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Adaptive T–S fuzzy-neural modeling and control for general MIMO unknown nonaffine nonlinear systems using projection update laws☆

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ABSTRACT

This paper describes a novel design of an on-line Takagi–Sugeno (T–S) fuzzy-neural controller for a class of general multiple input multiple output (MIMO) systems with unknown nonlinear functions and external disturbances. Instead of modeling the unknown systems directly, the T–S fuzzy-neural model approximates a virtual linearized system (VLS) of a real system with modeling errors and external disturbances. Compared with previous approaches, the main contribution of this paper is an investigation of more general MIMO unknown systems using on-line adaptive T–S fuzzy-neural controllers. In this paper, we also use projection update laws, which generalize the projection algorithm, to tune the adjustable parameters. This prevents parameter drift and ensures that the parameter matrix is bounded away from singularity. We prove that the closed-loop system controlled by the proposed controller is robust stable and the effect of all the modeling errors and external disturbances on the tracking error can be attenuated. Finally, two examples covering four cases are simulated in order to confirm the effectiveness and applicability of the proposed approach in this paper.

1. Introduction

The Takagi–Sugeno (T–S) fuzzy approach (Takagi & Sugeno, 1985) has been extensively used to model nonlinear systems. Two methods are often employed to construct T–S fuzzy models, namely approximated modeling (Tanaka, Iwasaki, & Wang, 2001) using local linearization techniques, and exact modeling (Lian, Chiang, Chiu, & Liu, 2001) using nonlinear combination techniques. In Cao, Rees, and Feng (1997) and Feng, Cao, Rees, and Chak (1997), the authors proved that the T–S fuzzy system can approximate any continuous function to any precision. Many studies (Chien, Wang, Li, & Su, 2006; Lam & Leung, 2007; Lee, Kuo & Wang, 2004) combining fuzzy logic with neural networks, called fuzzy-neural networks, have been carried out to improve the efficiency of function approximation. This paper is focused on stability analysis and controller design of on-line T–S fuzzy-neural control for general unknown nonaffine nonlinear systems.

Recently, some adaptive control schemes for nonlinear systems via fuzzy-neural networks have been proposed (Leu, Wang, & Lee, 2005; Wang, Cheng, & Leu, 2004). By using the well-known off-line tuning algorithms, an initial fuzzy-neural model with adjustable parameters can be constructed for unknown nonlinear systems. However, the derived fuzzy-neural model with the off-line tuned parameters cannot cope with parameter changes arising from external disturbances (Park & Cho, 2004). Thus, off-line algorithms cannot be applied to situations where real-time processing is required, such as adaptive control and signal processing. In Hou, Liao, and Yan (2007), Lam and Leung (2007) and Wang, Tanaka, and Griffin (1996), the stabilization problem for the systems represented in T–S fuzzy-neural models was addressed, but studies concerning tracking controller design based on T–S fuzzy-neural models for unknown nonlinear systems are relatively few. In Lin, Wang, and Lee (2006) and Park and Cho (2004), the authors only consider the stabilization problem for affine systems. Moreover, theoretical justification development presented in Leu, Wang, and Lee (1999) and Wang et al. (1996) is valid only for SISO nonlinear systems and so is hardly practical in real applications such as the trajectory control of robot manipulators and space vehicles.

In Park and Cho (2004) and Zheng, Wang, and Lee (2002), adaptive fuzzy controllers were developed for a nonlinear dynamical system. Unfortunately, the singularity of the parameter matrix was not discussed. It is well known that for a general update law, the denominator of the multinomial may be singular at some

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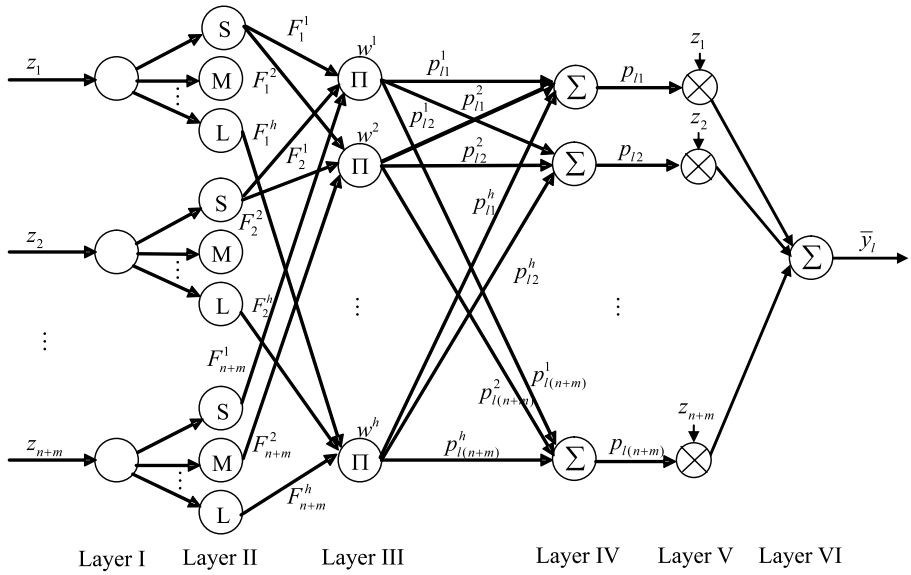

Fig. 1. Configuration of a fuzzy-neural approximator.

time which leads to infeasibility of the controller design. Thus, further improvement for the design algorithm is required, not only to attenuate the effects caused by the unmodeled dynamics, external disturbances, and modeling errors by using adaptive update laws, but also to tune the adjustable parameters for preventing parameter drift and ensuring the parameter matrix is bounded away from singularity.

It is therefore the objective of this paper to develop a novel method for more general MIMO unknown systems by using on-line adaptive T–S fuzzy-neural control. We prove that the closed-loop system controlled by the proposed controller is robust stable and the outputs of the system can asymptotically track the desired output trajectories.

Section 2 describes the T–S fuzzy-neural model and fuzzy-neural networks. Section 3 introduces the T–S fuzzy-neural model for virtual linearized systems (VLS). Section 4 presents a controller design for on-line modeling and robust tracking by using projection update laws. Several examples are illustrated in Section 5. Conclusions are drawn in Section 6.

2. T–S fuzzy-neural model

There are several ways to model a system by applying fuzzy techniques, such as the Mamdani model, the Takagi–Sugeno (T–S) model and combinations of these models. Mamdani was the first practical application of fuzzy logic control that implemented Zadeh's fuzzy theory. The T–S model was first introduced in 1985 by the Japanese scholars Takagi and Sugeno (1985). T–S fuzzy systems are nonlinear systems described by a set of IF-THEN rules. Such a model can approximate a wide class of nonlinear systems.

Fig. 1 shows the configuration of the T–S fuzzy-neural model (Leu et al., 1999), which is a typical T–S fuzzy inference system (Takagi & Sugeno, 1985) constructed from a neural network structure. It has a total of six layers. The T–S fuzzy-neural model is essentially a multi-model approach in which a set of linear models are combined to describe the global behavior of the system (Leu et al., 1999; Yen, Wang, & Gillespie, 1998). The T–S fuzzy-neural model is appropriate for developing fuzzy-neural controllers because many systems can be expressed locally in some form of linear mathematical model. The T–S fuzzy-neural model approximates a nonlinear system with a combination of several linear systems. It is formed by fuzzy partitioning of the input space. The premise of a fuzzy implication indicates a fuzzy subspace of the input space and each consequent expresses a local input–output relation in the subspace corresponding to the premise part (Park & Cho, 2004). The T–S fuzzy-neural model is defined as

$$\begin{aligned} R^{(i)}: &\text{ If } z_1 \text{ is } F_1^i \text{ and } \cdots z_n \text{ is } F_n^i \text{ and } \cdots z_{n+m} \text{ is } F_{n+m}^i \\ &\text{Then } \bar{y}_l = p_{l1}^i z_1 + p_{l2}^i z_2 + \cdots + p_{l(n+m)}^i z_{n+m} \end{aligned} \tag{1}$$

where $\mathbf{z} = [z_1 \quad z_2 \quad \cdots \quad z_{n+m}]^T \in \Re^{n+m}$ is a vector of linguistic variables, $\bar{y}_l$ represents the output of the fuzzy-neural network, F_j^i $(j = 1, 2, \ldots, n+m)$ are fuzzy sets, and p_{lk}^i $(i = 1, 2, \ldots, h,\ l = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n+m)$ are adjustable parameters which are called the weighting factors.

3. T–S fuzzy-neural model for virtual linearized system (VLS)

Suppose that the general MIMO unknown nonaffine nonlinear system is

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}, \mathbf{u}) + d_{d1} \\ \dot{x}_2 &= f_2(\mathbf{x}, \mathbf{u}) + d_{d2} \\ &\vdots \\ \dot{x}_n &= f_n(\mathbf{x}, \mathbf{u}) + d_{dn} \end{aligned} \tag{2}$$

where $\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T \in \Re^n$ denotes the state vector; $\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_m]^T \in \Re^m$is the input vector; $\mathbf{d}_d = [d_{d1} \quad d_{d2} \quad \cdots \quad d_{dn}]^T$represents external disturbances; $f_i : \Re^{n+m} \to \Re^1$ $(i = 1, 2, \ldots, n)$ are unknown functions whose first derivatives with respect to $\mathbf{x}$ and $\mathbf{u}$ exist. Without loss of generality, we assume a solution for (2) exists. The control objective is to steer all the states in $\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T$ to asymptotically track the virtual desired variables $\mathbf{x}_r = [x_{r1} \quad x_{r2} \quad \cdots \quad x_{rn}]^T$.

Definition 1. Suppose a function f is continuous in the closed interval $[\bar{x},\ x]$ and differentiable in the interval's interior $(\bar{x},\ x)$, where $\bar{x} = t_1 x,\ 0 < t_1 < 1$. Then for some x^* between $(\bar{x},\ x)$, we have $f'(x^*) = (f(x) - f(\bar{x}))/(x - \bar{x})$. We call $\bar{x}$ a critical point and x^* a differential mean point of f on $(\bar{x},\ x)$. Fig. 2 shows an illustration of the mean value theorem (Grossman & Derrick, 1998).

