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Inverse spectral problems for non-selfadjoint second-order differential operators with Dirichlet boundary conditions

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Abstract
We study the inverse problem for non-selfadjoint Sturm-Liouville operators on a finite interval with possibly multiple spectra. We prove the uniqueness theorem and obtain constructive procedures for solving the inverse problem along with the necessary and sufficient conditions of its solvability and also prove the stability of the solution.
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1 Introduction

Inverse spectral problems consist of recovering operators from given spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences and engineering (see, for example, monographs [1–7] and the references therein). We study the inverse problem for the Sturm-Liouville operator corresponding to the boundary value problem $L = L(q(x), T)$ of the form

$$\ell y := -y'' + q(x)y = \lambda y, \quad 0 < x < T < \infty, \tag{1}$$

$$y(0) = y(T) = 0, \tag{2}$$

where $q(x) \in L_1(0, T)$ is a complex-valued function. The results for the non-selfadjoint operator (1), (2) that we obtain in this paper are crucial in studying inverse problems for Sturm-Liouville operators on graphs with cycles. Here also lies the main reason of considering the case of Dirichlet boundary conditions (2) and arbitrary length T of the interval.

For the selfadjoint case, *i.e.*, when $q(x)$ is a real-valued function, the inverse problem of recovering L from its spectral characteristics was investigated fairly completely. As the most fundamental works in this direction we mention [8, 9], which gave rise to the so-called transformation operator method having become an important tool for studying inverse problems for selfadjoint Sturm-Liouville operators. The inverse problems for *non-selfadjoint* operators are more difficult for investigation. Some aspects of the inverse problem theory for non-selfadjoint Sturm-Liouville operators were studied in [10–14] and other papers.

In the present paper, we use the method of spectral mappings [7], which is effective for a wide class of differential and difference operators including non-selfadjoint ones. The method of spectral mappings is connected with the idea of the contour integration method and reduces the inverse problem to the so-called main equation of the inverse problem, which is a linear equation in the Banach space of bounded sequences. We prove the uniqueness theorem of the inverse problem, obtain algorithms for constructing its solution together with the necessary and sufficient conditions of its solvability. In general, by sufficiency one should require solvability of the main equation. Therefore, we also study those cases when the solvability of the main equation can be proved or easily checked, namely, selfadjoint case, the case of finite perturbations of the spectral data and the case of small perturbations. The study of the latter case allows us to prove also the stability of the inverse problem.

In the next section, we introduce the spectral data, study their properties and give the formulation of the inverse problem. In Section 3, we prove the uniqueness theorem. In Section 4 we derive the main equation and prove its solvability. Further, using the solution of the main equation, we provide an algorithm for solving the inverse problem. In Section 5, we obtain another algorithm, which we use in Section 6 for obtaining necessary and sufficient conditions of solvability of the inverse problem and for proving its stability.

2 Generalized spectral data. Inverse problem

Let $\{\lambda_n\}_{n\geq 1}$ be the spectrum of the boundary value problem (1), (2). In the self-adjoint case, the potential $q(x)$ is determined uniquely by specifying the classical discrete spectral data $\{\lambda_n, \alpha_n\}_{n\geq 1}$, where α_n are weight numbers determined by the formula

$$\alpha_n = \int_0^\pi S^2(x, \lambda_n)\,dx, \tag{3}$$

while $S(x,\lambda)$ is a solution of equation (1) satisfying the initial conditions

$$S(0,\lambda) = 0, \qquad S'(0,\lambda) = 1. \tag{4}$$

In the non-selfadjoint case, there may be a finite number of multiple eigenvalues and, hence, for unique determination of the Sturm-Liouville operator, one should specify some additional information. In the present section, we introduce the so-called generalized weight numbers, as was done for the case of operator (1) with Robin boundary conditions (see [11, 12]) and study the properties of the generalized spectral data.

Let the function $\psi(x,\lambda)$ be a solution of equation (1) under the conditions

$$\psi(T,\lambda) = 0, \qquad \psi'(T,\lambda) = -1. \tag{5}$$

For every fixed $x \in [0, T]$, the functions $S(x,\lambda)$, $\psi(x,\lambda)$ and their derivatives with respect to x are entire in λ. The eigenvalues λ_n, $n \geq 1$ of the problem L coincide with the zeros of its characteristic function

$$\Delta(\lambda) := \langle \psi(x,\lambda), S(x,\lambda)\rangle = \psi(0,\lambda) = S(T,\lambda), \tag{6}$$

where $\langle y,z\rangle := yz' - y'z$. It is known (see, *e.g.*, [2]) that the spectrum $\{\lambda_n\}_{n\geq 1}$ has the asymptotics

$$\rho_n := \sqrt{\lambda_n} = \frac{\pi n}{T} + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad \kappa_n = o(1), \tag{7}$$

where

$$\omega = Q(T), \qquad Q(x) = \frac{1}{2}\int_0^x q(t)\,dt.$$

Denote by m_n the multiplicity of the eigenvalue λ_n ($\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m_n-1}$) and put $\mathbb{S} = \{n : n-1 \in \mathbb{N}, \lambda_{n-1} \neq \lambda_n\} \cup \{1\}$. Note that by virtue of (7) for sufficiently large n, we have $m_n = 1$. Denote

$$S_\nu(x,\lambda) = \frac{1}{\nu!}\frac{d^\nu}{d\lambda^\nu}S(x,\lambda), \qquad \psi_\nu(x,\lambda) = \frac{1}{\nu!}\frac{d^\nu}{d\lambda^\nu}\psi(x,\lambda).$$

Hence, for $\nu \geq 1$ we have

$$\left.\begin{aligned} &\ell S_\nu(x,\lambda) = \lambda S_\nu(x,\lambda) + S_{\nu-1}(x,\lambda), \qquad S_\nu(0,\lambda) = S'_\nu(0,\lambda) = 0, \\ &\ell\psi_\nu(x,\lambda) = \lambda\psi_\nu(x,\lambda) + \psi_{\nu-1}(x,\lambda), \qquad \psi_\nu(T,\lambda) = \psi'_\nu(T,\lambda) = 0. \end{aligned}\right\} \tag{8}$$

Moreover, for $n \in \mathbb{S}$ formula (6) yields

$$S_\nu(T,\lambda_n) = \psi_\nu(0,\lambda_n) = \frac{1}{\nu!}\Delta^{(\nu)}(\lambda_n) = 0, \quad \nu = \overline{0, m_n - 1}. \tag{9}$$

Put

$$S_{n+\nu}(x) = S_\nu(x,\lambda_n), \qquad \psi_{n+\nu}(x) = \psi_\nu(x,\lambda_n), \quad n \in \mathbb{S}, \nu = \overline{0, m_n - 1}. \tag{10}$$

Thus, $\{S_n(x)\}_{n\geq 1}$, $\{\psi_n(x)\}_{n\geq 1}$ are complete systems of eigen- and the associated functions of the boundary value problem L. Together with the eigenvalues λ_n we consider generalized weight numbers α_n, $n \geq 1$, determined in the following way:

$$\alpha_{k+\nu} = \int_0^T S_{k+\nu}(x)S_{k+m_k-1}(x)\,dx, \quad k \in \mathbb{S}, \nu = \overline{0, m_k - 1}. \tag{11}$$

We note that the numbers α_n for sufficiently large n coincide with the classical weight numbers (4) for the selfadjoint Sturm-Liouville operator.

Definition 1 The numbers $\{\lambda_n, \alpha_n\}_{n\geq 1}$ are called the generalized spectral data of L.

Consider the following inverse problem.

Inverse Problem 1 Given the generalized spectral data $\{\lambda_n, \alpha_n\}_{n\geq 1}$, find $q(x)$.

Let the functions $C(x,\lambda)$, $\Phi(x,\lambda)$ be solutions of equation (1) under the conditions

$$C(0,\lambda) = \Phi(0,\lambda) = 1, \qquad C'(0,\lambda) = \Phi(T,\lambda) = 0.$$

The functions $\Phi(x,\lambda)$ and $M(\lambda) := \Phi'(0,\lambda)$ are called the Weyl solution and the Weyl function for L, respectively. According to (6), we have

$$\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)} = C(x,\lambda) + M(\lambda)S(x,\lambda), \tag{12}$$

$$\langle S(x,\lambda), \Phi(x,\lambda)\rangle \equiv -1, \tag{13}$$

$$M(\lambda) = -\frac{d(\lambda)}{\Delta(\lambda)}, \qquad d(\lambda) := \langle \psi(x,\lambda), C(x,\lambda)\rangle = -\psi'(0,\lambda) = C(T,\lambda). \tag{14}$$

The function $d(\lambda)$ is the characteristic function of the boundary value problem for the equation (1) with the boundary conditions $y'(0) = y(T) = 0$. Let $\{\mu_n\}_{n\geq 0}$ be its spectrum. Clearly, $\{\lambda_n\}_{n\geq 1} \cap \{\mu_n\}_{n\geq 0} = \emptyset$. Thus, $M(\lambda)$ is a meromorphic function with poles in λ_n, $n \geq 1$, and zeros in μ_n, $n \geq 0$. Moreover, (see, *e.g.*, [2])

$$\Delta(\lambda) = \frac{T^3}{\pi^2}\prod_{n=1}^{\infty}\frac{\lambda_n - \lambda}{n^2}, \qquad d(\lambda) = \frac{T^2}{\pi^2}\prod_{n=0}^{\infty}\frac{\mu_n - \lambda}{(n+1/2)^2}. \tag{15}$$

Let $\lambda = \rho^2$ and put $\tau = \operatorname{Im}\rho$. Using the known method (see, *e.g.*, [3]), one can prove the following asymptotics.

Lemma 1 (i) *For $|\rho| \to \infty$, the following asymptotics holds*

$$\left.\begin{aligned} S(x,\lambda) &= \frac{\sin\rho x}{\rho} - Q(x)\frac{\cos\rho x}{\rho^2} + \frac{1}{2\rho^2}\int_0^x q(t)\cos\rho(x-2t)\,dt \\ &\quad + O\left(\frac{1}{\rho^3}\exp(|\tau|x)\right), \\ S'(x,\lambda) &= \cos\rho x + Q(x)\frac{\sin\rho x}{\rho} - \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)\,dt \\ &\quad + O\left(\frac{1}{\rho^2}\exp(|\tau|x)\right), \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} \psi(x,\lambda) &= \frac{\sin\rho(T-x)}{\rho} - \big(Q(T)-Q(x)\big)\frac{\cos\rho(T-x)}{\rho^2} \\ &\quad + o\left(\tfrac{1}{\rho^2}\exp(|\tau|(T-x))\right), \\ \psi'(x,\lambda) &= -\cos\rho(T-x) - \big(Q(T)-Q(x)\big)\frac{\sin\rho(T-x)}{\rho} \\ &\quad + o\left(\tfrac{1}{\rho}\exp(|\tau|(T-x))\right) \end{aligned}\right\} \tag{17}$$

uniformly with respect to $x \in [0,T]$.

