Applied Mathematics Letters 26 (2013) 1175–1181

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Inverse spectral problems for Sturm–Liouville operators with partial information

Yu Ping Wang [a,*], Chung Tsun Shieh [b], Yan Ting Ma [a]

[a] Department of Applied Mathematics, Nanjing Forestry University, Nanjing, 210037, Jiangsu, People's Republic of China
[b] Department of Mathematics, Tamkang University, Danshui Dist. New Taipei City, 25137, Taiwan, ROC

ARTICLE INFO

Article history:
Received 7 May 2013
Received in revised form 9 May 2013
Accepted 2 July 2013

Keywords:
Inverse spectral problem
Sturm–Liouville operator
Potential
Information
Spectrum

ABSTRACT

In this paper, we study the inverse spectral problems for Sturm–Liouville operators with Robin boundary conditions and show that if the potential q on the interval $[0, \alpha]$ for some $\alpha \in [0, 1)$ is given a priori, then the potential q on the whole interval $[0, 1]$ can be uniquely determined by a subset of pairs of eigenvalues and the weight numbers of the corresponding eigenvalues or by parts of two spectra.

1. Introduction

Consider the following Sturm–Liouville operator $L := L(q, h_0, h_1)$ defined by

$$Ly := -y'' + q(x)y = \lambda y, \quad 0 < x < 1 \tag{1.1}$$

with the boundary conditions

$$U_0(y) := y'(0, \lambda) - h_0 y(0, \lambda) = 0, \tag{1.2}$$

$$U_1(y) := y'(1, \lambda) - h_1 y(1, \lambda) = 0, \tag{1.3}$$

where $h_\xi \in \mathbf{R}$ for $\xi = 0, 1$ and q is a real-valued function and $q \in L^2(0, 1)$.

Numerous research results for the problem (1.1)–(1.3) have been established by renowned mathematicians, notably, Borg [1] and Levinson [2], who independently showed that two spectra $\{\lambda_n, \mu_n\}$ uniquely determined the potential q and coefficients h_0, h_1 of the boundary conditions. Hochstadt and Lieberman [3] initiated the study of the so-called "half inverse problem" for the problem (1.1)–(1.3) and proved that if h_0 is known a priori and q is prescribed on the interval $[1/2, 1]$, then one spectrum is enough to determine the potential function uniquely. After that, the half inverse problems for differential operators were investigated by many authors [3–14]. In particular, Suzuki [4] showed by some examples that the fixed boundary condition (1.2) is necessary for the Hochstadt–Lieberman Theorem and that one spectrum cannot uniquely determine the potential q if q is prescribed on $\left[0, \frac{1}{2} - \varepsilon\right]$ for $0 < \varepsilon < \frac{1}{2}$. Marchenko [15] adopted an alternative approach to inverse spectral theory via the Weyl m-function and proved that the Weyl m-function of the Sturm–Liouville operator uniquely determined the coefficients h_0, h_1 of the boundary conditions as well as the potential q. In fact, a lot of related

* Corresponding author. Tel.: +13951082678.
E-mail addresses: ypwang@njfu.com.cn (Y.P. Wang), ctshieh@math.tku.edu.tw (C.T. Shieh), 657853573@qq.com (Y.T. Ma).

0893-9659/$ – see front matter
http://dx.doi.org/10.1016/j.aml.2013.07.003

results have been obtained by this approach [6,10,11,15,16,12–14,17–23] and one of the interesting results was achieved by Gesztesy and Simon (see [12, Theorem 1.3]), which is a generalization of the Hochstadt–Lieberman Theorem [3]. They used the Weyl m-function to study the inverse spectral problem of the problem (1.1)–(1.3) with prescribed partial information on the potential and parts of one spectrum. Wei and Xu [20] further demonstrated that if q is prescribed on the interval $[0, \alpha]$ for some $\alpha \in [0, 1)$, then parts of $\{(\lambda_n, \kappa_n)\}$ (see Section 2) are sufficient to determine the potential (see [20, Theorem 4.1]). Since the spectral data $\{\lambda_{1n}, \lambda_{2n}\}_{n=0}^{\infty}$, $\{(\lambda_n, \kappa_n)\}_{n=0}^{\infty}$ and $\{(\lambda_n, \alpha_n)\}_{n=0}^{\infty}$ are equivalent [23], this paper will address the inverse problems for the problem (1.1)–(1.3) via parts of $\{\lambda_{1n}, \lambda_{2n}\}_{n=0}^{\infty}$ or $\{(\lambda_n, \alpha_n)\}_{n=0}^{\infty}$. Therefore, two uniqueness theorems for the problem (1.1)–(1.3) are established on the basis of partial information on the potential and a subset of pairs of eigenvalues and the weight numbers of the corresponding eigenvalues or on the basis of partial information on the potential and parts of two spectra. The techniques used here are based on the Weyl m-function and methods developed in Refs. [12,20].

