresponding MSEs in parentheses are presented for different censoring schemes.

From Table 1, we observe that the MLEs and AMLEs behave almost similar in terms of absolute biases and MSE values. This holds true for all presented schemes. Bayes estimates obtained from Lindley's approach using MLEs and AMLEs respectively behave like MLEs and AMLEs. However, it seems that these Bayes estimates are marginally good. Noninformative MCMC estimates also compete quite well with these estimates. Among proper Bayes estimates of μ the one's obtained from Lindley's approach using MLEs performs really well. The corresponding MCMC estimates are also good. For estimating τ, the estimates from Lindley's approach using AMLEs show good performance. In this case, performance of the corresponding MCMC estimates is highly noticeable. Also, with the increase in effective sample size, absolute biases and MSE values of all estimates decrease significantly and this holds true for all proposed censoring schemes.

In Table 2, the average lengths and coverage probabilities for 95% interval estimates are presented for both the parameters μ and τ. We tabulate these results using different censoring schemes. We observe that the average lengths of asymptotic confidence intervals for μ using AMLEs is marginally wider than the corresponding lengths obtained using MLEs. However, in case of the parameter τ, the average lengths obtained using MLEs and AMLEs are almost alike. The interval estimates of μ obtained using the noninformative prior (Prior 1) compete quite well with those of the corresponding asymptotic estimates in terms of average lengths. On the other hand for τ, the performance of HPD intervals under prior 1 is not so satisfactory. We further observe that the HPD intervals (under Prior 2) show superior behaviour to all its competitors. This holds true for both the parameters μ and τ. Also, the average lengths of all confidence intervals tend to decrease with the increase in effective sample size. In the table, we also tabulate the coverage probabilities for the interval estimates of μ and τ. One can see that the most of the coverage probabilities lie below nominal level of 95% for different censoring schemes. Balakrishnan et al. [5] indicated that when the effective sample size is relatively small then the coverage probabilities may be unsatisfactory. In fact, we observe that (not reported in the table) the corresponding coverage probabilities do improve with the increase in effective sample size.

Table 1. Average and MSE values (in parentheses) of all estimates of μ and τ.

