Planning Progressive Type-I Interval Censoring Life Tests With Competing Risks

Shuo-Jye Wu, *Associate Member, IEEE*, and Syuan-Rong Huang

***Abstract*—In this article, we investigate some reliability and quality problems when the competing risks data are progressive type-I interval censored with binomial removals. The failure times of the individual causes are assumed to be statistically independent and exponentially distributed with different parameters. We obtain the estimates of the unknown parameters through a maximum likelihood method, and also derive the Fisher's information matrix. The optimal lengths of the inspection intervals are determined under two different criteria. The reliability sampling plans are established under given producer's and customer's risks. A Monte Carlo simulation is conducted to evaluate the performance of the estimators, and also some numerical results are presented.**

***Index Terms*—D-optimality, exponential distribution, maximum likelihood method, multiple failure modes, reliability sampling plan, variance-optimality.**

ABBREVIATION

MLE	maximum likelihood estimate(or)
TET	total experimental time

NOTATION

X_i	failure time of the i-th unit, $i = 1, \ldots, n$
X_{ij}	lifetime of the i-th unit under risk j, $j = 1, \ldots, s$
$f_j(\cdot)$	probability density function of X_{ij}
$f(x_i, j)$	joint probability density function of risk j and failure time of the i-th unit
$F(x_i, j)$	joint cumulative distribution function of risk j and failure time of the i-th unit
$F(\cdot)$	cumulative distribution function of X_i
λ_j	parameter of the exponential distribution under risk j, $j = 1, \ldots, s$
λ^*	total of hazard rates for all risks
θ	mean of the exponential distribution
n	number of test units
s	number of risks
k	number of inspections
τ_i	the i-th inspection time
n_{ij}	number of failures at the i-th stage due to risk j
n_{i+}	total number of failures at the i-th stage
n_{+j}	total number of failures due to risk j
n_{++}	total number of failures observed in a life test
r_i	number of removals at the i-th stage
m_i	number of non-removed surviving units at the beginning of the i-th stage
p_i	probability of a unit to be removed at the i-th stage
q_{ij}	probability of failure in $(\tau_{i-1}, \tau_i]$ due to risk j
q_i	probability of failure in $(\tau_{i-1}, \tau_i]$
$L(\cdot)$	likelihood function
$I(\cdot)$	Fisher's information
α	producer's risk
β	consumer's risk
d	lower specification limit, *i.e.*, the critical point for accepting a lot
z_γ	γ percentile of a standard statistical normal distribution

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S.-J. Wu is with the Department of Statistics, Tamkang University, Tamsui, New Taipei City 25137, Taiwan (e-mail: shuo@stat.tku.edu.tw).

S.-R. Huang is with the Department of Management Sciences, Tamkang University, Tamsui, New Taipei City 25137, Taiwan (e-mail: oklaok@gmail.com).

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I. INTRODUCTION

A product usually consists of many different components with various risk factors so that a product may fail due to one of several causes, called failure modes or competing risks. In certain applications, product lifetime is defined to be the earliest occurrence among all these risks. Nelson [25, Chapter 7] enumerated engineering situations when a product fails because of two or more risks. For instance, fatigue specimens of a certain sintered super-alloy can fail from a surface defect or an interior one. In ball bearing assemblies, a ball or the race can fail. A cylindrical fatigue specimen can fail in the cylindrical portion, in the fillet (or radius), or in the grip. A semiconductor device can fail at a junction or at a lead. Some other situations in engineering when competing risks occurred can be found in Kim and Bai [18], and Craiu and Lee [11].

In reliability analysis, ignoring the information on causes of failure may result in incorrect inference when improving the reliability of the products. Thus, the data for these competing risks models consist of the failure time, and an indicator variable denoting the specific cause of failure of the product. Cox [10] proposed the latent failure model to analyze the data with multiple failure modes. The cause of failure may be assumed to be statistically independent, or statistically dependent. In most situations, it is usually assumed that these competing risks are statistically independent. Although the assumption of statistical dependence may be more realistic, there are some identifiability issues in the underlying model. Kalbfleisch and Prentice [17], and Crowder [13] pointed out that, without the information of the covariates, it is not possible using data to test the assumption of statistical independence of failure times. To avoid the problem of model identifiability, we consider such latent failure time model formulations, and suppose the latent failure times are statistically independent. Recently, some studies about competing risks have been proposed by several authors including Balasooriya and Low [5], Bunea and Mazzuchi [7], Balakrishnan and Han [4], Pareek *et al.* [27], Han and Balakrishnan [15], Pascual [28], Hiu and Qiu [21], and Cramer and Schmiedt [12].

In addition to competing risks, the life test is important for manufacturers to evaluate product reliability in the competitive market. Censoring is also very common in life tests. Censoring usually applies when the exact lifetimes are known for only a portion of the products, and the remainder of the lifetimes are known only to exceed certain values, or to be within an interval of time. The most common censoring schemes are type-I censoring, and type-II censoring. One important characteristic of these two censoring schemes is that they do not allow for units to be removed from the test at points other than the final termination point. The allowance of removing surviving units from the test before the final termination point is desirable, as in the case of studies of wear in which the study of the actual aging process requires units to be fully disassembled at different stages of the experiment. When a compromise between the reduced time of experimentation and the observation of at least some extreme lifetimes is sought, such an allowance is also desirable. Therefore, a generalization of conventional censoring called progressive censoring has arisen. An account of progressive censoring can be found in the monograph by Balakrishnan and Aggarwala [3], or in the review article by Balakrishnan [2].

In practice, it is often impossible to continuously observe the testing process, even with censoring. The test units might be able to be inspected intermittently. That is, we can only count the number of failures in a time interval instead of measuring failure time exactly. Such a life test is called interval censoring, and data collected through this test are called grouped data. The issues regarding interval censoring have been investigated by several authors such as Cheng and Chen [8], Aggarwala [1], Xiong and Ming [33], and Yang and Tse [34].

One of the purposes of this paper is to explore the choice of the inspection interval length under progressive type-I interval censoring. We propose two selection criteria which enable one to choose the optimal value of the length. One is to minimize the asymptotic variance of the mean lifetime, and the other is to minimize the determinant of the variance-covariance matrix of the estimators of model parameters. Some related research results can be found in, for example, Lui *et al.* [22], Tse *et al.* [30], Wu *et al.* [32], Lin *et al.* [20], and Wu and Huang [31].

The second purpose of this paper is to study the acceptance sampling plan under progressive type-I interval censoring with competing risks. In an acceptance sampling plan, the experimenter determines whether a batch of products could be accepted or not by inspecting some products randomly selected from a batch. If the inspected characteristic of the products is the lifetime, a life test should be performed. Acceptance sampling plans used to determine the acceptability of a batch of products, with respect to their lifetimes, are known as reliability sampling plans. In the literature, reliability sampling plans have been studied by many researchers such as Fernández [14], Jun *et al.* [16], Chung and Seo [9], and Peréz-González and Fernández [29].

The rest of this paper is organized as follows. Some formulae and assumptions about competing risks are described in Section II. The maximum likelihood estimators (MLEs) and the Fisher's information matrix are derived in Section III. The optimal lengths of the inspection interval under two different selection criteria are investigated in Section IV. Given producer's and customer's risks, a reliability sampling plan is proposed in Section V. Some numerical results and an example are presented in Section VI. Extensions are provided in Section VII. Conclusions and discussions are made in Section VIII.

II. Model and Assumptions

Suppose that, in a life test, a test unit may fail in s different modes. We assume that the latent failure times follow an exponential distribution with different parameters λ_j, $j = 1, \ldots, s$. To avoid some identifiability issues, we assume $X_{i1}, \ldots, X_{is}$ are statistically independent. Further, we have $X_i = \min\{X_{i1}, \ldots, X_{is}\}$. Thus,

$$f(x_i, j) = \lambda_j \, e^{-\lambda^* x_i}, \quad x_i > 0,$$
and
$$F(x_i, j) = \frac{\lambda_j}{\lambda^*}\left[1 - e^{-\lambda^* x_i}\right], \quad x_i > 0,$$

where $\lambda^* = \sum_{l=1}^{s} \lambda_l$ is the total of the constant hazard functions for all risks.

Meeker and Escobar [23, p.79] indicated that the exponential distribution is a popular distribution for some kinds of electronic components such as capacitors and high-quality integrated circuits. Pal *et al.* [26, p.152] pointed out that the failure time of electric bulbs, batteries, appliances, and transistors, etc., can often be modeled with an exponential distribution.

Let us consider the competing risks data with progressive type-I interval censoring. Suppose that n units are simultaneously placed on a life test. Units are only inspected at pre-determined times $\tau_1, \tau_2, \ldots, \tau_k$, where $0 < \tau_1 < \tau_2 < \ldots < \tau_k$. At the i-th stage, let n_{ij} be the number of units known to have failed in the interval $(\tau_{i-1}, \tau_i]$ due to risk j, and let r_i be the number of surviving units being withdrawn from the test at time τ_i, for $i = 1, \ldots, k$, and $j = 1, \ldots, s$, where $\tau_0 = 0$. Thus, the observed data from a progressive type-I interval censoring with competing risks are $\{n_{ij}, r_i; i = 1, \ldots, k, j = 1, \ldots, s\}$.

For the given $r_1, \ldots, r_{i-1}$, we have the fact that

$$n_{i1}, \ldots, n_{is} | n_{i-1,1}, \ldots, n_{i-1,s}, \ldots, n_{11}, \ldots, n_{1s}, r_{i-1}, \ldots, r_1 \sim \text{multinomial}(m_i, q_{i1}, \ldots, q_{is}, 1 - q_i),$$
where
$$q_{ij} = \frac{F(\tau_i, j) - F(\tau_{i-1}, j)}{1 - F(\tau_{i-1})} = \frac{\lambda_j}{\lambda^*}\left[1 - e^{-\lambda^*(\tau_i - \tau_{i-1})}\right]$$

is the probability that a unit fails in the time interval $(\tau_{i-1}, \tau_i]$ due to risk j, and

$$q_i = \frac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})} = 1 - e^{-\lambda^*(\tau_i - \tau_{i-1})}$$

is the probability that a unit fails in the time interval $(\tau_{i-1}, \tau_i]$, for $i = 1, \ldots, k$ and $j = 1, \ldots, s$.

There are several different ways to decide the number of removals r_i at each stage. Here we assume that a unit being removed from the life test is statistically independent of the others but with probability p_i at the i-th stage. Then, we have

$$r_i | n_{i1}, \ldots, n_{is}, \ldots, n_{11}, \ldots, n_{1s}, r_{i-1}, \ldots, r_1 \sim \text{binomial}(m_i - n_{i+}, p_i),$$

where $n_{i+} = \sum_{j=1}^{s} n_{ij}$ is the total number of failures at the i-th stage, and $m_i = n - \sum_{l=1}^{i-1} n_{l+} - \sum_{l=1}^{i-1} r_l$ is the number of non-removed surviving units at the beginning of the i-th stage, $i = 1, \ldots, k$.