(ii) *Fix $\delta > 0$. Then for sufficiently large $|\lambda|$*

$$|\Delta(\lambda)| \geq \frac{C_\delta}{|\rho|}\exp(|\tau|T), \quad \lambda \in G_\delta, \tag{18}$$

where $G_\delta = \{\lambda = \rho^2 : |\rho - \pi k/T| \geq \delta, k \in \mathbb{Z}\}$.

Using (7), (10), (11) and (16), one can calculate

$$\alpha_n = \frac{T^3}{2\pi^2 n^2}\left(1 + \frac{\kappa_n}{n}\right), \quad \kappa_n = o(1), n \to \infty. \tag{19}$$

Fix $k \in \mathbb{S}$. According to (14), the function $M(\lambda)$ has a representation

$$M(\lambda) = \sum_{\nu=0}^{m_k-1} \frac{M_{k+\nu}}{(\lambda - \lambda_k)^{\nu+1}} + M_k^0(\lambda), \tag{20}$$

where $M_{k+m_k-1} \neq 0$, and the function $M_k^0(\lambda)$ is regular in a vicinity of λ_k. The sequence $\{M_n\}_{n\geq 1}$ is called *the Weyl sequence* for L. By virtue of (14), (17) and (18), the following estimate holds

$$M(\lambda) = O(\rho), \quad |\lambda| \to \infty, \lambda \in G_\delta. \tag{21}$$

Moreover, according to (6), (14), (16) and (17), for each fixed $\delta > 0$, we have

$$M(\rho^2) = i\rho + o(1), \quad |\rho| \to \infty, \arg\rho \in [\delta, \pi - \delta]. \tag{22}$$

The maximum modulus principle together with (7), (20) and (21) give

$$|M_n| \leq Cn^2. \tag{23}$$

Choose $\omega > 0$ such that $\Delta(\pm i\omega) \neq 0$ and put

$$\beta(\lambda) := \frac{\lambda}{\lambda^2 + \omega^2}, \qquad b_{n+\nu} := \frac{1}{\nu!}\beta^{(\nu)}(\lambda_n), \quad n \in \mathbb{S}, \nu = \overline{0, m_n - 1}.$$

According to (7) and (23), we get

$$\left(\frac{1}{\lambda - \lambda_n} + b_n\right) M_n = O\left(\frac{1}{n^2}\right), \quad n \to \infty,$$

and hence the series

$$N(\lambda) := \sum_{n\in\mathbb{S}} \sum_{\nu=0}^{m_n-1} \left(\frac{1}{(\lambda - \lambda_n)^{\nu+1}} + b_{n+\nu}\right) M_{n+\nu} \tag{24}$$

converges absolutely and uniformly in λ on bounded sets.

Theorem 1 *The following representation holds*

$$M(\lambda) = N(\lambda) + a, \quad a = \lim_{\tau\to+\infty}\left(i\tau - N(-\tau^2)\right). \tag{25}$$

Proof Consider the contour integral

$$I_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N} \left(\frac{1}{\lambda - \mu} + \beta(\mu)\right) M(\mu)\, d\mu, \quad \lambda \in \operatorname{int}\Gamma_N,$$

where the contour $\Gamma_N := \{\mu : |\mu| = (\pi(N + 1/2)/T)^2\}$, $N \in \mathbb{N}$ has the counterclockwise circuit. According to (7), we have $\Gamma_N \subset G_\delta$ for sufficiently large N and sufficiently small fixed $\delta > 0$. By virtue of (21), we obtain the estimate

$$\left(\frac{1}{\lambda - \mu} + \beta(\mu)\right) M(\mu) = O\left(\mu^{-\frac{3}{2}}\right)$$

uniformly with respect to λ in bounded subsets of $\mathbb{C}$, and hence

$$\lim_{N\to\infty} I_N(\lambda) = 0. \tag{26}$$

On the other hand, using the residue theorem [15], we calculate

$$I_N(\lambda) = -M(\lambda) + \sum_{n\in\mathbb{S},\lambda_n\in\operatorname{int}\Gamma_N} \left(\operatorname*{Res}_{\mu=\lambda_n} \frac{M(\mu)}{\lambda-\mu} + \operatorname*{Res}_{\mu=\lambda_n} \big(M(\mu)\beta(\mu)\big) \right) + b,$$
$$\lambda \in \operatorname{int}\Gamma_N \setminus \{\lambda_n\}_{n\geq 1}, \tag{27}$$

where

$$b = \operatorname*{Res}_{\mu=i\omega} \big(M(\mu)\beta(\mu)\big) + \operatorname*{Res}_{\mu=-i\omega} \big(M(\mu)\beta(\mu)\big) = \frac{M(i\omega)+M(-i\omega)}{2}.$$

Further, we calculate

$$\operatorname*{Res}_{\mu=\lambda_n} \frac{M(\mu)}{\lambda-\mu} = \sum_{\nu=0}^{m_n-1} \frac{M_{n+\nu}}{(\lambda-\lambda_n)^{\nu+1}}, \qquad \operatorname*{Res}_{\mu=\lambda_n} \big(M(\mu)\beta(\mu)\big) = \sum_{\nu=0}^{m_n-1} b_{n+\nu} M_{n+\nu}.$$

Substituting this into (27) and using (26), we obtain $M(\lambda) = N(\lambda) + b$. By virtue of (22), we get $b = a$ and arrive at (25). □

Theorem 2 *The coefficients M_n and the generalized weight numbers α_n determine each other uniquely by the formula*

$$\sum_{j=0}^{\nu} \alpha_{n+\nu-j} M_{n+m_n-j-1} = -\delta_{\nu,0}, \quad n \in \mathbb{S}, \nu = \overline{0, m_n - 1}. \tag{28}$$

Proof Using (10), (14) and (20), one can calculate

$$\sum_{j=0}^{\nu} M_{n+m_n-j-1} \Delta_{m_n+\nu-j,n} = \psi'_{n+\nu}(0), \quad n \in \mathbb{S}, \nu = \overline{0, m_n - 1}, \tag{29}$$

where $\Delta_{p,n} = \Delta^{(p)}(\lambda_n)/(p!)$. Obviously, $\psi_n(x) = \psi'_n(0) S_n(x)$, $n \in \mathbb{S}$. Moreover, by virtue of (8) and (10), induction gives

$$\psi_{n+\nu}(x) = \sum_{j=0}^{\nu} \psi'_{n+j}(0) S_{n+\nu-j}(x), \quad n \in \mathbb{S}, \nu = \overline{0, m_n - 1}. \tag{30}$$

Further, since

$$-S''(x,\lambda) + q(x)S(x,\lambda) = \lambda S(x,\lambda), \qquad -\psi''(x,\mu) + q(x)\psi(x,\mu) = \mu\psi(x,\mu),$$

we get

$$\big(S(x,\lambda)\psi'(x,\mu) - \psi(x,\mu)S'(x,\lambda)\big)' = (\lambda-\mu)S(x,\lambda)\psi(x,\mu),$$

and (4), (5) and (6) yield

$$\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}=-\int_0^T S(x,\lambda)\psi(x,\mu)\,dx.$$

Hence,

$$\frac{d}{d\lambda}\Delta(\lambda)=-\int_0^T S(x,\lambda)\psi(x,\lambda)\,dx,$$

and we calculate

$$\Delta_{m_n+\nu,n}=-\frac{1}{m_n+\nu}\sum_{j=0}^{m_n+\nu-1}\int_0^T \psi_j(x,\lambda_n)S_{m_n+\nu-1-j}(x,\lambda_n)\,dx,\quad \nu\geq 0.$$

Using (8) and (10) and integrating by parts, we obtain

$$\Delta_{m_n+\nu,n}=-\int_0^T \psi_{n+\nu}(x)S_{n+m_n-1}(x)\,dx,\quad n\in\mathbb{S},\nu=\overline{0,m_n-1}. \tag{31}$$

Substituting (30) in (31) and taking (11) into account, we arrive at

$$\Delta_{m_n+\nu,n}=-\sum_{j=0}^{\nu}\alpha_{n+\nu-j}\psi'_{n+j}(0),\quad n\in\mathbb{S},\nu=\overline{0,m_n-1}. \tag{32}$$

Finally, substituting (32) in (29), we get

$$\sum_{j=0}^{\nu}\psi'_{n+\nu-j}(0)\sum_{k=0}^{j}\alpha_{n+j-k}M_{n+m_n-k-1}=-\psi'_{n+\nu}(0),\quad n\in\mathbb{S},\nu=\overline{0,m_n-1}.$$

Since $\psi'_n(0)\neq 0$, $n\in\mathbb{S}$, by induction we obtain (28). □

According to (19) and (28), we have the asymptotics

$$M_n=-\frac{2\pi^2 n^2}{T^3}\left(1+\frac{\kappa_n}{n}\right),\quad \kappa_n=o(1). \tag{33}$$

Consider the following inverse problems.

Inverse Problem 2 Given the spectra $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 0}$, construct the function $q(x)$.

Inverse Problem 3 Given the Weyl function $M(\lambda)$, construct the function $q(x)$.

Remark 1 According to (14), (15), (24), (25) and (28), inverse Problems 1-3 are equivalent. The numbers $\{\lambda_n, M_n\}_{n\geq 1}$ can also be used as spectral data.

3 The uniqueness theorem

We agree that together with L we consider a boundary value problem $\tilde{L} = L(\tilde{q}(x), \tilde{T})$ of the same form but with another potential. If a certain symbol γ denotes an object related to L, then this symbol with tilde $\tilde{\gamma}$ denotes the analogous object related to $\tilde{L}$ and $\hat{\gamma} := \gamma - \tilde{\gamma}$.