This article is organized as follows. In Section 2, we present preliminaries for Sturm–Liouville operators. In Section 3, we will prove the validity of main results.

2. Preliminaries

In this section, we present preliminaries for Sturm–Liouville operators with Robin boundary conditions.

Let $S_1(x, \lambda)$, $S_2(x, \lambda)$, $u_-(x, \lambda)$ and $u_+(x, \lambda)$ be solutions of Eq. (1.1) which satisfy the initial conditions:

$$S_1(0, \lambda) = 0, \qquad S_1'(0, \lambda) = 1, \qquad S_2(0, \lambda) = 1, \qquad S_2'(0, \lambda) = 0, \tag{2.1}$$

$$u_-(0, \lambda) = 1, \qquad u_-'(0, \lambda) = h_0, \qquad u_+(1, \lambda) = 1, \qquad u_+'(1, \lambda) = h_1. \tag{2.2}$$

Clearly, $U_0(u_-) = U_1(u_+) = 0$ and

$$u_-(x, \lambda) = S_2(x, \lambda) + h_0 S_1(x, \lambda), \tag{2.3}$$

$$\begin{aligned} u_+(x, \lambda) &= S_2(1 - x, \lambda) - h_1 S_1(1 - x, \lambda) \\ &= U_1(S_1) S_2(x, \lambda) - U_1(S_2) S_1(x, \lambda). \end{aligned} \tag{2.4}$$

It is easy to show that

$$\int_0^1 (yL(z) - zL(y)) = [y, z](1) - [y, z](0), \tag{2.5}$$

where $[y, z](x) := y(x)z'(x) - y'(x)z(x)$ is the Wronskian of y and z.

Let

$$\Delta(\lambda) := [u_+, u_-](x, \lambda). \tag{2.6}$$

Therefore $\Delta(\lambda)$ is independent of x and

$$\Delta(\lambda) = U_1(u_-) = -U_0(u_+), \tag{2.7}$$

which is called $\Delta(\lambda)$ the characteristic function of the operator L, that is, zero values of $\Delta(\lambda)$ coincide with the eigenvalues of L.

We know that all zeros $\lambda_n (n \in \mathbf{N}_0 = \{0, 1, 2, \ldots\})$ of $\Delta(\lambda)$ are real and simple and satisfy the following asymptotic formula [23]:

$$\sqrt{\lambda_n} = n\pi + \frac{\omega}{n\pi} + \frac{c_n}{n}, \tag{2.8}$$

where $\omega = h_0 - h_1 + \frac{1}{2}\int_0^1 q(x)dx$, $\{c_n\} \in l^2$.

Let $\sigma(L) = \{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of the operator L and $u_-(x, \lambda_n)$ and $u_+(x, \lambda_n)$ be eigenfunctions of the corresponding eigenvalue λ_n, then there exists κ_n such that

$$u_+(x, \lambda_n) = \kappa_n u_-(x, \lambda_n), \tag{2.9}$$

where κ_n is called the normalization constant of the corresponding eigenvalue λ_n. Hence $\kappa_n \neq 0, \infty$ and

$$u_+(0, \lambda_n) = \kappa_n. \tag{2.10}$$

Define the weight numbers α_n by

$$\alpha_n := \int_0^1 u_+^2(x, \lambda_n)dx. \tag{2.11}$$

The numbers $\{\lambda_n, \alpha_n\}$ are called the spectral data of L. The following relation holds

$$\kappa_n \alpha_n = -\dot{\Delta}(\lambda_n), \tag{2.12}$$

where $\dot{\Delta}(\lambda_n) = \frac{d\Delta(\lambda)}{d\lambda}|_{\lambda=\lambda_n}$.