						Bayes (Prior 1)			Bayes (Prior2)		
						Lindley			Lindley		
n	m	Scheme	Parameter	MLE	AMLE	MLE	AMLE	MCMC	MLE	AMLE	MCMC
15	6	$(9, 0^{*5})$	μ	−0.07450	−0.08378	0.07994	0.02283	0.04665	−0.01466	0.02305	−0.03714
				(0.15067)	(0.15105)	(0.14559)	(0.13755)	(0.14978)	(0.09327)	(0.08965)	(0.10672)
			τ	0.89334	0.91943	1.19466	1.26226	1.40242	0.93727	1.01486	0.93368
				(0.23472)	(0.23912)	(0.40479)	(0.43841)	(0.72984)	(0.04185)	(0.04853)	(0.10249)
15	6	$(0^{*5}, 9)$	μ	−0.10312	−0.09236	0.03385	0.03070	0.03422	0.03098	0.02891	−0.02598
				(0.14152)	(0.13737)	(0.12346)	(0.11987)	(0.21433)	(0.18363)	(0.15024)	(0.08716)
			τ	0.82040	0.84304	1.17183	1.18064	1.18025	0.88503	0.86086	0.88374
				(0.38194)	(0.36614)	(0.37034)	(0.40177)	(0.41269)	(0.28139)	(0.32469)	(0.11498)
15	9	$(6, 0^{*8})$	μ	−0.02349	−0.03182	0.01122	0.02456	0.01182	−0.00987	−0.01014	−0.01019
				(0.10021)	(0.10069)	(0.10004)	(0.09865)	(0.09987)	(0.07808)	(0.07648)	(0.08076)
			τ	0.91649	0.93492	1.16231	1.19627	1.26159	0.94403	0.97971	0.92831
				(0.17697)	(0.18138)	(0.29485)	(0.30358)	(0.39803)	(0.06312)	(0.06369)	(0.09285)
15	9	$(0^{*8}, 6)$	μ	−0.0417	−0.04097	0.01422	0.03831	−0.04610	−0.01403	0.01327	−0.02427
				(0.08794)	(0.08762)	(0.08324)	(0.08413)	(0.10674)	(0.06477)	(0.11883)	(0.07260)
			τ	0.87960	0.88257	1.20588	1.23179	1.39418	0.93470	0.96879	0.91869
				(0.24854)	(0.24574)	(0.40219)	(0.43106)	(0.44635)	(0.02827)	(0.03057)	(0.11352)
20	8	$(12, 0^{*7})$	μ	−0.05453	−0.06497	0.05116	0.06052	0.02713	−0.02128	−0.0293	−0.02702
				(0.11377)	(0.11512)	(0.11153)	(0.10776)	(0.11551)	(0.08210)	(0.08369)	(0.08331)
			τ	0.94103	0.96412	1.17874	1.23007	1.25715	0.94938	1.00917	0.93924
				(0.19070)	(0.19582)	(0.31006)	(0.33379)	(0.36667)	(0.06224)	(0.07037)	(0.09113)
20	8	$(0^{*7}, 12)$	μ	−0.0786	−0.07599	0.03162	−0.05539	−0.01057	−0.01308	−0.04397	−0.07873
				(0.10345)	(0.10232)	(0.08087)	(0.084939)	(0.15955)	(0.08235)	(0.08757)	(0.08067)
			τ	0.86010	0.86704	1.16926	1.42074	1.34669	0.89540	1.22736	0.88012
				(0.29664)	(0.29219)	(0.21906)	(0.17277)	(0.14255)	(0.17211)	(0.13211)	(0.12019)
20	10	$(10, 0^{*9})$	μ	−0.03165	−0.04105	0.02744	−0.03179	0.02466	−0.01113	−0.01822	−0.01117
				(0.09101)	(0.09147)	(0.09026)	(0.08854)	(0.09201)	(0.07190)	(0.07056)	(0.07254)
			τ	0.93059	0.95119	1.13859	1.17721	1.21326	0.94386	0.98613	0.93888
				(0.15558)	(0.16083)	(0.23977)	(0.25401)	(0.30026)	(0.06538)	(0.06888)	(0.08735)
20	10	$(0^{*9}, 10)$	μ	−0.04931	−0.04869	0.02028	0.06198	−0.06533	−0.02139	0.04248	−0.04007
				(0.07729)	(0.07711)	(0.06844)	(0.07679)	(0.11234)	(0.05279)	(0.14937)	(0.06912)
			τ	0.88874	0.89066	1.18179	1.22831	1.22196	0.91891	0.98563	0.90789
				(0.23398)	(0.23191)	(0.29338)	(0.37161)	(0.36695)	(0.02117)	(0.02864)	(0.11941)
20	10	(1^{*10})	μ	−0.04081	−0.04418	0.03830	0.04282	0.03930	−0.01771	0.01805	−0.03399
				(0.08096)	(0.08099)	(0.07644)	(0.07483)	(0.08791)	(0.05827)	(0.06444)	(0.06393)
			τ	0.89969	0.90456	1.16013	1.18751	1.29279	0.93543	0.97170	0.93041
				(0.19768)	(0.19805)	(0.28050)	(0.30872)	(0.49620)	(0.04167)	(0.04809)	(0.09832)
25	10	$(15, 0^{*9})$	μ	−0.04212	−0.05258	0.02548	−0.03529	0.02542	−0.01797	−0.02663	−0.01862
				(0.08981)	(0.09054)	(0.08829)	(0.08595)	(0.09316)	(0.06964)	(0.06842)	(0.07118)
			τ	0.93854	0.96181	1.12926	1.17516	1.19248	0.94296	0.99483	0.94653
				(0.14942)	(0.15537)	(0.22072)	(0.23988)	(0.26978)	(0.06498)	(0.07122)	(0.08078)
25	10	$(0^{*9}, 15)$	μ	−0.07115	−0.07036	0.06270	−0.06457	−0.04932	−0.02884	−0.03143	−0.02547
				(0.08357)	(0.08308)	(0.06388)	(0.06504)	(0.14583)	(0.05362)	(0.05397)	(0.08360)
			τ	0.87639	0.87856	1.15066	1.2899	1.13208	0.89028	1.09379	0.88767
				(0.23625)	(0.23500)	(0.18486)	(0.38867)	(0.48823)	(0.04485)	(0.23322)	(0.14363)
25	15	$(10, 0^{*14})$	μ	−0.01784	−0.02530	0.02117	−0.02819	0.02724	−0.01473	−0.01925	−0.01319
				(0.06222)	(0.06258)	(0.06196)	(0.06183)	(0.06383)	(0.05401)	(0.05398)	(0.05078)
			τ	0.95416	0.97189	1.10871	1.13612	1.13971	0.95846	0.98731	0.96010
				(0.11119)	(0.11543)	(0.15823)	(0.16670)	(0.17125)	(0.06392)	(0.06590)	(0.07404)
25	15	$(0^{*14}, 10)$	μ	−0.02150	−0.02167	0.01381	0.01972	−0.08664	−0.0126	−0.01317	−0.01207
				(0.05156)	(0.05156)	(0.04997)	(0.04974)	(0.07342)	(0.04206)	(0.04158)	(0.05918)
			τ	0.92542	0.92526	1.13589	1.14265	1.09375	0.94405	0.95423	0.90838
				(0.15098)	(0.15085)	(0.21222)	(0.21799)	(0.30884)	(0.05454)	(0.05683)	(0.09837)

