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ON THE DEAD-CORE RATES FOR A PARABOLIC EQUATION WITH STRONG ABSORPTION

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Abstract

In this survey, we discuss the dead-core rates for a family of parabolic equations with strong absorption. This includes the standard heat equation, fast diffusion equation and slow diffusion equation. It is surprising that, even in the case of one space dimension, the dead-core rate can be either self-similar or non-self-similar. Some open problems are also given.

1. Introduction

In the study of singularity formation, one of the main issues is to determine the temporal asymptotic behavior of the singular solution at a singular point. The typical singularities of great interests are, for examples, blow-up (the solution becomes unbounded), extinction (the solution becomes identically zero), quenching (the solution reaches zero) and dead-core (the solution reaches zero). The difference between quenching and dead-core is that certain derivative of the solution blows up for quenching, but the solution is regular for dead-core.

We are interested in the case that a solution develops a singularity in *finite time*. Then it is important to determine the (temporal) singular rate of the given solution. Usually, we can classify the rate(s) into two classes,

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namely, whether it is self-similar or not. For a given equation, a special solution is *self-similar* if it is invariant under certain re-scaling of both independent and dependent variables. Hence the self-similar rate of a given equation is intrinsic. We say that the singular rate is self-similar, if it is the same as that of the self-similar solution.

A well-studied equation for blow-up is

$$u_t = \Delta u + u^p, \quad p > 1. \quad (1.1)$$

See, for examples, [39, 9, 10, 11, 28]. For extinction, the well-known equation is

$$u_t = \Delta u - u^p, \quad 0 < p < 1.$$

See, e.g., [8, 31, 20, 21]. For the equation

$$u_t = \Delta u - u^{-q}, \quad q > 0,$$

we have the quenching (cf. [12, 13, 5, 25]). The above mentioned works exhibit that the singular rates are all self-similar. Also, it is worthy to mention here that the self-similar rates of the above 3 equations are the same as that of the corresponding spatially independent ordinary differential equations.

For the dead-core, we study the following initial boundary value problem:

$$\begin{cases} u_t = (u^m)_{xx} - u^p, & x \in (-1, 1), \quad t > 0, \\ u(\pm 1, t) = k, & t > 0, \\ u(x, 0) = u_0(x), & x \in [-1, 1], \end{cases} \quad (1.2)$$

where $k > 0$, $0 < p < 1$, $p < m$. It is always assumed that the initial data u_0 satisfies

$$u_0 \in C([-1, 1]), \quad u_0 > 0 \text{ in } [-1, 1], \quad u_0(\pm 1) = k. \quad (1.3)$$

Problem (1.2) with $m = 1$ arises from the modeling of an isothermal reaction-diffusion process (cf. [2, 36]). Here u is the concentration of the reactant. The reactant becomes inactive (so that a dead-core is developed) whenever

u reaches zero. The reaction is of strong absorption, since the reaction rate $-u^{p-1} \rightarrow -\infty$ as $u \rightarrow 0$. The boundary condition means that the reactant is injected with a fixed amount on the boundary. For another application, we refer to [27, 24] in which (1.2) is the description of thermal energy transport in plasmas. For $m \neq 1$, we have the fast diffusion when $0 < m < 1$ and the slow diffusion when $m > 1$.

It is clear that problem (1.2) admits a unique local classical positive solution u . We set

$$T = T(u_0) := \inf \{t > 0 \mid \Lambda(t) = 0\} > 0, \quad \Lambda(t) := \min_{|x| \leq 1} u(x, t).$$

For suitable initial data, we have $T(u_0) < \infty$ in which the solution develops a dead-core in finite time (cf. [2, 36, 3]). It seems that there is nothing special for the temporal dead-core rate than the other singularities mentioned above. Therefore, nobody pays attention to this question in the last century. It turns out that the time asymptotic behavior of the solution u as $t \rightarrow T^-$ when $T < \infty$ exhibits a very interesting phenomenon. Our main purpose of this paper is to survey the recent developments of the dead-core rates.

