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Results in Mathematics

Inverse Problems for Sturm–Liouville Equations with Boundary Conditions Linearly Dependent on the Spectral Parameter from Partial Information

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Abstract. In this paper, we study the inverse spectral problems for Sturm–Liouville equations with boundary conditions linearly dependent on the spectral parameter and show that the potential of such problem can be uniquely determined from partial information on the potential and parts of two spectra, or alternatively, from partial information on the potential and a subset of pairs of eigenvalues and the normalization constants of the corresponding eigenvalues.

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1. Introduction

Consider the following boundary value problem $L := L(q, U_0, U_1)$ of the form

$$Ly = -y'' + q(x)y = \lambda y, \quad 0 < x < 1 \tag{1.1}$$

with the boundary conditions

$$U_0(y) := (\lambda - h_1)y'(0, \lambda) + (h_2\lambda - h_3)y(0, \lambda) = 0, \tag{1.2}$$

$$U_1(y) := (\lambda - H_1)y'(1, \lambda) - (H_2\lambda - H_3)y(1, \lambda) = 0, \tag{1.3}$$

where q is a real-valued function and $q \in L^2[0, 1], h_l, H_l \in \mathbf{R}, l = 1, 2, 3$, such that

$$r_0 = h_3 - h_1h_2 > 0 \textit{ and } r_1 = H_1H_2 - H_3 > 0.$$

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Sturm–Liouville equations with boundary conditions linearly or nonlinearly dependent on the spectral parameter were addressed by many authors (see [1–8]). Such problems often arise from physical problems, for example, vibration of a string, quantum mechanics and geophysics. Binding et al. [2] discussed the boundary value problem L and obtained oscillation and comparison results as well as the asymptotic estimates of eigenvalues, which can be considered as extension of Fulton's results [1]. Chernozhukova and Freiling [5] explored the inverse problem for Sturm–Liouville equations with boundary conditions polynomially dependent on the spectral parameter and showed that if coefficient functions $R_{0k}(\lambda), k = 0, 1$, of the boundary condition are known a priori, then the potential q and coefficient functions $R_{1k}(\lambda)$ of the boundary condition can be uniquely determined by the Weyl function, where $R_{\xi k}(\lambda) = \Sigma_{j=0}^{r_{\xi k}} a_{\xi kj}\lambda^{r_{\xi k}-j}, r_{\xi 1} = r_{\xi 0} \geq 0, a_{\xi 10} = 1, \xi, k = 0, 1$, are arbitrary polynomials of degree $r_{\xi 1}$ with complex coefficients such that $R_{\xi 1}(\lambda)$ and $R_{\xi 0}(\lambda)$ have no common zeros. Freiling and Yurko [6] studied three inverse problems for Sturm–Liouville equations with boundary conditions polynomially dependent on the spectral parameter from Weyl function, or from discrete spectral data or from two spectra and provided procedures for reconstructing this differential operator from the above spectral data, respectively.

Numerous research results for Eq. (1.1) with Robin boundary conditions have been established by renowned mathematicians, notably, Borg [9] and Levinson [10], respectively showed that two spectra $\{\lambda_n, \mu_n\}$ uniquely determined the potential q and coefficients $h, H \in \mathbf{R}$ of the boundary conditions. Hochstadt and Lieberman [11] initiated the study of the so-called "half inverse problem" for Eq. (1.1) with Robin boundary conditions, they proved that if coefficients h, H of the boundary conditions are given a priori and q is prescribed on the interval $[1/2, 1]$, then one spectrum is enough to determine the potential q uniquely. After that, half inverse problems for differential operators were respectively investigated by many authors (see [11–18]), in particular, Castillo [12] or Suzuki [13] independently showed that coefficient h of the boundary condition is necessary for the Hochstadt–Lieberman Theorem by an example. An alternative approach to inverse spectral theory is via the Weyl m-function. Marchenko [19] proved that the Weyl m-function of the Sturm–Liouville operator uniquely determined the coefficients h, H of the boundary conditions as well as the potential q. A lots of related results were obtained by this approach (see [4,7,19–26]) or the method of spectral mappings (see [5,6,14,15,17,27,28]). One of the interesting results was achieved by Gesztesy and Simon [21], they used the Weyl m-function to study the inverse spectral problem of Eq. (1.1) with Robin boundary conditions from prescribed partial information on the potential and parts of one spectrum, which is a generalization of the Hochstadt–Lieberman Theorem [11].