Table 2. Average interval length and coverage probability (in parentheses) for μ and τ.

			Average asymptotic interval length				Average HPD interval length			
			MLE		AMLE		Prior 1		Prior 2	
n	m	Scheme	μ	τ	μ	τ	μ	τ	μ	τ
15	6	$(9, 0^{*5})$	1.38135 (0.8906)	1.76097 (0.7905)	1.42023 (0.9222)	1.74599 (0.7997)	1.67596 (0.944)	2.91635 (0.944)	1.29798 (0.9425)	1.35903 (0.9285)
15	6	$(0^{*5}, 9)$	1.20592 (0.8186)	2.07104 (0.6973)	1.4577 (0.9285)	2.142 (0.7389)	1.122046 (0.7945)	4.84457 (0.8745)	0.89412 (0.8795)	1.39173 (0.939)
15	9	$(6, 0^{*8})$	1.17001 (0.9115)	1.55059 (0.8123)	1.19655 (0.9346)	1.54147 (0.8171)	1.36392 (0.954)	2.26529 (0.9485)	1.137 (0.95)	1.27703 (0.935)
15	9	$(0^{*8}, 6)$	1.03473 (0.8835)	1.77073 (0.785)	1.09152 (0.9288)	1.77546 (0.7908)	0.95916 (0.837)	2.86307 (0.899)	0.81868 (0.8765)	1.29945 (0.911)
20	8	$(12, 0^{*7})$	1.23036 (0.9126)	1.56941 (0.8252)	1.25308 (0.9307)	1.55126 (0.8326)	1.41441 (0.9425)	2.24901 (0.9375)	1.1705 (0.9465)	1.2678 (0.931)
20	8	$(0^{*7}, 12)$	1.09609 (0.8524)	1.90973 (0.7557)	1.24509 (0.9376)	1.95121 (0.786)	1.10326 (0.903)	2.15583 (0.7375)	0.88751 (0.9345)	1.09639 (0.818)
20	10	$(10, 0^{*10})$	1.11617 (0.9155)	1.4545 (0.8311)	1.13622 (0.9342)	1.44085 (0.8365)	1.26264 (0.95)	1.98369 (0.947)	1.0732 (0.943)	1.2024 (0.92)
20	10	$(0^{*10}, 10)$	0.98281 (0.8841)	1.723 (0.7915)	1.04597 (0.9311)	1.73291 (0.8047)	0.97238 (0.9021)	1.7868 (0.7865)	0.92228 (0.93412)	1.04615 (0.8155)
20	10	(1^{*10})	1.01183 (0.8969)	1.60016 (0.8003)	1.05744 (0.9316)	1.60343 (0.8094)	1.0771 (0.8985)	2.42263 (0.9265)	0.86449 (0.8945)	1.23971 (0.9085)
25	10	$(15, 0^{*9})$	1.11928 (0.9189)	1.42981 (0.8411)	1.13464 (0.9361)	1.41089 (0.8487)	1.23256 (0.945)	1.83796 (0.938)	1.06628 (0.9475)	1.18769 (0.922)
25	10	$(0^{*9}, 15)$	1.00945 (0.8735)	1.77415 (0.7899)	1.10633 (0.8829)	1.80098 (0.8182)	0.85931 (0.8485)	1.39777 (0.6615)	0.82279 (0.8955)	0.94275 (0.76)
25	15	$(10, 0^{*14})$	0.94129 (0.9299)	1.25625 (0.8667)	0.95334 (0.9415)	1.24773 (0.8705)	1.0168 (0.9375)	1.52525 (0.949)	0.90939 (0.9425)	1.07588 (0.9235)
25	15	$(0^{*14}, 10)$	0.83371 (0.9111)	1.44355 (0.841)	0.86084 (0.9355)	1.44604 (0.8452)	0.79683 (0.9025)	1.14917 (0.7465)	0.78257 (0.93578)	0.86471 (0.747)