Until the work of Herrero and Velazquez [22] in 1994, all singular rates of classical parabolic problems found are self-similar. The work [22] gives the first example of non-self-similar singularity for (1.1) with p sufficiently large and spatial dimension $n \geq 11$. See also the later developments by Mizoguchi [29, 30] and others. It is sometimes called the *Type II* singularity for a non-self-similar singularity. The self-similar rate is then called the type I. Another example of type II singularity is about the boundary gradient blow-up for the equation

$$u_t - u_{xx} = |u_x|^p, \quad p > 2,$$

under Dirichlet boundary conditions (cf. [15, 23, 26]). An important difference of self-similar and non-self-similar singularities is that there is a unique rate (independent of initial data) for a self-similar singularity, but there are infinitely many different rates (depending on initial data) for a non-self-similar singularity. Indeed, the rates of non-self-similar singularity are related to the spectrum of the linearized operator around the (singular) steady state, if it exists.

This paper is organized as follows. In the next section, we shall describe the results for dead-core rates for all $m > 0$. This includes the cases of heat equation, fast diffusion equation and slow diffusion equation. Then, in §3, we give some ideas of the proof for the non-self-similar dead-core rate for the fast diffusion equation. Finally, we give some open problems in §4.

2. Main Results

2.1. The case $m = 1$

For a solution u of the first equation in (1.2) defined for $(x, t) \in \mathbf{R} \times (-\infty, 0)$, let

$$u_\lambda(x, t) := \lambda^a u(\lambda^b x, \lambda t), \quad \lambda > 0, t < 0.$$

By the definition of (backward) self-similarity, u is self-similar if and only if $u = u_\lambda$ for all $\lambda > 0$. Hence we must have $a = -1/(1-p)$ and $b = 1/2 + (m-1)/[2(1-p)]$. Taking $\lambda = 1/(-t)$, the self-similar solution u is written in the form

$$u(x, t) = (-t)^{-a} \varphi(x/(-t)^b), \quad \varphi(y) := u(y, -1),$$

and the (temporal) self-similar rate of (1.2) is given by $-a$. In fact, any solution of ODE $u' = -u^p$, $0 < p < 1$, with $u(0) > 0$ tends to zero in finite time T in the rate $(T-t)^\beta$, where $\beta := 1/(1-p) = -a$. Hence the self-similar rate of (1.2) is the same as the corresponding ODE rate.

For $m = 1$, surprisingly it was shown in [18] that the dead-core rate is not self-similar, i.e. its order is not the same as for the corresponding ODE. Actually, from [18] we have

$$\lim_{t \rightarrow T} \frac{\Lambda(t)}{(T-t)^\beta} = 0 \tag{2.1}$$

for solutions of (1.2) with monotone symmetric initial data. Then the natural question is about the exact rate for a given solution.

Later, the exact rates for solutions with general initial data were derived in [17] using the exact solutions constructed in [19] and the braid group theory. More precisely, let u be a solution of (1.2) with $k \neq k_p$ such that

u develops a dead-core in the finite time T . Suppose that u_0 is radially non-decreasing. Then there exists $\gamma_n > 0$, for each $n \in \mathbf{N}$, depending on the initial datum u_0 , such that

$$c_1(T-t)^{\beta+2\beta\gamma_n} \leq u(0,t) \leq c_2(T-t)^{\beta+2\beta\gamma_n}, \quad t \in (0,T),$$

for some positive constants c_1, c_2 . Note that the above result holds for general higher space dimension when the domain is a ball (cf. [32]).