Theorem 3 *If $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$, $n \geq 1$, then $L = \tilde{L}$, i.e., $T = \tilde{T}$, $q(x) = \tilde{q}(x)$ a.e. on $(0, T)$. Thus, the specification of the generalized spectral data $\{\lambda_n, \alpha_n\}_{n\geq 1}$ determines the potential uniquely.*

Proof By virtue of (7), we have $T = \tilde{T}$. According to Remark 1, it is sufficient to prove that if $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$. Define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{bmatrix} \tilde{S}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{S}'(x,\lambda) & \tilde{\Phi}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} S(x,\lambda) & \Phi(x,\lambda) \\ S'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}. \tag{34}$$

Using (13) and (34), we calculate

$$\left.\begin{aligned} P_{j1}(x,\lambda) &= \Phi^{(j-1)}(x,\lambda)\tilde{S}'(x,\lambda) - S^{(j-1)}(x,\lambda)\tilde{\Phi}'(x,\lambda), \\ P_{j2}(x,\lambda) &= S^{(j-1)}(x,\lambda)\tilde{\Phi}(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\tilde{S}(x,\lambda), \end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned} S(x,\lambda) &= P_{11}(x,\lambda)\tilde{S}(x,\lambda) + P_{12}(x,\lambda)\tilde{S}'(x,\lambda), \\ \Phi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda). \end{aligned}\right\} \tag{36}$$

It follows from (12) and (35) that

$$\begin{aligned} P_{11}(x,\lambda) &= 1 + \frac{1}{\Delta(\lambda)}\big(\psi(x,\lambda)\big(\tilde{S}'(x,\lambda) - S'(x,\lambda)\big) - S(x,\lambda)\big(\tilde{\psi}'(x,\lambda) - \psi'(x,\lambda)\big)\big), \\ P_{12}(x,\lambda) &= \frac{1}{\Delta(\lambda)}\big(S(x,\lambda)\tilde{\psi}(x,\lambda) - \psi(x,\lambda)\tilde{S}(x,\lambda)\big), \\ P_{21}(x,\lambda) &= \frac{1}{\Delta(\lambda)}\big(\psi'(x,\lambda)\tilde{S}'(x,\lambda) - S'(x,\lambda)\tilde{\psi}'(x,\lambda)\big), \\ P_{22}(x,\lambda) &= 1 + \frac{1}{\Delta(\lambda)}\big(S'(x,\lambda)\big(\tilde{\psi}(x,\lambda) - \psi(x,\lambda)\big) - \psi'(x,\lambda)\big(\tilde{S}(x,\lambda) - S(x,\lambda)\big)\big). \end{aligned}$$

By virtue of (16)-(18), this yields

$$P_{11}(x,\lambda) = 1 + O\left(\frac{1}{\rho}\right), \qquad P_{12}(x,\lambda) = O\left(\frac{1}{\rho}\right), \quad |\lambda| \to \infty, \lambda \in G_\delta, \tag{37}$$

$$P_{22}(x,\lambda) = 1 + O\left(\frac{1}{\rho}\right), \qquad P_{21}(x,\lambda) = O(1), \quad |\lambda| \to \infty, \lambda \in G_\delta, \tag{38}$$

uniformly with respect to $x \in [0, T]$. On the other hand, according to (12) and (35), we get

$$\begin{aligned} P_{11}(x,\lambda) &= C(x,\lambda)\tilde{S}'(x,\lambda) - S(x,\lambda)\tilde{C}'(x,\lambda) + \hat{M}(\lambda)S(x,\lambda)\tilde{S}'(x,\lambda), \\ P_{12}(x,\lambda) &= S(x,\lambda)\tilde{C}(x,\lambda) - \tilde{S}(x,\lambda)C(x,\lambda) - \hat{M}(\lambda)S(x,\lambda)\tilde{S}(x,\lambda). \end{aligned}$$

Thus, if $\hat{M}(\lambda) \equiv 0$, then for each fixed x, the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ. Together with (37) this yields $P_{11}(x,\lambda) \equiv 1$, $P_{12}(x,\lambda) \equiv 0$. Substituting into (36), we get $S(x,\lambda) \equiv \tilde{S}(x,\lambda)$ and consequently $L = \tilde{L}$. □

4 Main equation. Solution of the inverse problem

Let the spectral data $\{\lambda_n, \alpha_n\}_{n\ge1}$ of $L = L(q(x), T)$ be given. We choose an arbitrary model boundary value problem $\tilde{L} = L(\tilde{q}(x), T)$ (*e.g.*, one can take $\tilde{q}(x) \equiv 0$). Introduce the numbers ξ_n, $n \ge 1$ by the formulae

$$\left.\begin{aligned} &\xi_{k+\nu} := |\rho_k - \tilde{\rho}_k| + \frac{1}{k^2}\sum_{p=\nu}^{m_k-1} |M_{k+p} - \tilde{M}_{k+p}|, k \in \mathbb{S}\cap\tilde{\mathbb{S}}, m_k = \tilde{m}_k, \nu = \overline{0, m_k-1}, \\ &\xi_n := 1 \quad \text{for the rest of } n. \end{aligned}\right\} \tag{39}$$

According to (7) and (33), we have

$$\xi_n = O\left(\frac{1}{n}\right), \quad n \to \infty. \tag{40}$$

Denote

$$\begin{aligned} &\lambda_{n,0} := \lambda_n, \qquad \lambda_{n,1} := \tilde{\lambda}_n, \qquad M_{n,0} := M_n, \qquad M_{n,1} := \tilde{M}_n, \\ &\mathbb{S}_0 := \mathbb{S}, \qquad \mathbb{S}_1 := \tilde{\mathbb{S}}, \qquad m_{n,0} := m_n, \qquad m_{n,1} := \tilde{m}_n, \\ &S_{k+\nu,i}(x) := S_\nu(x, \lambda_{k,i}), \qquad \tilde{S}_{k+\nu,i}(x) := \tilde{S}_\nu(x, \lambda_{k,i}), \quad k \in \mathbb{S}_i, \nu = \overline{0, m_{k,i}-1}, i = 0,1, \\ &D(x,\lambda,\mu) := \frac{\langle S(x,\lambda), S(x,\mu)\rangle}{\lambda - \mu} = \int_0^x S(t,\lambda), S(t,\mu)\,dt, \\ &D_{\nu,\eta}(x,\lambda,\mu) := \frac{1}{\nu!\eta!}\frac{\partial^{\nu+\eta}}{\partial\lambda^\nu \mu^\eta} D(x,\lambda,\mu). \end{aligned}$$

For $i,j = 0,1$, $n \in \mathbb{S}_i$ put

$$A_{n+\nu,i}(x,\lambda) := \sum_{p=\nu}^{m_{n,i}-1} M_{n+p,i} D_{0,p-\nu}(x,\lambda,\lambda_{n,i}), \qquad P_{n+\nu,i;k,j}(x) := \frac{1}{\nu!}\frac{\partial^\nu}{\partial\lambda^\nu} A_{k,j}(x,\lambda)\Big|_{\lambda=\lambda_{n,i}},$$

where $k \ge 1$, $\nu = \overline{0, m_{n,i}-1}$. Analogously, we define $\tilde{D}(x,\lambda,\mu)$, $\tilde{D}_{\nu,\eta}(x,\lambda,\mu)$, $\tilde{A}_{n,i}(x,\lambda)$ and $\tilde{P}_{n,i;k,j}(x)$, $n,k \ge 1$, $i,j = 0,1$, replacing S with $\tilde{S}$ in the definitions above.

By the same way as in [2], using (7), (16), (33) and Schwarz's lemma [15], we get the such estimates as $\pm \operatorname{Re}\rho \ge 0$, $n,k \ge 1$, $i,j = 0,1$, $\nu = 0,1$

$$\left|S_{n,i}^{(\nu)}(x)\right| \le Cn^{\nu-1}, \qquad \left|S_{n,0}^{(\nu)}(x) - S_{n,1}^{(\nu)}(x)\right| \le C\xi_n n^{\nu-1}, \tag{41}$$

$$\left.\begin{aligned} &\left|D(x,\lambda,\lambda_{k,j})\right| \le \frac{C\exp(|\tau|x)}{|\rho|k(|\rho \mp \pi k/T| + 1)}, \\ &\left|D(x,\lambda,\lambda_{k,0}) - D(x,\lambda,\lambda_{k,1})\right| \le \frac{C\xi_k \exp(|\tau|x)}{|\rho|k(|\rho \mp \pi k/T| + 1)}, \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned}
&\left|P_{n,i;k,j}(x)\right| \le \frac{Ck}{(|n-k|+1)n}, \qquad \left|P^{(\nu+1)}_{n,i;k,j}(x)\right| \le C\left(\nu k + \frac{k^{\nu+1}}{n}\right),\\
&\left|P_{n,i;k,0}(x) - P_{n,i;k,1}(x)\right| \le \frac{Ck\xi_k}{(|n-k|+1)n},\\
&\left|P^{(\nu+1)}_{n,i;k,0}(x) - P^{(\nu+1)}_{n,i;k,1}(x)\right| \le C\xi_k\left(\nu k + \frac{k^{\nu+1}}{n}\right),\\
&\left|P_{n,0;k,j}(x) - P_{n,1;k,j}(x)\right| \le \frac{Ck\xi_n}{(|n-k|+1)n},\\
&\left|P^{(\nu+1)}_{n,0;k,j}(x) - P^{(\nu+1)}_{n,1;k,j}(x)\right| \le C\xi_n\left(\nu k + \frac{k^{\nu+1}}{n}\right),\\
&\left|P_{n,0;k,0}(x) - P_{n,1;k,0}(x) - P_{n,0;k,1}(x) + P_{n,1;k,1}(x)\right| \le \frac{Ck\xi_n\xi_k}{(|n-k|+1)n},\\
&\left|P^{(\nu+1)}_{n,0;k,0}(x) - P^{(\nu+1)}_{n,1;k,0}(x) - P^{(\nu+1)}_{n,0;k,1}(x) + P^{(\nu+1)}_{n,1;k,1}(x)\right| \le C\xi_n\xi_k\left(\nu k + \frac{k^{\nu+1}}{n}\right).
\end{aligned}\right\} \tag{43}$$

The analogous estimates are also valid for $\tilde{S}_{n,i}(x)$, $\tilde{D}(x,\lambda,\lambda_{k,j})$, $\tilde{P}_{n,i;k,j}(x)$.

Lemma 2 *The following relation holds*

$$\tilde{S}_{n,i}(x) = S_{n,i}(x) - \sum_{k=1}^{\infty}\big(\tilde{P}_{n,i;k,0}(x)S_{k,0}(x) - \tilde{P}_{n,i;k,1}(x)S_{k,1}(x)\big), \quad n \ge 1, i = 0,1, \tag{44}$$

where the series converges absolutely and uniformly with respect to $x \in [0,T]$.