By using the mean value theorem (Vidyasagar, 1993), there are points x_{ij}^* $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n)$ and u_{ik}^* $(i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m)$ in the linear segments joining x_j to $\bar{x}_j$ $(j = 1, 2, \ldots, n)$ and u_k to $\bar{u}_k$ $(k = 1, 2, \ldots, m)$ for every

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} f_1(\bar{\mathbf{x}},\bar{\mathbf{u}}) \\ f_2(\bar{\mathbf{x}},\bar{\mathbf{u}}) \\ \vdots \\ f_n(\bar{\mathbf{x}},\bar{\mathbf{u}}) \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \mathbf{x}_\xi + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \mathbf{u}_\xi + \mathbf{d}_d \\
&= \begin{bmatrix} f_1(\bar{\mathbf{x}},\bar{\mathbf{u}})(x_1-\bar{x}_1)^{-1}+a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & f_2(\bar{\mathbf{x}},\bar{\mathbf{u}})(x_2-\bar{x}_2)^{-1}+a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & f_n(\bar{\mathbf{x}},\bar{\mathbf{u}})(x_n-\bar{x}_n)^{-1}+a_{nn} \end{bmatrix} \mathbf{x}_\xi \\
&\quad + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \mathbf{u}_\xi + \mathbf{d}_d \\
&= \begin{bmatrix} a'_{11} & a'_{12} & \dots & a'_{1n} \\ a'_{21} & a'_{22} & \dots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \dots & a'_{nn} \end{bmatrix} \mathbf{x}_\xi + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \mathbf{u}_\xi + \mathbf{d}_d \\
&= \mathbf{A}\mathbf{x}_\xi + \mathbf{B}\mathbf{u}_\xi + \mathbf{d}_d
\end{aligned} \tag{3}
$$

Box I.

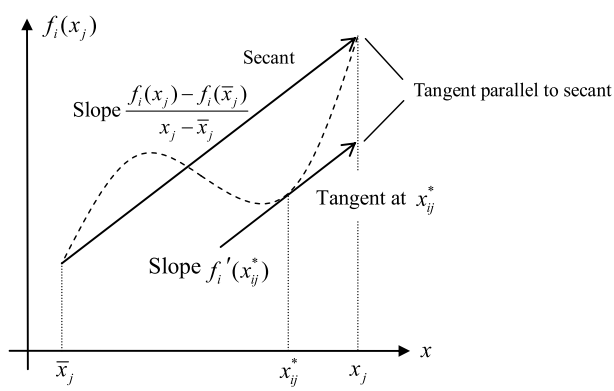

Fig. 2. The illustration of the mean value theorem.

function f_i ($i = 1, 2, \dots, n$). Therefore, the unknown nonaffine nonlinear functions can be formed as follows:

$$
\begin{aligned}
f_i(\mathbf{x},\mathbf{u}) &= f_i(\bar{\mathbf{x}},\bar{\mathbf{u}}) + \frac{\partial f_i(\mathbf{x}_i^*,\mathbf{u}_i^*)}{\partial x_1}(x_1-\bar{x}_1) + \frac{\partial f_i(\mathbf{x}_i^*,\mathbf{u}_i^*)}{\partial x_2}(x_2-\bar{x}_2) \\
&\quad + \dots + \frac{\partial f_i(\mathbf{x}_i^*,\mathbf{u}_i^*)}{\partial x_n}(x_n-\bar{x}_n) + \frac{\partial f_i(\mathbf{x}_i^*,\mathbf{u}_i^*)}{\partial u_1}(u_1-\bar{u}_1) \\
&\quad + \frac{\partial f_i(\mathbf{x}_i^*,\mathbf{u}_i^*)}{\partial u_2}(u_2-\bar{u}_2) + \dots + \frac{\partial f_i(\mathbf{x}_i^*,\mathbf{u}_i^*)}{\partial u_m}(u_m-\bar{u}_m)
\end{aligned}
$$

where $\mathbf{x}_i^* = [x_{i1}^* \; x_{i2}^* \; \cdots \; x_{in}^*]^T$ and $\mathbf{u}_i^* = [u_{i1}^* \; u_{i2}^* \; \cdots \; u_{im}^*]^T$ are mean points of f_i ($i = 1, 2, \dots, n$). Note that we can get different mean value points $\mathbf{x}_i^*$ ($i = 1, 2, \dots, n$) and $\mathbf{u}_i^*$ ($i = 1, 2, \dots, n$) for different functions.

Thus, we can transform the real system (2) into (3) by a virtual linearized system (VLS) as shown in Box I, where $\bar{\mathbf{x}} = \begin{bmatrix}\bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n\end{bmatrix}^T = t_1\mathbf{x}$ and $\bar{\mathbf{u}} = \begin{bmatrix}\bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_m\end{bmatrix}^T = t_2\mathbf{u}$ with $0 < t_1, t_2 < 1$ are vectors of critical points, $\mathbf{x}_\xi = [x_{\xi 1} \; x_{\xi 2} \; \cdots \; x_{\xi n}]^T = \mathbf{x} - \bar{\mathbf{x}}$, $\mathbf{u}_\xi = [u_{\xi 1} \; u_{\xi 2} \; \cdots \; u_{\xi m}]^T = \mathbf{u} - \bar{\mathbf{u}}$, $a_{ij} = \partial f_i(\mathbf{x}_i^*, \mathbf{u}_i^*)/\partial x_j$, and $b_{ik} = \partial f_i(\mathbf{x}_i^*, \mathbf{u}_i^*)/\partial u_k$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, m$. We can choose the parameters of t_1 and t_2 to find the values of $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$.

Remark 1. The virtual linearized system (VLS) models the unknown nonlinear system (2). Because the nonlinear functions of the general systems (2) are unknown, traditional T–S fuzzy control methods can rarely model and control them. Instead of modeling the unknown systems (2) directly, the T–S fuzzy-neural model in (1) (or Fig. 1) is used to approximate the virtual linearized system (VLS) in (3) in Box I which is used to model the unknown nonlinear system (2).

From (1) and Fig. 1, the coefficient, p_{lk} ($l = 1, 2, \dots, n$, $k = 1, 2, \dots, n+m$), of the T–S fuzzy-neural model is

$$
p_{lk} = \frac{\sum_{i=1}^{h} p_{lk}^i \left(\prod_{j=1}^{n+m} \mu_{F_j^i}(z_j) \right)}{\sum_{i=1}^{h} \left(\prod_{j=1}^{n+m} \mu_{F_j^i}(z_j) \right)} \tag{4}
$$

where $\mu_{F_j^i}(z_j)$ is the value of the membership function. For the tuning of the weighting factors p_{lk}^i, we define

$$
w^i \equiv \frac{\prod_{j=1}^{n+m} \mu_{F_j^i}(z_j)}{\sum_{i=1}^{h} \left(\prod_{j=1}^{n+m} \mu_{F_j^i}(z_j) \right)}, \quad i = 1, 2, \dots, h. \tag{5}
$$

The antecedent part of the fuzzy implication describes the conditions of the state deviations and input deviations $[\mathbf{x}_\xi^T, \mathbf{u}_\xi^T]^T$. The consequent part of the fuzzy implication represents the virtual linearized system (VLS) in (3) in Box I. For the purpose of approximating the virtual linearized system (VLS) in (3) in Box I, the ith fuzzy implication (1) can be described as

$$
\begin{aligned}
R^{(i)} &: \text{If } x_{\xi 1} \text{ is } F_1^i \text{ and } \cdots x_{\xi n} \text{ is } F_n^i \text{ and } u_{\xi 1} \text{ is } F_{n+1}^i \\
&\quad \text{and } \cdots u_{\xi m} \text{ is } F_{n+m}^i \\
&\quad \text{Then } \dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}^i \mathbf{x}_\xi + \hat{\mathbf{B}}^i \mathbf{u}_\xi
\end{aligned} \tag{6}
$$

where

$$
\hat{\mathbf{A}}^i = \begin{bmatrix} p_{11}^i & p_{12}^i & \dots & p_{1n}^i \\ p_{21}^i & p_{22}^i & \dots & p_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}^i & p_{n2}^i & \dots & p_{nn}^i \end{bmatrix}
$$

and

$$\hat{\mathbf{B}}^i = \begin{bmatrix} p^i_{1(n+1)} & p^i_{1(n+2)} & \cdots & p^i_{1(n+m)} \\ p^i_{2(n+1)} & p^i_{2(n+2)} & \cdots & p^i_{2(n+m)} \\ \vdots & \vdots & \ddots & \vdots \\ p^i_{n(n+1)} & p^i_{n(n+2)} & \cdots & p^i_{n(n+m)} \end{bmatrix}$$
$$= \begin{bmatrix} \hat{\mathbf{B}}^i_1 & \hat{\mathbf{B}}^i_2 & \cdots & \hat{\mathbf{B}}^i_m \end{bmatrix}. \tag{7}$$

After applying (4), (5) and some commonly used defuzzification strategies, the real system (2) or the VLS (3) in Box I becomes

$$\dot{\mathbf{x}} = \dot{\hat{\mathbf{x}}} + \mathbf{d}_d + \mathbf{d}_f = \sum_{i=1}^{h} w^i \left\{ \hat{\mathbf{A}}^i \mathbf{x}_\xi + \hat{\mathbf{B}}^i \mathbf{u}_\xi \right\} + \mathbf{d}_d + \mathbf{d}_f$$
$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \mathbf{x}_\xi$$
$$+ \begin{bmatrix} p_{1(n+1)} & p_{1(n+2)} & \cdots & p_{1(n+m)} \\ p_{2(n+1)} & p_{2(n+2)} & \cdots & p_{2(n+m)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n(n+1)} & p_{n(n+2)} & \cdots & p_{n(n+m)} \end{bmatrix} \mathbf{u}_\xi + \mathbf{d}_d + \mathbf{d}_f \tag{8}$$

where $\mathbf{d}_f = \left(\mathbf{A} - \sum_{i=1}^{h} w^i \hat{\mathbf{A}}^i\right) \mathbf{x}_\xi + \left(\mathbf{B} - \sum_{i=1}^{h} w^i \hat{\mathbf{B}}^i\right) \mathbf{u}_\xi$, and p_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n+m)$ is used to approximate a'_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n)$ and b_{ij} $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ of the virtual linearized system (VLS) in (3) in Box I which exactly equals the unknown nonlinear system (2). In this paper, (3) in Box I is a state equation and (8) is an approximate state equation. $\hat{\mathbf{x}}$ is the estimation of the state vector $\mathbf{x}$. Considering approximation error $\mathbf{d}_f$, (3) and (6), we can obtain (8)

$$\dot{\mathbf{x}} = \dot{\hat{\mathbf{x}}} + \mathbf{d}_d + \mathbf{d}_f.$$

4. On-line modeling and robust tracking controller design using projection

To design a robust controller for (2), the following assumptions are required.