6.2. *Data analysis*

In this section, we analyse a real data set to illustrate the implementation of proposed estimation methods. For this purpose, we use the data set given in [20] which represents the number of million revolutions before failure for 23 ball bearings. The corresponding observations are

17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84	51.96	54.12
55.56	67.80	68.64	68.64	68.88	84.12	93.12	98.64	105.12	105.84
127.92	128.04	173.40							

We first make an inference whether a $LN(\mu, \tau)$ distribution will fit the given data. For comparison purposes, we have also taken into consideration fitting log-logistic and Weibull distributions with both having, say, a shape parameter μ and a scale parameter τ. We used the method of maximum likelihood for estimating the unknown parameters of each of these models. The negative log-likelihood criterion (NLC), Kolmogorov–Smirnov (K–S) statistic, chi-squared (χ^2) statistic, Akaike's information criterion (AIC), and Bayesian information criterion (BIC) are employed to judge the goodness of fit. We refer to Krishna and Malik [21] and Dey and Kundu [22] for a review on these criteria. The observed and expected frequencies for different groups for χ^2 statistic are reported for all distributions in Table 3. We have compared different proposed criteria in Table 4. Based on the reported values, we observe that lognormal distribution fits the data reasonably well. So we analyse the given data set using this distribution.

Table 3. The observed and expected frequencies.

		Expected frequencies		
Intervals	Observed frequencies	Lognormal	Log-logistic	Weibull
0–35	3	2.92	2.69	3.55
35–55	7	6.09	5.95	4.54
55–80	5	6.43	6.96	6.04
80–100	3	3.15	3.18	3.85
100-	5	4.41	4.22	5.02

Table 4. Goodness-of-fit tests for given distributions.

Model	$\hat{\mu}$	$\hat{\tau}$	NLC	AIC	BIC	K–S	χ^2
lognormal	4.15038	0.27215	113.1286	230.2571	232.5281	0.0897	0.54
log-logistic	4.15880	0.29881	113.3730	230.7460	233.0170	0.0943	0.92
Weibull	2.10184	0.01221	113.6920	231.3839	233.6549	0.1510	1.78

Table 5. Generated progressively censored samples.

	Observations											
Scheme	$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$	$x_{(6)}$	$x_{(7)}$	$x_{(8)}$	$x_{(9)}$	$x_{(10)}$	$x_{(11)}$	$x_{(12)}$
$(11, 0^{*11})$	17.88	68.64	68.64	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.40
$(0, 11, 0^{*10})$	17.88	28.92	68.64	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.40
$(0^{*2}, 11, 0^{*9})$	17.88	28.92	33.00	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.40
$(0^{*11}, 11)$	17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84	51.96	54.12	55.56	67.80

Table 6. Estimates of μ and τ.

				Bayes		
				Lindley		
Scheme	Parameter	MLE	AMLE	MLE	AMLE	MCMC Prior 1
$(11, 0^{*11})$	μ	4.44525	4.41193	4.45131	4.45773	4.44829
	τ	0.28984	0.34178	0.35184	0.42472	0.35838
$(0, 11, 0^{*10})$	μ	4.41371	4.40625	4.44058	4.46989	4.44453
	τ	0.33856	0.34606	0.40493	0.43868	0.42478
$(0^{*2}, 11, 0^{*9})$	μ	4.39162	4.38686	4.42771	4.45773	4.42867
	τ	0.37353	0.3774	0.45255	0.48194	0.45942
$(0^{*11}, 11)$	μ	4.18420	4.18406	4.25525	4.34307	4.06632
	τ	0.31769	0.31762	0.44753	0.49545	0.31213

We generate four different progressively type-II censored samples of size $m = 12$ from $n = 23$ observations using four different censoring schemes and data are listed in Table 5. In Table 6, we tabulate all estimates of both unknown parameters μ and τ. MLEs are computed using the EM algorithm. AMLEs are also presented in the table. Noninformative prior is taken into consideration for producing the Bayes estimates against squared error loss. The method of Lindley is used to compute two different Bayes estimates using MLEs and AMLEs, respectively. Noninformative MCMC estimates are also obtained and presented in the table. In Table 7, the 95% approximate confidence intervals using asymptotic distributions of MLEs and AMLEs are tabulated. The corresponding HPD intervals of μ and τ are also given in Table 7.