Proof Let real numbers a, b be such that $a < \min \operatorname{Re}\lambda_{n,i}$, $b > \max|\operatorname{Im}\lambda_{n,i}|$, $n \ge 1$, $i = 0,1$. In the λ-plane consider a closed contour $\gamma_N := \partial\,\Xi_N$ (with a counterclockwise circuit), where $\Xi_N = \{\lambda : a \le \operatorname{Re}\lambda \le (N+1/2)^2\pi^2/T^2, |\operatorname{Im}\lambda| \le b\}$. By the standard method (see [2]), using (12), (35)-(37) and Cauchy's integral formula [15], we obtain the representation

$$\tilde{S}(x,\lambda) = S(x,\lambda) - \frac{1}{2\pi i}\int_{\gamma_N}\hat{M}(\mu)\tilde{D}(x,\lambda,\mu)S(x,\mu)\,d\mu + \varepsilon_N(x,\lambda), \tag{45}$$

where

$$\lim_{N\to\infty}\frac{\partial^\nu}{\partial\lambda^\nu}\varepsilon_N(x,\lambda) = 0, \quad \nu \ge 0,$$

uniformly with respect to $x \in [0,T]$ and λ on bounded sets. Calculating the integral in (45) by the residue theorem and using (20), we get

$$\frac{1}{2\pi i}\int_{\gamma_N}\hat{M}(\mu)\tilde{D}(x,\lambda,\mu)S(x,\mu)\,d\mu = \sum_{k=1}^{N}\big(\tilde{A}_{k,0}(x,\lambda)S_{k,0}(x) - \tilde{A}_{k,1}(x,\lambda)S_{k,1}(x)\big)$$

for sufficiently large N. Taking the limit in (45) as $N \to \infty$, we obtain

$$\tilde{S}(x,\lambda) = S(x,\lambda) - \sum_{k=1}^{\infty}\big(\tilde{A}_{k,0}(x,\lambda)S_{k,0}(x) - \tilde{A}_{k,1}(x,\lambda)S_{k,1}(x)\big). \tag{46}$$

Differentiating this with respect to λ, the corresponding number of times and then taking $\lambda = \lambda_{n,i}$, we arrive at (44). □

Analogously to (46), one can obtain the following relation

$$\tilde{\Phi}(x,\lambda) = \Phi(x,\lambda) - \sum_{k=1}^{\infty}\big(\tilde{F}_{k,0}(x,\lambda)S_{k,0}(x) - \tilde{F}_{k,1}(x,\lambda)S_{k,1}(x)\big), \tag{47}$$

where

$$\tilde{F}_{n+\nu,i}(x,\lambda) := \sum_{p=\nu}^{m_{n,i}-1} M_{n+p,i}\tilde{G}_{p-\nu}(x,\lambda,\lambda_{n,i}), \quad n \in \mathbb{S}_i, \nu = \overline{0, m_{n,i}-1}, i = 0,1,$$

$$\tilde{G}_\nu(x,\lambda,\mu) := \frac{1}{\nu!}\frac{d^\nu}{d\mu^\nu}\tilde{G}(x,\lambda,\mu),$$

$$\tilde{G}(x,\lambda,\mu) := \frac{\langle\tilde{\Phi}(x,\lambda),\tilde{S}(x,\mu)\rangle}{\lambda-\mu} = \frac{1}{\lambda-\mu} + \int_0^x \tilde{\Phi}(t,\lambda)\tilde{S}(t,\mu)\,dt.$$

For each fixed $x \in [0,T]$, the relation (44) can be considered as a system of linear equations with respect to $S_{n,i}(x)$, $n \geq 1$, $i = 0,1$. But the series in (44) converges only with brackets, *i.e.*, the terms in them cannot be dissociated. Therefore, it is inconvenient to use (44) as a main equation of the inverse problem. Below, we will transfer (44) to a linear equation in the Banach space of bounded sequences (see (53)).

Let w be the set of indices $u = (n,i)$, $n \geq 1$, $i = 0,1$. For each fixed $x \in [0,T]$, we define the vector

$$\phi(x) = \big[\phi_u(x)\big]_{u\in w}^T = \big[\phi_{n,0}(x), \phi_{n,1}(x)\big]_{n\geq 1}^T$$

(where T is the sign for transposition) by the formula

$$\begin{bmatrix}\phi_{n,0}(x)\\ \phi_{n,1}(x)\end{bmatrix} = n\begin{bmatrix}\chi_n & -\chi_n\\ 0 & 1\end{bmatrix}\begin{bmatrix}S_{n,0}(x)\\ S_{n,1}(x)\end{bmatrix}, \qquad \chi_n = \begin{cases}\xi_n^{-1}, & \xi_n \neq 0,\\ 0, & \xi_n = 0.\end{cases}$$

Note that if $\phi_{n,0}$, $\phi_{n,1}$ are given, then $S_{n,0}$, $S_{n,1}$ can be found by the formula

$$\begin{bmatrix}S_{n,0}(x)\\ S_{n,1}(x)\end{bmatrix} = \frac{1}{n}\begin{bmatrix}\xi_n & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\phi_{n,0}(x)\\ \phi_{n,1}(x)\end{bmatrix}. \tag{48}$$

Consider also a block-matrix

$$H(x) = \big[H_{u;v}(x)\big]_{u,v\in w} = \begin{bmatrix}H_{n,0;k,0}(x) & H_{n,0;k,1}(x)\\ H_{n,1;k,0}(x) & H_{n,1;k,1}(x)\end{bmatrix}_{n,k\geq 1}, \quad u = (n,i), v = (k,j),$$

where

$$\begin{bmatrix}H_{n,0;k,0}(x) & H_{n,0;k,1}(x)\\ H_{n,1;k,0}(x) & H_{n,1;k,1}(x)\end{bmatrix} = \frac{n}{k}\begin{bmatrix}\chi_n & -\chi_n\\ 0 & 1\end{bmatrix}\begin{bmatrix}P_{n,0;k,0}(x) & P_{n,0;k,1}(x)\\ P_{n,1;k,0}(x) & P_{n,1;k,1}(x)\end{bmatrix}\begin{bmatrix}\xi_k & 1\\ 0 & -1\end{bmatrix}.$$

Analogously, we introduce $\tilde{\phi}_{n,i}(x)$, $\tilde{\phi}(x)$ and $\tilde{H}_{n,i;k,j}(x)$, $\tilde{H}(x)$ by the replacement of $S_{n,i}(x)$, $P_{n,i;k,j}(x)$ in the preceding definitions with $\tilde{S}_{n,i}(x)$, $\tilde{P}_{n,i;k,j}(x)$, respectively. Using (41) and (43), we get the estimates

$$\begin{aligned} &\left|\phi_{n,i}^{(\nu)}(x)\right| \le Cn^{\nu}, \qquad \left|H_{n,i;k,j}(x)\right| \le \frac{C\xi_k}{|n-k|+1}, \\ &\left|H_{n,i;k,j}^{(\nu+1)}(x)\right| \le C\xi_k(n+k)^{\nu}, \quad \nu = 0,1, \end{aligned} \tag{49}$$

$$\begin{aligned} &\left|\tilde{\phi}_{n,i}^{(\nu)}(x)\right| \le Cn^{\nu}, \qquad \left|\tilde{H}_{n,i;k,j}(x)\right| \le \frac{C\xi_k}{|n-k|+1}, \\ &\left|\tilde{H}_{n,i;k,j}^{(\nu+1)}(x)\right| \le C\xi_k(n+k)^{\nu}, \quad \nu = 0,1, \end{aligned} \tag{50}$$

$$\left|\tilde{H}_{n,i;k,j}(x) - \tilde{H}_{n,i;k,j}(x_0)\right| \le C\xi_k|x-x_0|, \quad x,x_0 \in [0,T]. \tag{51}$$

Consider the Banach space B of bounded sequences $a = [a_u]_{u\in w}^T$ with the norm $\|a\|_B = \sup_{u\in w}|a_u|$. It follows from (49) and (50) that for each fixed $x \in [0,T]$, the operators $H(x)$ and $\tilde{H}(x)$, acting from B to B, are linear bounded ones and

$$\left\|H(x)\right\|_{B\to B}, \left\|\tilde{H}(x)\right\|_{B\to B} \le C \sup_{n\ge 1}\sum_{k=1}^{\infty}\frac{\xi_k}{|n-k|+1} < \infty. \tag{52}$$

Theorem 4 *For each fixed $x \in [0,T]$, the vector $\phi(x) \in B$ satisfies the equation*

$$\tilde{\phi}(x) = \left(I - \tilde{H}(x)\right)\phi(x) \tag{53}$$

in the Banach space B, where I is the identity operator.

Proof We rewrite (44) in the form

$$\begin{bmatrix} \tilde{S}_{n,0}(x) \\ \tilde{S}_{n,1}(x) \end{bmatrix} = \begin{bmatrix} S_{n,0}(x) \\ S_{n,1}(x) \end{bmatrix} - \sum_{k=1}^{\infty}\begin{bmatrix} \tilde{P}_{n,0;k,0}(x) & -\tilde{P}_{n,0;k,1}(x) \\ \tilde{P}_{n,1;k,0}(x) & -\tilde{P}_{n,1;k,1}(x) \end{bmatrix}\begin{bmatrix} S_{k,0}(x) \\ S_{k,1}(x) \end{bmatrix}, \quad n \ge 1.$$

Substituting here (48), and taking into account our notations, we arrive at

$$\tilde{\phi}_{n,i}(x) = \phi_{n,i}(x) - \sum_{(k,j)\in w}\tilde{H}_{n,i;k,j}(x)\phi_{k,j}(x), \quad (n,i) \in w,$$

which is equivalent to (53). □

For each fixed $x \in [0,T]$, the relation (53) can be considered as a linear equation with respect to $\phi(x)$. This equation is called the *main equation* of the inverse problem. Thus, the nonlinear inverse problem is reduced to the solution of the linear equation. Let us prove the unique solvability of the main equation.