Assumption 1 (*Chien et al., 2006; Wang et al., 2004*). Let $\mathbf{x}_\xi$ and $\mathbf{u}_\xi$ belong to the compact sets $\mathbf{U}_x$ and $\mathbf{U}_u$, respectively, where

$$\mathbf{U_x} = \{\mathbf{x} \in R^n : \|\mathbf{x}\| \le m_{\mathbf{x}} < \infty\} \quad \text{and}$$
$$\mathbf{U_u} = \left\{\mathbf{u} \in R^m : \|\mathbf{u}\| \le m_{\mathbf{u}} < \infty\right\}$$

and $m_{\mathbf{x}}$, $m_{\mathbf{u}}$ are design parameters. We define $\phi_{lj} = [p^1_{lj}\ p^2_{lj}\ \cdots\ p^h_{lj}]$, $l = 1, \ldots, n,\ j = 1, 2, \ldots, n+m$. It is known that the optimal adjustable parameters ϕ^*_{lj} lie in some convex regions

$$M_{\phi_{lj}} = \left\{\phi_{lj} \in R^h : \|\phi_{lj}\| \le m_{\phi_{lj}}\right\},$$
$$l = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n+m$$

where the radii $m_{\phi_{lj}}$ are constant, and

$$\phi^*_{lj} = \arg \min_{\phi_{lj} \in M_{\phi_{lj}}} \left[\sup_{\mathbf{x}_\xi \in \mathbf{U_x}, \mathbf{u}_\xi \in \mathbf{U_u}} \left| p^i_{lj}(\mathbf{x}_\xi, \mathbf{u}_\xi) - \hat{p}^i_{lj}(\mathbf{x}_\xi, \mathbf{u}_\xi | \phi_{lj}) \right| \right],$$
$$l = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n+m.$$

According to Assumption 1, we define the optimal adjustable matrices as

$$\mathbf{A}^{*i} = \begin{bmatrix} p^{*i}_{11} & p^{*i}_{12} & \cdots & p^{*i}_{1n} \\ p^{*i}_{21} & p^{*i}_{22} & \cdots & p^{*i}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p^{*i}_{n1} & p^{*i}_{n2} & \cdots & p^{*i}_{nn} \end{bmatrix} \quad \text{and}$$

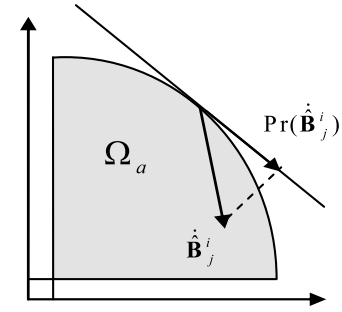

Fig. 3. Illustration of the projection update law for preventing parameter drift.

$$\mathbf{B}^{*i} = \begin{bmatrix} p^{*i}_{1(n+1)} & p^{*i}_{1(n+2)} & \cdots & p^{*i}_{1(n+m)} \\ p^{*i}_{2(n+1)} & p^{*i}_{2(n+2)} & \cdots & p^{*i}_{2(n+m)} \\ \vdots & \vdots & \ddots & \vdots \\ p^{*i}_{n(n+1)} & p^{*i}_{n(n+2)} & \cdots & p^{*i}_{n(n+m)} \end{bmatrix}.$$

Lemma 1 (*Vidyasagar, 1993*). *Suppose that a matrix $\mathbf{\Lambda} \in \Re^{n\times n}$ is given. For every symmetric positive definite matrix $\mathbf{Q} \in \Re^{n\times n}$, the Lyapunov matrix equation $\mathbf{\Lambda}^T\mathbf{\Gamma} + \mathbf{\Gamma}\mathbf{\Lambda} = -\mathbf{Q}$ has a unique solution for $\mathbf{\Gamma} = \mathbf{\Gamma}^T > 0$ if and only if $\mathbf{\Lambda} \in \Re^{n\times n}$ is a Hurwitz matrix.*

Let $\mathbf{e} = \mathbf{x} - \mathbf{x}_r = [e_1\ \ e_2\ \ \cdots\ \ e_n]^T$ denoted the tracking error for the state variables and a coefficient matrix is represented as follows:

$$\mathbf{\Lambda} = \begin{bmatrix} -\lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_n \end{bmatrix} \tag{9}$$

where the coefficients, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are selected such that the matrix $\mathbf{\Lambda}$ is a Hurwitz matrix. From (3) in Box I, based on the certainty equivalence approach, a control input can be chosen as

$$\mathbf{u}_\xi = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T(-\mathbf{A}\mathbf{x}_\xi + \boldsymbol{\omega} - \mathbf{u}_s) \tag{10}$$

where $\boldsymbol{\omega} = \begin{bmatrix}\omega_1 & \omega_2 & \cdots & \omega_n\end{bmatrix}^T = \dot{\mathbf{x}}_r + \mathbf{\Lambda}\mathbf{e}$ and $\mathbf{u}_s$ is an error compensator which is designed to compensate for $\mathbf{d}_d$. But the right side of (3) in Box I is unknown, so we replace $\mathbf{A}$ and $\mathbf{B}$ by $\sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i$ and $\sum_{i=1}^{h} w^i\hat{\mathbf{B}}^i$ in (8), respectively. Moreover, we define $\Omega = \sum_{i=1}^{h} w^i\hat{\mathbf{B}}^i$. The inverse of the square matrix $\Omega^T\Omega$ can be derived as

$$\begin{aligned}\left(\Omega^T\Omega\right)^{-1} &= \left[\alpha\mathbf{I} - (-\Omega^T\Omega + \alpha\mathbf{I})\right]^{-1} \\ &= \alpha^{-1}\mathbf{I} + \alpha^{-2}(-\Omega^T\Omega + \alpha\mathbf{I}) + \alpha^{-3}(-\Omega^T\Omega + \alpha\mathbf{I})^2 \\ &\quad + \alpha^{-4}(-\Omega^T\Omega + \alpha\mathbf{I})^3 + \cdots \\ &= \alpha^{-1}\sum_{k=0}^{N}\left(\alpha^{-1}(-\Omega^T\Omega + \alpha\mathbf{I})\right)^k + \mathbf{d}_\Omega \end{aligned} \tag{11}$$

where α is required to be sufficiently large so that all eigenvalues of $\alpha^{-1}(-\Omega^T\Omega + \alpha\mathbf{I})$ have magnitudes less than 1 (Chen, 1999) and

$$\mathbf{d}_\Omega = \alpha^{-1}\sum_{k=N+1}^{\infty}\left(\alpha^{-1}(-\Omega^T\Omega + \alpha\mathbf{I})\right)^k.$$

From (10) and (11), we redesign the fuzzy-neural control input as

$$\mathbf{u} = \left(\alpha^{-1}\sum_{k=0}^{N}(\alpha^{-1}(-\Omega^T\Omega + \alpha\mathbf{I}))^k\right)\Omega^T$$
$$\times\left(-\sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \dot{\mathbf{x}}_r + \mathbf{\Lambda}\mathbf{e} - \mathbf{u}_s\right) + (1-\beta)\mathbf{d}_\Delta + \beta\mathbf{u}_\Delta + \bar{\mathbf{u}}$$

where

$$\mathbf{d}_\Delta = \mathbf{d}_\Omega \Omega^T \left(-\sum_{i=1}^{h} w^i \hat{\mathbf{A}}^i \mathbf{x}_\xi + \dot{\mathbf{x}}_r + \mathbf{\Lambda e} - \mathbf{u}_s \right) = [d_{\Delta 1} \quad d_{\Delta 2} \quad \cdots \quad d_{\Delta m}]^T.$$

If $\left|\Omega\Omega^T\right| > \varepsilon$, we set $\beta = 0$. The fuzzy-neural control input becomes

$$\begin{aligned}\mathbf{u} &= \left(\alpha^{-1}\sum_{k=0}^{N}(\alpha^{-1}(-\Omega^T\Omega + \alpha\mathbf{I}))^k\right)\Omega^T \\ &\quad \times \left(-\sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \dot{\mathbf{x}}_r + \mathbf{\Lambda e} - \mathbf{u}_s\right) + \mathbf{d}_\Delta + \bar{\mathbf{u}} \\ &= (\Omega^T\Omega)^{-1}\Omega^T\left(-\sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \dot{\mathbf{x}}_r + \mathbf{\Lambda e} - \mathbf{u}_s\right) + \bar{\mathbf{u}} \qquad (12)\end{aligned}$$

where ε is a small positive constant.

If $\left|\Omega\Omega^T\right| \le \varepsilon$, we choose $\beta = 1$. The fuzzy-neural control input becomes

$$\begin{aligned}\mathbf{u} &= \left(\left(\alpha^{-1}\sum_{k=0}^{N}(\alpha^{-1}(-\Omega^T\Omega + \alpha\mathbf{I}))^k\right)\Omega^T \right. \\ &\quad \left. \times \left(-\sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \dot{\mathbf{x}}_r + \mathbf{\Lambda e} - \mathbf{u}_s\right) + \mathbf{d}_\Delta\right) \\ &\quad + (\mathbf{u}_\Delta - \mathbf{d}_\Delta) + \bar{\mathbf{u}} \\ &= \left(\alpha^{-1}\sum_{k=0}^{N}(\alpha^{-1}(-\Omega^T\Omega + \alpha\mathbf{I}))^k\right)\Omega^T \\ &\quad \times \left(-\sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \dot{\mathbf{x}}_r + \mathbf{\Lambda e} - \mathbf{u}_s\right) + \mathbf{u}_\Delta + \bar{\mathbf{u}} \qquad (13)\end{aligned}$$

where $\mathbf{u}_\Delta$ is designed to estimate $\mathbf{d}_\Delta$.

Remark 2. We consider two cases based on the determinant of $(\Omega^T\Omega)$. If $\left|\Omega^T\Omega\right| > \varepsilon$, (12) can be calculated. However, if $\left|\Omega^T\Omega\right| \le \varepsilon$, we use $\mathbf{u}_\Delta$ to approximate $\mathbf{d}_\Delta$. Then the controller $\mathbf{u}$ becomes (13). In second case, we use projection algorithms (Ioannou & Datta, 1991; Luenberger, 1969; Wang, Leu, & Hsu, 2001) which can be applied to solve the parameter drift problem. The principal idea behind such approaches is to project the directions of adaptations (i.e. $\dot{\hat{\mathbf{A}}}^i$, $\dot{\hat{\mathbf{B}}}^i$, $\dot{\mathbf{q}}_k$, $\dot{\mathbf{q}}_{\Delta j}$), whenever they has the tendency to move into or stay at $\left|\Omega^T\Omega\right| < \varepsilon$, so that the determinant of $(\Omega^T\Omega)$ can become larger or equal ε.

Fig. 3 shows the use of the projection update law for preventing parameter drift. If the parameter vector is on the boundary of the constraint set Ω_a but moving toward the inside of Ω_a, then project the gradient vector onto the tangent of Ω_a.