III. Maximum Likelihood Estimation

Based on the competing risk data from a progressive type-I interval censoring scheme, the likelihood function can be written as the equation shown at the bottom of the page. Thus, the likelihood equation for λ_j is

$$\frac{\partial \ln L(\lambda_1, \ldots, \lambda_s)}{\partial \lambda_j} = \frac{n_{+j}}{\lambda_j} + \sum_{i=1}^{k} \left[n_{i+} \left(\frac{\tau_i - \tau_{i-1}}{q_i} - \frac{1}{\lambda^*} \right) - m_i(\tau_i - \tau_{i-1}) \right] = 0,$$

where $n_{+j} = \sum_{i=1}^{k} n_{ij}$ is the total number of failures due to risk j, $j = 1, \ldots, s$. From the likelihood equations, we determine that the solution to the likelihood equations must satisfy

$$\hat{\lambda}_j = \frac{n_{+j}}{n_{++}} \hat{\lambda}^*, \quad j = 1, 2, \ldots, s, \tag{1}$$

where $n_{++} = \sum_{i=1}^{k} \sum_{j=1}^{s} n_{ij}$ is the total number of failures observed in the life test, and $\hat{\lambda}^* = \sum_{j=1}^{s} \hat{\lambda}_j$. Computing the unique value for $\hat{\lambda}^*$ is defined below.

Plugging (1) back to the likelihood equations, one can get

$$\sum_{i=1}^{k} \frac{n_{i+}}{q_i} (\tau_i - \tau_{i-1}) - \sum_{i=1}^{k} m_i (\tau_i - \tau_{i-1}) = 0. \tag{2}$$

Because the left-hand side of (2) is a function of $\hat{\lambda}^*$, we can write it down as $g(\hat{\lambda}^*)$. It is easy to show that $\lim_{\hat{\lambda}^* \to 0} g(\hat{\lambda}^*) = \infty$, $\lim_{\hat{\lambda}^* \to \infty} g(\hat{\lambda}^*) = \sum_{i=1}^{k} (n_{i+} - m_i)(\tau_i - \tau_{i-1}) < 0$, and $(\partial/\partial\hat{\lambda}^*) g(\hat{\lambda}^*) = -\sum_{i=1}^{k} \left[(n_{i+}/q_i^2)(1-q_i)(\tau_i - \tau_{i-1})^2 \right] < 0$. These results imply that $g(\hat{\lambda}^*)$ is a strictly decreasing function of $\hat{\lambda}^*$, and hence $g(\hat{\lambda}^*) = 0$ has a unique solution of $\hat{\lambda}^*$. Because (2) has a unique solution of $\hat{\lambda}^*$, one can solve this equation first, and then use (1) to get the MLEs $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_s$.

In the special case where the inspection time intervals are of equal length, say $\tau_i - \tau_{i-1} = \tau$, we can find the explicit form for the MLE of λ^* as

$$\hat{\lambda}^* = \frac{\ln\left(\sum_{i=1}^{k} m_i \right) - \ln\left(\sum_{i=1}^{k} m_i - n_{++} \right)}{\tau}. \tag{3}$$

Hence, the MLEs of λ_j, $j = 1, \ldots, s$, can also be obtained in explicit forms.

Under some mild regularity conditions, the asymptotic normality of the MLEs can be easily established. The elements of the Fisher's information matrix are

$$E\left(-\frac{\partial^2 \ln L(\lambda_1, \ldots, \lambda_s)}{\partial \lambda_j \partial \lambda_l} \right) = \begin{cases} \frac{E(n_{+j})}{\lambda_j^2} + \sum\limits_{i=1}^{k} E(n_{i+}) \left[\frac{(\tau_i - \tau_{i-1})^2 (1-q_i)}{q_i^2} - \frac{1}{\lambda^{*2}} \right], & j = l, \\ \sum\limits_{i=1}^{k} E(n_{i+}) \left[\frac{(\tau_i - \tau_{i-1})^2 (1-q_i)}{q_i^2} - \frac{1}{\lambda^{*2}} \right], & j \neq l. \end{cases} \tag{4}$$

To obtain $E(n_{+j})$ and $E(n_{i+})$ in (4), let us compute the expectations of m_i, n_{ij}, and r_i for $i = 1, \ldots, k$ and $j = 1, \ldots, s$. When $i = 1$, we have $m_1 = n$, $n_{11}, \ldots, n_{1s} \sim \text{multinomial}(m_1, q_{11}, \ldots, q_{1s}, 1 - q_1)$, and $r_1 | n_{11}, \ldots, n_{1s} \sim \text{binomial}(m_1 - n_{1+}, p_1)$. Therefore,

$$E(m_1) = n, \; E(n_{1j}) = m_1 q_{1j} = n q_{1j}, \quad j = 1, \ldots, s,$$
$$E(n_{1+}) = n q_1, \text{ and } E(r_1) = E[(m_1 - n_{1+}) p_1] = n(1 - q_1) p_1.$$

$$\begin{aligned} L(\lambda_1, \ldots, \lambda_s) &\propto \prod_{i=1}^{k} f(n_{i1}, \ldots, n_{is} | n_{i-1,1}, \ldots, n_{i-1,s}, \ldots, n_{11}, \ldots, n_{1s}, r_{i-1}, \ldots, r_1) \\ &\quad \times f(r_i | n_{i1}, \ldots, n_{is}, \ldots, n_{11}, \ldots, n_{1s}, r_{i-1}, \ldots, r_1) \\ &\propto \prod_{i=1}^{k} \left(\prod_{j=1}^{s} q_{ij}^{n_{ij}} \right) (1 - q_i)^{m_i - n_{i+}} \\ &= \prod_{i=1}^{k} \prod_{j=1}^{s} \left(\frac{\lambda_j}{\lambda^*} \frac{q_i}{1 - q_i} \right)^{n_{ij}} (1 - q_i)^{m_i}. \end{aligned}$$

When $i=2$, we know that $m_2 = m_1 - n_{1+} - r_1$,

$$n_{21},\ldots,n_{2s}|n_{11},\ldots,n_{1s},r_1 \sim \text{multinomial}(m_2,\ q_{21},\ldots,q_{2s},1-q_2),$$

and $r_2|n_{21},\ldots,n_{2s},n_{11},\ldots,n_{1s},r_1 \sim \text{binomial}(m_2 - n_{2+},p_2)$. Hence,

$$\begin{aligned}E(m_2) &= E(m_1-n_{1+}-r_1) = n(1-q_1)(1-p_1),\\ E(n_{2j}) &= E(m_2)q_{2j} = n(1-q_1)(1-p_1)q_{2j},\quad j=1,\ldots,s,\\ E(n_{2+}) &= n(1-q_1)(1-p_1)q_2,\\ &\text{and}\\ E(r_2) &= E[(m_2-n_{2+})p_2] = n(1-q_1)(1-p_1)(1-q_2)p_2.\end{aligned}$$

It is easy to obtain that, by induction,

$$\begin{aligned}E(m_i) &= n\prod_{l=1}^{i-1}(1-q_l)(1-p_l),\\ E(n_{ij}) &= n\prod_{l=1}^{i-1}[(1-q_l)(1-p_l)]q_{ij},\\ &\text{and}\\ E(r_i) &= n\prod_{l=1}^{i-1}[(1-q_l)(1-p_l)](1-q_i)p_i;\end{aligned}$$

and also that

$$\begin{aligned}E(n_{i+}) &= n\prod_{l=1}^{i-1}[(1-q_l)(1-p_l)]q_i,\\ &\text{and}\\ E(n_{+j}) &= n\sum_{i=1}^{k}\prod_{l=1}^{i-1}[(1-q_l)(1-p_l)]q_{ij},\end{aligned}$$

for $i=1,\ldots,k$, and $j=1,\ldots,s$. Hence, (4) can be written as equation (5) at the bottom of the page.

To implement the life test conveniently, experimenters may choose that the inspection time intervals are of equal length, and the probabilities of removals at each stage are the same (*i.e.*, $\tau_i - \tau_{i-1} = \tau$, and $p_i = p$). The elements of the Fisher's information matrix can be simplified as

$$E\left(-\frac{\partial^2 \ln L(\lambda_1,\ldots,\lambda_s)}{\partial\lambda_j\partial\lambda_l}\right) = \begin{cases} nq\frac{1-[(1-q)(1-p)]^k}{1-(1-q)(1-p)}\left[\frac{\tau^2(1-q)}{q^2}+\frac{\lambda^*-\lambda_j}{\lambda^{*2}\lambda_j}\right], & j=l,\\ nq\frac{1-[(1-q)(1-p)]^k}{1-(1-q)(1-p)}\left[\frac{\tau^2(1-q)}{q^2}-\frac{1}{\lambda^{*2}}\right], & j\neq l,\end{cases}$$

where $q = 1-e^{-\lambda^*\tau}$.

IV. Optimal Inspection Length

In this section, we investigate the optimal lengths of the inspection intervals in a progressive type-I interval censoring with competing risks. For simplicity of discussion, we only consider the special case where $\tau_i - \tau_{i-1} = \tau$, $p_i = p$, and $s=2$ causes of failure in the life test.

From Section III, we know the Fisher's information matrix with two competing risks is

$$I(\lambda_1,\lambda_2) = nq\frac{1-[(1-q)(1-p)]^k}{1-(1-q)(1-p)}\begin{pmatrix}O_{11} & O_{12}\\ O_{21} & O_{22}\end{pmatrix},$$

where

$$O_{11} = \frac{\tau^2(1-q)}{q^2}+\frac{\lambda^*-\lambda_1}{\lambda^{*2}\lambda_1},\ O_{22} = \frac{\tau^2(1-q)}{q^2}+\frac{\lambda^*-\lambda_2}{\lambda^{*2}\lambda_2},$$

and

$$O_{12} = O_{21} = \frac{\tau^2(1-q)}{q^2} - \frac{1}{\lambda^{*2}}.$$

The variance-covariance matrix is the inverse of the Fisher's information matrix,

$$I^{-1}(\lambda_1,\lambda_2) = \frac{1-(1-q)(1-p)}{nq(1-[(1-q)(1-p)]^k)}\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix},$$

where

$$\gamma_{11} = \frac{O_{22}}{O_{11}O_{22}-O_{12}^2},\quad \gamma_{22} = \frac{O_{11}}{O_{11}O_{22}-O_{12}^2},$$

$$\text{and}\quad \gamma_{12} = \frac{-O_{12}}{O_{11}O_{22}-O_{12}^2}.$$

Then, by asymptotic normality of the MLE, one can obtain that $\hat{\lambda}^*$ is asymptotically normally distributed with mean λ^*, and variance V^*/n, where

$$V^* = \frac{1-(1-q)(1-p)}{q(1-[(1-q)(1-p)]^k)}(\gamma_{11}+\gamma_{22}+2\gamma_{12}).$$

Note that the mean lifetime θ is the reciprocal of λ^*. By the invariance property of the MLE and the Delta method, the distribution of the estimator $\hat{\theta} = 1/\hat{\lambda}^*$ of the mean lifetime is approximately normal with mean θ, and variance $V^*/n\lambda^{*4}$.