Theorem 5 *For each fixed $x \in [0,T]$, the operator $I - \tilde{H}(x)$ has a bounded inverse operator, namely $I + H(x)$, i.e., the main equation* (53) *is uniquely solvable.*

Proof Acting in the same way as in Lemma 2 and using (37) and (38), we obtain

$$D(x,\lambda,\mu) - \tilde{D}(x,\lambda,\mu) = \frac{1}{2\pi i}\int_{\gamma_N}\tilde{D}(x,\lambda,\xi)\hat{M}(\xi)D(x,\xi,\mu)\,d\xi + \varepsilon_N^1(x,\lambda,\mu),$$

where

$$\lim_{N\to\infty}\frac{\partial^{\nu+j}}{\partial\lambda^\nu\,\partial\mu^j}\varepsilon_N^1(x,\lambda,\mu)=0,\quad \nu,j\geq 0,$$

uniformly with respect to $x\in[0,T]$ and λ, μ on bounded sets. Calculating the integral by the residue theorem and passing to the limit as $N\to\infty$, we obtain

$$D(x,\lambda,\mu)-\tilde{D}(x,\lambda,\mu)=\sum_{p=0}^{1}(-1)^p\sum_{l\in\mathbb{S}_p}\sum_{\nu=0}^{m_{l,p}-1}\tilde{A}_{l+\nu,p}(x,\lambda)D_{\nu,0}(x,\lambda_{l,p},\mu).$$

According to the definition of $P_{n,i;k,j}(x)$, $\tilde{P}_{n,i;k,j}(x)$, we arrive at

$$P_{n,i;k,j}(x)-\tilde{P}_{n,i;k,j}(x)=\sum_{l=1}^{\infty}\bigl(\tilde{P}_{n,i;l,0}(x)P_{l,0;k,j}(x)-\tilde{P}_{n,i;l,1}(x)P_{l,1;k,j}(x)\bigr),\quad n,k\geq 1, i,j=0,1.$$

Further, taking the definition of $H_{n,i;k,j}(x)$, $\tilde{H}_{n,i;k,j}(x)$ into account, we get

$$H_{n,i;k,j}(x)-\tilde{H}_{n,i;k,j}(x)=\sum_{(l,p)\in w}\tilde{H}_{n,i;l,p}(x)H_{l,p;k,j}(x),\quad (n,i),(k,j)\in w,$$

which is equivalent to $(I-\tilde{H}(x))(I+H(x))=I$. Symmetrically, one gets

$$\bigl(I+H(x)\bigr)\bigl(I-\tilde{H}(x)\bigr)=I.$$

Hence the operator $(I-\tilde{H}(x))^{-1}$ exists, and it is a linear bounded operator. □

Using the solution of the main equation, one can construct the function $q(x)$. Thus, we obtain the following algorithm for solving the inverse problem.

Algorithm 1 *Let the spectral data* $\{\lambda_n,\alpha_n\}_{n\geq1}$ *be given. Then*

(i) *construct* M_n, $n\geq 1$, *by solving the linear systems* (28);

(ii) *choose* $\tilde{L}$ *and calculate* $\tilde{\phi}(x)$ *and* $\tilde{H}(x)$;

(iii) *find* $\phi(x)$ *by solving equation* (53);

(iv) *choose* $n\in\mathbb{S}$ (*e.g.*, $n=1$) *and construct* $q(x)$ *by the formula*

$$q(x)=\frac{\phi''_{n,1}(x)}{\phi_{n,1}(x)}+\lambda_n.$$

Remark 2 In the particular case, when $\lambda_n=\tilde{\lambda}_n$, $\alpha_n=\tilde{\alpha}_n$ for $n>N$ (let for definiteness $N+1\in\mathbb{S}\cap\tilde{\mathbb{S}}$) according to (44) and the definition of $S_{n,i}(x)$, $\tilde{P}_{n,i;k,j}(x)$, the main equation becomes the linear algebraic system

$$\tilde{S}_{n,i}(x)=S_{n,i}(x)-\sum_{k=1}^{N}\bigl(\tilde{P}_{n,i;k,0}(x)S_{k,0}(x)-\tilde{P}_{n,i;k,1}(x)S_{k,1}(x)\bigr),\quad n=\overline{1,N}, i=0,1,\tag{54}$$

whose determinant does not vanish for any $x\in[0,T]$ by virtue of Theorem 5.

In the next section for the case $q(x) \in L_2(0,T)$, we give another algorithm, which is used in Section 6 for obtaining the necessary and sufficient conditions for the solvability of the inverse problem.

5 Algorithm 2

Here and in the sequel, we assume that $q(x) \in L_2(0,T)$. It is known that then $\{\kappa_n\} \in l_2$ in formulae (7), (19) and (33). We agree that in the sequel one and the same symbol $\{\kappa_n\}$ denotes different sequences in l_2. Let us choose the model boundary value problem $L = L(\tilde{q}(x), T)$, so that $\tilde{\omega} = \omega$ (for example, one can take $\tilde{q}(x) \equiv 2\omega/T$). Then besides (40), according to (7), (33) and (39), we have

$$\xi_n = \frac{\kappa_n}{n}, \qquad \Omega := \left(\sum_{n=1}^{\infty}(n\xi_n)^2\right)^{\frac{1}{2}} < \infty, \qquad \sum_{n=1}^{\infty}\xi_n < \infty. \tag{55}$$

Denote

$$\tilde{B}_{n+\nu,i}(x) := \sum_{p=\nu}^{m_{n,i}-1} M_{n+p,i}\tilde{S}_{n+p-\nu,i}(x), \quad n \in S_i, \nu = \overline{0, m_{n,i}-1}, i = 0,1, \tag{56}$$

$$\varepsilon_0(x) := \sum_{k=1}^{\infty}\big(\tilde{B}_{k,0}(x)S_{k,0}(x) - \tilde{B}_{k,1}(x)S_{k,1}(x)\big), \qquad \varepsilon(x) := 2\varepsilon_0'(x). \tag{57}$$

It is obvious that

$$\tilde{A}_{n,i}'(x,\lambda) = \tilde{S}(x,\lambda)\tilde{B}_{n,i}(x), \quad n \geq 1, i = 0,1. \tag{58}$$

Lemma 3 *The series in* (57) *converges absolutely and uniformly on* $[0,T]$ *and allows termwise differentiation. The function* $\varepsilon_0(x)$ *is absolutely continuous, and* $\varepsilon(x) \in L_2(0,T)$.

Proof It is sufficient to prove for the case $m_{n,i} = 1$, $n \geq 1$, $i = 0,1$. We rewrite $\varepsilon_0(x)$ to the form $\varepsilon_0(x) = A_1(x) + A_2(x)$, where

$$\left.\begin{aligned} A_1(x) &= \sum_{k=1}^{\infty}(M_{k,0} - M_{k,1})\tilde{S}_{k,0}(x)S_{k,0}(x), \\ A_2(x) &= \sum_{k=1}^{\infty}M_{k,1}\big((\tilde{S}_{k,0}(x) - \tilde{S}_{k,1}(x))S_{k,0}(x) + \tilde{S}_{k,1}(x)(S_{k,0}(x) - S_{k,1}(x))\big). \end{aligned}\right\} \tag{59}$$

It follows from (33), (41) and (55) that the series in (59) converges absolutely and uniformly on $[0,T]$, and

$$\left|A_j(x)\right| \leq C\sum_{k=0}^{\infty}\xi_k \leq C\Omega, \quad j = 1,2.$$

Furthermore, using the asymptotic formulae (7), (16) and (33), we calculate

$$A_1'(x) = \sum_{k=1}^{\infty}(M_{k,0} - M_{k,1})\frac{d}{dx}\big(\tilde{S}_{k,0}(x)S_{k,0}(x)\big) = \sum_{k=1}^{\infty}\left(\kappa_k \sin 2kx + O\left(\frac{\kappa_k}{k}\right)\right).$$

Hence $A_1(x) \in W_2^1[0,T]$. Similarly, we get $A_2(x) \in W_2^1[0,T]$, and consequently $\varepsilon_0(x) \in W_2^1[0,T]$. □

Lemma 4 *The following relation holds*

$$q(x) = \tilde{q}(x) + \varepsilon(x). \tag{60}$$

Proof Differentiating (46) twice with respect to x and using (57) and (58), we get

$$\tilde{S}'(x,\lambda) = S'(x,\lambda) - \varepsilon_0(x)\tilde{S}(x,\lambda) - \sum_{k=1}^{\infty}\bigl(\tilde{A}_{k,0}(x,\lambda)S'_{k,0}(x) - \tilde{A}_{k,1}(x,\lambda)S'_{k,1}(x)\bigr),$$

$$\begin{aligned}\tilde{S}''(x,\lambda) = S''(x,\lambda) &- \sum_{k=1}^{\infty}\Bigl(\bigl(\tilde{S}(x,\lambda)\tilde{B}_{k,0}(x)\bigr)'S_{k,0}(x) - \bigl(\tilde{S}(x,\lambda)\tilde{B}_{k,1}(x)\bigr)'S_{k,1}(x)\Bigr)\\ &- 2\tilde{S}(x,\lambda)\sum_{k=1}^{\infty}\bigl(\tilde{B}_{k,0}(x)S'_{k,0}(x) - \tilde{B}_{k,1}(x)S'_{k,1}(x)\bigr) - \sum_{k=1}^{\infty}\bigl(\tilde{A}_{k,0}(x,\lambda)S''_{k,0}(x)\\ &- \tilde{A}_{k,1}(x,\lambda)S''_{k,1}(x)\bigr).\end{aligned}$$

Using (1) and (8), we replace here the second derivatives, and then replace $S(x,\lambda)$ using (46). This yields

$$\begin{aligned}&\hat{q}(x)\tilde{S}(x,\lambda)\\ &\quad= 2\tilde{S}(x,\lambda)\sum_{k=1}^{\infty}\bigl(\tilde{B}_{k,0}(x)S'_{k,0}(x) - \tilde{B}_{k,1}(x)S'_{k,1}(x)\bigr)\\ &\qquad+ \sum_{k=1}^{\infty}\Bigl(\bigl(\tilde{S}(x,\lambda)\tilde{B}_{k,0}(x)\bigr)'S_{k,0}(x) - \bigl(\tilde{S}(x,\lambda)\tilde{B}_{k,1}(x)\bigr)'S_{k,1}(x)\Bigr)\\ &\qquad+ \sum_{k=1}^{\infty}\bigl((\lambda-\lambda_{k,0})\tilde{A}_{k,0}(x,\lambda)S_{k,0}(x) - (\lambda-\lambda_{k,1})\tilde{A}_{k,1}(x,\lambda)S_{k,1}(x)\bigr) - \mathcal{A}(x,\lambda),\end{aligned} \tag{61}$$

where

$$\mathcal{A}(x,\lambda) = \sum_{j=0}^{1}(-1)^j\sum_{m_{k,j}\ge 2}\sum_{\nu=0}^{m_{k,j}-2}\tilde{A}_{k+\nu+1,j}(x,\lambda)S_{k+\nu,j}(x).$$

Using (1) and (8) for $j = 0,1$, $k \in \mathbb{S}_j$, $\nu = \overline{0, m_{k,j}-1}$, we calculate

$$\bigl(\tilde{S}(x,\lambda)\tilde{B}_{k+\nu,j}(x)\bigr)' + (\lambda-\lambda_{k,j})\tilde{A}_{k+\nu,j}(x,\lambda) = 2\tilde{S}(x,\lambda)\tilde{B}'_{k+\nu,j}(x) + (1-\delta_{\nu,m_{k,j}-1})\tilde{A}_{k+\nu+1,j}(x,\lambda).$$

Applying this relation, we get

$$\begin{aligned}&\sum_{k=1}^{\infty}\Bigl(\bigl(\tilde{S}(x,\lambda)\tilde{B}_{k,0}(x)\bigr)'S_{k,0}(x) - \bigl(\tilde{S}(x,\lambda)\tilde{B}_{k,1}(x)\bigr)'S_{k,1}(x)\Bigr)\\ &\qquad+ \sum_{k=1}^{\infty}\bigl((\lambda-\lambda_{k,0})\tilde{A}_{k,0}(x,\lambda)S_{k,0}(x) - (\lambda-\lambda_{k,1})\tilde{A}_{k,1}(x,\lambda)S_{k,1}(x)\bigr)\\ &\quad= 2\tilde{S}(x,\lambda)\sum_{k=1}^{\infty}\bigl(\tilde{B}'_{k,0}(x)S_{k,0}(x) - \tilde{B}'_{k,1}(x)S_{k,1}(x)\bigr) + \mathcal{A}(x,\lambda),\end{aligned}$$

which together with (57) and (61) gives (60). □

Thus, we obtain the following algorithm for solving the inverse problem.