Suppose that the adaptive law of $\hat{\mathbf{B}}^i = \left[\hat{\mathbf{B}}_1^i \quad \hat{\mathbf{B}}_2^i \quad \cdots \quad \hat{\mathbf{B}}_m^i\right]$ is chosen as

$$\dot{\hat{\mathbf{B}}}_j^i = \begin{cases} \eta_2 w^i \mathbf{e} u_{\xi j}, & \text{if } \left|\Omega^T\Omega\right| > \varepsilon \text{ or} \\ \quad \left(\left|\Omega^T\Omega\right| = \varepsilon \text{ and } \hat{\mathbf{B}}_j^{iT} w^i \mathbf{e} u_{\xi j} \le 0\right), \\ \Pr(\eta_2 w^i \mathbf{e} u_{\xi j}), & \text{if } \left|\Omega^T\Omega\right| = \varepsilon \text{ and } \hat{\mathbf{B}}_j^{iT} w^i \mathbf{e} u_{\xi j} > 0, \end{cases}$$
$$i = 1, 2, \ldots, h, \; j = 1, 2, \ldots, m$$

where the projection operator (Ioannou & Datta, 1991; Luenberger, 1969; Wang et al., 2001) is given as

$$\Pr(\eta_2 w^i \mathbf{e} u_{\xi j}) = \eta_2 w^i \mathbf{e} u_{\xi j} - \eta_2 \frac{\hat{\mathbf{B}}_j^{iT} w^i \mathbf{e} u_{\xi j}}{\left\|\hat{\mathbf{B}}_j^i\right\|^2}\hat{\mathbf{B}}_j^i.$$

From $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_r$ and substituting (12) and (13) for (8), the error dynamic equation of the VLS becomes

$$\begin{aligned}\dot{\mathbf{e}} &= \dot{\mathbf{x}} - \dot{\mathbf{x}}_r \\ &= \dot{\hat{\mathbf{x}}} + \mathbf{d}_d + \mathbf{d}_f - \dot{\mathbf{x}}_r \\ &= \sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \sum_{i=1}^{h} w^i\hat{\mathbf{B}}^i\mathbf{u}_\xi + \mathbf{d}_d + \left(\mathbf{A} - \sum_{i=1}^{h} w^i\mathbf{A}^{*i}\right)\mathbf{x}_\xi \\ &\quad + \left(\mathbf{B} - \sum_{i=1}^{h} w^i\mathbf{B}^{*i}\right)\mathbf{u}_\xi + \sum_{i=1}^{h} w^i(\mathbf{A}^{*i} - \hat{\mathbf{A}}^i)\mathbf{x}_\xi \\ &\quad + \sum_{i=1}^{h} w^i(\mathbf{B}^{*i} - \hat{\mathbf{B}}^i)\mathbf{u}_\xi - \dot{\mathbf{x}}_r \\ &= \sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \left(-\sum_{i=1}^{h} w^i\hat{\mathbf{A}}^i\mathbf{x}_\xi + \dot{\mathbf{x}}_r + \mathbf{\Lambda e} - \mathbf{u}_s\right) \\ &\quad + \sum_{i=1}^{h} w^i(\mathbf{A}^{*i} - \hat{\mathbf{A}}^i)\mathbf{x}_\xi + \sum_{i=1}^{h} w^i(\mathbf{B}^{*i} - \hat{\mathbf{B}}^i)\mathbf{u}_\xi \\ &\quad + \left(\mathbf{d}_d + \left(\mathbf{A} - \sum_{i=1}^{h} w^i\mathbf{A}^{*i}\right)\mathbf{x}_\xi + \left(\mathbf{B} - \sum_{i=1}^{h} w^i\mathbf{B}^{*i}\right)\mathbf{u}_\xi\right) \\ &\quad - \dot{\mathbf{x}}_r - \beta\Omega\mathbf{d}_\Delta + \beta\Omega\mathbf{u}_\Delta \\ &= \mathbf{\Lambda e} + \sum_{i=1}^{h} w^i\tilde{\mathbf{A}}^i\mathbf{x}_\xi + \sum_{i=1}^{h} w^i\tilde{\mathbf{B}}^i\mathbf{u}_\xi + \tilde{\mathbf{d}} - \mathbf{u}_s \\ &\quad - \beta\Omega\mathbf{d}_\Delta + \beta\Omega\mathbf{u}_\Delta \qquad (14)\end{aligned}$$

where $\tilde{\mathbf{A}}^i = \mathbf{A}^{*i} - \hat{\mathbf{A}}^i$, $\tilde{\mathbf{B}}^i = \mathbf{B}^{*i} - \hat{\mathbf{B}}^i$, and $\tilde{\mathbf{d}} = \mathbf{d}_d + \left(\mathbf{A} - \sum_{i=1}^{h} w^i\mathbf{A}^{*i}\right)\mathbf{x}_\xi + \left(\mathbf{B} - \sum_{i=1}^{h} w^i\mathbf{B}^{*i}\right)\mathbf{u}_\xi = [\tilde{d}_1 \; \tilde{d}_2 \; \cdots \; \tilde{d}_n]^T$. We define $\mathbf{u}_s$ (the error compensator) and $\mathbf{e}_\Delta$ as

$$\begin{aligned}\mathbf{u}_s &= \begin{bmatrix} \text{sign}(e_{\Delta 1}) & 0 & \cdots & 0 \\ 0 & \text{sign}(e_{\Delta 2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \text{sign}(e_{\Delta n}) \end{bmatrix}\hat{\mathbf{d}} \\ &= \text{Diag}[\text{sign}(\mathbf{e}_\Delta)]\hat{\mathbf{d}} \qquad (15)\end{aligned}$$

and

$$\mathbf{e}_\Delta = \mathbf{e}^{\mathbf{T}}\mathbf{\Gamma} = [e_{\Delta 1} \quad e_{\Delta 2} \quad \cdots \quad e_{\Delta n}] \qquad (16)$$

where $\hat{\mathbf{d}} = [\hat{d}_1 \quad \hat{d}_2 \quad \cdots \quad \hat{d}_n]^T$ and $\mathbf{\Gamma} > 0$ is a Lyapunov matrix. The fuzzy implications are defined to obtain $\hat{\mathbf{d}}$, which is the estimate of $\tilde{\mathbf{d}}$, as follows:

$$\begin{aligned}&\text{If } e_{\Delta 1} \text{ is } F_{e_{\Delta 1}}^i \text{ and } \|\mathbf{x}\| \text{ is } F_{\|\mathbf{x}\|}^i \quad \text{then } \hat{d}_1 = q_1^i \\ &\text{If } e_{\Delta 2} \text{ is } F_{e_{\Delta 2}}^i \text{ and } \|\mathbf{x}\| \text{ is } F_{\|\mathbf{x}\|}^i \quad \text{then } \hat{d}_2 = q_2^i \\ &\vdots \\ &\text{If } e_{\Delta n} \text{ is } F_{e_{\Delta n}}^i \text{ and } \|\mathbf{x}\| \text{ is } F_{\|\mathbf{x}\|}^i \quad \text{then } \hat{d}_n = q_n^i.\end{aligned} \qquad (17)$$

After applying some commonly used defuzzification strategies, we have

$$\begin{aligned}\hat{d}_k &= \frac{\sum_{i=1}^{h_{\hat{d}}} q_k^i \left(\mu_{F_{e_{\Delta k}}^i}(e_{\Delta k})\mu_{F_{\|\mathbf{x}\|}^i}(\|\mathbf{x}\|)\right)}{\sum_{i=1}^{h_{\hat{d}}} \left(\mu_{F_{e_{\Delta k}}^i}(e_{\Delta k})\mu_{F_{\|\mathbf{x}\|}^i}(\|\mathbf{x}\|)\right)} \\ &= \sum_{i=1}^{h_{\hat{d}}} \tau_k^i q_k^i, \quad k = 1, 2, \ldots, n \qquad (18)\end{aligned}$$

where

$$\mathbf{q}_k = [q_k^1 \quad q_k^2 \quad \cdots \quad q_k^{h_{\hat{d}}}]^T, \quad k = 1, 2, \ldots, n$$

and

$$\boldsymbol{\tau}_k = [\tau_k^1 \quad \tau_k^2 \quad \cdots \quad \tau_k^{h_{\hat{d}}}]^T$$

$$= \left[\frac{\mu_{F^1_{e_{\Delta k}}}(e_{\Delta k})\mu_{F^1_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)}{\sum_{i=1}^{h_{\hat{d}}}\left(\mu_{F^i_{e_{\Delta k}}}(e_{\Delta k})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)} \times \frac{\mu_{F^2_{e_{\Delta k}}}(e_{\Delta k})\mu_{F^2_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)}{\sum_{i=1}^{h_{\hat{d}}}\left(\mu_{F^i_{e_{\Delta k}}}(e_{\Delta k})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)} \cdots \times \frac{\mu_{F^{h_{\hat{d}}}_{e_{\Delta k}}}(e_{\Delta k})\mu_{F^{h_{\hat{d}}}_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)}{\sum_{i=1}^{h_{\hat{d}}}\left(\mu_{F^i_{e_{\Delta k}}}(e_{\Delta k})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)} \right]^T, \quad k = 1, 2, \ldots, n. \tag{19}$$

Moreover, we define $\mathbf{u}_\Delta$ and $\bar{\mathbf{e}}_\Delta$ as

$$\mathbf{u}_\Delta = \begin{bmatrix} \mathrm{sign}(\bar{e}_{\Delta 1}) & 0 & \cdots & 0 \\ 0 & \mathrm{sign}(\bar{e}_{\Delta 2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{sign}(\bar{e}_{\Delta m}) \end{bmatrix} \hat{\mathbf{d}}_\Delta$$

$$= \mathrm{Diag}[\mathrm{sign}(\bar{\mathbf{e}}_\Delta)]\hat{\mathbf{d}}_\Delta \tag{20}$$

and

$$\bar{\mathbf{e}}_\Delta = \mathbf{e}^{\mathbf{T}}\boldsymbol{\Gamma}\Omega = [\bar{e}_{\Delta 1} \quad \bar{e}_{\Delta 2} \quad \cdots \quad \bar{e}_{\Delta m}] \tag{21}$$

where $\hat{\mathbf{d}}_\Delta = [\hat{d}_{\Delta 1} \quad \hat{d}_{\Delta 2} \quad \cdots \quad \hat{d}_{\Delta m}]^T$ and $\boldsymbol{\Gamma} > 0$ is a Lyapunov matrix. The fuzzy implications are defined to obtain $\hat{\mathbf{d}}_\Delta$, which is the estimate of $\mathbf{d}_\Delta$, as follows:

$$\begin{aligned} &\text{If } \bar{e}_{\Delta 1} \text{ is } F^i_{\bar{e}_{\Delta 1}} \text{ and } \|\mathbf{x}\| \text{ is } F^i_{\|\mathbf{x}\|} \quad \text{then } \hat{d}_{\Delta 1} = q^i_{\Delta 1} \\ &\text{If } \bar{e}_{\Delta 2} \text{ is } F^i_{\bar{e}_{\Delta 2}} \text{ and } \|\mathbf{x}\| \text{ is } F^i_{\|\mathbf{x}\|} \quad \text{then } \hat{d}_{\Delta 2} = q^i_{\Delta 2} \\ &\vdots \\ &\text{If } \bar{e}_{\Delta m} \text{ is } F^i_{\bar{e}_{\Delta m}} \text{ and } \|\mathbf{x}\| \text{ is } F^i_{\|\mathbf{x}\|} \quad \text{then } \hat{d}_{\Delta m} = q^i_{\Delta m}. \end{aligned} \tag{22}$$