2.2. The case $m \neq 1$

We assume further that

$$u_0 \in C^2([-1,1]), \quad (u_0^m)'' \leq u_0^p \quad \text{in } [-1,1], \quad (2.2)$$

$$u_0 \quad \text{is even and nondecreasing in } |x| \quad \text{and} \quad T(u_0) < \infty. \quad (2.3)$$

It then follows from the strong maximum principle that $u_t < 0$ in $Q_T := (-1,1) \times (0,T)$, $T = T(u_0)$, $u(-x,t) = u(x,t)$ for $(x,t) \in Q_T$, and $u_x > 0$ in $(0,1) \times (0,T)$.

For the steady states, we have the followings:

- (1.2) admits a unique steady state $U_k \in C^2([-1,1])$ for each given $k > 0$.
- U_k is an even and nondecreasing function of $|x|$ and it is a nondecreasing function of k .
- There exists $k_0 = k_0(m,p) > 0$ such that: if $k \in (0, k_0)$ then U_k vanishes on an interval of positive length, if $k = k_0$ then U_k vanishes only at $x = 0$, and if $k > k_0$ then U_k is positive.

The following theorem shows that the dead-core occurs in finite time.

Theorem 2.1. *Assume $0 < p < 1$, $m > p$ and (1.3). Then $T(u_0) < \infty$ for any u_0 when $0 < k < k_0$. For $k \geq k_0$, for any $\eta, M > 0$ there exists $\delta = \delta(\eta, M) > 0$ such that $T(u_0) < \infty$ whenever $\|u_0\|_\infty \leq M$ and $u_0 \leq \delta$ on a subinterval of $[-1,1]$ of length η .*

This theorem was proved in [16] for $0 < m < 1$ and [4] for $m > 1$. The proof is by constructing a suitable super-self-similar solution (cf. [35]).

The following theorem shows that the dead-core rate is not self-similar for $0 < m < 1$.

Theorem 2.2 ([16]). *Let $k > 0$, $0 < p < m < 1$ and assume (1.3), (2.2) and (2.3) hold.*

Then (2.1) holds.

To prove Theorem 2.2, we use the following self-similar variables:

$$y = \frac{x}{(T-t)^\alpha}, \quad s = -\ln(T-t), \quad z(y, s) := \left[\frac{u(x, t)}{(T-t)^\beta} \right]^m,$$

where

$$\alpha = \frac{m-p}{2(1-p)}, \quad \beta = \frac{1}{1-p}.$$

Then z satisfies the equation

$$\gamma z^{\gamma-1} z_s = z_{yy} - \alpha \gamma y z^{\gamma-1} z_y + \beta z^\gamma - z^q \quad \text{in } \Omega, \quad (2.4)$$

where $\gamma := 1/m$, $q := p/m$, and

$$\Omega := \{(y, s) : |y| < e^{\alpha s}, -\ln(T) =: s_0 < s < \infty\}.$$

The boundary and initial conditions are transformed into

$$z(e^{\alpha s}, s) = k^m e^{\beta m s}, \quad s > s_0, \quad (2.5)$$

$$z(y, s_0) = z_0(y) := T^{-\beta m} u_0^m(y T^\alpha), \quad y \in [-T^{-\alpha}, T^{-\alpha}]. \quad (2.6)$$

Then Theorem 2.2 follows from the following more general theorem.

Theorem 2.3 ([16]). *Under the assumptions of Theorem 2.2, the corresponding global solution z of (2.4)-(2.6) satisfies*

$$\lim_{s \rightarrow \infty} z(y, s) = V_1(y) := k_{p,m} |y|^{\frac{2m}{m-p}},$$

where $k_{p,m} = \left(\frac{(m-p)^2}{2m(m+p)}\right)^{\frac{m}{m-p}}$, uniformly on $\{|y| < R\}$, for any $R > 0$ fixed.

Note that V_1 is a stationary solution of (2.4). In fact, each stationary solution of (2.4) corresponds to a self-similar solution of $u_t = (u^m)_{xx} - u^p$ in $\mathbb{R} \times (-\infty, T)$.