We denote $\lambda = \rho^2$ and $\tau = |Im\rho|$ and then we have the asymptotic formulae of $u_-(x,\lambda)$ and $u_+(x,\lambda)$ as follows

$$u_-(x,\lambda) = \cos\rho x + O\left(\frac{e^{\tau x}}{\rho}\right), \quad 0 \le x \le 1,$$
$$u'_-(x,\lambda) = -\rho\sin\rho x + O(e^{\tau x}), \quad 0 \le x \le 1, \tag{2.13}$$

$$u_+(x,\lambda) = \cos\rho(1-x) + O\left(\frac{e^{\tau(1-x)}}{\rho}\right), \quad 0 \le x \le 1,$$
$$u'_+(x,\lambda) = \rho\sin\rho(1-x) + O(e^{\tau(1-x)}), \quad 0 \le x \le 1. \tag{2.14}$$

Consequently, for sufficiently large $|\lambda|$, we obtain the following asymptotic formula of the characteristic function $\Delta(\lambda)$ of the operator L:

$$\Delta(\lambda) = -\rho\sin\rho + O(e^{\tau}). \tag{2.15}$$

Let $G_\delta := \{\rho : |\rho - k\pi| > \delta, k \in \mathbf{Z}\}$, where $\mathbf{Z} := \{0, \pm1, \pm2, \ldots\}$ and δ sufficiently small, then there exists a constant C_δ such that for sufficiently large $|\lambda|$,

$$|\Delta(\lambda)| \ge C_\delta|\rho|e^{\tau}, \quad \forall\rho \in G_\delta. \tag{2.16}$$

We define the Weyl m-function $m(x,\lambda)$ by

$$m(x,\lambda) = \frac{u'_+(x,\lambda)}{u_+(x,\lambda)}.$$

From Ref. [15] we get the following asymptotic formulae:

$$m(x,\lambda) = i\rho + o(1), \qquad \frac{1}{m(x,\lambda)} = -\frac{i}{\rho} + o\left(\frac{1}{\rho^2}\right) \tag{2.17}$$

uniformly in $x \in [0, 1-\delta]$ for $\delta > 0$ as $|\lambda| \to \infty$ in any sector $\varepsilon < Arg(\lambda) < \pi - \varepsilon$ for $\varepsilon > 0$. The following two lemmas are important for proofs of the main results in this paper.

Lemma 2.1 (*[15]*). *Let $m(\alpha,\lambda)(\alpha \in [0,1))$ be the Weyl m-function of the problem (1.1)–(1.3). Then $m(\alpha,\lambda)$ uniquely determines coefficient h_1 of the boundary condition as well as q on the interval $[\alpha, 1]$.*

Lemma 2.2 (*[12, Proposition B.6]*). *Let $f(z)$ be an entire function such that*

(1) $\sup_{|z|=R_k} |f(z)| \le C_1 \exp(C_2 R_k^{\alpha})$ *for some* $0 < \alpha < 1$, *some sequence* $R_k \to \infty$ *as* $k \to \infty$ *and* $C_1, C_2 > 0$;
(2) $\lim_{|x|\to\infty} |f(ix)| = 0$.

Then $f \equiv 0$.

3. Main results and proofs

In this section, we study the uniqueness theorems for Sturm–Liouville operators with Robin boundary conditions and intend to reconstruct the operator on the basis of arbitrary partial information on the potential q and a subset of eigenvalues and the corresponding weight numbers or on the basis of arbitrary partial information on the potential q and parts of two spectra. The techniques used here are analogous to the methods developed in Refs. [12,20].

Consider the following Sturm–Liouville operators $L_{kj} := L(q_k, h_{0j}, h_{1k})$ defined by

$$L_k u_k := -u''_k + q_k(x)u_k = \lambda u_k, \quad x \in (0,1) \tag{3.1}$$

with boundary conditions

$$U_{0j}(u_k) := u'_k(0,\lambda) - h_{0j}u_k(0,\lambda) = 0, \tag{3.2}$$
$$U_{1k}(u_k) := u'_k(1,\lambda) - h_{1k}u_k(1,\lambda) = 0, \tag{3.3}$$

where $h_{0j}, h_{1k} \in \mathbf{R}$, $h_{01} \ne h_{02}$ and $q_k \in L^2((0,1),\mathbf{R})$ for $k, j = 1, 2$.