Algorithm 2 *Let the spectral data* $\{\lambda_n, \alpha_n\}_{n\geq 1}$ *be given. Then*
(i) *construct* M_n, $n \geq 1$, *by solving the linear systems* (28);
(ii) *choose* $\tilde{L}$ *so that* $\omega = \tilde{\omega}$ *and calculate* $\tilde{\phi}(x)$ *and* $\tilde{H}(x)$;
(iii) *find* $\phi(x)$ *by solving equation* (53), *and calculate* $S_{n,j}(x)$, $n \geq 1$, $j = 0, 1$, *by* (48);
(iv) *calculate* $q(x)$ *by formulae* (56), (57) *and* (60).

6 Necessary and sufficient conditions

In the present section, we obtain necessary and sufficient conditions for the solvability of the inverse problem. In the general non-selfadjoint case, they must include the requirement of the solvability of the main equation. In Section 7, some important cases will be considered when the solvability of the main equation can be proved by sufficiency, namely, the selfadjoint case, the case of finite-dimensional perturbations of the spectral data and the case of small perturbations.

Theorem 6 *For complex numbers* $\{\lambda_n, \alpha_n\}_{n\geq 1}$ *to be the spectral data of a certain boundary value problem* $L(q(x), T)$ *with* $q(x) \in L_2(0, T)$, *it is necessary and sufficient that*
(i) *the relations* (7) *and* (19) *hold with* $\{\kappa_n\} \in l_2$;
(ii) $\alpha_n \neq 0$ *for all* $n \in \mathbb{S}$;
(iii) (*Condition* S) *for each* $x \in [0, T]$, *the linear bounded operator* $I - \tilde{H}(x)$, *acting from* B *to* B, *has a bounded inverse one*. *Here* $\tilde{L}$ *is chosen so that* $\tilde{\omega} = \omega$.

The boundary value problem $L = L(q(x), T)$ *can be constructed by Algorithms* 1 *and* 2.

The necessity part of the theorem was proved above; here, we prove the sufficiency. We note that sufficiency condition (ii) of the theorem allows to solve linear systems (28) for finding M_n, $n \geq 1$, which are used for constructing the main equation. Moreover, we have

$$M_{n+m_n-1} \neq 0, \quad n \in \mathbb{S}. \tag{62}$$

Let $\phi(x) = [\phi_u(x)]_{u\in w}$ be the solution of the main equation (53). Denote

$$H(x) = \big[H_{u;v}(x)\big]_{u,v\in w} := \big(I - \tilde{H}(x)\big)^{-1} - I,$$

i.e.,

$$\big(I - \tilde{H}(x)\big)\big(I + H(x)\big) = \big(I + H(x)\big)\big(I - \tilde{H}(x)\big) = I. \tag{63}$$

Similarly to Lemma 1.6.7 in [2] using (51) and (53), one can prove the following assertion.

Lemma 5 *For* $n, k \geq 1$, $i, j, \nu = 0, 1$, $x \in [0, T]$, *the following relations hold*

$$\phi_{n,i}(x) \in C^1[0, T], \qquad \big|\phi_{n,i}^{(\nu)}(x)\big| \leq Cn^{\nu}, \tag{64}$$

$$\big|\phi_{n,i}^{(\nu)}(x) - \tilde{\phi}_{n,i}^{(\nu)}(x)\big| \leq C\Omega\eta_n^{1-\nu}, \tag{65}$$

$$\big|H_{n,i;k,j}(x)\big| \leq C\xi_k\left(\frac{1}{|n-k|+1} + \Omega\eta_n\right), \tag{66}$$

$$\big|H_{n,i;k,j}(x)\big| \le C\xi_k\left(\frac{1}{|n-k|+1} + \Omega\eta_k\right), \tag{67}$$

$$\big|H'_{n,i;k,j}(x)\big| \le C\xi_k, \tag{68}$$

where Ω *is defined in* (55) *and*

$$\eta_n := \left(\sum_{k=1}^{\infty} \frac{1}{k^2(|n-k|+1)^2}\right)^{1/2}.$$

We define the functions $S_{n,i}(x)$ by formulae (48), and according to (64), we get (41). Then (44) is also valid. By virtue of (48), (65) and Lemma 5, we have

$$\big|S^{(\nu)}_{n,i}(x) - \tilde{S}^{(\nu)}_{n,i}(x)\big| \le \frac{C}{n}\Omega\eta_n^{1-\nu}, \tag{69}$$

Furthermore, we construct the functions $S(x,\lambda)$ and $\Phi(x,\lambda)$ via (46) and (47) and the function $q(x)$ by formulae (56), (57) and (60). Clearly,

$$S_\nu(x,\lambda_{n,i}) = S_{n+\nu,i}(x), \quad n \in \mathbb{S}_i, \nu = \overline{0, m_{n,i}-1}, i = 0,1. \tag{70}$$

Analogously to Lemma 1.6.8 in [2] using (41) and (69), one can prove the following assertion.

Lemma 6 $q(x) \in L_2(0,T)$.

Lemma 7 *For* $i = 0,1$, $n \in \mathbb{S}_i$, $\nu = \overline{1, m_{n,i}-1}$ *the following relations hold*

$$\ell S_{n,i}(x) = \lambda_n S_{n,i}(x), \qquad \ell S_{n+\nu,i}(x) = \lambda_n S_{n+\nu,i}(x) + S_{n+\nu-1,i}(x), \tag{71}$$

$$\ell S(x,\lambda) = \lambda S(x,\lambda), \qquad \ell\Phi(x,\lambda) = \lambda\Phi(x,\lambda), \tag{72}$$

$$S(0,\lambda) = 0, \qquad S'(0,\lambda) = 1, \qquad \Phi(0,\lambda) = 1, \qquad \Phi(T,\lambda) = 0. \tag{73}$$

Proof (1) According to the estimates (42), the series in (46) is termwise differentiable with respect to x, and hence $S(0,\lambda) = 0$, $S'(0,\lambda) = 1$. By virtue of (70), we have $S_{n,j}(0) = 0$, $(n,j) \in w$. Thus, formula (47) gives $\Phi(0,\lambda) = 1$.

(2) In order to prove (71) and (72), we first assume that

$$\Omega_1 := \left(\sum_{k=0}^{\infty} \big(k^2\xi_k\big)^2\right)^{1/2} < \infty. \tag{74}$$

Differentiating (63) twice, we obtain

$$H''(x) = \big(I + H(x)\big)\tilde{H}''(x)\big(I + H(x)\big) + 2\big(I + H(x)\big)\tilde{H}'(x)H'(x). \tag{75}$$

It follows from (50), (66) and (67) that the series in (75) converges absolutely and uniformly for $x \in [0,T]$, $H_{n,i;k,j}(x) \in C^2[0,T]$, and

$$\big|H''_{n,i;k,j}(x)\big| \le C\xi_k(n+k). \tag{76}$$

Solving the main equation (53), we infer

$$\phi_{n,i}(x) = \tilde{\phi}_{n,i}(x) + \sum_{k,j} H_{n,i;k,j}(x)\tilde{\phi}_{k,j}(x), \quad x \in [0,T], (n,i),(k,j) \in w. \tag{77}$$

According to (50) and (66), the series in (77) converges absolutely and uniformly for $x \in [0,T]$. Further, using (77), we calculate

$$\tilde{\ell}\phi_{n,i}(x) = \tilde{\ell}\tilde{\phi}_{n,i}(x) + \sum_{k,j} H_{n,i;k,j}(x)\tilde{\ell}\tilde{\phi}_{k,j}(x) - 2\sum_{k,j} H'_{n,i;k,j}(x)\tilde{\phi}'_{k,j}(x) - \sum_{k,j} H''_{n,i;k,j}(x)\tilde{\phi}_{k,j}(x),$$

where according to (50), (67), (68) and (76), the series converges absolutely and uniformly for $x \in [0,T]$ and

$$\tilde{\ell}\phi_{n,i}(x) \in C[0,T], \qquad \left|\tilde{\ell}\phi_{n,i}(x)\right| \le Cn^2, \quad (n,i) \in w.$$

On the other hand, it follows from the proof of Lemma 3 and from (74) that $q(x) - \tilde{q}(x) \in C[0,T]$; hence

$$\ell\phi_{n,i}(x) \in C[0,T], \qquad \left|\ell\phi_{n,i}(x)\right| \le Cn^2, \quad (n,i) \in w.$$

Together with (48) this implies that

$$\ell S_{n,i}(x) \in C[0,T], \qquad \left|\ell S_{n,i}(x)\right| \le Cn, \qquad \left|\ell S_{n,0}(x) - \ell S_{n,1}(x)\right| \le Cn\xi_n, \quad (n,i) \in w.$$

Using (44), (57) and (60), we get

$$\begin{aligned}\tilde{\ell}\tilde{S}_{n,i}(x) &= \ell S_{n,i}(x) - \sum_{k=1}^{\infty}\big(\tilde{P}_{n,i;k,0}(x)\ell S_{k,0}(x) - \tilde{P}_{n,i;k,1}(x)\ell S_{k,1}(x)\big) \\ &\quad - \sum_{k=1}^{\infty}\big(\langle\tilde{S}_{n,i}(x),\tilde{B}_{k,0}(x)\rangle S_{k,0}(x) - \langle\tilde{S}_{n,i}(x),\tilde{B}_{k,1}(x)\rangle S_{k,1}(x)\big), \quad (n,i) \in w.\end{aligned} \tag{78}$$