We study two selection criteria which enable us to determine the optimal value of τ.

A. Variance-Optimality

The mean lifetime is an important characteristic for analyzing the reliability of products. To estimate the mean lifetime of the products more precisely, we need to select the optimal value of the inspection interval which minimizes the asymptotic variance

$$E\left(-\frac{\partial^2 \ln L(\lambda_1,\ldots,\lambda_s)}{\partial\lambda_j\partial\lambda_l}\right) = \begin{cases} n\sum_{i=1}^{k}\prod_{l=1}^{i-1}[(1-q_l)(1-p_l)]\left[\frac{(\tau_i-\tau_{i-1})^2(1-q_i)}{q_i}+\frac{q_i(\lambda^*-\lambda_j)}{\lambda^{*2}\lambda_j}\right], & j=l,\\ n\sum_{i=1}^{k}\prod_{l=1}^{i-1}[(1-q_l)(1-p_l)]\left[\frac{(\tau_i-\tau_{i-1})^2(1-q_i)}{q_i}-\frac{q_i}{\lambda^{*2}}\right], & j\neq l.\end{cases} \quad (5)$$

TABLE I
AVERAGE ESTIMATES AND ESTIMATED RISKS (IN PARENTHESES) OF MLEs, $\lambda_1 = 1, \lambda_2 = 0.5$

			$p = 0.05$		$p = 0.1$		$p = 0.25$	
n	k	τ	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
10	3	0.5	1.0930 (0.2680)	0.5544 (0.1218)	1.0963 (0.2730)	0.5475 (0.1254)	1.1006 (0.2953)	0.5507 (0.1336)
		1.0	1.0649 (0.1747)	0.5389 (0.0937)	1.0638 (0.1724)	0.5337 (0.0918)	1.0575 (0.1741)	0.5231 (0.0903)
		1.5	0.9030 (0.0853)	0.4545 (0.0533)	0.9033 (0.0860)	0.4518 (0.0537)	0.8941 (0.0876)	0.4503 (0.0541)
	5	0.5	1.1252 (0.2653)	0.5593 (0.1255)	1.1164 (0.2654)	0.5582 (0.1247)	1.1081 (0.2789)	0.5570 (0.1280)
		1.0	1.0632 (0.1653)	0.5340 (0.0895)	1.0714 (0.1728)	0.5342 (0.0898)	1.0614 (0.1745)	0.5249 (0.0880)
		1.5	0.9053 (0.0839)	0.4500 (0.0539)	0.9034 (0.0845)	0.4495 (0.0533)	0.8973 (0.0874)	0.4506 (0.0536)
	7	0.5	1.1230 (0.2695)	0.5655 (0.1219)	1.1125 (0.2633)	0.5575 (0.1196)	1.1168 (0.2931)	0.5602 (0.1319)
		1.0	1.0624 (0.1639)	0.5322 (0.0881)	1.0639 (0.1668)	0.5328 (0.0888)	1.0617 (0.1767)	0.5271 (0.0898)
		1.5	0.9041 (0.0855)	0.4496 (0.0529)	0.8999 (0.0850)	0.4529 (0.0527)	0.9019 (0.0863)	0.4515 (0.0530)
15	3	0.5	1.0692 (0.1615)	0.5329 (0.0731)	1.0651 (0.1670)	0.5347 (0.0762)	1.0643 (0.1705)	0.5324 (0.0825)
		1.0	1.0848 (0.1529)	0.5427 (0.0719)	1.0777 (0.1518)	0.5398 (0.0685)	1.0846 (0.1559)	0.5384 (0.0721)
		1.5	0.9818 (0.0814)	0.4930 (0.0455)	0.9869 (0.0795)	0.4914 (0.0460)	0.9784 (0.0826)	0.4882 (0.0450)
	5	0.5	1.0768 (0.1543)	0.5410 (0.0727)	1.0754 (0.1595)	0.5383 (0.0747)	1.0754 (0.1738)	0.5339 (0.0793)
		1.0	1.0860 (0.1515)	0.5392 (0.0673)	1.0821 (0.1528)	0.5413 (0.0695)	1.0774 (0.1519)	0.5406 (0.0742)
		1.5	0.9854 (0.0794)	0.4923 (0.0451)	0.9811 (0.0789)	0.4933 (0.0452)	0.9783 (0.0826)	0.4917 (0.0461)
	7	0.5	1.0799 (0.1566)	0.5384 (0.0698)	1.0836 (0.1624)	0.5378 (0.0725)	1.0761 (0.1735)	0.5406 (0.0801)
		1.0	1.0859 (0.1477)	0.5460 (0.0705)	1.0816 (0.1492)	0.5412 (0.0700)	1.0819 (0.1520)	0.5375 (0.0719)
		1.5	0.9774 (0.0788)	0.4936 (0.0444)	0.9828 (0.0796)	0.4901 (0.0457)	0.9832 (0.0829)	0.4862 (0.0464)
30	3	0.5	1.0319 (0.0667)	0.5168 (0.0330)	1.0337 (0.0670)	0.5144 (0.0330)	1.0299 (0.0749)	0.5138 (0.0358)
		1.0	1.0514 (0.0780)	0.5236 (0.0334)	1.0516 (0.0794)	0.5267 (0.0349)	1.0505 (0.0823)	0.5257 (0.0357)
		1.5	1.0543 (0.0769)	0.5259 (0.0339)	1.0531 (0.0786)	0.5263 (0.0336)	1.0506 (0.0776)	0.5249 (0.0329)
	5	0.5	1.0355 (0.0630)	0.5172 (0.0309)	1.0375 (0.0667)	0.5178 (0.0332)	1.0321 (0.0713)	0.5172 (0.0354)
		1.0	1.0558 (0.0816)	0.5274 (0.0342)	1.0465 (0.0776)	0.5275 (0.0341)	1.0497 (0.0790)	0.5257 (0.0352)
		1.5	1.0483 (0.0772)	0.5223 (0.0338)	1.0457 (0.0757)	0.5229 (0.0337)	1.0524 (0.0789)	0.5278 (0.0344)
	7	0.5	1.0359 (0.0627)	0.5170 (0.0301)	1.0411 (0.0683)	0.5166 (0.0307)	1.0342 (0.0704)	0.5179 (0.0346)
		1.0	1.0546 (0.0780)	0.5227 (0.0338)	1.0547 (0.0807)	0.5253 (0.0357)	1.0522 (0.0809)	0.5294 (0.0360)
		1.5	1.0521 (0.0755)	0.5265 (0.0334)	1.0498 (0.0744)	0.5260 (0.0343)	1.0494 (0.0785)	0.5237 (0.0336)

of the estimator of the mean lifetime. Note that the asymptotic variance of the estimator of the mean lifetime is

$$Var(\hat{\theta}) = \frac{1-(1-q)(1-p)}{n\lambda^{*4}q(1-[(1-q)(1-p)]^k)}(\gamma_{11}+\gamma_{22}+2\gamma_{12}).$$

Thus, the optimal length of the inspection interval τ is determined by minimizing

$$h_V(\tau) = \frac{1-(1-q)(1-p)}{\lambda^{*4}(1-[(1-q)(1-p)]^k)}\frac{q}{\tau^2(1-q)}.$$

B. D-Optimality

It is well known that the determinant $|I^{-1}(\lambda_1,\lambda_2)|$ is proportional to the volume of the asymptotic joint confidence region for $(\lambda_1,\lambda_2)'$ so that minimizing this determinant is equivalent to minimizing the volume of the confidence region. A small value of the determinant of the asymptotic variance-covariance matrix $I^{-1}(\lambda_1,\lambda_2)$ results in a high joint precision of the estimators of λ_1 and λ_2. Therefore, the optimal value of the inspection length τ is obtained by minimizing

$$h_D(\tau)=\left[\frac{1-(1-q)(1-p)}{1-[(1-q)(1-p)]^k}\right]^2\frac{\lambda^{*2}}{\tau^2(1-q)}\left(\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}+2\right)^{-1}.$$

V. SAMPLING PLANS WITH COMPETING RISKS

To illustrate the reliability sampling plans with competing risks, the number of causes of failure is assumed to be $s = 2$. We also suppose that the lengths of the inspection interval are equal (*i.e.*, $\tau_i - \tau_{i-1} = \tau$), and the removal probabilities at each stage are all the same (*i.e.*, $p_i = p$) in this section.

Suppose products whose mean lifetimes are smaller than a lower specification limit, d, are not of acceptable quality to the consumer. From a batch of products, n units are randomly selected, and are tested under a progressive type-I interval censoring life test. Comparing the estimate $\hat{\theta}$ of mean lifetime θ with d, the batch is either accepted if $\hat{\theta} > d$, or rejected if $\hat{\theta} \le d$. Thus, the quality parameter is acceptable if it is greater than θ_0, and a quality parameter less than θ_1 is considered reject-able, where $\theta_0 > \theta_1$. The values of θ_0 and θ_1 are decided by an agreement between the producer and the consumer. Hence, the reliability sampling plan tests the simple hypothesis system $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ with a desired power $1-\beta$ at α-level. Therefore, we have

$$P\left(\hat{\theta} > d \,\middle|\, \theta = \theta_0\right) = 1-\alpha,$$
$$\text{and}$$
$$P\left(\hat{\theta} > d \,\middle|\, \theta = \theta_1\right) = \beta,$$

where α is the producer's risk, and β is the consumer's risk.

Note that θ is the reciprocal of $\lambda^* = \lambda_1 + \lambda_2$. Without loss of generality, we assume that $\lambda_1 \ge \lambda_2$, and $\lambda_2 = c\lambda_1$, where $0 < c \le 1$. Then, we have $\lambda^* = (1+c)\lambda_1$. Now, set $\theta_0 = 1/(1+c)\lambda_{1(0)}$, and $\theta_1 = 1/(1+c)\lambda_{1(1)}$, where $\lambda_{1(0)} \le \lambda_{1(1)}$.