Theorem 1.1 ([21, Theorem 1.3]). *Let $\sigma(L') = \{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of Eq.* (1.1) *with Robin boundary conditions. If q is prescribed on the interval*

$[0, \frac{1}{2} + \frac{\alpha}{2}]$ *for some* $\alpha \in [0,1)$*, then the potential* q *a.e. on the whole interval* $[0,1]$ *and coefficient* H *of the boundary condition can be uniquely determined by coefficient* h *of the boundary condition and a subset* $S \subseteq \sigma(L')$ *satisfying*

$$\sharp\{\lambda \in S|\lambda \leq t)\} \geq (1-\alpha)\sharp\{\lambda \in \sigma(L')|\lambda \leq t)\} + \frac{\alpha}{2}, \tag{1.4}$$

for all sufficiently large $t \in \mathbf{R}^+$*.*

Theorem 1.1 implies that partial information on the potential (more than one half of the interval) and parts of one spectrum are sufficient to determine the potential on the whole interval. Suzuki [13] showed that one spectrum cannot uniquely determine the potential q if q is prescribed on $[0, \frac{1}{2} - \varepsilon]$ for $0 < \varepsilon < \frac{1}{2}$ by an example. Hence, people may interest in the following inverse problem:

IP-1: *Assuming that* q *is prescribed on* $[0, \frac{1}{2} - \varepsilon], 0 < \varepsilon < \frac{1}{2}$*. What extra conditions can ensure the unique determination of the potential?*

The purpose of this paper is to solve IP-1 for the problem (1.1)–(1.3). We first establish some uniqueness theorems for Sturm–Liouville equations with boundary conditions linearly dependent on the spectral parameter from partial information of the potential and parts of two spectra. And then we obtain two uniqueness theorems for the problem (1.1)–(1.3) from partial information of the potential and a subset of pairs of eigenvalues and the normalization constants of the corresponding eigenvalues. The techniques used here are based on the Weyl function and methods developed in Refs. [6,21] and [23].

This article is organized as follows: In Sect. 2, we present preliminaries for Sturm–Liouville equations with boundary conditions linearly dependent on the spectral parameter. We state main results and prove the main results in Sect. 3.

2. Preliminaries

In this section, we present preliminaries for the boundary value problem L.

Let $S_1(x,\lambda)$, $S_2(x,\lambda)$, $u_-(x,\lambda)$ and $u_+(x,\lambda)$ be solutions of Eq. (1.1) which satisfy the initial conditions:

$$S_1(0,\lambda) = 0, S_1'(0,\lambda) = 1, S_2(0,\lambda) = 1, S_2'(0,\lambda) = 0, \tag{2.1}$$

$$u_-(0,\lambda) = \lambda - h_1,\ u_-'(0,\lambda) = h_3 - h_2\lambda, \tag{2.2}$$

$$u_+(1,\lambda) = \lambda - H_1,\ u_+'(1,\lambda) = H_2\lambda - H_3. \tag{2.3}$$

Clearly, $U_0(u_-) = U_1(u_+) = 0$ and

$$u_-(x,\lambda) = (\lambda - h_1)S_2(x,\lambda) + (h_3 - h_2\lambda)S_1(x,\lambda), \tag{2.4}$$

$$u_+(x,\lambda) = U_1(S_1)S_2(x,\lambda) - U_1(S_2)S_1(x,\lambda). \tag{2.5}$$

Let

$$\Delta(\lambda) := [u_+, u_-](x, \lambda), \tag{2.6}$$

where $[y, z](x) := y(x)z'(x) - y'(x)z(x)$ is the Wronskian of y and z. Therefore $\Delta(\lambda)$ is independent of the variable x and

$$\Delta(\lambda) = U_1(u_-) = -U_0(u_+), \tag{2.7}$$

which is called the characteristic function of the boundary value problem L.

It is easy to show that all zeros $\lambda_n, n \in \mathbf{N_0}$, of $\Delta(\lambda)$ are real and simple. Let $\sigma(L) = \{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of L. Since $u_-(x, \lambda_n)$ and $u_+(x, \lambda_n)$ are eigenfunctions of the corresponding eigenvalue λ_n, then there exists κ_n such that

$$u_+(x, \lambda_n) = \kappa_n u_-(x, \lambda_n), \tag{2.8}$$

where κ_n is called the normalization constant of the corresponding eigenvalue λ_n. Hence $\kappa_n \neq 0, \infty$ and

$$\begin{aligned} &u_+(0, \lambda_n) = \kappa_n(\lambda_n - h_1), \quad \lambda_n - h_1 \neq 0, \\ &u_+(0, \lambda_n) = 0,\ u'_+(0, \lambda_n) = \kappa_n(h_3 - h_2\lambda), \quad \lambda_n - h_1 = 0. \end{aligned} \tag{2.9}$$

From Ref. [6], we have the following asymptotic formulae

$$\begin{aligned} u_-(x, \lambda) &= \rho^2 \cos \rho x + O(\rho e^{\tau x}), \\ u'_-(x, \lambda) &= -\rho^3 \sin \rho x + O(\rho^2 e^{\tau x}), \end{aligned} \tag{2.10}$$

$$\begin{aligned} u_+(x, \lambda) &= \rho^2 \cos \rho(1 - x) + O(\rho e^{\tau(1-x)}), \\ u'_+(x, \lambda) &= \rho^3 \sin \rho(1 - x) + O(\rho^2 e^{\tau(1-x)}), \end{aligned} \tag{2.11}$$

and

$$\Delta(\lambda) = -\rho^5 \sin \rho + O(\rho^4 e^{\tau}), \tag{2.12}$$

where $\rho = \sqrt{\lambda}, \tau = |Im\rho|$.