Table 7. Interval estimates for μ and τ.

Scheme	Parameter	Asymp. C. I. using MLE	Asymp. C. I. using AMLE	HPD C. I. using Prior 1
$(11, 0^{*11})$	μ	(4.14285, 4.74766)	(3.99604, 4.82782)	(4.11469, 4.79374)
	τ	(0.06598, 0.5137)	(0.10815, 0.57541)	(0.13049, 0.71376)
$(0, 11, 0^{*10})$	μ	(4.09968, 4.72774)	(3.99235, 4.82016)	(4.104, 4.79734)
	τ	(0.09613, 0.581)	(0.10008, 0.59205)	(0.16642, 0.80115)
$(0^{*2}, 11, 0^{*9})$	μ	(4.06777, 4.71547)	(3.96737, 4.80635)	(4.05578, 4.76457)
	τ	(0.10592, 0.64115)	(0.10279, 0.65201)	(0.1572, 0.88241)
$(0^{*11}, 11)$	μ	(3.90683, 4.46157)	(3.77884, 4.58928)	(3.93598, 4.14932)
	τ	(0.03549, 0.59988)	(0.00924, 0.626)	(0.1858, 0.84407)

7. Optimal censoring

In previous sections, we considered point and interval estimations of unknown parameters μ and τ of the $LN(\mu, \tau)$ distribution when samples are obtained using progressive type-II censoring. We derived different estimates using several known censoring schemes. However, in various reliability and life testing studies, it is desirable for practical considerations to select the optimum progressive censoring scheme from a class of possible schemes. That is, for a given n and m, one needs to select R_i, $i = 1, 2, \ldots, m$ (with $\sum_{i=1}^{m} R_i = n - m$) which is optimal in the sense that the selected scheme will provide maximum information about the unknown parameters. The problem of comparing two different censoring schemes has received much interest among various researchers, see for example, Ng et al.,[23] Kundu,[10] and Pradhan and Kundu.[8] We consider several different criteria to compare two different censoring schemes. In particular, we will utilize the following optimality criteria and one may refer to Ng et al. [23] for more discussions on these criteria.

Criterion I: Minimizing the determinant of the variance–covariance matrix $V(\theta)$ of the MLEs.
Criterion II: Minimizing the trace of the variance–covariance matrix $V(\theta)$ of the MLEs.
Criteria III, IV, and V: Minimizing the variance of logarithm of the pth quantile when $p = 0.5$, 0.90, and 0.95, respectively.

Further, note that the MLE of logarithm of the p-th quantile of the $LN(\mu, \tau)$ distribution is given by

$$\ln T_p = \hat{\mu} + \sqrt{\hat{\tau}}\,\Phi^{-1}(p),$$

with corresponding asymptotic variance

$$\mathrm{Var}[\hat{\mu} + \sqrt{\hat{\tau}}\,\Phi^{-1}(p)] = V_{11}(n, R) + \left(\frac{\Phi^{-1}(p)}{2\sqrt{\hat{\tau}}}\right)^2 V_{22}(n, R) + \left(\frac{\Phi^{-1}(p)}{\sqrt{\hat{\tau}}}\right) V_{12}(n, R).$$

Unfortunately, this information measure (minimizing the variance) depends on p. Gupta and Kundu [11] and Pradhan and Kundu [8] proposed an information measure as follows:

$$I_W = \int_0^1 (\mathrm{Var}[\hat{\mu} + \sqrt{\hat{\tau}}\,\Phi^{-1}(p)])_p W(p)\,\mathrm{d}p,$$

where $W(p)$ is a nonnegative function satisfying

$$\int_0^1 W(p)\,\mathrm{d}p = 1.$$

This information measure represents the average asymptotic variance of the quantile estimators over all quantile points and, hence is independent of p. Without loss of generality, one might choose $W(p) = 1$, for $0 < p < 1$. Hence, we have the following optimality criterion.

Criterion VI: Minimizing I_W.