TABLE II
AVERAGE ESTIMATES AND ESTIMATED RISKS (IN PARENTHESES) OF MLEs, $\lambda_1 = 1$, $\lambda_2 = 0.1$

			$p=0.05$		$p=0.1$		$p=0.25$	
n	k	τ	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
10	3	0.5	1.0995 (0.2199)	0.1117 (0.0190)	1.0925 (0.2241)	0.1107 (0.0191)	1.0970 (0.2392)	0.1107 (0.0215)
		1.0	1.1128 (0.1931)	0.1116 (0.0164)	1.1084 (0.1867)	0.1130 (0.0164)	1.1041 (0.1904)	0.1109 (0.0167)
		1.5	1.0338 (0.0982)	0.1033 (0.0124)	1.0356 (0.0973)	0.1041 (0.0124)	1.0311 (0.1016)	0.1044 (0.0125)
	5	0.5	1.1057 (0.1978)	0.1121 (0.0165)	1.0995 (0.2132)	0.1111 (0.0174)	1.0912 (0.2288)	0.1096 (0.0190)
		1.0	1.1125 (0.1826)	0.1119 (0.0160)	1.1222 (0.1899)	0.1112 (0.0153)	1.1062 (0.1891)	0.1079 (0.0155)
		1.5	1.0425 (0.0958)	0.1028 (0.0126)	1.0380 (0.0961)	0.1029 (0.0121)	1.0337 (0.1006)	0.1027 (0.0126)
	7	0.5	1.1080 (0.1970)	0.1118 (0.0162)	1.1112 (0.2074)	0.1123 (0.0174)	1.1133 (0.2385)	0.1116 (0.0190)
		1.0	1.1199 (0.1830)	0.1138 (0.0161)	1.1120 (0.1820)	0.1128 (0.0160)	1.1044 (0.1844)	0.1130 (0.0166)
		1.5	1.0393 (0.0962)	0.1051 (0.0128)	1.0358 (0.0955)	0.1037 (0.0127)	1.0216 (0.0988)	0.1022 (0.0125)
15	3	0.5	1.0575 (0.1236)	0.1048 (0.0107)	1.0609 (0.1267)	0.1066 (0.0115)	1.0586 (0.1362)	0.1068 (0.0126)
		1.0	1.0854 (0.1334)	0.1081 (0.0100)	1.0925 (0.1379)	0.1091 (0.0104)	1.0776 (0.1320)	0.1078 (0.0108)
		1.5	1.0728 (0.1038)	0.1068 (0.0094)	1.0742 (0.1049)	0.1064 (0.0093)	1.0658 (0.1071)	0.1075 (0.0098)
	5	0.5	1.0620 (0.1172)	0.1054 (0.0099)	1.0686 (0.1213)	0.1058 (0.0103)	1.0652 (0.1315)	0.1074 (0.0119)
		1.0	1.0910 (0.1330)	0.1083 (0.0099)	1.0937 (0.1353)	0.1096 (0.0102)	1.0799 (0.1312)	0.1095 (0.0107)
		1.5	1.0745 (0.1028)	0.1073 (0.0090)	1.0745 (0.1048)	0.1063 (0.0092)	1.0695 (0.1069)	0.1075 (0.0095)
	7	0.5	1.0773 (0.1182)	0.1081 (0.0097)	1.0716 (0.1193)	0.1065 (0.0101)	1.0663 (0.1299)	0.1065 (0.0113)
		1.0	1.0949 (0.1316)	0.1096 (0.0100)	1.0852 (0.1339)	0.1084 (0.0101)	1.0832 (0.1338)	0.1082 (0.0102)
		1.5	1.0702 (0.1041)	0.1071 (0.0092)	1.0821 (0.1078)	0.1073 (0.0092)	1.0649 (0.1039)	0.1069 (0.0094)
30	3	0.5	1.0270 (0.0548)	0.1026 (0.0052)	1.0297 (0.0550)	0.1027 (0.0052)	1.0258 (0.0609)	0.1040 (0.0058)
		1.0	1.0417 (0.0549)	0.1043 (0.0044)	1.0381 (0.0527)	0.1043 (0.0046)	1.0403 (0.0593)	0.1043 (0.0047)
		1.5	1.0586 (0.0665)	0.1060 (0.0045)	1.0535 (0.0668)	0.1052 (0.0045)	1.0513 (0.0651)	0.1048 (0.0046)
	5	0.5	1.0307 (0.0490)	0.1035 (0.0044)	1.0329 (0.0501)	0.1040 (0.0048)	1.0313 (0.0573)	0.1047 (0.0056)
		1.0	1.0403 (0.0521)	0.1039 (0.0042)	1.0458 (0.0562)	0.1050 (0.0044)	1.0367 (0.0555)	0.1045 (0.0048)
		1.5	1.0540 (0.0647)	0.1064 (0.0045)	1.0513 (0.0640)	0.1063 (0.0046)	1.0529 (0.0657)	0.1039 (0.0045)
	7	0.5	1.0334 (0.0463)	0.1023 (0.0043)	1.0341 (0.0496)	0.1043 (0.0047)	1.0326 (0.0569)	0.1039 (0.0053)
		1.0	1.0382 (0.0510)	0.1046 (0.0043)	1.0415 (0.0523)	0.1048 (0.0044)	1.0440 (0.0571)	0.1040 (0.0048)
		1.5	1.0552 (0.0656)	0.1059 (0.0045)	1.0543 (0.0649)	0.1051 (0.0044)	1.0525 (0.0662)	0.1043 (0.0045)

From the asymptotic normality of the MLE $\hat{\theta}$, we can obtain

$$n = \left[\frac{(1+c)(z_\beta\sqrt{V_1} - z_{1-\alpha}\sqrt{V_0})}{\lambda_{1(0)}^{-1} - \lambda_{1(1)}^{-1}} \right]^2, \tag{6}$$

and

$$d = \frac{\frac{z_{1-\alpha}\sqrt{V_0}}{\lambda_{1(1)}} - \frac{z_\beta\sqrt{V_1}}{\lambda_{1(0)}}}{(1+c)(z_{1-\alpha}\sqrt{V_0} - z_\beta\sqrt{V_1})}, \tag{7}$$

where V_0, and V_1 are $nVar(\hat{\theta})$ evaluated at θ_0, and θ_1, respectively. Here z_γ is the $1-\gamma$ percentile of a standard normal distribution.

VI. NUMERICAL RESULTS

In this section, some numerical results are presented. We first conduct a Monte Carlo simulation to study the performance of the MLEs in terms of the average estimates and the estimated risks in Section VI-A. Secondly, a numerical study for investigating the optimal length of the inspection interval is presented in Section VI-B. Thirdly, in Section VI-C, some tables are provided for illustrating the use of the reliability sampling plan with competing risks. Finally, in Section VI-D, we apply the proposed methods to a real data set presented in Boardman and Kendell [6].

A. Performance of Parameter Estimates

Here, we present a Monte Carlo simulation to evaluate the performance of the parameter estimations. We only consider the case of $s = 2$, which is two causes of failure in a life test. We set $(\lambda_1, \lambda_2) = (1, 0.5), (1, 0.1), (1, 0.01)$, $n = 10, 15, 30$, $k = 3, 5, 7$, $\tau = 0.5, 1.0, 1.5$, and $p = 0.05, 0.1, 0.25$. Under these settings, the average estimates, and the estimated risks of the MLEs based on 10000 simulated samples are listed in Tables I–III. The tables show that the average estimates of the MLEs are all close to the true parameters for different combinations of n, k, τ, and p. Even though the sample size n is small, the MLEs still perform well in terms of the average estimates. Large values of n or k lead to small values of the estimated risks of the MLEs. As the proportion of removals p increases, the estimated risks of the MLEs become a little bit large, especially for the small sample size.

B. Optimal Plans

To obtain the optimal value of τ, we need to take the first derivative of $h_V(\tau)$ or $h_D(\tau)$ with respect to τ. Then, the optimal value for these criteria must satisfy

$$\frac{d}{d\tau}h_V(\tau) = \frac{(1-q)(1-p)}{1-(1-q)(1-p)} - \frac{k[(1-q)(1-p)]^k}{1-[(1-q)(1-p)]^k} - \frac{2}{\tau\lambda^*} + \frac{1}{q} = 0,$$

or

$$\frac{d}{d\tau}h_D(\tau) = \frac{(1-q)(1-p)}{1-(1-q)(1-p)} - \frac{k[(1-q)(1-p)]^k}{1-[(1-q)(1-p)]^k} - \frac{1}{\tau\lambda^*} + \frac{1}{2} = 0.$$

The optimal values cannot be solved to explicit forms. However, the equations for solving τ are all functions of $\lambda^*\tau$. Therefore,

TABLE III
AVERAGE ESTIMATES AND ESTIMATED RISKS (IN PARENTHESES) OF MLES, $\lambda_1 = 1, \lambda_2 = 0.01$

			$p = 0.05$		$p = 0.1$		$p = 0.25$	
n	k	τ	λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
10	3	0.5	1.0984 (0.2065)	0.0109 (0.0016)	1.0857 (0.2093)	0.0111 (0.0017)	1.0941 (0.2255)	0.0111 (0.0020)
		1.0	1.1154 (0.1859)	0.0107 (0.0014)	1.1152 (0.1933)	0.0116 (0.0016)	1.1106 (0.1996)	0.0111 (0.0015)
		1.5	1.0581 (0.1069)	0.0102 (0.0012)	1.0568 (0.1070)	0.0102 (0.0012)	1.0532 (0.1115)	0.0111 (0.0014)
	5	0.5	1.1049 (0.1888)	0.0118 (0.0016)	1.0941 (0.1940)	0.0110 (0.0016)	1.1071 (0.2206)	0.0115 (0.0020)
		1.0	1.1205 (0.1812)	0.0107 (0.0014)	1.1205 (0.1822)	0.0115 (0.0015)	1.1156 (0.1915)	0.0111 (0.0015)
		1.5	1.0641 (0.1058)	0.0106 (0.0012)	1.0603 (0.1051)	0.0103 (0.0012)	1.0563 (0.1090)	0.0104 (0.0012)
	7	0.5	1.1162 (0.1967)	0.0105 (0.0013)	1.1077 (0.1929)	0.0115 (0.0016)	1.1068 (0.2149)	0.0107 (0.0017)
		1.0	1.1242 (0.1787)	0.0116 (0.0016)	1.1313 (0.1908)	0.0108 (0.0014)	1.1181 (0.1868)	0.0109 (0.0015)
		1.5	1.0667 (0.1052)	0.0107 (0.0013)	1.0652 (0.1039)	0.0108 (0.0013)	1.0529 (0.1082)	0.0106 (0.0013)
15	3	0.5	1.0532 (0.1138)	0.0108 (0.0010)	1.0536 (0.1245)	0.0107 (0.0011)	1.0693 (0.1395)	0.0104 (0.0012)
		1.0	1.0782 (0.1207)	0.0101 (0.0008)	1.0725 (0.1182)	0.0109 (0.0009)	1.0771 (0.1309)	0.0102 (0.0009)
		1.5	1.0811 (0.1062)	0.0106 (0.0008)	1.0793 (0.1090)	0.0112 (0.0009)	1.0715 (0.1106)	0.0106 (0.0009)
	5	0.5	1.0650 (0.1089)	0.0102 (0.0008)	1.0671 (0.1123)	0.0113 (0.0010)	1.0656 (0.1239)	0.0105 (0.0011)
		1.0	1.0863 (0.1145)	0.0103 (0.0008)	1.0856 (0.1226)	0.0102 (0.0008)	1.0851 (0.1265)	0.0110 (0.0009)
		1.5	1.0866 (0.1101)	0.0104 (0.0008)	1.0811 (0.1100)	0.0104 (0.0008)	1.0749 (0.1091)	0.0109 (0.0009)
	7	0.5	1.0678 (0.1036)	0.0108 (0.0009)	1.0701 (0.1113)	0.0106 (0.0009)	1.0697 (0.1267)	0.0109 (0.0011)
		1.0	1.0905 (0.1214)	0.0109 (0.0009)	1.0863 (0.1188)	0.0108 (0.0009)	1.0824 (0.1244)	0.0110 (0.0010)
		1.5	1.0818 (0.1051)	0.0106 (0.0009)	1.0836 (0.1073)	0.0109 (0.0009)	1.0772 (0.1093)	0.0104 (0.0008)
30	3	0.5	1.0269 (0.0518)	0.0103 (0.0005)	1.0289 (0.0521)	0.0108 (0.0005)	1.0252 (0.0588)	0.0106 (0.0006)
		1.0	1.0373 (0.0493)	0.0103 (0.0004)	1.0345 (0.0502)	0.0105 (0.0004)	1.0401 (0.0535)	0.0104 (0.0004)
		1.5	1.0576 (0.0609)	0.0106 (0.0004)	1.0510 (0.0588)	0.0104 (0.0004)	1.0502 (0.0611)	0.0107 (0.0004)
	5	0.5	1.0278 (0.0462)	0.0102 (0.0004)	1.0287 (0.0476)	0.0104 (0.0004)	1.0292 (0.0561)	0.0109 (0.0005)
		1.0	1.0363 (0.0456)	0.0106 (0.0004)	1.0424 (0.0489)	0.0107 (0.0004)	1.0377 (0.0520)	0.0104 (0.0004)
		1.5	1.0559 (0.0596)	0.0105 (0.0004)	1.0523 (0.0597)	0.0102 (0.0004)	1.0481 (0.0612)	0.0105 (0.0004)
	7	0.5	1.0358 (0.0446)	0.0102 (0.0004)	1.0345 (0.0445)	0.0103 (0.0004)	1.0348 (0.0539)	0.0102 (0.0005)
		1.0	1.0393 (0.0472)	0.0105 (0.0004)	1.0425 (0.0504)	0.0106 (0.0004)	1.0392 (0.0513)	0.0104 (0.0004)
		1.5	1.0544 (0.0578)	0.0104 (0.0004)	1.0533 (0.0589)	0.0108 (0.0004)	1.0533 (0.0615)	0.0105 (0.0004)