We have the following uniqueness theorem.

Theorem 3.1. *Let $\sigma(L_k) = \{\lambda_{kn}\}_{n=0}^{\infty} (k = 1, 2)$ be the spectrum of Eq. (3.1) with boundary conditions (1.2) and (3.3) and $S = \{\lambda_{1n}\}_{n\in\Lambda} \subseteq \sigma(L_1) \bigcap \sigma(L_2)$, $\Lambda \subseteq \mathbf{N}_0$ and coefficient h_0 of the boundary condition be given a priori. Suppose the following conditions*

(1) $q_1 = q_2$ *a.e. on the interval* $[0,\alpha]$ *for some* $\alpha \in [0,1)$,

(2) $\alpha_{1n} = \alpha_{2n}$ *for all* $n \in \Lambda$ *and the inequality*

$$\sharp\{\lambda \in S | \lambda \le t\} \ge (1-\alpha)\sharp\{\lambda \in \sigma(L_1) | \lambda \le t\} + \frac{\alpha - 1}{2} \tag{3.4}$$

holds for all sufficiently large $t \in \mathbf{R}^+$.

are satisfied. Then $q_1 = q_2$ *on* $[0, 1]$ *and* $h_{11} = h_{12}$.

Proof. Let $u_{k+}(x, \lambda)(k = 1, 2)$ be the solution of Eq. (3.1) for q_k under the terminal conditions $u_{k+}(1, \lambda) = 1$ and $u'_{k+}(1, \lambda) = h_{1k}$. By Green's formula, we have

$$\begin{aligned}\int_0^1 Q(x)u_{1+}(x, \lambda)u_{2+}(x, \lambda)dx &= [u_{1+}, u_{2+}](1, \lambda) - [u_{1+}, u_{2+}](0, \lambda)\\ &= F(1, \lambda) - F(0, \lambda),\end{aligned} \tag{3.5}$$

where $Q(x) = q_2(x) - q_1(x)$ and

$$F(x, \lambda) = [u_{1+}, u_{2+}](x, \lambda). \tag{3.6}$$

From $Q(x) = 0$ on $[0, \alpha]$ together with the terminal conditions $u_{k+}(1, \lambda)$ and $u'_{k+}(1, \lambda)(k = 1, 2)$, we get

$$F(0, \lambda) = h_{12} - h_{11} - \int_\alpha^1 Q(x)u_{1+}(x, \lambda)u_{2+}(x, \lambda)dx. \tag{3.7}$$

Denote the entire functions $\Delta_k(\lambda)$ by

$$\Delta_k(\lambda) = -U_0(u_+) = -[u'_{k+}(0, \lambda) - h_0 u_{k+}(0, \lambda)] \quad (k = 1, 2). \tag{3.8}$$

Then

$$\begin{aligned}F(0, \lambda) &= [u_{1+}, u_{2+}](0, \lambda)\\ &= \begin{vmatrix} u_{1+}(0, \lambda) & u_{2+}(0, \lambda) \\ u'_{1+}(0, \lambda) & u'_{2+}(0, \lambda)\end{vmatrix}\\ &= -\begin{vmatrix} u_{1+}(0, \lambda) & u_{2+}(0, \lambda) \\ \Delta_1(\lambda) & \Delta_2(\lambda)\end{vmatrix}.\end{aligned} \tag{3.9}$$

$$\frac{dF(0, \lambda)}{d\lambda} = -\left(\begin{vmatrix} \dfrac{du_{1+}(0, \lambda)}{d\lambda} & \dfrac{du_{2+}(0, \lambda)}{d\lambda} \\ \Delta_1(\lambda) & \Delta_2(\lambda)\end{vmatrix} + \begin{vmatrix} u_{1+}(0, \lambda) & u_{2+}(0, \lambda) \\ \dot{\Delta}_1(\lambda) & \dot{\Delta}_2(\lambda)\end{vmatrix} \right). \tag{3.10}$$