Therefore, the asymptotic formula of eigenvalues $\sqrt{\lambda_n}$ of the boundary value problem L is

$$\sqrt{\lambda_n} = (n - 2)\pi + \frac{\omega}{n\pi} + \frac{l_n}{n}, \tag{2.13}$$

where $\omega = \frac{1}{2}\int_0^1 q(x)dx - h_2 - H_2$ and $\{l_n\} \in l^2$.

Denote $G_{\delta'} := \{\rho : |\rho - k\pi| \geq \delta', k \in \mathbf{Z}\}$ for small sufficiently $\delta' > 0$, then there exists a constant $C_{\delta'}$ such that

$$|\Delta(\lambda)| \geq C_{\delta'}|\rho|^5 e^{\tau}, \quad \rho \in G_{\delta'},\ |\rho| \gg 1. \tag{2.14}$$

Let $\Phi(x, \lambda)$ be the solution of Eq. (1.1) under the condition $U_1(\Phi) = 1, U_0(\Phi) = 0$. Put $M(\lambda) := \Phi(0, \lambda)$, which is called the Weyl function for L. We have

$$\Phi(x,\lambda) = -\frac{u_+(x,\lambda)}{\Delta(\lambda)} = \frac{1}{\lambda - h_1}(S_1(x,\lambda) + M(\lambda)u_-(x,\lambda)), \tag{2.15}$$

where

$$M(\lambda) = -\frac{u_+(0,\lambda)}{\Delta(\lambda)}, \tag{2.16}$$

which is called the Weyl function of the boundary value problem L. Consequently the Weyl function $M(\lambda)$ is meromorphic with simple poles in the points $\lambda = \lambda_n$ for $n \in \mathbf{N_0}$.

The following two lemmas are important for us to prove main results in this paper.

Lemma 2.1 ([6]). *Let $M(\lambda)$ be the Weyl function of the boundary value problem L. If coefficients $h_l, l = 1,2,3,$ of the boundary condition are given a priori, then $M(\lambda)$ uniquely determines coefficients $H_l, l = 1,2,3,$ of the boundary condition as well as q (a.e.) on the interval $[0,1]$.*

Lemma 2.2 ([21, *Proposition B.6*]). *Let $f(z)$ be an entire function such that*

(1) $\sup_{|z|=R_k}|f(z)| \le C_1 exp(C_2R_k^\alpha)$ *for some $0<\alpha<1$, some sequence $R_k \to \infty$ as $k\to\infty$ and $C_1, C_2 > 0$;*

(2) $\lim_{|x|\to\infty}|f(ix)| = 0, x \in \mathbf{R}$.

Then $f \equiv 0$.

3. Inverse Problems with Partial Information

In this section, we study the uniqueness theorems for Sturm–Liouville equations with boundary conditions linearly dependent on the spectral parameter. The authors intent to recover the boundary value problem L from partial information on the potential q and parts of two spectra or from partial information on the potential q and a subset of eigenvalues and the corresponding normalization constants.

Denote the boundary value problems $L_{kj} := L_{kj}(q_k, U_{0j}, U_{1k}), k, j = 1,2,$ of the form

$$L_{kj}u_k := -u_k'' + q_k(x)u_k = \lambda u_k \ , x \in (0,1), \tag{3.1}$$

with the boundary conditions

$$U_{0j}(u_k) := (\lambda - h_{1j})u_k'(0,\lambda) + (h_{2j}\lambda - h_{3j})u_k(0,\lambda) = 0, \tag{3.2}$$

$$U_{1k}(u_k) := (\lambda - H_{1k})u_k'(1,\lambda) - (H_{2k}\lambda - H_{3k})u_k(1,\lambda) = 0, \tag{3.3}$$

where q_k are real-valued functions with $q_k \in L^2[0,1], h_{lj}, H_{lk} \in \mathbf{R}, l = 1,2,3, k, j = 1,2,$ are such that

$$r_{0j} = h_{3j} - h_{1j}h_{2j} > 0,\ r_{1k} = H_{1k}H_{2k} - H_{3k} > 0,\ \frac{h_{21}\lambda - h_{31}}{\lambda - h_{11}} \ne \frac{h_{22}\lambda - h_{32}}{\lambda - h_{12}}. \tag{3.4}$$

And let $\sigma(L_{kj})=\{\lambda_{kjn}\}_{n=0}^{\infty}, k,j=1,2$, be the spectrum of L_{kj}. Since $\frac{h_{21}\lambda-h_{31}}{\lambda-h_{11}} \neq \frac{h_{22}\lambda-h_{32}}{\lambda-h_{12}}$, we can easily prove $\sigma(L_{k1})\cap\sigma(L_{k2})=\emptyset$ for $k=1$ or $k=2$.