As mentioned by Pradhan and Kundu,[8] although the total number of sampling schemes are finite, they can be quite large. So far one does not have any efficient algorithm to search the optimal schemes from all possible progressive censoring schemes. Since for given n and m, all the censoring schemes $(R_1, R_2, \ldots, R_m)$ with $R_1 + R_2 + \cdots + R_m = n - m$ will belong to the convex hull generated by the points $(n - m, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, n - m)$. Pradhan and Kundu [8] suggested that a sub-optimal censoring scheme can be obtained by selecting the optimal censoring scheme among these extreme points on the convex hull. The expected test time is also an important factor in reliability studies because the time required to complete an experiment has direct effect on the associated cost. Under progressive type-II censoring, the time required to complete the test is the time to observe the mth failure $X_{(m)}$. For a given censoring scheme, the total expected test time can be computed as

$$E(X_{(m)}) = c_{m-1} \sum_{i=1}^{m} a_{i,m} \int_0^1 \mathrm{e}^{\mu+\sqrt{\tau}\Phi^{-1}(v)} (1 - v)^{r_i - 1}\, \mathrm{d}v.$$

In Table 8, we present the optimal censoring scheme for different n and m when $\mu = 0$ and $\tau = 1$. In this table, E_i represents that $n - m$ is at the ith position in the extreme points on convex hull. For example, if $n = 10$ and $m = 5$, then the optimal plan $(0, 5, 0, 0, 0)$ is denoted by E_2. The corresponding expected test time is also reported in Table 8. It can be observed that Criteria IV

Table 8. Optimal progressive type-II censoring plan.

(n, m)	Optimality criterion	Optimal plan	$E(X_{(m)})$
(15,6)	I, II	E_2	3.92863
	III	E_6	0.75380
	IV, V	E_1	4.21752
	VI	E_3	3.55590
(15,9)	I,VI	E_4	4.74556
	II	E_3	4.94981
	III	E_9	1.24207
	IV, V	E_1	5.24957
(20,6)	I,VI	E_2	3.85302
	II,IV, V	E_1	4.17873
	III	E_4	2.89175
(20,8)	I,VI	E_3	4.41981
	II	E_2	4.68157
	III	E_8	0.75937
	IV, V	E_1	4.89950
(20,10)	I	E_3	5.16293
	II	E_2	5.33940
	III	E_{10}	0.97673
	IV, V	E_1	5.49033
	VI	E_4	4.95334
(25,8)	I	E_3	4.34496
	II, VI	E_2	4.63246
	III	E_5	3.56059
	IV, V	E_1	4.87543
(25,10)	I, VI	E_3	5.08116
	II	E_2	5.28642
	III	E_9	2.39079
	IV, V	E_1	5.46461
(25,12)	I, VI	E_4	5.51178
	II	E_3	5.68445
	III	E_{12}	5.21235
	IV, V	E_1	5.96856

Table 9. Optimal progressive censoring plan for real data.

	Scheme	Criterion I	II	III	IV	V	VI	$E(X_{(m)})$
1	$(11, 0^{*11})$	0.000245	0.033505	0.021957	0.045085	0.057595	0.031918	173.40
		(1.379592)	(1.202209)	(0.915289)	(1.438372)	(1.510999)	(1.127201)	
2	$(0, 11, 0^{*10})$	0.000365	0.040463	0.024714	0.054492	0.069856	0.036299	173.40
		(0.926027)	(0.995477)	(0.813182)	(1.190065)	(1.245791)	(0.991156)	
3	$(0^{*2}, 11, 0^{*9})$	0.000498	0.047089	0.026913	0.062688	0.080757	0.040177	173.40
		(0.678714)	(0.855401)	(0.746739)	(1.034472)	(1.077628)	(0.895487)	
4	$(0^{*11}, 11)$	0.000338	0.040280	0.020097	0.064849	0.087026	0.035978	67.80
		(1)	(1)	(1)	(1)	(1)	(1)	

and V always give the same optimal plan E_1; that is, when $n - m$ items are removed at the time of the first failure observed. The advantage of this type of censoring is that if the items put on a test are very costly, the removed surviving items on early stage can be used for the other purposes. However, the expected test time is maximum in this case, as expected. The Criterion III mostly provides the optimal plan E_m which is type-II censoring. For large n and small effective sample size m, the optimal plan given by Criterion III slightly shifts to the left of the convex hull. For example, when $n = 25$ and $m = 12$, the optimal plan under Criterion III is E_{12}; but for $n = 25$ and $m = 10$, the plan becomes E_9, and for $n = 25$ and $m = 8$ plan is E_5. The expected test time in this case is minimum but items on the test can not be saved on the early stage hence it may increase the cost of test. Furthermore, we observe that the optimal plan given by Criterion VI slightly shifts to the right of the convex hull with the increase in the effective sample size m. Also, similar behaviour is observed in most of the cases for Criteria I and II. Finally, optimal plans given by Criteria I, II and VI can be used to balance between the cost of expected test time and cost of failure of items.