For all $\lambda_{1n} \in S$, we obtain

$$\begin{aligned}\left.\frac{dF(0, \lambda)}{d\lambda}\right|_{\lambda=\lambda_{1n}} &= -\left(\left.\begin{vmatrix} \dfrac{du_{1+}(0, \lambda)}{d\lambda} & \dfrac{du_{2+}(0, \lambda)}{d\lambda} \\ \Delta_1(\lambda) & \Delta_2(\lambda)\end{vmatrix}\right|_{\lambda=\lambda_{1n}} + \begin{vmatrix} u_{1+}(0, \lambda_{1n}) & u_{2+}(0, \lambda_{1n}) \\ \dot{\Delta}_1(\lambda_{1n}) & \dot{\Delta}_2(\lambda_{1n})\end{vmatrix} \right)\\ &= \kappa_{2n}\dot{\Delta}_1(\lambda_{1n}) - \kappa_{1n}\dot{\Delta}_2(\lambda_{1n}).\end{aligned} \tag{3.11}$$

By virtue of the assumption $\alpha_{1n} = \alpha_{2n}$ of Theorem 3.1 for $n \in \Lambda$ together with (2.12), this yields

$$\kappa_{2n}\dot{\Delta}_1(\lambda_{1n}) - \kappa_{1n}\dot{\Delta}_2(\lambda_{1n}) = 0.$$

Hence

$$F(0, \lambda_{1n}) = 0 \quad \text{and} \quad \left.\frac{dF(0, \lambda)}{d\lambda}\right|_{\lambda=\lambda_{1n}} = 0. \tag{3.12}$$

Therefore, we see that all $\lambda_{1n} \in S$ are zeros of $F(0, \lambda)$ of order at least 2. Since $Q(x) = 0$ on $[0, \alpha]$, we have $F(0, \lambda) = F(\alpha, \lambda)$. Hence all $\lambda_{1n} \in S$ are also zeros of $F(\alpha, \lambda)$ of order at least 2. Without loss of generality, we assume all eigenvalues $\lambda_{kn} > 1$ of the problem (3.1), (1.2) and (3.3). By virtue of Ref. [23, p. 8], there exists a constant c_k such that

$$\Delta_k(\lambda) = c_k \prod_{n \in \mathbf{N}_0} \left(1 - \frac{\lambda}{\lambda_{kn}}\right) \quad (k = 1, 2). \tag{3.13}$$

Define the functions

$$G_S(\lambda) = \prod_{\lambda_{1n} \in S} \left(1 - \frac{\lambda}{\lambda_{1n}}\right) \tag{3.14}$$

and

$$K(\lambda) = \frac{F(\alpha,\lambda)}{G_S^2(\lambda)}. \tag{3.15}$$

Then, $K(\lambda)$ is an entire function in λ. Note that

$$\begin{aligned} F(\alpha,\lambda) &= u_{1+}(\alpha,\lambda)u'_{2+}(\alpha,\lambda) - u'_{1+}(\alpha,\lambda)u_{2+}(\alpha,\lambda) \\ &= u'_{1+}(\alpha,\lambda)u'_{2+}(\alpha,\lambda)(m_1^{-1}(\alpha,\lambda) - m_2^{-1}(\alpha,\lambda)), \end{aligned} \tag{3.16}$$

where $m_k(\alpha,\lambda) = \frac{u'_{k+}(\alpha,\lambda)}{u_{k+}(\alpha,\lambda)} (k = 1, 2)$. From (2.14), (2.17), (3.16) and (3.8), we get

$$|F_1(\alpha,\lambda)| = o(e^{2\tau(1-\alpha)}) \tag{3.17}$$

and

$$|\Delta_k(\lambda)| = O(\rho e^{\tau}). \tag{3.18}$$

For convenience, we denote

$$N_{G_S}(t) = \sharp\{\lambda_{1n} \in S | \lambda_{1n} \le t\}, \qquad N_{\Delta_1}(t) = \sharp\{\lambda_{1n} \in \sigma(L_1) | \lambda_{1n} \le t\}.$$