For the slow diffusion case, let $0 < p < 1 < m$,

$$\sigma_0 = \frac{(m-1)}{m(m-1+2p)},$$

and assume that

$$k^{m-p} \leq \sigma_0^2 \frac{m(m+p)}{2}.$$

Recall

$$\beta = \frac{1}{1-p}, \quad \alpha = \frac{m-p}{2(1-p)}$$

Then the dead-core rate is self-similar for some solutions for the case $m > 1$. More precisely, we have

Theorem 2.4 ([4]). *Let u be a solution of (1.2) with u_0 satisfying (1.3), (2.2) and (2.3).*

In addition, we assume that the initial datum u_0 satisfies

$$0 \leq (u_0)_x \leq \sigma_0 x u_0^{1+p-m} \text{ for } 0 \leq x \leq 1.$$

If $m+p \geq 2$, then

$$\lim_{t \rightarrow T} u(x, t)(T-t)^{-\beta} = \beta^{-\beta}$$

uniformly on $\{|x| \leq C(T-t)^\alpha\}$ for any $C > 0$. In the case $m+p < 2$, the ω -limit set is not empty.

3. Ideas of the Proof of Theorem 2.3

This section is devoted to the proof of Theorem 2.3. There are three key ingredients of the proof, namely,

- (1) Derive some a priori estimates of z from above and below;

- (2) Construct a Lyapunov functional by the method of Zelenyak [40] (see also [14]);
- (3) Classify all the possible steady states for z on the whole real line (a Liouville's type theorem).

First, we derive some a priori estimates for global solution z of (2.4) as follows.

Lemma 3.1 ([16]). *Under the assumptions of Theorem 2.2, there hold*

$$\begin{aligned} z(y, s) &\leq C(1 + |y|)^{\frac{2}{1-q}}, \\ |z_y(y, s)| &\leq C(1 + |y|)^{\frac{1+q}{1-q}}, \quad \text{if } |y| \geq 1, \\ |z_y(y, s)| &\leq C|y|, \quad \text{if } |y| \leq 1, \end{aligned}$$

in $\{(y, s) : |y| < e^{\alpha s}, -\ln T < s < \infty\}$ for some constant $C > 0$.

Furthermore, we have

Lemma 3.2 ([16]). *Let $t_0 \in (0, T)$. Then, under the assumptions of Theorem 2.2, there exists $c > 0$ such that $u(x, t) \geq c|x|^{2/(m-p)}$ for $x \in (-1, 1)$, $t_0 < t < T$.*

In terms of similarity variables, it follows from Lemma 3.2 that

$$z(y, s) \geq D^*|y|^\delta \quad \text{in } \Omega_0 := \{(y, s) : |y| < e^{\alpha s}, -\ln(T-t_0) < s < \infty\} \quad (3.1)$$

for some positive constant D^* , where $\delta := \frac{2m}{m-p} = \frac{2}{1-q}$.

Next, we turn to the construction of a Lyapunov functional. For the semilinear case, i.e. $m = \gamma = 1$ in (2.4), we have

$$z_s = z_{yy} + a(y)z_y + g(z) \quad \text{in } \Omega,$$

where $a(y) = -y/2$ and $g(z) = \beta z - z^q$. Define

$$\begin{aligned} \rho = \rho(y) &= \exp \left\{ \int^y a(\xi) d\xi \right\} = e^{-y^2/4}, \\ E[z](s) &= \frac{1}{2} \int z_y^2 \rho dy - \int G \rho dy, \quad G(z) := \int^z g(\xi) d\xi. \end{aligned}$$

Then, by ignoring the boundary integral, we can easily derive

$$\frac{dE[z](s)}{ds} = - \int \rho z_s^2 dy < 0.$$

This gives a Lyapunov functional for the semilinear case.