We obtain the following Theorem 3.1.

Theorem 3.1. *Let $\sigma(L_{kj}) = \{\lambda_{kjn}\}_{n=0}^{\infty}, k,j = 1,2$, be as that defined above, $(\alpha_0,\alpha_1,\alpha_2) \in [0,1) \times [0,1/2] \times [0,1/2]$, $A_j \subseteq \sigma(L_{1j}) \bigcap \sigma(L_{2j})$ for $j = 1$ or 2. Suppose the coefficients $h_{l1}, l = 1,2,3$, of the boundary conditions are given a priori and the following conditions:*

(1) $\alpha_0 - \alpha_1 - \alpha_2 \geq 0$,

(2) $q_1 = q_2$ *on the interval* $[0,\alpha_0]$,

(3) there are two nonnegative integers β_1 and β_2 with $\beta_1+\beta_2 = 6$ and two nonnegative small numbers ε_1 and ε_2 with $\varepsilon_1+\varepsilon_2 > 0$ so that the inequalities

$$\sharp\{\lambda \in A_j | \lambda \leq t)\} \geq (1-2\alpha_j)\sharp\{\lambda \in \sigma(L_{1j}) | \lambda \leq t)\} + \frac{10\alpha_j - \beta_j + \varepsilon_j}{2}, \quad j = 1,2 \tag{3.5}$$

holds for $t \gg 1$,

are satisfied, then

$$q_1 = q_2 \text{ a.e. on } [0,1] \text{ and } H_{l1} = H_{l2}, \quad l = 1,2,3.$$

Proof. Let $u_{k+}(x,\lambda), k = 1,2$, be the solution of Eq. (3.1) with the terminal conditions $u_{k+}(1,t) = \lambda - H_{1k}$ and $u'_{k+}(1,t) = H_{2k}\lambda - H_{3k}$. By Green's formula, we have

$$\begin{aligned}\int_0^1 & Q(x)u_{1+}(x,\lambda)u_{2+}(x,\lambda)dx \\ &= [u_{1+},u_{2+}](1,\lambda) - [u_{1+},u_{2+}](0,\lambda) \\ &= F(1,\lambda) - F(0,\lambda),\end{aligned}$$

where $Q(x) = q_2(x) - q_1(x)$,

$$F(\alpha_0,\lambda) := [u_{1+},u_{2+}](\alpha_0,\lambda)$$

and

$$\Delta_{kj}(\lambda) = -[(\lambda - h_{1j})u'_{k+}(0,\lambda) + (h_{2j}\lambda - h_{3j})u_{k+}(0,\lambda)] \text{ for } k,j=1,2. \tag{3.6}$$

Note that

$$\begin{aligned}F(0,\lambda) &= u_{1+}(0,\lambda)u'_{2+}(0,\lambda) - u'_{1+}(0,\lambda)u_{2+}(0,\lambda) \\ &= \frac{1}{\lambda - h_{1j}}[u_{1+}(0,\lambda)U_{0j}(u_{2+}) - u_{2+}(0,\lambda)U_{0j}(u_{1+})] \\ &= \frac{1}{h_{2j}\lambda - h_{3j}}[u'_{2+}(0,\lambda)U_{0j}(u_{1+}) - u'_{1+}(0,\lambda)U_{0j}(u_{2+})].\end{aligned} \tag{3.7}$$

From Lemma 1 in Ref. [6] and (3.7), we see that $F(0,\lambda_{1jn}) = 0$ for $\lambda_{1jn} \in \sigma(L_{1j})\bigcap\sigma(L_{2j})$ hold. The assumption (2) yields to

$$F(0,\lambda) = F(\alpha_0,\lambda). \tag{3.8}$$

Hence

$$F(\alpha_0,\lambda_{1jn}) = 0, \forall\lambda_{1jn} \in A_j \subseteq \sigma(L_{1j})\bigcap\sigma(L_{2j}), \quad j = 1,2.$$

Without loss of generality, we assume all eigenvalues $\lambda_{kjn} \neq 0$ in this section. And denote $t_0 := \liminf_{n\in\mathbf{N}_0}\{|\lambda_{11n}|,|\lambda_{12n}|\}$. By virtue of Ref. [6, Lemma 4], this yields

$$\Delta_{kj}(\lambda) = c_{kj}\prod_{n\in\mathbf{N_0}}(1-\frac{\lambda}{\lambda_{kjn}}), \quad k,j = 1,2, \tag{3.9}$$

where

$$c_{kj} = \pi\prod_{n=0}^{2}\lambda_{kjn}\prod_{n=3}^{\infty}\frac{\lambda_{kjn}}{(n-2)^2}$$

are constant. From (2.10)–(2.11), we have

$$|F(\alpha_0,\lambda)| = O(\rho^4 e^{2\tau(1-\alpha_0)}) \tag{3.10}$$

and

$$|\Delta_{1j}(\lambda)| = O(\rho^5 e^{\tau}). \tag{3.11}$$

Define

$$G_j(\lambda) = \prod_{\lambda_{1jn}\in A_j}\left(1-\frac{\lambda}{\lambda_{1jn}}\right) \text{ for } j = 1,2 \tag{3.12}$$

and

$$K(\lambda) = \frac{F(\alpha_0,\lambda)}{G_1(\lambda)G_2(\lambda)}. \tag{3.13}$$