By virtue of (3.4), we obtain

$$N_{G_S}(t) \ge (1-\alpha)N_{\Delta_1}(t) + \frac{\alpha - 1}{2}. \tag{3.19}$$

Since $\Delta_1(\lambda)$ is an entire function in λ of order $\frac{1}{2}$, there exists a positive constant C such that

$$N_{G_S}(t) \le N_{\Delta_1}(t) \le Ct^{\frac{1}{2}}. \tag{3.20}$$

From the above assumption $\lambda_{1n} > 1$ for all $n \ge 0$, we get $N_{G_S}(1) = N_{\Delta_1}(1) = 0$. For a fixed real number y and $|y| \gg 1$, we have

$$\begin{aligned} \ln|G_S(iy)| &= \frac{1}{2}\ln G_S(iy)\overline{G_S(iy)} = \frac{1}{2}\sum_{\lambda_{1n}\in S} \ln\left(1 - \frac{iy}{\lambda_{1n}}\right)\left(1 + \frac{iy}{\lambda_{1n}}\right) \\ &= \frac{1}{2}\sum_{\lambda_{1n}\in S} \ln\left(1 + \frac{y^2}{(\lambda_{1n})^2}\right) = \frac{1}{2}\int_1^\infty \ln\left(1 + \frac{y^2}{t^2}\right) \mathrm{d}N_{G_S}(t) \\ &= \frac{1}{2}\ln\left(1 + \frac{y^2}{t^2}\right) N_{G_S}(t)\,|_1^\infty - \frac{1}{2}\int_1^\infty N_{G_S}(t)\mathrm{d}\left[\ln\left(1 + \frac{y^2}{t^2}\right)\right]. \end{aligned} \tag{3.21}$$

Since

$$\ln\left(1 + \frac{y^2}{t^2}\right) = O\left(\frac{1}{t^2}\right), \quad \text{as } t \to \infty,$$

we obtain

$$\lim_{t\to\infty} \ln\left(1 + \frac{y^2}{t^2}\right) N_{G_S}(t) = 0$$

and

$$\lim_{t\to\infty} \ln\left(1 + \frac{y^2}{t^2}\right) N_{\Delta_1}(t) = 0.$$

By assumption (2) of Theorem 3.1, there exists a constant $t_0 \ge 1$ and C_1 such that

$$N_{G_S} = \begin{cases} N_{G_S}(t) \ge (1-\alpha)N_{\Delta_1}(t) + \dfrac{\alpha - 1}{2}, & t \ge t_0, \\ N_{G_S}(t) \ge (1-\alpha)N_{\Delta_1}(t) - C_1, & t < t_0. \end{cases}$$

Consequently, from (3.21) together with the following relation

$$\frac{y^2}{t^3 + ty^2} = -\frac{d}{dt}\left(\frac{1}{2}\ln\left(1 + \frac{y^2}{t^2}\right)\right),$$

we get

$$\begin{aligned}\ln|G_S(\mathrm{i}y)| &= \int_1^{\infty}\frac{y^2}{t^3+ty^2}N_{G_S}(t)\mathrm{d}t\\ &= \int_1^{t_0}\frac{y^2}{t^3+ty^2}N_{G_S}(t)\mathrm{d}t+\int_{t_0}^{\infty}\frac{y^2}{t^3+ty^2}N_{G_S}(t)\mathrm{d}t\\ &\geq (1-\alpha)\int_1^{\infty}\frac{y^2}{t^3+ty^2}N_{\Delta_{1j}}(t)\mathrm{d}t+\frac{\alpha-1}{2}\int_1^{\infty}\frac{y^2}{t^3+ty^2}\mathrm{d}t-\left(\frac{\alpha-1}{2}+C_1\right)\int_1^{t_0}\frac{y^2}{t^3+ty^2}dt\\ &= (1-\alpha)\ln|\Delta_1(\mathrm{i}y)|+\frac{\alpha-1}{4}\ln(1+y^2)+\frac{\alpha-1+2C_1}{4}\ln\frac{1+y^2}{t_0^2+y^2}+\frac{\alpha-1+2C_1}{4}\ln t_0.\end{aligned}\tag{3.22}$$

This implies

$$|G_S(\mathrm{i}y)|\geq C_0|\Delta_1(iy)|^{1-\alpha}(1+y^2)^{\frac{\alpha-1}{2}},\tag{3.23}$$

where C_0 is constant.