For the quasilinear case, i.e., $\gamma \neq 1$, it seems that we can follow the semilinear case to define

$$E[z](s) = \frac{1}{2} \int z_y^2 \rho dy - \int G \rho dy,$$

where ρ is re-defined by

$$\rho(y, z(y, s)) = \exp \left\{ -\alpha\gamma \int^y \xi z^{\gamma-1}(\xi, s) d\xi \right\}.$$

However, since ρ also depends on s , we cannot get any useful information about dE/ds .

Therefore, for the quasilinear case, following an idea of Zylyenyak [40] we define

$$\rho(y, z(y, s), z_y(y, s)) := \exp \left\{ -\alpha\gamma \int_0^y \xi \psi(\xi; y, z(y, s), z_y(y, s))^{\gamma-1} d\xi \right\},$$

where $\psi(\xi; y, v, w)$ is the solution of the following *backward problem*:

$$\begin{aligned} \psi_{\xi\xi} - \alpha\gamma\xi\psi^{\gamma-1}\psi_{\xi} + \beta z^{\gamma} - z^q &= 0, & \xi \in (0, y], \\ \psi(y; y, v, w) &= v, & \psi_{\xi}(y; y, v, w) = w. \end{aligned}$$

For our case, in order to guarantee the backward solution to exist up to $\xi = 0$, we need some modifications of the term $\beta z^{\gamma} - z^q$ to be defined as follows. Following [14], we take a smooth and nonincreasing function ζ on \mathbb{R} such that

$$\zeta(\eta) = 0, \quad \eta \geq 2, \quad \zeta(\eta) = 1, \quad \eta \leq 1, \quad 0 \leq \zeta(\eta) \leq 1, \quad \eta \in (1, 2).$$

Then we define

$$\widehat{g}(\xi, v) = g(v) \left[1 - \zeta \left(\frac{2v}{D^* \xi^{\delta} \zeta(\xi) + D^* [1 - \zeta(\xi)]} \right) \right] - v^{-1} \zeta \left(\frac{2v}{D^* \xi^{\delta} \zeta(\xi) + D^* [1 - \zeta(\xi)]} \right),$$

where the constant D^* is defined in (3.1). Without loss of generality, we may assume that $D^* < \kappa/(2^\delta)$. Note that $\widehat{g}(\xi, v) = g(v)$ for all ξ whenever $v \geq \kappa$.

With this modification of g , we have the following lemma.

Lemma 3.3 ([16]). *Let $\psi(\xi; y, v, w)$ be defined as the solution of the backward problem:*

$$\psi_{\xi\xi} - \alpha\gamma\xi\psi^{\gamma-1}\psi_\xi + \widehat{g}(\xi, \psi) = 0, \quad \xi < y, \tag{3.2}$$

$$\psi(y; y, v, w) = v, \quad \psi_\xi(y; y, v, w) = w, \tag{3.3}$$

where $v > 0$ and $w \in \mathbb{R}$. Then the solution ψ of (3.2)-(3.3) can be continued backward to $\xi = 0$.

Then we can construct a suitable Lyapunov functional as follows. Define

$$E[z](s) := \int_0^{R(s)} \Phi(y, z(y, s), z_y(y, s))dy, \quad R(s) := e^{\alpha s},$$

$$\Phi(y, v, w) := \int_0^w (w - \sigma)P(y, v, \sigma)d\sigma - \int_\kappa^v \widehat{g}(y, \mu)P(y, \mu, 0)d\mu,$$

$$P(y, v, w) := \exp \left\{ -\alpha\gamma \int_0^y \xi\psi(\xi; y, v, w)^{\gamma-1}d\xi \right\},$$

Since $z(y, s) \geq D^*|y|^\delta$ in Ω_0 , z also satisfies the equation

$$\gamma z^{\gamma-1}z_s = z_{yy} - \alpha\gamma y z^{\gamma-1}z_y + \widehat{g}(y, z) \quad \text{in } \Omega_0.$$

Then we have

$$\frac{d}{ds}E[z](s) = -\gamma \int_0^{R(s)} P(y, z(y, s), z_y(y, s))z^{\gamma-1}(y, s)|z_s(y, s)|^2dy + J_1(s),$$

where J_1 satisfies the property $\int_{s_0}^\infty |J_1(s)|ds < \infty$. Thus we have constructed a Lyapunov functional for (2.4).