Therefore, $K(\lambda)$ is an entire function in λ. Denote

$$N_{G_j}(t) = \sharp\{\lambda_{1jn} \in A_j|\lambda_{1jn} \leq t\},\ N_{\Delta_{1j}}(t) = \sharp\{\lambda_{1jn} \in \sigma(L_{1j})|\lambda_{1jn} \leq t\}.$$

Then the assumption (3) leads

$$N_{G_j}(t) \geq (1-2\alpha_j)N_{\Delta_{1j}}(t) + \frac{10\alpha_j - \beta_j + \varepsilon_j}{2}$$

and there exists constants $t_j \geq t_0$ and C_{1j} such that

$$N_{G_j} = \begin{cases} N_{G_j}(t) \geq (1-2\alpha_j)N_{\Delta_{1j}}(t) + \frac{10\alpha_j-\beta_j+\varepsilon_j}{2}, & t \geq t_j, \\ N_{G_j}(t) \geq (1-2\alpha_j)N_{\Delta_{1j}}(t) - C_{1j}, & t < t_j. \end{cases} \tag{3.14}$$

Since $\Delta_{1j}(\lambda)$ is an entire function in λ of order $\frac{1}{2}$, there exists a positive constant C such that

$$N_{G_j}(t) \leq N_{\Delta_{1j}}(t) \leq Ct^{\frac{1}{2}}, \tag{3.15}$$

and $N_{G_j}(t_0) = N_{\Delta_{1j}}(t_0) = 0$.

For a fixed real number y and $|y| \gg 1$, we have

$$\begin{aligned}
\ln|G_j(iy)| &= \frac{1}{2}\ln G_j(iy)\overline{G_j(iy)} = \frac{1}{2}\sum_{\lambda_{1jn}\in A_j}\ln\left(1-\frac{iy}{\lambda_{1jn}}\right)\left(1+\frac{iy}{\lambda_{1jn}}\right)\\
&= \frac{1}{2}\sum_{\lambda_{1jn}\in A_j}\ln\left(1+\frac{y^2}{(\lambda_{1jn})^2}\right) = \frac{1}{2}\int_{t_0}^{\infty}\ln\left(1+\frac{y^2}{t^2}\right)\mathrm{d}N_{G_j}(t)\\
&= \frac{1}{2}\ln(1+\frac{y^2}{t^2})N_{G_j}(t)|_{t_0}^{\infty} - \frac{1}{2}\int_{t_0}^{\infty}N_{G_j}(t)\mathrm{d}\left[\ln\left(1+\frac{y^2}{t^2}\right)\right]. \qquad (3.16)
\end{aligned}$$

Since

$$\ln\left(1+\frac{y^2}{t^2}\right) = O\left(\frac{1}{t^2}\right), \text{ as } t\to\infty,$$

we obtain

$$\lim_{t\to\infty}\ln\left(1+\frac{y^2}{t^2}\right)N_{G_j}(t) = 0$$

and

$$\lim_{t\to\infty}\ln\left(1+\frac{y^2}{t^2}\right)N_{\Delta_{1j}}(t) = 0.$$

Consequently, from (3.16) together with the following relation

$$\frac{y^2}{t^3+ty^2} = -\frac{d}{dt}\left(\frac{1}{2}\ln\left(1+\frac{y^2}{t^2}\right)\right),$$

we have

$$\begin{aligned}
&\ln|G_j(\mathrm{i}y)|\\
&= \int_{t_0}^{\infty}\frac{y^2}{t^3+ty^2}N_{G_j}(t)\mathrm{d}t = \int_{t_0}^{t_j}\frac{y^2}{t^3+ty^2}N_{G_j}(t)\mathrm{d}t + \int_{t_j}^{\infty}\frac{y^2}{t^3+ty^2}N_{G_j}(t)\mathrm{d}t\\
&\geq (1-2\alpha_j)\int_{t_0}^{\infty}\frac{y^2}{t^3+ty^2}N_{\Delta_{1j}}(t)\mathrm{d}t + \frac{10\alpha_j-\beta_j+\varepsilon_j}{2}\int_{t_0}^{\infty}\frac{y^2}{t^3+ty^2}\mathrm{d}t\\
&\quad -\left(\frac{10\alpha_j-\beta_j+\varepsilon_j}{2}+C_{1j}\right)\int_{t_0}^{t_j}\frac{y^2}{t^3+ty^2}dt\\
&= (1-2\alpha_j)\ln|\Delta_{1j}(\mathrm{i}y)| + \frac{10\alpha_j-\beta_j+\varepsilon_j}{4}\ln\left(1+\frac{y^2}{t_0^2}\right)\\
&\quad + \frac{10\alpha_j-\beta_j+\varepsilon_j+2C_{1j}}{4}\ln\frac{t_0^2+y^2}{t_j^2+y^2} + \frac{10\alpha_j-\beta_j+\varepsilon_j+2C_{1j}}{2}\ln|\frac{t_j^2}{t_0^2}|. \qquad (3.17)
\end{aligned}$$