Similarly, using (46) and (47), we calculate

$$\begin{aligned}\tilde{\ell}\tilde{S}(x,\lambda) &= \ell S(x,\lambda) - \sum_{k=1}^{\infty}\big(\tilde{A}_{k,0}(x,\lambda)\ell S_{k,0}(x) - \tilde{A}_{k,1}(x,\lambda)\ell S_{k,1}(x)\big) \\ &\quad - \sum_{k=1}^{\infty}\big(\langle\tilde{S}(x,\lambda),\tilde{B}_{k,0}(x)\rangle S_{k,0}(x) - \langle\tilde{S}(x,\lambda),\tilde{B}_{k,1}(x)\rangle S_{k,1}(x)\big),\end{aligned} \tag{79}$$

$$\begin{aligned}\tilde{\ell}\tilde{\Phi}(x,\lambda) &= \ell\Phi(x,\lambda) - \sum_{k=1}^{\infty}\big(\tilde{F}_{k,0}(x,\lambda)\ell S_{k,0}(x) - \tilde{F}_{k,1}(x,\lambda)\ell S_{k,1}(x)\big) \\ &\quad - \sum_{k=1}^{\infty}\big(\langle\tilde{\Phi}(x,\lambda),\tilde{B}_{k,0}(x)\rangle S_{k,0}(x) - \langle\tilde{\Phi}(x,\lambda),\tilde{B}_{k,1}(x)\rangle S_{k,1}(x)\big).\end{aligned} \tag{80}$$

For $n \in \mathbb{S}_i$ and $i = 0, 1$, it follows from (78) that

$$\begin{aligned}
\lambda_{n,i}\tilde{S}_{n,i}(x) &= \ell S_{n,i}(x) - \sum_{k=1}^{\infty}\big(\tilde{P}_{n,i;k,0}(x)\ell S_{k,0}(x) - \tilde{P}_{n,i;k,1}(x)\ell S_{k,1}(x)\big) \\
&\quad - \sum_{k=1}^{\infty}\big((\lambda_{n,i} - \lambda_{k,0})\tilde{P}_{n,i;k,0}(x)S_{k,0}(x) - (\lambda_{n,i} - \lambda_{k,1})\tilde{P}_{n,i;k,1}(x)S_{k,1}(x)\big) \\
&\quad + \sum_{j=0}^{1}(-1)^j \sum_{m_{k,j}>1} \sum_{s=0}^{m_{k,j}-2} \tilde{P}_{n,i;k+s+1,j}(x)S_{k+s,j}(x),
\end{aligned}$$

$$\begin{aligned}
\lambda_{n,i}\tilde{S}_{n+\nu,i}(x) + \tilde{S}_{n+\nu-1,i}(x) &= \ell S_{n+\nu,i}(x) - \sum_{k=1}^{\infty}\big(\tilde{P}_{n+\nu,i;k,0}(x)\ell S_{k,0}(x) - \tilde{P}_{n+\nu,i;k,1}(x)\ell S_{k,1}(x)\big) \\
&\quad - \sum_{k=1}^{\infty}\Big(\big((\lambda_{n,i} - \lambda_{k,0})\tilde{P}_{n+\nu,i;k,0}(x) + \tilde{P}_{n+\nu-1,i;k,0}(x)\big)S_{k,0}(x) \\
&\quad - \big((\lambda_{n,i} - \lambda_{k,1})\tilde{P}_{n+\nu,i;k,1}(x) + \tilde{P}_{n+\nu-1,i;k,1}(x)\big)S_{k,1}(x)\Big) \\
&\quad + \sum_{j=0}^{1}(-1)^j \sum_{m_{k,j}>1} \sum_{s=0}^{m_{k,j}-2} \tilde{P}_{n+\nu,i;k+s+1,j}(x)S_{k+s,j}(x), \quad \nu = \overline{1, m_{n,i} - 1},
\end{aligned}$$

and, consequently, we arrive at

$$\gamma_{n,i}(x) = \sum_{k=1}^{\infty}\big(\tilde{P}_{n,i;k,0}(x)\gamma_{k,0}(x) - \tilde{P}_{n,i;k,1}(x)\gamma_{k,1}(x)\big), \quad (n,i) \in w, \tag{81}$$

where for $l \in \mathbb{S}_i$, $\nu = \overline{1, m_{l,i} - 1}$

$$\gamma_{l,i}(x) := \ell S_{l,i}(x) - \lambda_{l,i}S_{l,i}(x), \qquad \gamma_{l+\nu,i}(x) := \ell S_{l+\nu,i}(x) - \lambda_{l,i}S_{l+\nu,i}(x) - S_{l+\nu-1,i}(x).$$

Using (81), we get

$$\beta_{n,i}(x) = \sum_{k,j}\tilde{H}_{n,i;k,j}(x)\beta_{k,j}(x), \quad (n,i),(k,j) \in w, \tag{82}$$

where

$$\beta_{n,1}(x) = n\gamma_{n,1}(x), \qquad \beta_{n,0}(x) = n\chi_n\big(\gamma_{n,0}(x) - \gamma_{n,1}(x)\big).$$

Since $|\gamma_{n,i}(x)| \le Cn$, $|\gamma_{n,0}(x) - \gamma_{n,1}(x)| \le Cn\xi_n$, we have

$$\big|\beta_{n,i}(x)\big| \le Cn^2. \tag{83}$$

It follows from (50), (74), (82) and (83) that $|\beta_{n,i}(x)| \le C$. Then, by virtue of Condition S in Theorem 6, $\beta_{n,i}(x) = 0$, and consequently $\gamma_{n,i}(x) = 0$. Thus, we obtain (71).

Furthermore, since

$$\big\langle \tilde{S}(x,\lambda), \tilde{B}_{n+\nu,i}(x)\big\rangle = \begin{cases} (\lambda - \lambda_{n,i})\tilde{A}_{n+m_{n,i}-1,i}(x,\lambda), & \nu = m_{n,i} - 1, \\ (\lambda - \lambda_{n,i})\tilde{A}_{n+\nu,i}(x,\lambda) - \tilde{A}_{n+\nu+1,i}(x,\lambda), & \nu = \overline{0, m_{n,i} - 2}, \end{cases}$$

formula (79) gives

$$\lambda\tilde{S}(x,\lambda) = \ell S(x,\lambda) - \lambda\sum_{k=1}^{\infty}\big(\tilde{A}_{k,0}(x,\lambda)S_{k,0}(x) - \tilde{A}_{k,1}(x,\lambda)S_{k,1}(x)\big).$$

From this, by virtue of (46), it follows that $\ell S(x,\lambda) = \lambda S(x,\lambda)$. Analogously, using (80) we obtain $\ell\Phi(x,\lambda) = \lambda\Phi(x,\lambda)$. Thus, (71) and (72) are proved for the case when (74) is fulfilled.

Denote $\Delta(\lambda) := S(T,\lambda)$. It follows from (46) and (47) for $x = T$ that

$$\tilde{\Delta}(\lambda) = \Delta(\lambda) - \sum_{k=0}^{\infty}\big(\tilde{A}_{k,0}(T,\lambda)\Delta_{k,0} - \tilde{A}_{k,1}(T,\lambda)\Delta_{k,1}\big), \tag{84}$$

$$0 = \Phi(T,\lambda) - \sum_{k=0}^{\infty}\big(\tilde{F}_{k,0}(T,\lambda)\Delta_{k,0} - \tilde{F}_{k,1}(T,\lambda)\Delta_{k,1}\big), \tag{85}$$

where $\Delta_{n+\nu,i} = \Delta^{(\nu)}(\lambda_{n,i})/\nu!$, $n \in \mathbb{S}_i$, $\nu = \overline{0, m_{n,i}-1}$. Differentiating (84) with respect to λ an appropriate number of times and substituting $\lambda = \lambda_{n1}$, we get

$$\Delta_{n,1} = \sum_{k=1}^{\infty}\big(\tilde{P}_{n,1;k,0}(T)\Delta_{k,0} - \tilde{P}_{n,1;k,1}(T)\Delta_{k,1}\big). \tag{86}$$

Let us show that

$$\tilde{P}_{n,1;k,1}(T) = -\delta_{n,k}. \tag{87}$$

Indeed, for $n,k \in \tilde{\mathbb{S}}$, $\nu = \overline{0,\tilde{m}_n-1}$, $s = \overline{0,\tilde{m}_k-1}$ we have

$$\tilde{P}_{n+\nu,1;k+s,1}(T) = \sum_{p=0}^{\tilde{m}_n-1-s}\tilde{M}_{k+p+s}\tilde{D}_{\nu,p}(T,\tilde{\lambda}_n,\tilde{\lambda}_k). \tag{88}$$

Moreover, according to (9), we have $\langle\tilde{S}_\nu(x,\tilde{\lambda}_n),\tilde{S}(x,\tilde{\lambda}_k)\rangle|_{x=T} = 0$, and hence

$$(\tilde{\lambda}_n - \tilde{\lambda}_k)\tilde{D}_{\nu,p}(T,\tilde{\lambda}_n,\tilde{\lambda}_k) + \tilde{D}_{\nu-1,p}(T,\tilde{\lambda}_n,\tilde{\lambda}_k) - \tilde{D}_{\nu,p-1}(T,\tilde{\lambda}_n,\tilde{\lambda}_k) = 0, \tag{89}$$

where $D_{\alpha,\beta}(T,\tilde{\lambda}_n,\tilde{\lambda}_k) = 0$ for negative α or β. For $\tilde{\lambda}_n \neq \tilde{\lambda}_k$ solving the system (89), we obtain $D_{\nu,p}(T,\tilde{\lambda}_n,\tilde{\lambda}_k) = 0$, which together with (88) gives $P_{n+\nu,1;k+s,1}(x) = 0$ for $n \neq k$. If $\tilde{\lambda}_n = \tilde{\lambda}_k$, then (11) and (89) give

$$\tilde{D}_{\nu,p}(T,\tilde{\lambda}_n,\tilde{\lambda}_n) = \begin{cases} 0, & 0 \le \nu + p \le \tilde{m}_n - 2, \\ \tilde{\alpha}_{n+\nu+p-\tilde{m}_n+1}, & \tilde{m}_n - 1 \le \nu + p \le 2\tilde{m}_n - 2. \end{cases} \tag{90}$$

According to (88) and (90), we have $\tilde{P}_{n+\nu,1;n+s,1}(T) = 0$, $\nu < s$. Moreover, using (88), (90) and (28), we calculate

$$\tilde{P}_{n+\nu,1;n+s,1}(T) = \sum_{p=0}^{\nu-s}\tilde{\alpha}_{n+\nu-s-p}\tilde{M}_{n+\nu+p-\tilde{m}_n+1} = -\delta_{\nu-s,0}$$

and arrive at (87). Using (86) and (87), we get

$$\sum_{k=1}^{\infty} \tilde{P}_{n,1;k,0}(T)\Delta_{k,0} = 0.$$

Then, by virtue of Condition S, $\Delta_{k,0} = 0$, $k \geq 1$. Substituting this into (85) and using the relation $\tilde{F}_{k,1}(T,\lambda) = 0$, $k \geq 1$, we obtain $\Phi(T,\lambda) = 0$.