From (3.17), (3.18) and (3.23), we have

$$|K(iy)|=\left|\frac{F(\alpha,\mathrm{i}y)}{G_S^2(iy)}\right|=o(1)\tag{3.24}$$

for $|y|$ sufficiently large.

Applying Lemma 2.2, we obtain

$$K(\lambda)=0,\quad \forall\lambda\in\mathbf{C}.$$

Therefore

$$F(0,\lambda)=F(\alpha,\lambda)=0,\quad \forall\lambda\in\mathbf{C}.\tag{3.25}$$

By virtue of (3.25) together with (3.16), this yields

$$m_1(0,\lambda)=m_2(0,\lambda),\quad \forall\lambda\in\mathbf{C}.\tag{3.26}$$

This implies

$$q_1=q_2\quad \text{a.e. on } [0,1],\quad \text{and}\quad h_{11}=h_{12}.$$

The proof of Theorem 3.1 is now completed. □

Remark 1. From the proofs of Theorem 3.1, we provide an alternative proof for Theorem 1.2.2 or Theorem 1.2.4 (see [23, Theorem 1.2.2, p.21 and Theorem 1.2.4, p. 24]).

Let $\alpha=\frac{1}{2}$; then we have the following corollary.

Corollary 3.2. *Let $\sigma(L)=\{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of the problem (1.1)–(1.3). Assume that the coefficient h_0 of the boundary condition is given a priori and q on the interval $\left[0,\frac{1}{2}\right]$ is known a priori, then the even spectral data $\{(\lambda_{2n},\alpha_{2n})\}_{n=0}^{\infty}$ or the odd spectral data $\{(\lambda_{2n-1},\alpha_{2n-1})\}_{n=1}^{\infty}$ is sufficient to determine the potential q on the whole interval $[0,1]$ and coefficient h_1 of the boundary condition.*

In the rest parts of this section, we use partial information on the potential and parts of two spectra to establish the following uniqueness theorem for Sturm–Liouville operators.

Denote $\sigma(L_{kj})=\{\lambda_{kjn}\}_{n=0}^{\infty}(k,j=1,2)$ the spectrum of the problem (3.1)–(3.3). Since $h_{01}\neq h_{02}$, it is easy to prove $\sigma(L_{k1})\cap\sigma(L_{k2})=\emptyset$ for $k=1,2$. Applying the same arguments as that in the proof of Theorem 3.1, we can prove Theorem 3.3. We omit the details here. The readers can follow the proofs of Theorem 3.1 to reach the following conclusions.

Theorem 3.3. *Let $\sigma(L_{kj})$ be as that defined above, coefficients h_{01},h_{02} of the boundary conditions be given a prior, $S_j=\{\lambda_{1jn}\}_{n\in\Lambda_j}\subseteq\sigma(L_{1j})\bigcap\sigma(L_{2j})$, $\Lambda_j\subseteq\mathbf{N}_0$ for $k,j=1,2$ and $(\alpha_0,\alpha_1,\alpha_2)\in[0,1]\times[0,1/2]\times[0,1/2]$. Suppose the following conditions:*

(1) $q_1=q_2$ on the interval $[0,\alpha_0]$,
(2) $\alpha_0-\alpha_1-\alpha_2=0$ and the inequality

$$\sharp\{\lambda\in S_j|\lambda\leq t\}\geq(1-2\alpha_j)\sharp\{\lambda\in\sigma(L_{1j})|\lambda\leq t\}+\frac{2\alpha_j-1}{2}\tag{3.27}$$

holds for $t\gg1$ and $j=1,2$.

are satisfied, then

$$q_1 = q_2 \quad \text{a.e. on } [0, 1] \quad \text{and} \quad h_{11} = h_{12}.$$

Remark 2. (1) Theorem 3.3 is also true for all other types of separated boundary conditions.
(2) For the case $\alpha_0 = \alpha_1 = \alpha_2 = 0$, Theorem 3.3 leads to the Borg theorem [1].
(3) If $\alpha_0 = \frac{1+\alpha}{2}$, $\alpha_1 = \frac{\alpha}{2}$, $\alpha_2 = \frac{1}{2}$, Theorem 3.3 leads to the Gesztesy–Simon theorem [16].
(4) Let $\alpha_0 = \frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$, then Theorem 3.3 leads to the Hochstadt–Lieberman theorem [3].