For sufficiently large $y \in \mathbf{R}$, (3.17) implies

$$|G_j(\mathrm{i}y)| \geq C_{01j}|\Delta_{1j}(iy)|^{1-2\alpha_j}|y|^{10\alpha_j-\beta_j+\varepsilon_j}, \quad j=1,2, \tag{3.18}$$

where C_{01j} are constant. Therefore for sufficiently large $y \in \mathbf{R}$, we have

$$|G_1(iy)G_2(iy)| \geq C_{011}C_{012}|y|^{4+\varepsilon_1+\varepsilon_2}e^{2\tau(1-\alpha_1-\alpha_2)}. \tag{3.19}$$

Hence

$$|K(iy)| = \frac{F(\alpha_0, iy)}{G_{11}(iy)G_{12}(iy)} = O\left(\frac{1}{|y|^{\varepsilon_1+\varepsilon_2}}\right). \tag{3.20}$$

Analogous to the proof of Ref. [21], we can prove the condition (1) for the entire function $K(\lambda)$ in Lemma 2.2. From Lemma 2.2, we obtain

$$K(\lambda) = 0, \ \forall\lambda \in \mathbf{C}.$$

i.e.,

$$F(\alpha_0, \lambda) = 0, \ \forall\lambda \in \mathbf{C}.$$

Consequently

$$F(0, \lambda) = 0, \ \forall\lambda \in \mathbf{C}.$$

i.e.,

$$u_{1+}(0,\lambda)u'_{2+}(0,\lambda) - u'_{1+}(0,\lambda)u_{2+}(0,\lambda) = 0, \ \forall\lambda \in \mathbf{C}. \tag{3.21}$$

Henceforth

$$\frac{u_{1+}(0,\lambda)}{(h_{12}\lambda - h_{13})u_{1+}(0,\lambda) - (\lambda - h_{11})u'_{1+}(0,\lambda)} = \frac{u_{2+}(0,\lambda)}{(h_{12}\lambda - h_{13})u_{2+}(0,\lambda) - (\lambda - h_{11})u'_{2+}(0,\lambda)}$$

or equivalently

$$M_1(\lambda) = M_2(\lambda), \ \forall\lambda \in \mathbf{C}. \tag{3.22}$$

From Lemma 2.1, we conclude that

$$q_1 = q_2 \ \textit{a.e. on } [0,1] \textit{ and } H_{l1} = H_{l2}, \quad l = 1,2,3.$$

This completes the proof of Theorem 3.1 □

Let us present an example of Theorem 3.1 to the readers. If we take $\alpha_0 = \frac{1}{2}, \alpha_1 = 0, \alpha_2 = \frac{1}{2}, \beta_1 = 1, \beta_2 = 5, \varepsilon_1 = 1$ and $\varepsilon_2 = 0$, then Theorem 3.1 turns to the Hochstadt-Lieberman type theorem as follows.

Corollary 3.2. *Let $\sigma(L_{k1}) = \{\lambda_{k1n}\}_{n=0}^{\infty}$ be as that defined in Theorem 3.1 for $k = 1$ and 2, and coefficients $h_{l1}, l = 1,2,3,$ of the boundary condition are given a priori. If $q_1 = q_2$ on the interval $[0, \frac{1}{2}]$ and*

$$\lambda_{11n} = \lambda_{21n}, \ \forall n \geq 0,$$

then

$$q_1 = q_2 \text{ a.e. on } [0,1] \text{ and } H_{l1} = H_{l2}, \quad l = 1,2,3.$$

Instead of using parts of two spectral sets and partial information on the potential, one can use a subset of eigenvalues and the corresponding normalization constants to establish the uniqueness theorems for the boundary value problem L. In the remaining of this section, we shall use the techniques in Ref. [23] to some uniqueness theorems.