In Table 9, we report all the values of proposed criteria for progressive type-II censoring schemes as listed in Table 5. Note that scheme 4 is the traditional type-II censoring. Thus, we also report the gain in efficiency (in parentheses) when the other censoring schemes is used in place of type-II censoring. From this table, it can be observed that scheme 1 is the optimal scheme for all proposed criteria except criterion III ($p = 0.50$). Similarly, except criteria IV ($p = 0.90$) and V ($p = 0.95$), type-II censoring is the optimal one when we compare schemes 2, 3 and 4. Scheme 2 can also be used instead of type-II censoring as there is a small difference in gain of efficiency for criteria II and VI.

8. Conclusions

In this paper we have considered estimation of unknown parameters μ and τ of a lognormal distribution when the data are collected under a progressive type-II censoring scheme. We have derived the MLEs and AMLEs of these unknown parameters. Bayes estimates are computed by using Lindley's method and importance sampling under squared error loss. The asymptotic confidence intervals and HPD intervals are also obtained. We have also studied the optimal censoring scheme under different criteria. The results for some small values of n and m are reported. Since the optimal scheme is searched among the extreme points on a convex hull, it may not be the global optimum. More work on establishing a searching algorithm is needed in the future. Finally, further investigation into other lifetime distributions as well as designing a test plan in presence of cost constraint are also interesting direction.

Acknowledgements

The authors thank the Editor, Associate Editor, and an anonymous referee for their suggestions that have led significant improvements in the presentation of the manuscript. The first author thanks the Council of Scientific and Industrial Research (CSIR) India for funding this research work.

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Appendix 1

For the two parameter case (λ_1, λ_2), the Lindley's approximation to the integral (9) suggest that

$$\hat{g}_{B1} = g(\hat{\lambda_1}, \hat{\lambda_2}) + \tfrac{1}{2}[A + l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}] + \rho_1 A_{12} + \rho_2 A_{21}, \tag{A1}$$

where

$$A = \sum_{i=1}^{2}\sum_{j=1}^{2} w_{ij}\sigma_{ij}, \quad l_{ij} = \frac{\partial^{i+j}L(\lambda_1, \lambda_2)}{\partial\lambda_1^i \partial\lambda_2^j},$$

$$\rho_i = \frac{\partial\rho}{\partial\lambda_i}, \quad w_i = \frac{\partial g}{\partial\lambda_i}, \quad w_{ij} = \frac{\partial^2 g}{\partial\lambda_i\partial\lambda_j},$$

$$\rho = \ln \pi(\lambda_1, \lambda_2), \quad A_{ij} = w_i\sigma_{ii} + w_j\sigma_{ji},$$
$$B_{ij} = (w_i\sigma_{ii} + w_j\sigma_{ij})\sigma_{ii}, \quad C_{ij} = 3w_i\sigma_{ii}\sigma_{ij} + w_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2).$$

Here $L(\cdot,\cdot)$ denotes the log-likelihood function, $\pi(\lambda_1, \lambda_2)$ denotes the corresponding prior distribution and σ_{ij} is the $(i,j)th$ element of the inverse of the Fisher information matrix. Note that the expressions in Equation (A1) are evaluated at the MLE $(\hat{\lambda}_1, \hat{\lambda}_2)$. For the case of our estimation problem with $(\lambda_1, \lambda_2) = (\mu, \tau)$, the approximate Bayes estimates of μ and τ are computed using the expression (A1). These estimates are given in Equations (11) and (12), respectively and the corresponding expressions are evaluated as

$$\rho_1 = \frac{b_1(a_1 - \mu)}{\tau}, \quad \rho_2 = \frac{1}{2\tau^2}(q_2 + b_1(\mu - a_1)^2) - \frac{1}{\tau}(p_2 + 1.5),$$