TABLE IV
OPTIMAL LENGTH OF THE INSPECTION INTERVAL WITH $\lambda^* = 1$

	$p = 0.05$		$p = 0.1$		$p = 0.25$	
k	τ_V	τ_D	τ_V	τ_D	τ_V	τ_D
2	1.2225	1.4732	1.2383	1.4981	1.2890	1.5768
3	1.0222	1.2128	1.0557	1.2650	1.1593	1.4228
4	0.8983	1.0608	0.9505	1.1422	1.1040	1.3691
5	0.8162	0.9658	0.8880	1.0770	1.0825	1.3530
6	0.7601	0.9055	0.8515	1.0445	1.0750	1.3488
7	0.7216	0.8679	0.8310	1.0297	1.0725	1.3478
8	0.6955	0.8456	0.8202	1.0236	1.0718	1.3476
9	0.6782	0.8331	0.8149	1.0211	1.0715	1.3476
10	0.6673	0.8266	0.8123	1.0202	1.0715	1.3475

one can set $\lambda^* = 1$, and then obtain the optimal value of τ for the given k and p. The optimal values of τ for the other values of λ^* can be computed as $\tilde{\tau}/\lambda^*$, where $\tilde{\tau}$ is the optimal value of τ with $\lambda^* = 1$.

Let $\tilde{\tau}_V$, and $\tilde{\tau}_D$ be the optimal lengths of the inspection intervals according to the variance-optimality, and D-optimality, respectively. Table IV presents the results for $k = 2(1)10$, and $p = 0.05, 0.1, 0.25$ when $\lambda^* = 1$. The findings are as follows. The optimal lengths under variance-optimality are always smaller than those under D-optimality. For both criteria, a large number of stages k results in a short length of the inspection interval. A larger removal probability p makes a longer optimal length.

C. Reliability Sampling Plan

To illustrate the use of the proposed sampling plan, we perform some numerical studies. Consider $(\alpha, \beta) = (0.05, 0.1)$, $\lambda_{1(0)} = 0.1, 0.5, 0.9$, $\xi = 0.5, 1.0, 1.5, 2.0, 3.0$, $k = 2(1)10$, $c = 0.2, 0.5, 0.8$, $p = 0.05, 0.1, 0.25$, and $\tau = 1$, where ξ satisfies $\lambda_{1(1)} = \lambda_{1(0)} + \lambda_{1(0)}\xi$. The number of test units n and the critical point d are listed in Tables V – VII.

From these tables, we find the following results. As the number of stages k increases, the number of test units n becomes fewer, and the value of the critical point d becomes large. However, the influence of the number of stages k becomes weak when the value of $\lambda_{1(0)}$ is getting large. If either the value of ξ, $\lambda_{1(0)}$, or c increases, the number of test units n , and the critical point d become small. A larger removal probability p leads to more test units in a life test; however, it has a slight effect on the critical point d.

In addition to the above observed phenomena, we can find another interesting result. Under the assumptions that $\lambda_2 = c\lambda_1$, and $\lambda_{1(1)} = (1+\xi)\lambda_{1(0)}$, (6), and (7) can be written as

$$n = \left[\left(\frac{1}{\lambda_{1(0)}\tau}\right)\left(\frac{\left(\frac{z_\beta\sqrt{W_1}}{(1+\xi)^2} - z_{1-\alpha}\sqrt{W_0}\right)}{\left(\frac{(1+c)\xi}{(1+\xi)}\right)}\right)\right]^2,$$

and

$$d = \left(\frac{1}{\lambda_{1(0)}}\right)\left[\frac{\left(\frac{z_{1-\alpha}\sqrt{W_0}}{(1+\xi)} - \frac{z_\beta\sqrt{W_1}}{(1+\xi)^2}\right)}{\left((1+c)\left(z_{1-\alpha}\sqrt{W_0} - \frac{z_\beta\sqrt{W_1}}{(1+\xi)^2}\right)\right)}\right],$$

TABLE V
NUMBER OF TEST UNITS n AND CRITICAL POINT d FOR $c = 0.2$

		$\lambda_{1(0)} = 0.1$						$\lambda_{1(0)} = 0.5$						$\lambda_{1(0)} = 0.9$					
		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$	
ξ	k	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d
0.5	2	242	6.3997	248	6.3994	267	6.3986	81	1.2921	82	1.2919	87	1.2913	72	0.7239	73	0.7237	76	0.7232
	3	176	6.4070	184	6.4064	213	6.4044	70	1.2960	72	1.2955	79	1.2940	68	0.7255	69	0.7252	73	0.7243
	4	144	6.4139	154	6.4127	188	6.4090	66	1.2984	68	1.2976	76	1.2954	67	0.7261	68	0.7257	72	0.7246
	5	125	6.4201	136	6.4183	174	6.4128	64	1.2999	67	1.2989	75	1.2961	66	0.7263	68	0.7259	72	0.7247
	6	113	6.4259	125	6.4234	166	6.4157	63	1.3007	66	1.2996	75	1.2964	66	0.7264	68	0.7260	72	0.7248
	7	104	6.4312	117	6.4278	162	6.4179	63	1.3012	66	1.2999	74	1.2966	66	0.7264	68	0.7260	72	0.7248
	8	98	6.4360	112	6.4317	159	6.4196	62	1.3015	65	1.3001	74	1.2966	66	0.7265	68	0.7260	72	0.7248
	9	93	6.4403	108	6.4350	157	6.4208	62	1.3016	65	1.3002	74	1.2966	66	0.7265	68	0.7260	72	0.7248
	10	90	6.4443	105	6.4380	156	6.4218	62	1.3017	65	1.3003	74	1.2967	66	0.7265	68	0.7260	72	0.7248
1.0	2	87	5.1077	89	5.1072	96	5.1052	30	1.0521	31	1.0515	32	1.0499	28	0.6004	28	0.6001	29	0.5989
	3	64	5.1255	67	5.1240	77	5.1190	27	1.0604	27	1.0593	30	1.0558	27	0.6036	27	0.6030	28	0.6010
	4	52	5.1420	56	5.1391	68	5.1301	25	1.0654	26	1.0636	29	1.0586	26	0.6047	27	0.6039	28	0.6016
	5	46	5.1570	50	5.1525	63	5.1388	24	1.0681	25	1.0659	28	1.0598	26	0.6050	27	0.6042	28	0.6017
	6	41	5.1706	46	5.1642	61	5.1454	24	1.0696	25	1.0671	28	1.0604	26	0.6052	26	0.6043	28	0.6018
	7	38	5.1829	43	5.1744	59	5.1504	24	1.0704	25	1.0677	28	1.0606	26	0.6052	26	0.6043	28	0.6018
	8	36	5.1939	41	5.1832	58	5.1542	24	1.0708	25	1.0680	28	1.0607	26	0.6052	26	0.6043	28	0.6018
	9	35	5.2038	40	5.1908	57	5.1569	24	1.0710	25	1.0682	28	1.0607	26	0.6052	26	0.6043	28	0.6018
	10	33	5.2125	39	5.1972	57	5.1588	24	1.0711	25	1.0682	28	1.0607	26	0.6052	26	0.6043	28	0.6018
1.5	2	54	4.2173	55	4.2164	59	4.2135	19	0.8899	20	0.8891	20	0.8867	18	0.5208	19	0.5203	19	0.5186
	3	39	4.2433	41	4.2410	47	4.2335	17	0.9012	17	0.8995	19	0.8945	17	0.5248	18	0.5239	18	0.5212
	4	32	4.2671	35	4.2628	42	4.2494	16	0.9074	16	0.9050	18	0.8980	17	0.5261	17	0.5250	18	0.5219
	5	28	4.2887	31	4.2820	39	4.2616	15	0.9107	16	0.9077	18	0.8994	17	0.5265	17	0.5253	18	0.5221
	6	26	4.3082	28	4.2986	37	4.2709	15	0.9124	16	0.9091	18	0.9001	17	0.5266	17	0.5254	18	0.5221
	7	24	4.3255	27	4.3129	36	4.2777	15	0.9133	16	0.9098	18	0.9003	17	0.5267	17	0.5255	18	0.5222
	8	23	4.3409	26	4.3250	36	4.2827	15	0.9138	16	0.9101	18	0.9004	17	0.5267	17	0.5255	18	0.5222
	9	22	4.3544	25	4.3353	35	4.2862	15	0.9141	16	0.9103	18	0.9005	17	0.5267	17	0.5255	18	0.5222
	10	21	4.3662	24	4.3438	35	4.2887	15	0.9142	16	0.9104	18	0.9005	17	0.5267	17	0.5255	18	0.5222
2.0	2	40	3.5769	41	3.5758	45	3.5722	15	0.7749	15	0.7740	16	0.7709	14	0.4677	15	0.4671	15	0.4651
	3	30	3.6087	31	3.6058	36	3.5965	13	0.7878	13	0.7858	14	0.7797	14	0.4722	14	0.4711	14	0.4680
	4	24	3.6376	26	3.6323	32	3.6155	12	0.7946	13	0.7917	14	0.7835	13	0.4736	14	0.4723	14	0.4687
	5	21	3.6637	23	3.6553	29	3.6300	12	0.7981	12	0.7946	14	0.7850	13	0.4740	14	0.4727	14	0.4689
	6	19	3.6869	21	3.6750	28	3.6408	12	0.7999	12	0.7960	14	0.7856	13	0.4742	14	0.4728	14	0.4690
	7	18	3.7074	20	3.6917	27	3.6486	12	0.8008	12	0.7967	14	0.7859	13	0.4742	14	0.4728	14	0.4690
	8	17	3.7252	19	3.7056	27	3.6541	12	0.8013	12	0.7971	14	0.7860	13	0.4742	14	0.4728	14	0.4690
	9	16	3.7408	19	3.7172	27	3.6580	12	0.8015	12	0.7973	14	0.7861	13	0.4742	14	0.4729	14	0.4690
	10	16	3.7542	18	3.7268	26	3.6607	11	0.8017	12	0.7973	14	0.7861	13	0.4742	14	0.4729	14	0.4690
3.0	2	29	2.7295	30	2.7282	32	2.7237	11	0.6261	11	0.6248	12	0.6211	11	0.4091	12	0.4083	12	0.4058
	3	22	2.7678	23	2.7642	26	2.7525	10	0.6402	10	0.6377	11	0.6305	11	0.4142	11	0.4129	11	0.4091
	4	18	2.8023	19	2.7955	23	2.7746	9	0.6472	9	0.6439	10	0.6344	11	0.4157	11	0.4143	11	0.4100
	5	16	2.8328	17	2.8221	21	2.7910	9	0.6507	9	0.6468	10	0.6359	11	0.4162	11	0.4147	11	0.4102
	6	14	2.8593	16	2.8444	20	2.8027	9	0.6525	9	0.6482	10	0.6366	11	0.4164	11	0.4148	11	0.4102
	7	13	2.8822	15	2.8628	20	2.8110	9	0.6535	9	0.6489	10	0.6368	11	0.4165	11	0.4148	11	0.4102
	8	12	2.9017	14	2.8779	20	2.8168	9	0.6539	9	0.6493	10	0.6369	11	0.4165	11	0.4148	11	0.4102
	9	12	2.9183	14	2.8901	19	2.8207	9	0.6542	9	0.6495	10	0.6370	11	0.4165	11	0.4148	11	0.4102
	10	11	2.9323	13	2.8999	19	2.8233	9	0.6543	9	0.6495	10	0.6370	11	0.4165	11	0.4148	11	0.4102