After applying some commonly used defuzzification strategies, we have

$$\hat{d}_{\Delta j} = \frac{\sum_{i=1}^{h_{\hat{d}_\Delta}} q^i_{\Delta j}\left(\mu_{F^i_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)}{\sum_{i=1}^{h_{\hat{d}_\Delta}}\left(\mu_{F^i_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)}$$

$$= \sum_{i=1}^{h_{\hat{d}_\Delta}} \tau^i_{\Delta j} q^i_{\Delta j}, \quad j = 1, 2, \ldots, m \tag{23}$$

where

$$\mathbf{q}_{\Delta j} = [q^1_{\Delta j} \quad q^2_{\Delta j} \quad \cdots \quad q^{h_{\hat{d}_\Delta}}_{\Delta j}]^T, \quad j = 1, 2, \ldots, m$$

and

$$\boldsymbol{\tau}_{\Delta j} = [\tau^1_{\Delta j} \quad \tau^2_{\Delta j} \quad \cdots \quad \tau^{h_{\hat{d}_\Delta}}_{\Delta j}]^T$$

$$= \left[\frac{\mu_{F^1_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^1_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)}{\sum_{i=1}^{h_{\hat{d}_\Delta}}\left(\mu_{F^i_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)} \times \frac{\mu_{F^2_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^2_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)}{\sum_{i=1}^{h_{\hat{d}_\Delta}}\left(\mu_{F^i_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)} \cdots \times \frac{\mu_{F^{h_{\hat{d}_\Delta}}_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^{h_{\hat{d}_\Delta}}_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)}{\sum_{i=1}^{h_{\hat{d}_\Delta}}\left(\mu_{F^i_{\bar{e}_{\Delta j}}}(\bar{e}_{\Delta j})\mu_{F^i_{\|\mathbf{x}\|}}(\|\mathbf{x}\|)\right)} \right]^T, \quad j = 1, 2, \ldots, m. \tag{24}$$

Assumption 2 (*Chien et al., 2006; Wang et al., 2004*). $\left|\tilde{d}_k\right| \le \boldsymbol{\tau}_k^T \mathbf{q}_k^*$ $(k = 1, 2, \ldots, n)$ and $\left|d_{\Delta j}\right| \le \boldsymbol{\tau}_{\Delta j}^T \mathbf{q}_{\Delta j}^*$ $(j = 1, 2, \ldots, m)$, where $\mathbf{q}_k^*$ and $\mathbf{q}_{\Delta j}^*$ are the optimal adjustable vectors, $\mathbf{q}_k$ and $\mathbf{q}_{\Delta j}$ represent the estimates of $\mathbf{q}_k^*$ and $\mathbf{q}_{\Delta j}^*$, respectively.

On the basis of the above discussion, the following theorem can be obtained.

Theorem 1. *Consider the general MIMO unknown nonaffine nonlinear system (2), which is approximated as (8). If the controllers are designed as (12) and (13) with update laws*

$$\dot{\hat{\mathbf{A}}}^i = \eta_1 w^i \mathbf{e}\mathbf{x}_\xi^T, \quad i = 1, 2, \ldots, h \tag{25}$$

$$\dot{\hat{\mathbf{B}}}_j^i = \begin{cases} \eta_2 w^i \mathbf{e} u_{\xi j}, \\ \quad \text{if } \left|\Omega^T\Omega\right| > \varepsilon \text{ or } \left(\left|\Omega^T\Omega\right| = \varepsilon \text{ and } \hat{\mathbf{B}}_j^{iT} w^i \mathbf{e} u_{\xi j} \le 0\right) \\ \mathrm{Pr}(\eta_2 w^i \mathbf{e} u_{\xi j}) = \eta_2 w^i \mathbf{e} u_{\xi j} - \eta_2 \dfrac{\hat{\mathbf{B}}_j^{iT} w^i \mathbf{e} u_{\xi j}}{\|\hat{\mathbf{B}}_j^i\|^2}\hat{\mathbf{B}}_j^i, \\ \quad \text{if } \left|\Omega^T\Omega\right| = \varepsilon \text{ and } \hat{\mathbf{B}}_j^{iT} w^i \mathbf{e} u_{\xi j} > 0 \end{cases}$$

$$i = 1, 2, \ldots, h, \; j = 1, 2, \ldots, m \tag{26}$$

$$\dot{\mathbf{q}}_k = \gamma_1 \boldsymbol{\tau}_k \left|e_{\Delta k}\right|, \quad k = 1, 2, \ldots, n \tag{27}$$

$$\dot{\mathbf{q}}_{\Delta j} = -\gamma_2 \boldsymbol{\tau}_{\Delta j} \left|\bar{e}_{\Delta j}\right|, \quad j = 1, 2, \ldots, m \tag{28}$$

where η_1, η_2, γ_1 and γ_2 are positive constants, then the closed-loop system is robust stable and $\lim_{t\to\infty} \|\mathbf{e}(t)\| = 0$.

Proof. Consider the Lyapunov-like function candidate

$$\begin{aligned} v = {} & \frac{1}{2}\mathbf{e}^T\boldsymbol{\Gamma}\mathbf{e} + \frac{1}{2\eta_1}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{A}}^{iT}\boldsymbol{\Gamma}\tilde{\mathbf{A}}^i) + \frac{1}{2\eta_2}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{B}}^{iT}\boldsymbol{\Gamma}\tilde{\mathbf{B}}^i) \\ & + \frac{1}{2\gamma_1}\sum_{i=1}^{n}(\mathbf{q}_i^* - \mathbf{q}_i)^T(\mathbf{q}_i^* - \mathbf{q}_i) \\ & + \frac{\beta}{2\gamma_2}\sum_{j=1}^{m}(\mathbf{q}_{\Delta j}^* - \mathbf{q}_{\Delta j})^T(\mathbf{q}_{\Delta j}^* - \mathbf{q}_{\Delta j}). \end{aligned} \tag{29}$$

We define $\tilde{\mathbf{q}}_i = \mathbf{q}_i^* - \mathbf{q}_i$ $(i = 1, 2, \ldots, n)$ and $\tilde{\mathbf{q}}_{\Delta j} = \mathbf{q}_{\Delta j}^* - \mathbf{q}_{\Delta j}$ $(j = 1, 2, \ldots, m)$. Differentiating (29) with respect to time, we get

$$\begin{aligned} \dot{v} = {} & \frac{1}{2}\dot{\mathbf{e}}^T\boldsymbol{\Gamma}\mathbf{e} + \frac{1}{2}\mathbf{e}^T\boldsymbol{\Gamma}\dot{\mathbf{e}} + \frac{1}{2\eta_1}\sum_{i=1}^{h}\mathrm{tr}(\dot{\tilde{\mathbf{A}}}^{iT}\boldsymbol{\Gamma}\tilde{\mathbf{A}}^i) \\ & + \frac{1}{2\eta_1}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{A}}^{iT}\boldsymbol{\Gamma}\dot{\tilde{\mathbf{A}}}^i) + \frac{1}{2\eta_2}\sum_{i=1}^{h}\mathrm{tr}(\dot{\tilde{\mathbf{B}}}^{iT}\boldsymbol{\Gamma}\tilde{\mathbf{B}}^i) \\ & + \frac{1}{2\eta_2}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{B}}^{iT}\boldsymbol{\Gamma}\dot{\tilde{\mathbf{B}}}^i) - \frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i - \frac{\beta}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}. \end{aligned} \tag{30}$$

Inserting (14), (15) and (20) in the above equation yields

$$\begin{aligned}\dot{v} &= \frac{1}{2}\mathbf{e}^T(\mathbf{\Lambda}^T\mathbf{\Gamma}+\mathbf{\Gamma}\mathbf{\Lambda})\mathbf{e}+\mathbf{e}^T\mathbf{\Gamma}\sum_{i=1}^{h}w^i\tilde{\mathbf{A}}^i\mathbf{x}_\xi+\mathbf{e}^T\mathbf{\Gamma}\sum_{i=1}^{h}w^i\tilde{\mathbf{B}}^i\mathbf{u}_\xi\\ &\quad+\frac{1}{\eta_1}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{A}}^{iT}\mathbf{\Gamma}\dot{\tilde{\mathbf{A}}}^i)+\frac{1}{\eta_2}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{B}}^{iT}\mathbf{\Gamma}\dot{\tilde{\mathbf{B}}}^i)+\mathbf{e}^T\mathbf{\Gamma}\tilde{\mathbf{d}}\\ &\quad-\mathbf{e}^T\mathbf{\Gamma}\mathbf{u}_s-\beta\mathbf{e}^T\mathbf{\Gamma}\Omega\mathbf{d}_\Delta+\beta\mathbf{e}^T\mathbf{\Gamma}\Omega\mathbf{u}_\Delta\\ &\quad-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i-\frac{\beta}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}.\end{aligned}\tag{31}$$

From Lemma 1, substituting $\mathbf{\Lambda}^T\mathbf{\Gamma}+\mathbf{\Gamma}\mathbf{\Lambda}=-\mathbf{Q}$ in (31), we have

$$\begin{aligned}\dot{v} &= -\frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e}+\mathbf{e}^T\mathbf{\Gamma}\sum_{i=1}^{h}w^i\tilde{\mathbf{A}}^i\mathbf{x}_\xi+\mathbf{e}^T\mathbf{\Gamma}\sum_{i=1}^{h}w^i\tilde{\mathbf{B}}^i\mathbf{u}_\xi\\ &\quad+\frac{1}{\eta_1}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{A}}^{iT}\mathbf{\Gamma}\dot{\tilde{\mathbf{A}}}^i)+\frac{1}{\eta_2}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{B}}^{iT}\mathbf{\Gamma}\dot{\tilde{\mathbf{B}}}^i)+\mathbf{e}^T\mathbf{\Gamma}\tilde{\mathbf{d}}\\ &\quad-\mathbf{e}^T\mathbf{\Gamma}\mathbf{u}_s-\beta\mathbf{e}^T\mathbf{\Gamma}\Omega\mathbf{d}_\Delta\\ &\quad+\beta\mathbf{e}^T\mathbf{\Gamma}\Omega\mathbf{u}_\Delta-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i-\frac{\beta}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}.\end{aligned}\tag{32}$$