References

[1] G. Borg, Eine Umkehrung der Sturm–Liouvilleschen eigenwertaufgabe, Acta Math. 78 (1946) 1–96.
[2] N. Levinson, The inverse Sturm–Liouville problem, Mat. Tidsskr. 13 (1949) 25–30.
[3] H. Hochstadt, B. Lieberman, An inverse Sturm–Liouville problem wity mixed given data, SIAM J. Appl. Math. 34 (1978) 676–680.
[4] T. Suzuki, Inverse problems for heat equations on compact intervals and on circles I, J. Math. Soc. Japan 38 (1986) 39–65.
[5] L. Sakhnovich, Half inverse problems on the finite interval, Inverse Problems 17 (2001) 527–532.
[6] R. Hryniv, Ya. Mykytyuk, Half inverse spectral problems for Sturm–Liouville operators with singular potentials, Inverse Problems 20 (5) (2004) 1423–1444.
[7] H. Koyunbakan, E.S. Panakhov, Half inverse problem for diffusion operators on the finite interval, J. Math. Anal. Appl. 326 (2007) 1024–1030.
[8] O. Hald, Discontinuous inverse eigenvalue problems, Comm. Pure Appl. Math. 37 (5) (1984) 539–977.
[9] G.S. Wei, H.K. Xu, On the missing eigenvalue problem for an inverse Sturm–Liouville problem, J. Math. Pure Appl. 91 (2009) 468–475.
[10] S.A. Buterin, On half inverse problem for differential pencils with the spectral parameter in boundary conditions, Tamkang J. Math. 42 (3) (2011) 355–364.
[11] C.T. Shieh, S.A. Buterin, M. Ignatiev, On Hochstadt–Lieberman theorem for Sturm–Liouville operators, Far East J. Appl. Math. 52 (2) (2011) 131–146.
[12] F. Gesztesy, B. Simon, Inverse spectral analysis with partial information on the potential II the case of discrete spectrum, Trans. Amer. Math. Soc. 352 (2000) 2765–2787.
[13] R. del Rio, F. Gesztesy, B. Simon, Inverse spectral analysis with partial information on the potential III updating boundary conditions, Int. Math. Res. Not. IMRN 15 (1997) 751–758.
[14] F. Gesztesy, B. Simon, Inverse spectral analysis with partial information on the potential I the case of an a.c. component in the spectrum, Helv. Phys. Acta 70 (1997) 66–71.
[15] V.A. Marchenko, Some questions in the theory of one-dimensional linear differential operators of the second order I, Tr. Mosk. Mat. Obs. 1 (1952) 327–420 (in Russian);
Amer. Math. Soc. Transl. 101 (2) (1973) 1–104. English transl.
[16] W.N. Everitt, On a property of the *m*-coeffcient of a second-order linear differential equation, J. Lond. Math. Soc. 4 (1972) 443–457.
[17] G.S. Wei, H.K. Xu, Inverse spectral problem with partial information given on the potential and norming constants, Trans. Amer. Math. Soc. 364 (6) (2012) 3265–3288.
[18] L. Amour, J. Faupin, T. Raoux, Inverse spectral results for Schrödinger operators on the unit interval with partial information given on the potentials, J. Math. Phys. 50 (3) (2009) 033505(14pp).
[19] M. Horvath, On the inverse spectral theory of Schrödinger and Dirac operators, Trans. Amer. Math. Soc. 353 (2001) 4155–4171.
[20] G.S. Wei, H.K. Xu, Inverse spectral problem with partial information given on the potential and norming constants, Trans. Amer. Math. Soc. 364 (6) (2012) 3265–3288.
[21] G. Freiling, V.A. Yurko, Inverse problems for Sturm–Liouville equations with boundary conditions polynomially dependent on the spectral parameter, Inverse Problems 26 (6) (2010) 055003(17pp).
[22] Y.P. Wang, Borg-type theorem for the missing eigenvalue problem, Appl. Math. Lett. 26 (4) (2013) 452–456.
[23] G. Freiling, V.A. Yurko, Inverse Sturm–Liouville Problems and Their Applications, Nova Science Publishers, New York, 2001.