$$l_{30} = \frac{1}{\tau^{1.5}}\left\{\sum_{i=1}^{m} R_i\,(y_{(i)}^2 - 1)h(y_{(i)}) - 3\sum_{i=1}^{m} R_i y_{(i)} h^2(y_{(i)}) + 2\sum_{i=1}^{m} R_i h^3(y_{(i)})\right\},$$

$$l_{03} = \frac{1}{8\tau^3}\left\{-8m + 24\sum_{i=1}^{m} y_{(i)}^2 + \sum_{i=1}^{m} R_i(y_{(i)}^5 - 10y_{(i)}^3 + 15y_{(i)})h(y_{(i)})\right.$$
$$\left. - \sum_{i=1}^{m} R_i(3y_{(i)}^4 - 9y_{(i)}^2)h^2(y_{(i)}) + 2\sum_{i=1}^{m} R_i y_{(i)}^3 h^3(y_{(i)})\right\},$$

$$l_{21} = \frac{1}{2\tau^2}\left\{2m + \sum_{i=1}^{m} R_i(y_{(i)}^3 - 3y_{(i)})h(y_{(i)}) - \sum_{i=1}^{m} R_i(3y_{(i)}^2 - 2)h^2(y_{(i)})\right.$$
$$\left. +2\sum_{i=1}^{m} R_i y_{(i)} h^3(y_{(i)})\right\},$$

$$l_{12} = \frac{1}{4\tau^{5/2}}\left\{8\sum_{i=1}^{m} y_{(i)} + \sum_{i=1}^{m} R_i(y_{(i)}^4 - 6y_{(i)}^2 + 3)h(y_{(i)}) - \sum_{i=1}^{m} R_i(3y_{(i)}^3 - 5y_{(i)})h^2(y_{(i)})\right.$$
$$\left. +2\sum_{i=1}^{m} R_i y_{(i)}^2 h^3(y_{(i)})\right\}.$$

Appendix 2

Recall the samples (μ_i, τ_i), $i = 1, 2, \ldots, s$, as generated in Section 5.2. In this appendix, we briefly describe the method of Chen and Shao [19] for computing HPD intervals. Suppose that μ is the unknown parameter of interest, and $\pi(\mu|x)$, $\Pi(\mu|x)$ denote its posterior density and posterior distribution functions, respectively. If $\mu^{(p)}$ denotes the pth quantile of μ, then we have $\mu^{(p)} = \inf\{\mu : \Pi(\mu|x) \geq p; 0 < p < 1\}$. It can be observed that for a given μ^*, a simulation consistent estimator of $\Pi(\mu^*|x)$ can be obtained by

$$\Pi(\mu^*|x) = \frac{\sum_{i=1}^{s} 1_{\mu \leq \mu^*} h(\mu_i, \tau_i)}{\sum_{i=1}^{s} h(\mu_i, \tau_i)},$$

where $1_{\mu \leq \mu^*}$ is the indicator function. Let $\mu_{(i)}$ be the ordered values of μ_i. Then the corresponding estimate is obtain as

$$\hat{\Pi}(\mu^*|x) = \begin{cases} 0 & \text{if } \mu^* < \mu_{(1)}, \\ \sum_{j=1}^{i} w_j & \text{if } \mu_{(i)} \leq \mu^* < \mu_{(i+1)}, \\ 1 & \text{if } \mu^* \geq \mu_{(s)}, \end{cases}$$

where

$$w_i = \frac{h(\mu_{(i)}, \tau_{(i)})}{\sum_{i=1}^{s} h(\mu_{(i)}, \tau_{(i)})}, \quad i = 1, 2, \ldots, s.$$

Now, $\mu^{(p)}$ is estimated by

$$\hat{\mu}^{(p)} = \begin{cases} \mu_{(1)} & \text{if } p = 0, \\ \mu_{(i)} & \text{if } \sum_{j=1}^{i-1} w_j < p \leq \sum_{j=1}^{i} w_j. \end{cases}$$

To obtain a $100(1-p)\%$ confidence interval for μ, we consider the intervals of the form $(\hat{\mu}^{(j/s)}, \hat{\mu}^{((j+[(1-p)s])/s)}), i = 1, 2, \ldots, s - [(1-p)s]$ with $[u]$ denoting the greatest integer less than or equal to u. The interval with the smallest width is treated as the HPD interval. Similarly the HPD interval for the parameter τ can be constructed.