When the determinant of the matrix $\Omega^T\Omega$ is bigger than ε, we set $\beta=0$ and (32) becomes

$$\begin{aligned}\dot{v} &= \Delta+\mathbf{e}^T\mathbf{\Gamma}\tilde{\mathbf{d}}-\mathbf{e}^T\mathbf{\Gamma}\,\mathrm{Diag}[\mathrm{sign}(\mathbf{e}_\Delta)]\hat{\mathbf{d}}-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i\\ &= \Delta+\sum_{i=1}^{n}e_{\Delta i}\tilde{d}_i-\sum_{i=1}^{n}|e_{\Delta i}|\,\hat{d}_i-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i\\ &\le \Delta+\sum_{i=1}^{n}|e_{\Delta i}|\left|\tilde{d}_i\right|-\sum_{i=1}^{n}|e_{\Delta i}|\,\hat{d}_i-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i\\ &\le \Delta+\sum_{i=1}^{n}|e_{\Delta i}|\boldsymbol{\tau}_i^T\mathbf{q}_i^*-\sum_{i=1}^{n}|e_{\Delta i}|\,\boldsymbol{\tau}_i^T\mathbf{q}_i-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i\\ &= \Delta+\sum_{i=1}^{n}|e_{\Delta i}|\,\boldsymbol{\tau}_i^T\tilde{\mathbf{q}}_i-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i\\ &= \Delta+\left(|e_{\Delta 1}|\,\boldsymbol{\tau}_1^T-\frac{1}{\gamma_1}\dot{\mathbf{q}}_1^T\right)\tilde{\mathbf{q}}_1+\left(|e_{\Delta 2}|\,\boldsymbol{\tau}_2^T-\frac{1}{\gamma_1}\dot{\mathbf{q}}_2^T\right)\tilde{\mathbf{q}}_2+\cdots\\ &\quad+\left(|e_{\Delta n}|\boldsymbol{\tau}_n^T-\frac{1}{\gamma_1}\dot{\mathbf{q}}_n^T\right)\tilde{\mathbf{q}}_n\end{aligned}\tag{33}$$

where

$$\begin{aligned}\Delta &= -\frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e}+\mathbf{e}^T\mathbf{\Gamma}\sum_{i=1}^{h}w^i\tilde{\mathbf{A}}^i\mathbf{x}_\xi+\mathbf{e}^T\mathbf{\Gamma}\sum_{i=1}^{h}w^i\tilde{\mathbf{B}}^i\mathbf{u}_\xi\\ &\quad+\frac{1}{\eta_1}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{A}}^{iT}\mathbf{\Gamma}\dot{\tilde{\mathbf{A}}}^i)+\frac{1}{\eta_2}\sum_{i=1}^{h}\mathrm{tr}(\tilde{\mathbf{B}}^{iT}\mathbf{\Gamma}\dot{\tilde{\mathbf{B}}}^i)\\ &= -\frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e}+\mathrm{tr}\left(\sum_{i=1}^{h}w^i\tilde{\mathbf{A}}^{iT}\mathbf{\Gamma}\mathbf{e}\mathbf{x}_\xi^T-\sum_{i=1}^{h}\frac{\tilde{\mathbf{A}}^{iT}\mathbf{\Gamma}\dot{\hat{\mathbf{A}}}^i}{\eta_1}\right)\\ &\quad+\mathrm{tr}\left(\sum_{i=1}^{h}w^i\tilde{\mathbf{B}}^{iT}\mathbf{\Gamma}\mathbf{e}\mathbf{u}_\xi^T-\sum_{i=1}^{h}\frac{\tilde{\mathbf{B}}^{iT}\mathbf{\Gamma}\dot{\hat{\mathbf{B}}}^i}{\eta_2}\right).\end{aligned}\tag{34}$$

When the determinant of the matrix $\Omega^T\Omega$ is smaller than or equal ε, we set $\beta=1$ and (32) becomes

$$\begin{aligned}\dot{v} &= \Delta+\mathbf{e}^T\mathbf{\Gamma}\tilde{\mathbf{d}}-\mathbf{e}^T\mathbf{\Gamma}\,\mathrm{Diag}[\mathrm{sign}(\mathbf{e}_\Delta)]\hat{\mathbf{d}}-\mathbf{e}^T\mathbf{\Gamma}\Omega\mathbf{d}_\Delta+\mathbf{e}^T\mathbf{\Gamma}\Omega\mathbf{u}_\Delta\\ &\quad-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i-\frac{1}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}\\ &= \Delta+\sum_{i=1}^{n}e_{\Delta i}\tilde{d}_i-\sum_{i=1}^{n}|e_{\Delta i}|\,\hat{d}_i-\sum_{j=1}^{m}\bar{e}_{\Delta j}d_{\Delta j}+\sum_{j=1}^{m}\left|\bar{e}_{\Delta j}\right|\hat{d}_{\Delta j}\\ &\quad-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i-\frac{1}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}\\ &\le \Delta+\sum_{i=1}^{n}|e_{\Delta i}|\left|\bar{d}_i\right|-\sum_{i=1}^{n}|e_{\Delta i}|\,\hat{d}_i-\sum_{j=1}^{m}\left|\bar{e}_{\Delta j}\right|\left|d_{\Delta j}\right|\\ &\quad+\sum_{j=1}^{m}\left|\bar{e}_{\Delta j}\right|\hat{d}_{\Delta j}-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i-\frac{1}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}\\ &\le \Delta+\sum_{i=1}^{n}|e_{\Delta i}|\,\boldsymbol{\tau}_i^T\mathbf{q}_i^*-\sum_{i=1}^{n}|e_{\Delta i}|\,\boldsymbol{\tau}_i^T\mathbf{q}_i-\sum_{j=1}^{m}\left|\bar{e}_{\Delta j}\right|\boldsymbol{\tau}_{\Delta j}^T\mathbf{q}_{\Delta j}^*\\ &\quad+\sum_{j=1}^{m}\left|\bar{e}_{\Delta j}\right|\boldsymbol{\tau}_{\Delta j}^T\mathbf{q}_{\Delta j}-\frac{1}{\gamma_1}\sum_{i=1}^{n}\tilde{\mathbf{q}}_i^T\dot{\mathbf{q}}_i-\frac{1}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}\\ &= \Delta+\sum_{i=1}^{n}|e_{\Delta i}|\,\boldsymbol{\tau}_i^T\tilde{\mathbf{q}}_i-\sum_{j=1}^{m}\left|\bar{e}_{\Delta j}\right|\boldsymbol{\tau}_{\Delta j}^T\tilde{\mathbf{q}}_{\Delta j}-\frac{1}{\gamma_1}\sum_{i=1}^{n}\dot{\mathbf{q}}_i^T\tilde{\mathbf{q}}_i\\ &\quad-\frac{1}{\gamma_2}\sum_{j=1}^{m}\tilde{\mathbf{q}}_{\Delta j}^T\dot{\mathbf{q}}_{\Delta j}\\ &= \Delta+\left(|e_{\Delta 1}|\,\boldsymbol{\tau}_1^T-\frac{1}{\gamma_1}\dot{\mathbf{q}}_1^T\right)\tilde{\mathbf{q}}_1+\left(|e_{\Delta 2}|\,\boldsymbol{\tau}_2^T-\frac{1}{\gamma_1}\dot{\mathbf{q}}_2^T\right)\tilde{\mathbf{q}}_2+\cdots\\ &\quad+\left(|e_{\Delta n}|\,\boldsymbol{\tau}_n^T-\frac{1}{\gamma_1}\dot{\mathbf{q}}_n^T\right)\tilde{\mathbf{q}}_n-\left(|\bar{e}_{\Delta 1}|\,\boldsymbol{\tau}_{\Delta 1}^T+\frac{1}{\gamma_2}\dot{\mathbf{q}}_{\Delta 1}^T\right)\tilde{\mathbf{q}}_{\Delta 1}\\ &\quad-\left(|\bar{e}_{\Delta 2}|\,\boldsymbol{\tau}_{\Delta 2}^T+\frac{1}{\gamma_2}\dot{\mathbf{q}}_{\Delta 2}^T\right)\tilde{\mathbf{q}}_{\Delta 2}-\cdots\\ &\quad-\left(|\bar{e}_{\Delta m}|\,\boldsymbol{\tau}_{\Delta m}^T+\frac{1}{\gamma_2}\dot{\mathbf{q}}_{\Delta m}^T\right)\tilde{\mathbf{q}}_{\Delta m}\end{aligned}\tag{35}$$

where Δ is the same as (34). If we select $\dot{\hat{\mathbf{A}}}^i$, $\dot{\hat{\mathbf{B}}}^i$, $\dot{\mathbf{q}}_k$ and $\dot{\mathbf{q}}_{\Delta k}$ as (25)–(28), (33) and (35) become

$$\dot{v}=-\frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e}\le 0.\tag{36}$$

Eqs. (29) and (36) only guarantee that $\mathbf{e}(t)\in L_\infty$, but not that it converges. The boundedness of $\mathbf{e}(t)$ implies the boundedness of $\mathbf{x}(t)$. Since the operating states are finite, $\mathbf{x}_\xi$ is bounded. Based on Assumption 1 and the boundedness of $\mathbf{x}_\xi$, $\mathbf{u}_\xi$ is bounded. Therefore, $\dot{\mathbf{e}}(t)$ is bounded, i.e. $\dot{\mathbf{e}}(t)\in L_\infty$. Integrating both sides of (36) yields

$$v(t)-v(0)\le-\frac{1}{2}\lambda_{\min}(\mathbf{Q})\int_0^t\|\mathbf{e}(\tau)\|^2\,\mathrm{d}\tau\tag{37}$$

where $\lambda_{\min}(\mathbf{Q})>0$ is the minimum eigenvalue of $\mathbf{Q}$. When t tends to approach infinity, (37) becomes

$$\int_0^\infty\|\mathbf{e}(\tau)\|^2\,\mathrm{d}\tau\le\frac{v(0)-v(\infty)}{\frac{1}{2}\lambda_{\min}(\mathbf{Q})}.\tag{38}$$

Since the right side of (38) is bounded, $\mathbf{e}\in L_2$. Therefore, by using Barbalat's Lemma (Hornick, Stinchcombe, & White, 1989), we have $\|\mathbf{e}(t)\|\to 0$ as $t\to\infty$. This completes the proof. □

Fig. 4 shows the overall scheme of the T–S fuzzy-neural controller proposed in this paper. By using the proposed T–S fuzzy-neural model controller, the states of the general unknown nonlinear system with external disturbances can track the desired variables effectively. Moreover, adjustable parameters of the T–S