Theorem 3.3. *Let $\sigma(L_k) = \{\lambda_{kn}\}_{n=0}^{\infty}, k = 1,2$, be the spectrum of Eq. (3.1) with boundary conditions (1.2) and (3.3). Assuming that the coefficients $h_l, l = 1,2,3,$ of the boundary condition are given a priori. Suppose the following conditions*

(1) $q_1 = q_2$ a.e. on the interval $[0,\alpha]$ for some $\alpha \in [0,1)$,

(2) there exists a sufficiently small positive number ε and a set $A = \{\lambda_{1j}\}_{j\in\Lambda} \subseteq \sigma(L_1)\bigcap\sigma(L_2)$, where $\Lambda \subseteq \mathbf{N}_0$ so that

$$\kappa_{1j} = \kappa_{2j} \text{ for all } \lambda_{1j} \in A$$

and the equality

$$\sharp\{\lambda \in A|\lambda \le t)\} \ge (1-\alpha)\sharp\{\lambda \in \sigma(L_{11})|\lambda \le t)\} + \frac{5\alpha - 3 + 2\varepsilon}{4} \tag{3.23}$$

holds for all sufficiently large $t \in \mathbf{R}^+$.

are satisfied. Then $q_1 = q_2$ on [0,1] and $H_{l1} = H_{l2}, l = 1,2,3$.

Proof. Let $u_{k+}(x,\lambda)$ be the solution of Eq. (3.1) with the initial conditions $u_{k+}(1,\lambda) = \lambda - H_{1k}$ and $u'_{k+}(1,\lambda) = H_{2k}\lambda - H_{3k}$ for $k = 1,2$. For an arbitrary solution v_2 of Eq. (3.1) of the corresponding potential q_2, we denote

$$F_v(x,\lambda) := [u_{1+} - u_{2+}, v_2](x,\lambda) \tag{3.24}$$

and

$$\Delta_{0k}(\lambda) = -[(\lambda - h_1)u'_{k+}(0,\lambda) + (h_2\lambda - h_3)u_{k+}(0,\lambda)] \text{ for } k = 1,2. \tag{3.25}$$

Then

$$\begin{aligned}
F_v(0,\lambda) &= (u'_{1+}(1,\lambda) - u'_{2+}(1,\lambda))v_2(1,\lambda) - \int_0^1 (q_1 - q_2)u_{1+}v_2\,dx \\
&= [u_{1+} - u_{2+}, v_2](0,\lambda) \\
&= \frac{1}{\lambda - h_1}\begin{vmatrix} u_{1+}(0,\lambda) - u_{2+}(0,\lambda) & v_2(0,\lambda) \\ \Delta_{02}(\lambda) - \Delta_{01}(\lambda) & (\lambda - h_1)v'_2(0,\lambda) + (h_2\lambda - h_3)v_2(0,\lambda) \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{\lambda - h_1}(u_{1+}(0,\lambda) - u_{2+}(0,\lambda)) & \frac{1}{\lambda - h_1}v_2(0,\lambda) \\ \Delta_{02}(\lambda) - \Delta_{01}(\lambda) & (\lambda - h_1)v'_2(0,\lambda) + (h_2\lambda - h_3)v_2(0,\lambda) \end{vmatrix}.
\end{aligned} \tag{3.26}$$

If $\lambda_{1j} - h_1 \neq 0$ for some $j \in \mathbf{N}_0$, (3.26) together with (2.9) implies

$$\Delta_{02}(\lambda_{1j}) - \Delta_{01}(\lambda_{1j}) = 0 \textit{ and}$$
$$\frac{1}{\lambda_{1j} - h_1}(u_{1+}(0, \lambda_{1j}) - u_{2+}(0, \lambda_{1j})) = 0 \textit{ for all } \lambda_{1j} \in A. \tag{3.27}$$

If $\lambda_{1j} - h_1 = 0$ for some $j \in \mathbf{N}_0$, (2.9) yields

$$u_{1+}(0, \lambda_{1j}) = u_{2+}(0, \lambda_{1j}) = 0 \textit{ and } u'_{1+}(0, \lambda_{1j}) - u'_{2+}(0, \lambda_{1j}) = 0 \textit{ for all } \lambda_{1j} \in A. \tag{3.28}$$

Therefore, (3.27) and (3.28) implies

$$F_v(0, \lambda_{1j}) = 0 \textit{ for all } \lambda_{1j} \in A. \tag{3.29}$$

By virtue of the assumption (1) together with the Green's formula, we obtain

$$\begin{aligned} F_v(0, \lambda) &= (u'_{1+}(1, \lambda) - u'_{2+}(1, \lambda))v_2(1, \lambda) - \int_0^1 (q_1 - q_2)u_{1+}v_2 dx \\ &= (u'_{1+}(1, \lambda) - u'_{2+}(1, \lambda))v_2(1, \lambda) - \int_\alpha^1 (q_1 - q_2)u_{1+}v_2 dx \\ &= [u_{1+} - u_{2+}, v_2](\alpha, \lambda). \end{aligned} \tag{3.30}$$

Next, we will prove $u_{1+}(\alpha, \lambda) = u_{2+}(\alpha, \lambda)$ and $u'_{1+}(\alpha, \lambda) = u'_{2+}(\alpha, \lambda)$ for all $\lambda \in \mathbf{C}$.