where W_0, and W_1 are the values of $W = [(1-(1-q)(1-p))/(1-((1-q)(1-p))^k)](q/(1-q))$ evaluated at $\lambda_{1(0)}$, and $\lambda_{1(1)}$, respectively. One can see that, when c, ξ, k, and p are fixed, the value of n remains unchanged for any pair of $(\lambda_{1(0)}, \tau)$ such that $\lambda_{1(0)}\tau$ is a constant. However, the value of d will be multiplied by the ratio between two values of $\lambda_{1(0)}$. For example, from Table V, when $c = 0.2$, $\xi = 1.0$, $k = 5$, and $p = 0.05$, we have $n = 46$, and $d = 5.1570$ for $(\lambda_{1(0)}, \tau) = (0.1, 1)$. If we choose $(\lambda_{1(0)}, \tau) = (0.01, 10)$, then the value of n is still equal to 46 because $\lambda_{1(0)}\tau = 0.1$, and the value of d is 51.57 because $0.1/0.01 = 10$.

Note that, if the test units have a large mean lifetime, then one needs a longer experimental time to get enough information. However, in practice, there may be a limitation on total experimental time (TET). Thus, one might be interested in the influence of TET on n and d. Table VIII lists the values of n and d under the limitations that TETs are 1, 5, and 10. Because $\theta_0 = 1/(1+c)\lambda_{1(0)}$, it means that a small $\lambda_{1(0)}$ implies a large mean lifetime. From Table VIII, we can observe the following results. If TET is small, the number of test units n is increasing in the number of inspections k. One needs a large number of test units to conduct the reliability sampling plan when TET is short but the mean lifetime is long. If TET is medium, the relation between n and k is from increasing to decreasing when the mean lifetime is from large to small. If TET is large, and the mean lifetime is small, n becomes very large as k is small. In

TABLE VI
NUMBER OF TEST UNITS n AND CRITICAL POINT d FOR $c = 0.5$

		$\lambda_{1(0)} = 0.1$						$\lambda_{1(0)} = 0.5$						$\lambda_{1(0)} = 0.9$					
		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$	
ξ	k	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d
0.5	2	200	5.1230	205	5.1228	221	5.1220	75	1.0365	76	1.0363	80	1.0357	74	0.5817	75	0.5816	77	0.5812
	3	148	5.1301	155	5.1295	178	5.1275	68	1.0395	69	1.0390	74	1.0378	72	0.5827	73	0.5825	75	0.5818
	4	122	5.1366	130	5.1355	158	5.1319	65	1.0411	67	1.0405	73	1.0387	71	0.5830	72	0.5827	75	0.5820
	5	107	5.1425	117	5.1407	148	5.1353	64	1.0419	66	1.0412	72	1.0391	71	0.5830	72	0.5828	75	0.5820
	6	98	5.1478	108	5.1453	142	5.1379	63	1.0423	65	1.0415	72	1.0392	71	0.5831	72	0.5828	75	0.5820
	7	92	5.1525	102	5.1492	138	5.1398	63	1.0425	65	1.0417	72	1.0393	71	0.5831	72	0.5828	75	0.5820
	8	87	5.1567	98	5.1525	136	5.1412	63	1.0426	65	1.0417	72	1.0393	71	0.5831	72	0.5828	75	0.5820
	9	84	5.1604	95	5.1553	135	5.1422	63	1.0427	65	1.0417	72	1.0393	71	0.5831	72	0.5828	75	0.5820
	10	81	5.1636	93	5.1577	134	5.1429	63	1.0427	65	1.0418	72	1.0393	71	0.5831	72	0.5828	75	0.5820
1.0	2	72	4.0943	74	4.0937	80	4.0918	29	0.8488	29	0.8484	30	0.8469	30	0.4879	30	0.4876	31	0.4867
	3	54	4.1114	56	4.1099	64	4.1049	26	0.8551	26	0.8541	28	0.8512	29	0.4896	29	0.4892	30	0.4878
	4	45	4.1269	48	4.1241	57	4.1153	25	0.8582	26	0.8568	28	0.8529	29	0.4901	29	0.4896	30	0.4881
	5	39	4.1408	43	4.1364	54	4.1231	24	0.8597	25	0.8581	27	0.8535	28	0.4902	29	0.4897	30	0.4881
	6	36	4.1531	40	4.1469	52	4.1289	24	0.8604	25	0.8586	27	0.8538	28	0.4902	29	0.4897	30	0.4881
	7	34	4.1638	38	4.1557	51	4.1331	24	0.8607	25	0.8588	27	0.8538	28	0.4902	29	0.4897	30	0.4881
	8	32	4.1732	36	4.1631	50	4.1361	24	0.8608	25	0.8589	27	0.8539	28	0.4902	29	0.4897	30	0.4881
	9	31	4.1813	35	4.1692	49	4.1382	24	0.8609	25	0.8590	27	0.8539	28	0.4902	29	0.4897	30	0.4881
	10	30	4.1883	35	4.1742	49	4.1397	24	0.8609	25	0.8590	27	0.8539	28	0.4902	29	0.4897	30	0.4881
1.5	2	45	3.3858	46	3.3849	49	3.3821	18	0.7234	18	0.7227	19	0.7205	20	0.4300	20	0.4296	20	0.4284
	3	33	3.4108	35	3.4085	40	3.4011	16	0.7316	17	0.7302	18	0.7261	19	0.4322	19	0.4316	20	0.4298
	4	28	3.4332	30	3.4290	36	3.4158	16	0.7354	16	0.7336	18	0.7281	19	0.4328	19	0.4321	20	0.4301
	5	25	3.4530	27	3.4464	33	3.4267	16	0.7372	16	0.7350	17	0.7289	19	0.4329	19	0.4322	20	0.4301
	6	23	3.4703	25	3.4611	32	3.4346	15	0.7379	16	0.7356	17	0.7292	19	0.4329	19	0.4322	20	0.4301
	7	21	3.4852	23	3.4732	31	3.4403	15	0.7383	16	0.7358	17	0.7293	19	0.4329	19	0.4322	20	0.4301
	8	20	3.4980	23	3.4831	31	3.4442	15	0.7384	16	0.7360	17	0.7293	19	0.4329	19	0.4322	20	0.4301
	9	19	3.5088	22	3.4912	31	3.4468	15	0.7385	16	0.7360	17	0.7293	19	0.4329	19	0.4322	20	0.4301
	10	19	3.5179	22	3.4976	30	3.4487	15	0.7385	16	0.7360	17	0.7293	19	0.4329	19	0.4322	20	0.4301
2.0	2	34	2.8764	34	2.8753	37	2.8718	14	0.6355	14	0.6347	15	0.6320	16	0.3944	16	0.3939	17	0.3924
	3	25	2.9070	26	2.9041	30	2.8948	13	0.6448	13	0.6431	14	0.6382	16	0.3969	16	0.3961	16	0.3940
	4	21	2.9340	22	2.9287	27	2.9123	12	0.6489	13	0.6467	14	0.6404	15	0.3974	16	0.3966	16	0.3943
	5	19	2.9576	20	2.9494	25	2.9250	12	0.6507	12	0.6482	13	0.6412	15	0.3976	16	0.3968	16	0.3944
	6	17	2.9779	19	2.9665	24	2.9341	12	0.6515	12	0.6488	13	0.6414	15	0.3976	16	0.3968	16	0.3944
	7	16	2.9952	18	2.9804	24	2.9404	12	0.6518	12	0.6491	13	0.6415	15	0.3976	16	0.3968	16	0.3944
	8	15	3.0098	17	2.9916	23	2.9447	12	0.6520	12	0.6492	13	0.6416	15	0.3976	16	0.3968	16	0.3944
	9	15	3.0219	17	3.0005	23	2.9475	12	0.6521	12	0.6492	13	0.6416	15	0.3976	16	0.3968	16	0.3944
	10	14	3.0320	16	3.0076	23	2.9495	12	0.6521	12	0.6493	13	0.6416	15	0.3976	16	0.3968	16	0.3944
3.0	2	24	2.2022	25	2.2009	27	2.1964	11	0.5248	11	0.5237	11	0.5203	14	0.3645	14	0.3638	14	0.3619
	3	18	2.2389	19	2.2352	22	2.2236	10	0.5349	10	0.5329	10	0.5270	13	0.3674	13	0.3665	14	0.3638
	4	15	2.2707	16	2.2640	19	2.2436	9	0.5392	9	0.5366	10	0.5294	13	0.3680	13	0.3671	14	0.3642
	5	14	2.2978	15	2.2875	18	2.2576	9	0.5411	9	0.5382	10	0.5302	13	0.3682	13	0.3672	14	0.3643
	6	12	2.3204	14	2.3063	18	2.2673	9	0.5419	9	0.5389	10	0.5305	13	0.3683	13	0.3672	14	0.3643
	7	12	2.3391	13	2.3212	17	2.2738	9	0.5423	9	0.5391	10	0.5306	13	0.3683	13	0.3672	14	0.3643
	8	11	2.3545	12	2.3328	17	2.2781	9	0.5424	9	0.5393	10	0.5306	13	0.3683	13	0.3672	14	0.3643
	9	11	2.3670	12	2.3419	17	2.2809	9	0.5425	9	0.5393	10	0.5306	13	0.3683	13	0.3672	14	0.3643
	10	10	2.3772	12	2.3489	17	2.2827	9	0.5426	9	0.5393	10	0.5306	13	0.3683	13	0.3672	14	0.3643

addition, we can find that d is not affected by TET. The patterns and results for the other values of $\lambda_{1(0)}$, ξ, and c are very similar to those in Table VIII, and hence we do not present them here.