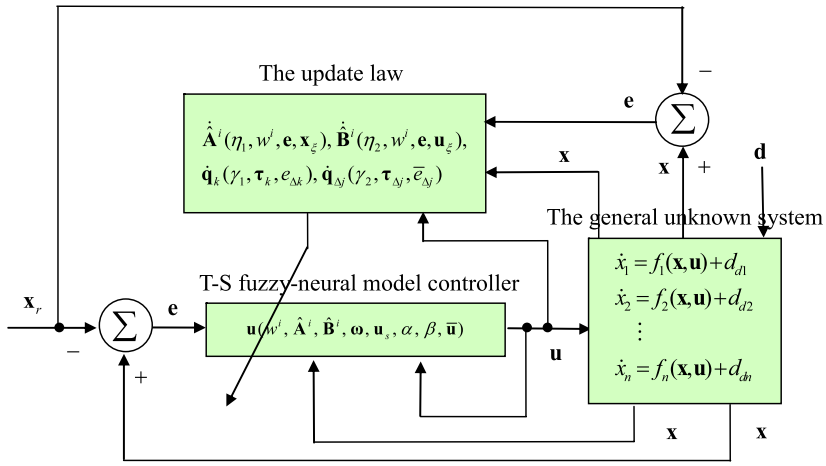

Fig. 4. The overall scheme of the T–S fuzzy-neural controller.

fuzzy-neural model cannot only be tuned to have a good tracking performance but also ensure the parameter matrix is bounded away from singularity through the proposed projection update laws. It is an on-line identification algorithm for the T–S fuzzy-neural model and a robust tracking controller for the general unknown systems.

5. Illustrative example

This section presents the simulation results of the proposed controller, showing that the tracking error of the closed-loop system can be made arbitrarily small. In addition, the simulation results confirm that the effect of modeling errors and external disturbances on the tracking error is attenuated efficiently by the proposed controller. The experiments use a personal computer with a Pentium-4 2.4 GHz CPU, 1 GB RAM, and the programming language MATLAB 7.0.4 to obtain the output responses of the closed-loop systems. Tables 1 and 2 illustrate the calculating time (constructing fuzzy sets, obtaining the control laws and update laws, etc.).

Example 1. Consider an MIMO nonaffine nonlinear system:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}, \mathbf{u}) + d_{d1} = (1 + \sin(x_1 x_2))(u_1 + \mathrm{e}^{u_1} - 3) + d_{d1} \\ \dot{x}_2 &= f_2(\mathbf{x}, \mathbf{u}) + d_{d2} = x_1 x_2 + 10\log(2 + x_1^2)u_2 + 2u_1 + d_{d2} \end{aligned} \tag{39}$$

where $f_i(\mathbf{x}, \mathbf{u})$, $i = 1, 2$ are unknown nonlinear functions, u_1 and u_2 are the control inputs, and both d_{d1} and d_{d2} are external disturbances which are assumed to be random values in the interval $[-0.1, 0.1]$. In this example, two different cases are simulated. The parameters t_1 and t_2 for critical points $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ are both 0.5 in case 1, and are both 0.75 in case 2.

Five fuzzy sets over the interval $[-6, 6]$ are defined for $\mathbf{x}_\xi = [x_{\xi 1}, x_{\xi 2}]^T$ with the term sets (PB, PS, Z, NS, NB) and three fuzzy sets over the interval $[-1400, 1400]$ for $\mathbf{u}_\xi = [u_{\xi 1}\ u_{\xi 2}]^T$. The design parameters are selected as $\eta = 0.003$, $\lambda_1 = 5$, $\lambda_2 = 6$ and $\mathbf{Q} = [1\ 0;\ 0\ 2]$. The initial states of the system are assumed to be $\mathbf{x}(0) = [2, -2]^T$ (case 1) and $\mathbf{x}(0) = [-1, \ -1]^T$ (case 2). We use the proposed control laws in (12) and (13) to control the state x_1 of the system to track the reference signal $x_{r1} = 1.5 - 4\mathrm{e}^{-0.5t}$ (case 1), $\sin(0.5t) + \cos(t)$ (case 2) and to control the state x_2 of the system to track the reference signal $x_{r2} = 2\mathrm{e}^{-0.5t}$ (case 1), $\cos(0.5t) + \sin(t)$ (case 2). Figs. 5 and 6 show the trajectories of the state vector $\mathbf{x}$, reference signal vector $\mathbf{x}_r$ and control input vector $\mathbf{u}$ (cases 1 and 2). The simulation results indicate that the effect of modeling errors, and external disturbances on the tracking error is attenuated efficiently by the proposed controller.

Table 1
Calculating time in Example 1.

Calculating time (time unit is second)	Case 1	Case 2
Constructing fuzzy sets	0.013	0.015
Obtaining the control laws	0.047	0.052
Obtaining the update laws	0.058	0.068

Example 2. Consider a third order MIMO nonlinear system:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}, \mathbf{u}) + d_{d1} = x_1 u_1 u_2 + 0.2u_3 + d_{d1} \\ \dot{x}_2 &= f_2(\mathbf{x}, \mathbf{u}) + d_{d2} = x_1 + x_2^2 + x_3 + 3u_1 + u_2 + d_{d2} \\ \dot{x}_3 &= f_3(\mathbf{x}, \mathbf{u}) + d_{d3} = x_1 + 2x_2 + 3x_1 x_3 \\ &\quad + u_1 + 2(2 + 0.5\sin(x_1))u_2 + d_{d3} \end{aligned} \tag{40}$$

where $f_i(\mathbf{x}, \mathbf{u})$, $i = 1, 2, 3$ are unknown nonlinear functions, u_1, u_2 and u_3 are the control inputs, and d_{d1}, d_{d2} and d_{d3} are external disturbances which are assumed to be sine waves with the amplitude ± 0.1 and the period 2π. In this example, two different cases are simulated. The parameters t_1 and t_2 for critical points $\bar{\mathbf{x}}$ and $\bar{\mathbf{u}}$ are 0.5 and 0.5, respectively, and are the same in both cases.

Five fuzzy sets over the interval $[-6, 6]$ are defined for $\mathbf{x}_\xi = [x_{\xi 1}, x_{\xi 2}, x_{\xi 3}]^T$ with the term sets (PB, PS, Z, NS, NB) and three fuzzy sets over the interval $[-1400, 1400]$ for $u_\xi = [u_{\xi 1}, u_{\xi 2}, u_{\xi 3}]^T$. The design parameters are selected as $\eta = 0.003$, $\lambda_1 = 15$, $\lambda_2 = 30$, $\lambda_2 = 60$ and $\mathbf{Q} = [50\ 0\ 0;\ 0\ 50\ 0;\ 0\ 0\ 50]$. The initial states of the system are assumed to be $\mathbf{x}(0) = [1.5, \ 1.5, \ 1]^T$ (case 1) and $\mathbf{x}(0) = [0, \ 0, \ 0]^T$ (case 2). We use the proposed control laws in (12) and (13) to control the state x_1 of the system to track the reference signal $x_{r1} = 1.5 - \mathrm{e}^{-0.5t}$ (case 1), $0.5 + \frac{1}{4}\cos(0.25\pi t) + \frac{1}{4}\sin(0.5\pi t)$ (case 2) and to control the state x_2 of the system to track the reference signal $x_{r2} = 0.5\mathrm{e}^{-0.5t}$ (case 1), $0.5 + \frac{1}{4}\sin(0.25\pi t) + \frac{1}{4}\sin(0.5\pi t)$ (case 2) and to control the state x_3 of the system to track the reference signal $x_{r3} = -0.25\mathrm{e}^{-0.5t}$ (case 1), $-0.5 - \frac{1}{4}\sin(0.25\pi t) - \frac{1}{4}\sin(0.5\pi t)$ (case 2). Figs. 7 and 8 show the trajectories of state vector $\mathbf{x}$, reference signal vector $\mathbf{x}_r$ and control input vector $\mathbf{u}$ (cases 1 and 2). The simulation results indicate that the effect of modeling errors and external disturbances on the tracking error is attenuated efficiently by the proposed controller.

6. Conclusion

The nonaffine nonlinear functions of general systems are unknown and traditional T–S fuzzy control methods can model and control them only with great difficulty. Therefore, we proposed the

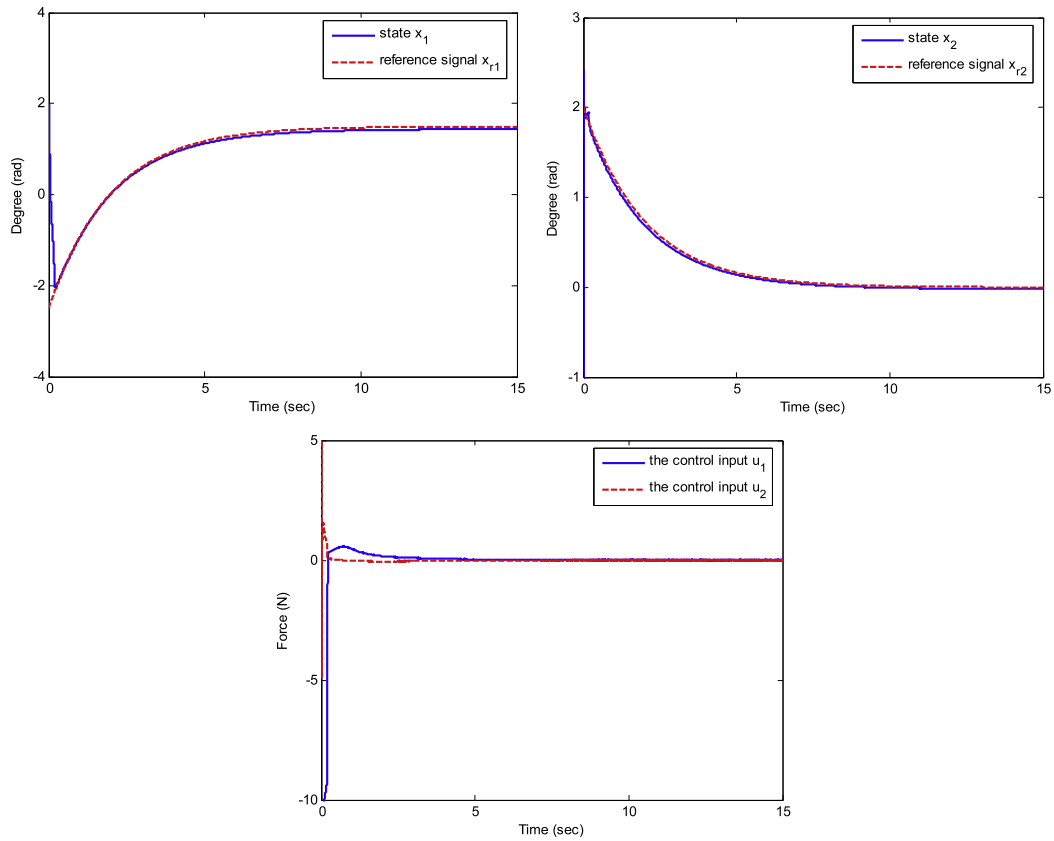

Fig. 5. The trajectories of state vector $\mathbf{x}$, reference signal vector $\mathbf{x}_r$ and control input vector $\mathbf{u}$ in Example 1 (case 1).