At first, we show that $u_{1+}(\alpha, \lambda) = u_{2+}(\alpha\,\lambda)$ holds. Let $v_2(x, \lambda) =: v_D(x, \lambda)$ be the solution of Eq. (3.1) with the conditions $v_D(\alpha, \lambda) = 0$ and $v'_D(\alpha, \lambda) = 1$. Then

$$F_{v_D}(0, \lambda) = u_{1+}(\alpha, \lambda) - u_{2+}(\alpha, \lambda). \tag{3.31}$$

Define the entire functions $G_A(\lambda)$ and $H_D(\lambda)$ by

$$G_A(\lambda) = \prod_{\lambda_{1n} \in A} (1 - \frac{\lambda}{\lambda_{1n}}) \textit{ and } H_D(\lambda) = \frac{F_{v_D}(0, \lambda)}{G_A(\lambda)}. \tag{3.32}$$

By (2.10) and (3.29), we have

$$u_{1+}(\alpha, iy) - u_{2+}(\alpha, iy) = O(|y|^{\frac{1}{2}} e^{(1-\alpha)|y|^{\frac{1}{2}}}). \tag{3.33}$$

for sufficiently large $|y|$. Analogous to the argument in the proof of Theorem 3.1, we have

$$|G_A(\mathrm{i}y)| \geq C_1 |\Delta_{01}(iy)|^{1-\alpha} |y|^{\frac{5\alpha - 3 + 2\varepsilon}{2}} \geq C_2 |y|^{1+\varepsilon} e^{(1-\alpha)|y|^{\frac{1}{2}}}, \tag{3.34}$$

where C_1, C_2 are positive constants. Hence

$$|H_D(iy)| = O\left(\frac{1}{|y|^{\frac{1+2\varepsilon}{2}}}\right). \tag{3.35}$$

By Lemma 2.2, we obtain

$$H_D(\lambda) = 0, \ \forall \lambda \in \mathbf{C}.$$

Therefore

$$u_{1+}(\alpha, \lambda) - u_{2+}(\alpha, \lambda) = 0, \quad \forall \lambda \in \mathbf{C}. \tag{3.36}$$

Secondly, we will prove $u'_{1+}(\alpha, \lambda) = u'_{2+}(\alpha, \lambda)$ for all $\lambda \in \mathbf{C}$. Let us take $v_2(x, \lambda) =: v_N(x, \lambda)$ be the solution of Eq. (3.1) with the initial conditions $v_N(\alpha, \lambda) = 1$ and $v'_N(\alpha, \lambda) = 0$. Then

$$F_{v_N}(0, \lambda) = u'_{2+}(\alpha, \lambda) - u'_{1+}(\alpha, \lambda). \tag{3.37}$$

Define the entire function $H_N(\lambda)$ by

$$H_N(\lambda) = \frac{F_{v_N}(0, \lambda)}{G_A(\lambda)}. \tag{3.38}$$

By (2.11) again, we have

$$|u'_{2+}(\alpha, iy) - u'_{1+}(\alpha, iy)| = O(|y| e^{(1-\alpha)|y|^{\frac{1}{2}}}). \tag{3.39}$$

Hence

$$|H_N(iy)| = O\left(\frac{1}{|y|^{\varepsilon}}\right). \tag{3.40}$$

Applying Lemma 2.2, we get

$$H_N(\lambda) = 0, \quad \forall \lambda \in \mathbf{C}. \tag{3.41}$$

Therefore

$$u'_{2+}(\alpha, \lambda) - u'_{1+}(\alpha, \lambda) = 0, \quad \forall \lambda \in \mathbf{C}. \tag{3.42}$$

(3.36) and (3.42) together with the assumption (1) leads to the conclusion

$$[u_{1+}, u_{2+}](0, \lambda) = [u_{1+}, u_{2+}](\alpha, \lambda) = 0, \quad \forall \lambda \in \mathbf{C}. \tag{3.43}$$

Hence

$$M_1(\lambda) = M_2(\lambda). \tag{3.44}$$

By Lemma 2.1, we conclude that

$$q_1 = q_2 \text{ on } [0, 1] \text{ and } H_{l1} = H_{l2} \text{ for } l = 1, 2, 3.$$

By now, the proof of Theorem 3.3 is completed. □

Remark. Theorem 3.3 is a generalization of Theorem 4.1 in Ref. [23].

Let $\alpha = \frac{1}{2}$ and $\varepsilon = \frac{1}{4}$, we have the following corollary.

Corollary 3.4. *Let $\sigma(L) = \{\lambda_j\}_{j=0}^{\infty}$ be the spectrum of the problem (1.1)–(1.3). Assume that the coefficients $h_l, l = 1, 2, 3,$ of the boundary condition are given a priori and q on the interval $[0, \frac{1}{2}]$ is known a priori, then the even spectral* *data* $\{(\lambda_{2j}, \kappa_{2j})\}_{j=0}^{\infty}$ *or the odd spectral data* $\{(\lambda_{2j-1}, \kappa_{2j-1})\}_{j=1}^{\infty}$ *is sufficient to determine the potential* q *on the whole interval* $[0, 1]$ *and coefficients* H_l *for* $l = 1, 2, 3$.

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