(3) Let us now consider the general case when instead of (74) only (55) holds. Put

$$\rho_{n,(l)} := \begin{cases} \rho_n, & n < l, \\ \tilde{\rho}_n, & n \geq l, \end{cases} \qquad M_{n,(l)} := \begin{cases} M_n, & n < l, \\ \tilde{M}_n, & n \geq l. \end{cases}$$

We agree that if the symbol γ denotes an object constructed with the help of the numbers $\{\rho_n, M_n\}_{n\geq 1}$, then the symbol $\gamma_{(l)}$ denotes the corresponding object, constructed with the help of $\{\rho_{n,(l)}, M_{n,(l)}\}_{n\geq 1}$. Then for all $l \geq 1$, we have

$$\Omega_{1,(l)} = \left(\sum_{n=1}^{\infty} \left(n^2 \xi_{n,(l)}\right)^2 \right)^{1/2} = \left(\sum_{n=1}^{l-1} \left(n^2 \xi_n\right)^2 \right)^{1/2} < \infty.$$

For each fixed $l \geq 1$, we solve the corresponding main equation

$$\tilde{\phi}_{(l)}(x) = \left(I - \tilde{H}_{(l)}(x)\right)\phi_{(l)}(x),$$

and construct the functions $S_{(l)}(x,\lambda)$ and the boundary value problem $L(q_{(l)}(x),T)$. Using Lemma 1.5.1 in [2], one can show that

$$\lim_{l\to\infty} \|q_{(l)} - q\|_{L_2} = 0, \qquad \lim_{l\to\infty} \max_{0\leq x\leq T} \left|S_{(l)}(x,\lambda) - S(x,\lambda)\right| = 0.$$

Denote by $S_0(x,\lambda)$ the solution of equation (1) under the initial conditions $S_0(0,\lambda) = 0$, $S_0'(0,\lambda) = 1$. According to Lemma 1.5.3 in [2], we obtain

$$\lim_{l\to\infty} \max_{0\leq x\leq T} \left|S_{(l)}(x,\lambda) - S_0(x,\lambda)\right| = 0.$$

Hence $S_0(x,\lambda) = S(x,\lambda)$, *i.e.*, $\ell S(x,\lambda) = \lambda S(x,\lambda)$. Similarly, we get $\ell\Phi(x,\lambda) = \lambda\Phi(x,\lambda)$.

Notice that we additionally proved that $\Delta^{(\nu)}(\lambda_n) = 0$, $\nu = \overline{0, m_n - 1}$, $n \in \mathbb{S}$, *i.e.*, $\{\lambda_n\}_{n\geq 1}$ is a spectrum of L. $\square$

Proof of Theorem 6 According to (72) and (73), the function $\Phi(x,\lambda)$ is the Weyl function for the constructed boundary value problem L. Choose n^* so that $m_n = \tilde{m}_n = 1$, $n \geq n^*$, and put. Differentiating (47) with respect to x and then substituting $x = 0$, we obtain

$$\begin{aligned} M(\lambda) = \tilde{M}(\lambda) &+ \sum_{k=n^*}^{\infty} \left(\frac{M_k}{\lambda - \lambda_k} - \frac{\tilde{M}_k}{\lambda - \tilde{\lambda}_k} \right) \\ &+ \sum_{\substack{k\in\mathbb{S} \\ k<n^*}} \sum_{\nu=0}^{m_k-1} \frac{M_{k+\nu}}{(\lambda - \lambda_k)^{\nu+1}} - \sum_{\substack{k\in\tilde{\mathbb{S}} \\ k<n^*}} \sum_{\nu=0}^{\tilde{m}_k-1} \frac{\tilde{M}_{k+\nu}}{(\lambda - \tilde{\lambda}_k)^{\nu+1}}, \end{aligned} \tag{91}$$

where the series converges uniformly with respect to λ in bounded sets. From (62) and (91), it follows that for each $n \in \mathbb{S}$, the number λ_n is a pole of the function $M(\lambda)$ of order m_n. Thus, $\{\lambda_n\}_{n\geq 1}$ is the spectrum, and $\{M_n\}_{n\geq 1}$ is the Weyl sequence of L. Consequently, $\{\lambda_n, \alpha_n\}_{n\geq 1}$ are the spectral data of L. □

7 Spacial cases and stability of the solution

The requirement that the main equation is uniquely solvable (Condition S in Theorem 6) is essential and cannot be omitted (see Example 1.6.1 in [2]). Condition S is difficult to check in the general case. We point out three cases, for which the unique solvability of the main equation can be proved or checked.

(1) *The selfadjoint case.* It is known that in the selfadjoint case, *i.e.*, when the function $q(x)$ is real-valued, the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 1}$ are real numbers, and

$$\lambda_n \neq \lambda_m \quad (n \neq m), \qquad \alpha_n > 0. \tag{92}$$

Let real numbers $\{\lambda_n, \alpha_n\}_{n\geq 1}$ having the asymptotics (7) and (19) with $\{\kappa_n\} \in l_2$ and satisfying (92) be given. Choose $\tilde{L}$, construct $\tilde{\phi}(x)$, $\tilde{H}(x)$ and consider the equation (53). Similarly to Lemma 1.6.6 in [2], one can prove the following assertion.

Lemma 8 *For each fixed $x \in [0, T]$, the operator $I - \tilde{H}(x)$, acting from B to B, has a bounded inverse operator. Thus, the main equation* (53) *has a unique solution $\phi(x) \in B$.*

By virtue of Theorem 6 and Lemma 8, the following theorem holds.

Theorem 7 *For real numbers $\{\lambda_n, \alpha_n\}_{n\geq 1}$ to be the spectral data of a certain selfadjoint boundary value problem $L(q(x), T)$ with $q(x) \in L_2(0, T)$, it is necessary and sufficient to satisfy the asymptotics* (7) *and* (19) *with $\{\kappa_k\} \in l_2$ and condition* (92).

(2) *Finite-dimensional perturbations of the spectral data.* Let a model boundary value problem $\tilde{L}$ with the spectral data $\{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n\geq 1}$ be given. We change a finite subset of these numbers. In other words, we consider numbers $\{\lambda_n, \alpha_n\}_{n\geq 1}$ such that $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$, $n > N$ for certain $N + 1 \in \tilde{\mathbb{S}}$ and arbitrary in the rest. Then for such spectral data, the main equation becomes the linear algebraic system (54), and Condition S is equivalent to the condition that the determinant of this system does not equal zero for each $x \in [0, T]$. Such perturbations are very popular in applications. We note that for the selfadjoint case the determinant of the system (54) is always nonzero.

(3) *Local solvability of the main equation.* For small perturbations of the spectral data, Condition S is fulfilled automatically. Let us for simplicity consider the case of simple spectra, *i.e.*, $\tilde{\mathbb{S}} = \mathbb{N}$. The following theorem is valid.

Theorem 8 *Let $\tilde{L} = L(\tilde{q}(x), T)$ be given. There exists $\delta > 0$ (which depends on $\tilde{L}$) such that if complex numbers $\{\lambda_n, \alpha_n\}_{n\geq 1}$ satisfy the condition $\Omega < \delta$, then there exists a unique boundary value problem $L(q(x), T)$ with $q(x) \in L_2(0, T)$, for which the numbers $\{\lambda_n, \alpha_n\}_{n\geq 1}$ are the spectral data, and*

$$\|q - \tilde{q}\|_{L_2(0,T)} < C\Omega, \tag{93}$$

where C depends only on $\tilde{L}$.

Proof Let C denote various constants, which depend only on $\tilde{L}$. Since $\Omega < \infty$, the asymptotical formulae (7) and (19) are fulfilled. Choose $\delta_0 \in (0,1)$ such that if $\Omega < \delta_0$ then $\alpha_n \neq 0$, $n \in \mathbb{S}$. According to (52), we have $\|\tilde{H}(x)\| \le C\Omega$. Choose $\delta \le \delta_0$ such that if $\Omega < \delta$, then $\|\tilde{H}(x)\| \le 1/2$ for $x \in [0,T]$. In this case, there exists $(I - \tilde{H}(x))^{-1}$. Thus, all conditions of Theorem 6 are fulfilled, and hence there exists a unique $q(x) \in L_2(0,T)$, such that the numbers $\{\lambda_n, \alpha_n\}_{n\ge1}$ are the spectral data of $L(q(x),T)$. Moreover, (41) and (69) are valid. Using (57), one can get (93). □

Theorem 8 gives the stability of Inverse Problem 1. Denote

$$\Omega' := \left(\sum_{n=1}^{\infty}\left(n\xi_n'\right)^2\right)^{\frac{1}{2}},$$

where the numbers ξ_n', $n \ge 1$ are determined by the formulae

$$\xi_{k+\nu}' := \frac{|\lambda_k - \tilde{\lambda}_k|}{k} + k^2 \sum_{p=0}^{m_k-1-\nu} |\alpha_{k+p} - \tilde{\alpha}_{k+p}|$$

for $k \in \mathbb{S} \cap \tilde{\mathbb{S}}$, $m_k = \tilde{m}_k$, $\nu = \overline{0, m_k - 1}$, and $\xi_n' := 1$ for other n. According to (7), (19) and (28), we have $C_1\Omega \le \Omega' \le C_2\Omega$, $C_1, C_2 > 0$, and hence (93) is equivalent to the estimate

$$\|q - \tilde{q}\|_{L_2(0,T)} < C\Omega'.$$

Similarly to [2], one can obtain the stability of the solution in the uniform norm and also the necessary and sufficient conditions of the solvability for the inverse problem, when $q(x)$ is in $W_2^N[0,T]$ or in $L_1(0,T)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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