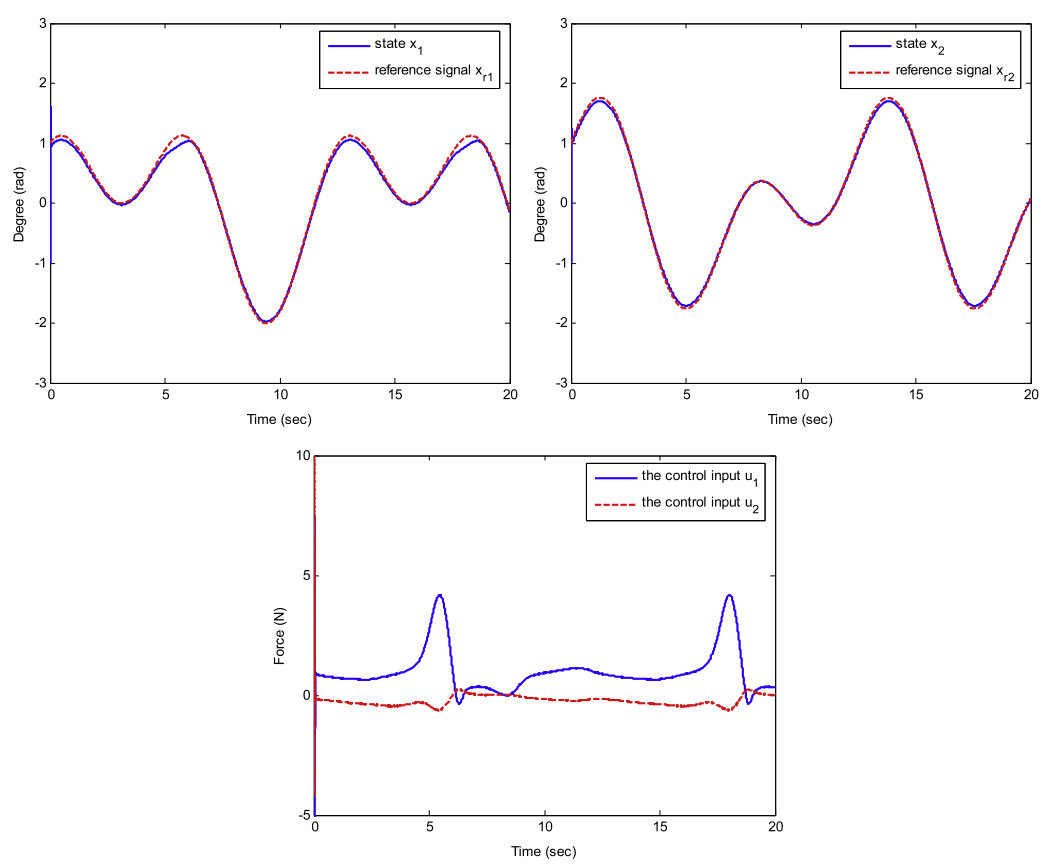

Fig. 6. The trajectories of state vector $\mathbf{x}$, reference signal vector $\mathbf{x}_r$ and control input vector $\mathbf{u}$ in Example 1 (case 2).

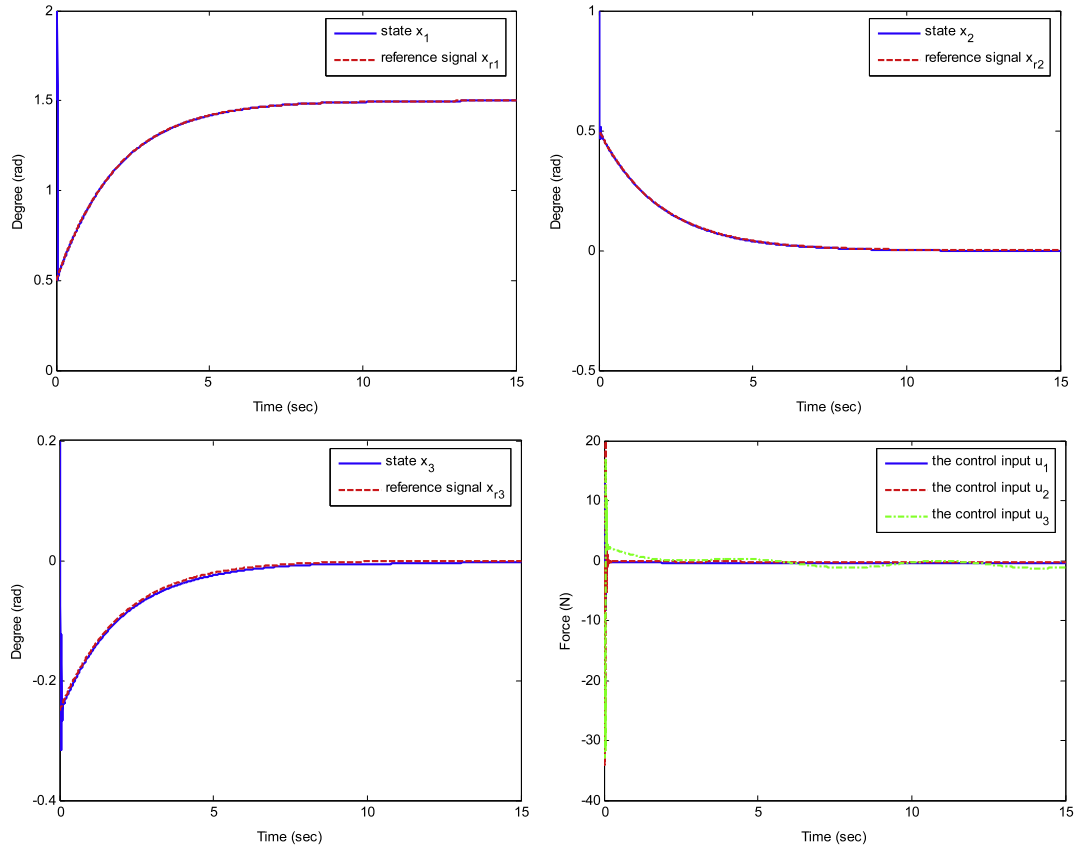

Fig. 7. The trajectories of state vector $\mathbf{x}$, reference signal vector $\mathbf{x}_r$ and control input vector $\mathbf{u}$ in Example 2 (case 1).

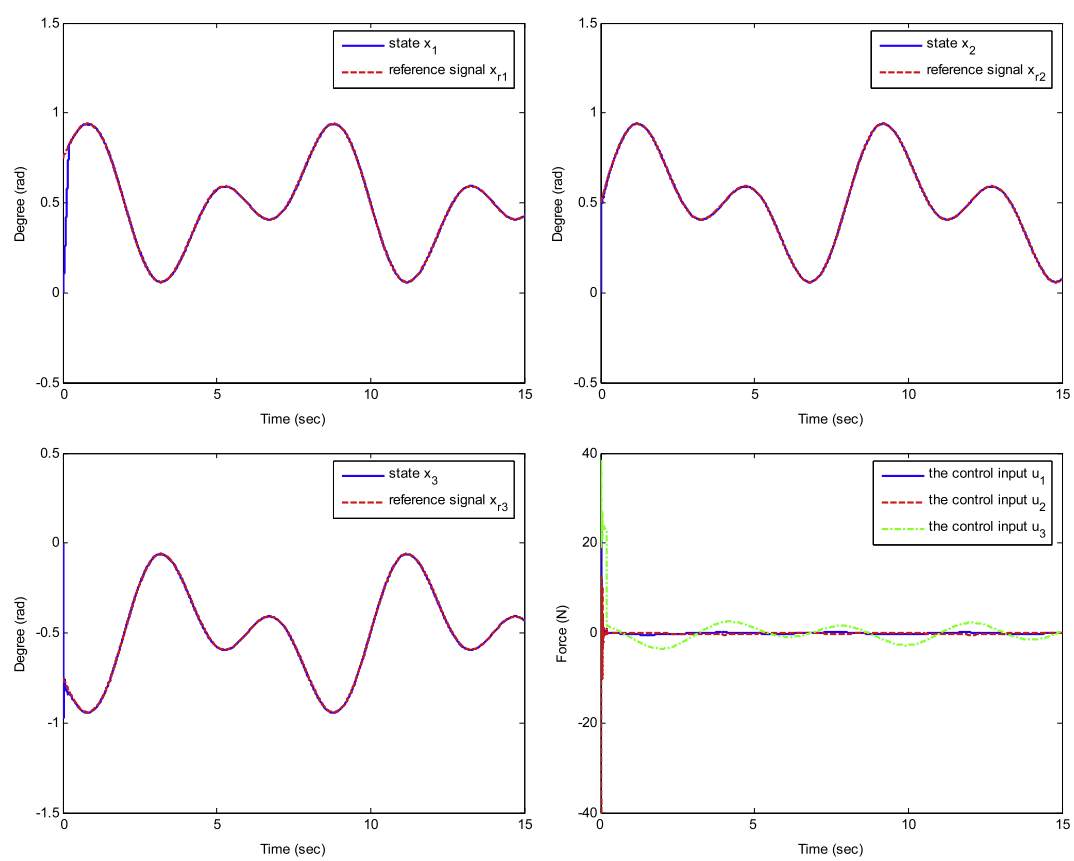

Fig. 8. The trajectories of state vector $\mathbf{x}$, reference signal vector $\mathbf{x}_r$ and control input vector $\mathbf{u}$ in Example 2 (case 2).

Table 2
Calculating time in Example 2.

Calculating time (time unit is second)	Case 1	Case 2
Constructing fuzzy sets	0.048	0.033
Obtaining the control laws	0.089	0.076
Obtaining the update laws	0.099	0.097

on-line identification algorithm for the virtual linearized system (VLS) and put a significant emphasis on the robust tracking controller design using an adaptive scheme for the general unknown systems. In addition, we use the proposed projection update law to tune the adjustable parameters to prevent parameter drift. The tracking error of the closed-loop system can be made arbitrarily small. Finally, simulation results are provided to demonstrate the robustness and applicability of the proposed control scheme.

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