Finally, we study the associated ordinary differential equation to (2.4) on the whole real line \mathbb{R} , namely,

$$V'' - \alpha\gamma y V^{\gamma-1}V' + \beta V^\gamma - V^q = 0, \quad y \in \mathbb{R}. \tag{3.4}$$

Recall that

$$\alpha = \frac{m-p}{2(1-p)}, \quad \beta = \frac{1}{1-p}, \quad \gamma = \frac{1}{m}, \quad q = \frac{p}{m}, \quad 0 < p < m < 1.$$

We proved that

Lemma 3.4 ([16]). *Let $V \in C^2(\mathbb{R})$ be a solution of (3.4) such that*

$$V = V(|y|), \quad \text{with } V' \geq 0, \quad V > 0 \quad \text{for all } y > 0,$$

and such that V is polynomially bounded. Then

$$V = V_1 := k_{p,m}|y|^{\frac{2m}{m-p}} \quad \text{or} \quad V = V_2 := \kappa$$

where $k_{p,m} = \left(\frac{(m-p)^2}{2m(m+p)}\right)^{\frac{m}{m-p}}$ and $\kappa = (1-p)^{\frac{m}{1-p}}$.

Proof of Theorem 2.3. Let s_j be a sequence with $s_j \rightarrow \infty$ as $j \rightarrow \infty$. We define $z_j(y, s) = z(y, s + s_j)$ for all $j \in \mathbb{N}$ and $(y, s) \in \Omega$. Define $\tilde{\Omega} := \{(y, s) \in \mathbb{R}^2 : y \neq 0\}$. Using a compactness argument, there exists a subsequence $\{j_l\}$ and a function $z_\infty \in C^{2,1}(\tilde{\Omega})$ such that $z_{j_l}(y, s) \rightarrow z_\infty(y, s)$ as $l \rightarrow \infty$, locally uniformly in $C^{2,1}(\tilde{\Omega})$. Moreover, z_∞ satisfies (2.4) in $\tilde{\Omega}$. Consider the Lyapunov functional $E[z](s)$ constructed as above. We deduce that $\partial_s z_\infty(y, s) = 0$ and $z_\infty = z_\infty(y)$ satisfies

$$z'' - \alpha\gamma y z^{\gamma-1} z' + \beta z^\gamma - z^q = 0, \quad y > 0. \quad (3.5)$$

Then we show that z_∞ can be extended to a symmetric C^2 solution of (3.5) on \mathbb{R} , in view of the symmetry of z in y . Therefore, the conclusion follows from the polynomial bound of z_∞ and Lemma 3.4. \square

4. Open Problems

As mentioned in the introduction, the classical Type II singularity for blow-up is found for spatial dimension $n \geq 11$. For the dead-core problem, there is already a rich phenomena for one spatial dimension. In the dead-core problem, the case when $m = 1$ is well understood. This includes the exact

dead-core rates for general symmetric initial data and higher dimensional radially symmetric case.

However, very little is known for the dead-core for quasilinear equations. When $0 < m < 1$, we only know that the dead-core rate is non-self-similar. It would be very interesting to determine the exact dead-core rates for general initial data. As in the works [19, 17], we need to construct a family of special solutions with the desired dead-core rates and to analyze the spectrum of the associated linearized operator around the singular steady state.

In the slow diffusion case (i.e., when $m > 1$), we have found that dead-core rate is self-similar for certain initial data. This is also surprised to us. We suspect that there should also be certain initial data which give non-self-similar dead-core rate(s) for the slow diffusion equation. If so, then it is also interesting to determine the exact dead-core rates. These questions are all left open.

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