D. Illustrative Example

To illustrate the use of the methods proposed in this article, the following example is discussed. Boardman and Kendell [6] presented a grouped data set which consists of the frequency of failure times of radio transmitters with two types of failures modes. The original data were studied by Mendenhall and Hader [24]. The sample was censored at 630 hours, and 44 of 369 transmitters had not failed. Cox [10] grouped the data by time of failure, and then tested the assumption that these data come from a double Poisson process. The χ^2 goodness-of-fit test indicated that this assumption cannot be rejected. For illustrative purpose, we generate a progressive type-I interval censored sample with $k = 6$, $\tau = 100$, and $p = 0.05$ from Table II in Boardman and Kendell [6]. The sample is listed in our Table IX. From (3), the MLE of λ^* is $\hat{\lambda}^* = 0.0045$. Hence, the MLEs of λ_1, and λ_2 can be computed as $\hat{\lambda}_1 = 0.0030$, and $\hat{\lambda}_2 = 0.0015$, respectively, by using (1).

To find the optimal length of the inspection interval in a future life test with progressive type-I interval censoring, we assume that the parameters (λ_1, λ_2) are equal to $(0.0030, 0.0015)$, our MLEs obtained previously. The optimal lengths of the inspection intervals according to the variance-optimality and D-op-

TABLE VII
NUMBER OF TEST UNITS n AND CRITICAL POINT d FOR $c = 0.8$

		$\lambda_{1(0)} = 0.1$						$\lambda_{1(0)} = 0.5$						$\lambda_{1(0)} = 0.9$					
		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$	
ξ	k	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d
0.5	2	172	4.2719	177	4.2717	190	4.2709	73	0.8660	74	0.8658	77	0.8652	79	0.4870	80	0.4869	82	0.4866
	3	129	4.2788	135	4.2782	154	4.2762	67	0.8683	68	0.8679	73	0.8668	78	0.4876	79	0.4874	81	0.4870
	4	108	4.2849	115	4.2838	139	4.2803	65	0.8693	67	0.8688	72	0.8674	78	0.4877	78	0.4875	80	0.4870
	5	96	4.2904	104	4.2886	130	4.2834	64	0.8698	66	0.8692	71	0.8676	78	0.4877	78	0.4875	80	0.4870
	6	89	4.2952	97	4.2928	126	4.2857	64	0.8700	66	0.8694	71	0.8677	78	0.4877	78	0.4875	80	0.4870
	7	83	4.2994	93	4.2962	123	4.2873	64	0.8701	66	0.8694	71	0.8677	78	0.4877	78	0.4875	80	0.4870
	8	80	4.3030	90	4.2990	122	4.2885	64	0.8701	66	0.8695	71	0.8677	78	0.4877	78	0.4875	80	0.4870
	9	77	4.3061	87	4.3014	121	4.2893	64	0.8701	66	0.8695	71	0.8677	78	0.4877	78	0.4875	80	0.4870
	10	75	4.3087	86	4.3033	120	4.2899	64	0.8701	66	0.8695	71	0.8677	78	0.4877	78	0.4875	80	0.4870
1.0	2	63	3.4186	64	3.4180	69	3.4161	28	0.7133	28	0.7129	29	0.7115	33	0.4133	33	0.4131	34	0.4124
	3	47	3.4351	49	3.4336	56	3.4287	26	0.7179	26	0.7171	28	0.7147	32	0.4142	32	0.4139	33	0.4130
	4	40	3.4497	42	3.4469	51	3.4382	25	0.7199	26	0.7188	27	0.7157	32	0.4144	32	0.4141	33	0.4131
	5	35	3.4624	38	3.4581	48	3.4453	25	0.7206	26	0.7194	27	0.7160	32	0.4145	32	0.4141	33	0.4131
	6	33	3.4734	36	3.4674	46	3.4503	25	0.7209	25	0.7197	27	0.7161	32	0.4145	32	0.4141	33	0.4131
	7	31	3.4827	34	3.4750	45	3.4538	25	0.7211	25	0.7198	27	0.7162	32	0.4145	32	0.4141	33	0.4131
	8	30	3.4906	33	3.4812	45	3.4563	25	0.7211	25	0.7198	27	0.7162	32	0.4145	32	0.4141	33	0.4131
	9	29	3.4972	32	3.4861	44	3.4579	25	0.7211	25	0.7198	27	0.7162	32	0.4145	32	0.4141	33	0.4131
	10	28	3.5027	32	3.4900	44	3.4590	25	0.7211	25	0.7198	27	0.7162	32	0.4145	32	0.4141	33	0.4131
1.5	2	39	2.8315	40	2.8306	42	2.8277	18	0.6126	18	0.6120	19	0.6101	23	0.3708	23	0.3705	23	0.3696
	3	29	2.8555	31	2.8532	35	2.8458	17	0.6186	17	0.6174	18	0.6140	22	0.3720	22	0.3716	23	0.3703
	4	25	2.8764	26	2.8722	31	2.8593	16	0.6209	17	0.6194	18	0.6153	22	0.3722	22	0.3718	23	0.3705
	5	22	2.8944	24	2.8880	30	2.8690	16	0.6218	16	0.6202	18	0.6157	22	0.3723	22	0.3718	23	0.3705
	6	20	2.9096	22	2.9008	29	2.8758	16	0.6221	16	0.6205	18	0.6158	22	0.3723	22	0.3718	23	0.3705
	7	19	2.9223	21	2.9110	28	2.8804	16	0.6223	16	0.6206	18	0.6158	22	0.3723	22	0.3718	23	0.3705
	8	19	2.9328	21	2.9190	28	2.8835	16	0.6223	16	0.6206	18	0.6158	22	0.3723	22	0.3718	23	0.3705
	9	18	2.9413	20	2.9253	28	2.8855	16	0.6223	16	0.6206	18	0.6158	22	0.3723	22	0.3718	23	0.3705
	10	18	2.9483	20	2.9302	27	2.8868	16	0.6223	16	0.6206	18	0.6158	22	0.3723	22	0.3718	23	0.3705
2.0	2	29	2.4094	30	2.4083	32	2.4048	14	0.5434	14	0.5426	15	0.5402	19	0.3480	19	0.3477	19	0.3465
	3	22	2.4387	23	2.4358	26	2.4266	13	0.5500	13	0.5486	14	0.5446	19	0.3494	19	0.3489	19	0.3474
	4	19	2.4638	20	2.4586	24	2.4425	13	0.5525	13	0.5508	14	0.5460	19	0.3496	19	0.3491	19	0.3475
	5	17	2.4850	18	2.4770	22	2.4537	12	0.5534	13	0.5516	14	0.5464	19	0.3497	19	0.3491	19	0.3476
	6	16	2.5027	17	2.4917	22	2.4613	12	0.5538	13	0.5519	14	0.5465	19	0.3497	19	0.3492	19	0.3476
	7	15	2.5171	16	2.5032	21	2.4664	12	0.5539	13	0.5520	14	0.5465	19	0.3497	19	0.3492	19	0.3476
	8	14	2.5288	16	2.5121	21	2.4697	12	0.5540	13	0.5520	14	0.5465	19	0.3497	19	0.3492	19	0.3476
	9	14	2.5382	15	2.5190	21	2.4718	12	0.5540	13	0.5520	14	0.5465	19	0.3497	19	0.3492	19	0.3476
	10	13	2.5457	15	2.5242	21	2.4732	12	0.5540	13	0.5520	14	0.5465	19	0.3497	19	0.3492	19	0.3476
3.0	2	21	1.8508	22	1.8495	23	1.8449	11	0.4598	11	0.4589	11	0.4559	18	0.3404	18	0.3400	18	0.3386
	3	16	1.8857	17	1.8820	19	1.8705	10	0.4672	10	0.4656	11	0.4608	18	0.3421	18	0.3414	18	0.3396
	4	14	1.9149	14	1.9083	17	1.8885	10	0.4699	10	0.4679	11	0.4622	18	0.3424	18	0.3417	18	0.3398
	5	12	1.9387	13	1.9288	16	1.9006	10	0.4709	10	0.4687	10	0.4627	18	0.3424	18	0.3417	18	0.3398
	6	11	1.9579	12	1.9446	16	1.9085	10	0.4713	10	0.4691	10	0.4628	18	0.3424	18	0.3418	18	0.3398
	7	11	1.9730	12	1.9566	15	1.9135	10	0.4714	10	0.4692	10	0.4628	18	0.3424	18	0.3418	18	0.3398
	8	10	1.9850	11	1.9656	15	1.9168	10	0.4715	10	0.4692	10	0.4628	18	0.3424	18	0.3418	18	0.3398
	9	10	1.9945	11	1.9723	15	1.9188	10	0.4715	10	0.4692	10	0.4628	18	0.3424	18	0.3418	18	0.3398
	10	10	2.0019	11	1.9774	15	1.9201	10	0.4715	10	0.4692	10	0.4628	18	0.3424	18	0.3418	18	0.3398

timality are equal to the values in Table IV divided by $\hat{\lambda}^* = 0.0045$. For example, when $p = 0.05$, and $k = 6$, one has $\tau_V = 0.7601/0.0045 = 168.9111$, and $\tau_D = 0.9055/0.0045 = 201.2222$.

Suppose that a quality engineer wants to design a reliability sampling plan for a particular lot of radio transmitters with a desired power $1 - \beta = 0.95$ at level $\alpha = 0.05$. Assume that the quality parameter is the mean lifetime. Thus, the quality parameter is acceptable if it is greater than or equal to 266.6667 (*i.e.*, $\lambda_{1(0)} = 0.0025$), and a quality parameter less than or equal to 177.7778 (*i.e.*, $\xi = 0.5$) is considered unacceptable. From the result discussed above, we have $k = 6$, $p = 0.05$, $c = 0.0015/0.0030 = 0.5$, and $\tau_D = 201.2222$. For the convenience of the engineer, one can choose the inspection length to be $\tau = 200$. Note that $\lambda_{1(0)}\tau = (0.0025)(200) = 0.5$. We can obtain $n = 63$ and $d = 1.0423$ from Table VI. Therefore, the engineer draws a random sample of size 63 from the lot, and puts them on a 6-stage progressively type-I interval censored life test. By comparing the values between the estimate $\hat{\theta}$ of the mean lifetime and the critical point $d = (1.0423)(0.5/0.0025) = 208.46$, we would accept the lot if $\hat{\theta} > 208.46$; otherwise, we would reject it.

VII. EXTENSIONS

In Section IV, we investigated the optimal lengths of inspection intervals under the special case of equal inspection

TABLE VIII
NUMBER OF TEST UNITS n AND CRITICAL POINT d FOR $c = 0.2$ AND $\xi = 1$ UNDER THE LIMITATION OF TET

		$\lambda_{1(0)} = 0.1$						$\lambda_{1(0)} = 0.5$						$\lambda_{1(0)} = 0.9$					
		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$		$p = 0.05$		$p = 0.1$		$p = 0.25$	
TET	k	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d	n	d
1	2	162	5.0873	166	5.0870	180	5.0860	43	1.0334	44	1.0331	47	1.0322	32	0.5825	32	0.5822	34	0.5813
	3	166	5.0869	175	5.0864	203	5.0847	44	1.0323	46	1.0318	52	1.0301	31	0.5805	33	0.5800	36	0.5786
	4	170	5.0866	183	5.0859	228	5.0836	45	1.0317	48	1.0310	57	1.0288	32	0.5797	34	0.5790	39	0.5770
	5	174	5.0864	192	5.0855	255	5.0826	46	1.0314	50	1.0305	63	1.0278	32	0.5792	35	0.5784	43	0.5759
	6	178	5.0862	201	5.0851	283	5.0817	47	1.0311	52	1.0300	69	1.0268	33	0.5789	36	0.5779	46	0.5751
	7	183	5.0860	210	5.0847	312	5.0808	48	1.0309	54	1.0296	75	1.0260	33	0.5786	37	0.5775	50	0.5743
	8	187	5.0858	220	5.0844	342	5.0800	48	1.0307	56	1.0293	82	1.0252	34	0.5784	38	0.5771	53	0.5736
	9	191	5.0856	230	5.0840	374	5.0792	49	1.0305	58	1.0289	88	1.0246	35	0.5782	40	0.5768	57	0.5730
	10	196	5.0855	240	5.0836	406	5.0785	51	1.0303	60	1.0285	95	1.0239	35	0.5780	41	0.5764	61	0.5724
5	2	43	5.1671	44	5.1657	47	5.1610	31	1.1076	31	1.1070	32	1.1050	63	0.6686	63	0.6684	64	0.6679
	3	44	5.1613	46	5.1588	52	5.1507	26	1.0822	27	1.0810	28	1.0774	35	0.6282	35	0.6278	36	0.6265
	4	45	5.1587	48	5.1551	57	5.1441	25	1.0728	26	1.0711	28	1.0661	29	0.6126	29	0.6120	30	0.6101
	5	46	5.1570	50	5.1525	63	5.1388	24	1.0681	25	1.0659	28	1.0598	26	0.6050	27	0.6042	28	0.6017
	6	47	5.1556	52	5.1502	69	5.1341	24	1.0654	25	1.0627	29	1.0556	25	0.6008	25	0.5997	27	0.5968
	7	48	5.1544	54	5.1482	75	5.1300	24	1.0635	26	1.0605	30	1.0525	24	0.5981	25	0.5969	27	0.5935
	8	48	5.1534	56	5.1463	82	5.1262	24	1.0622	26	1.0588	32	1.0501	24	0.5963	25	0.5949	28	0.5911
	9	49	5.1524	58	5.1444	88	5.1228	25	1.0611	27	1.0573	33	1.0480	24	0.5950	25	0.5934	28	0.5893
	10	51	5.1514	60	5.1427	95	5.1196	25	1.0602	27	1.0561	34	1.0462	24	0.5940	25	0.5923	29	0.5878
10	2	30	5.2603	31	5.2577	32	5.2495	81	1.2313	81	1.2311	81	1.2304	1246	0.8076	1246	0.8075	1247	0.8075
	3	30	5.2387	31	5.2341	34	5.2198	39	1.1453	39	1.1447	40	1.1428	142	0.7164	142	0.7163	142	0.7161
	4	30	5.2299	32	5.2236	37	5.2042	30	1.1115	31	1.1104	31	1.1074	63	0.6688	63	0.6686	64	0.6680
	5	31	5.2250	33	5.2169	40	5.1932	27	1.0950	27	1.0936	29	1.0896	43	0.6433	43	0.6429	44	0.6420
	6	31	5.2216	34	5.2119	43	5.1844	25	1.0857	26	1.0839	28	1.0791	35	0.6284	35	0.6279	36	0.6266
	7	32	5.2188	35	5.2077	47	5.1768	25	1.0799	25	1.0778	27	1.0722	31	0.6190	31	0.6185	32	0.6168
	8	32	5.2165	36	5.2039	50	5.1701	24	1.0760	25	1.0737	27	1.0673	28	0.6128	29	0.6121	30	0.6101
	9	33	5.2144	37	5.2004	53	5.1642	24	1.0732	25	1.0706	28	1.0637	27	0.6084	27	0.6076	29	0.6054
	10	33	5.2125	39	5.1972	57	5.1588	24	1.0711	25	1.0682	28	1.0607	26	0.6052	26	0.6043	28	0.6018

TABLE IX
PROGRESSIVELY TYPE-I INTERVAL CENSORED SAMPLE GENERATED FROM TABLE II IN BOARDMAN AND KENDELL (1970)

i	Time(h)	n_{i1}	n_{i2}	r_i
1	(0,100]	55	30	12
2	(100,200]	62	34	7
3	(200,300]	34	18	4
4	(300,400]	18	8	2
5	(400,500]	15	6	1
6	(500,600]	12	3	4

length and two competing risks. Now, we want to extend the idea to the general case where three or more risks and unequal inspection lengths are considered. Let $I(\lambda_1, \ldots, \lambda_s)$ be the Fisher's information matrix with s competing risks and unequal lengths of inspection intervals. The elements of $I(\lambda_1, \ldots, \lambda_s)$ can be found in (5). Based on the asymptotic properties of the MLE, $\sqrt{n}(\hat{\theta} - \theta)$ converges in distribution to the normal distribution with mean 0 and variance V^*/λ^{*^4}, where $V^* = \mathbf{1}'I^{-1}(\lambda_1, \ldots, \lambda_s)\mathbf{1}$, and $\mathbf{1} = (1, \ldots, 1)'$ is an $s\times$ 1 vector. Thus, the variance-optimality is to find the values of $\tau_1, \ldots, \tau_k$ such that V^*/λ^{*^4} is minimum. Lin *et al.* [20], and Lin *et al.* [19] suggested using the simulated annealing algorithm to determine the inspection times for any specified progressive type-I interval censoring plan. This suggestion can be applied to the optimality problem discussed here. For D-optimality, one only needs to change the objective function to the determinant $|I^{-1}(\lambda_1, \ldots, \lambda_s)|$, and then apply a simulated annealing algorithm to get the optimal inspection times.

In Section V, we studied the reliability sampling plans with two competing risks, and equal length of inspection interval. Here we extend the proposed plan to the general case with three or more risks and unequal lengths. Note that, with s competing risks, the mean lifetime θ is the reciprocal of $\lambda^* = \lambda_1 + \cdots + \lambda_s$. We can still assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$, and $\lambda_i = c_i\lambda_1$, where $0 < c_s \leq \cdots \leq c_2 \leq c_1 = 1$. Then, one has $\lambda^* = \sum_{i=1}^{s} c_i\lambda_1$. Equations (6), and (7) can be re-written as $n = [(\sum_{i=1}^{s} c_i)(z_\beta\sqrt{V_1} - z_{1-\alpha}\sqrt{V_0})/(\lambda_{1(0)}^{-1} - \lambda_{1(1)}^{-1})]^2$, and

$$d = \frac{\left(\frac{z_{1-\alpha}\sqrt{V_0}}{\lambda_{1(1)}} - \frac{z_\beta\sqrt{V_1}}{\lambda_{1(0)}}\right)}{\left(\sum_{i=1}^{s} c_i\right)\left(z_{1-\alpha}\sqrt{V_0} - z_\beta\sqrt{V_1}\right)},$$

where $\theta_0 = 1/(\sum_{i=1}^{s} c_i)\lambda_{1(0)}$, $\theta_1 = 1/(\sum_{i=1}^{s} c_i)\lambda_{1(1)}$, and V_0 and V_1 are V^*/λ^{*^4} evaluated at θ_0 and θ_1, respectively.

VIII. CONCLUSIONS AND DISCUSSIONS

In this article, we propose the progressive type-I interval censoring scheme with statistically independent competing risks when the numbers of removals follow binomial distributions. We use the maximum likelihood method to estimate the unknown parameters, and derive the Fisher's information matrix.

Based on the asymptotic distribution of the MLEs, we propose two criteria to decide the optimal lengths of the inspection intervals, and also establish the reliability sampling plans. From the Monte Carlo study, we find that the MLEs of the unknown parameters perform well even though the sample size is small. Here we only consider the case with two competing risks. However, the concept can be extended to the general cases with more than two risks.

Finally, there are some open questions which have not been investigated in this paper. First, one practical problem arising from designing a life test or establishing the reliability sampling plan is the restricted budget of the experiment. The size of the budget always affects the decisions of the number of test units, the number of inspections, and the length of the inspection intervals. Thus, the problem of obtaining a precise estimation of life parameters under a restricted experiment cost, and the problem of exploring the optimum values of decision variables with cost consideration in conducting a reliability sampling plan with desired producer's and consumer's risks are two important issues for the reliability analyst. Second, the cause of failure may be assumed to be statistically independent or dependent. In most situations, it is usually assumed that these competing risks are statistically independent. Although the assumption of dependence may be more realistic, there are some identifiability issues with the underlying model. Thus, the optimal life test and reliability sampling plan with dependent risks are two possible future works.

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Shuo-Jye Wu (A'10) is a Professor in the Department of Statistics at Tamkang University, Taiwan. He received his Ph.D. in Statistics from the University of Wisconsin-Madison in 1996. He is a member of ASA, a member of ICSA, and a member of IEEE. His research interests are in the development and application of statistical methodologies for problems in reliability.

Syuan-Rong Huang is a Ph.D. student in the Department of Management Sciences at Tamkang University, Taiwan. He received his Master degree in Statistics from Tamkang University in 2007. His research interest is industrial statistics.