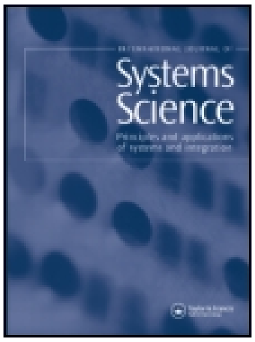

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In this article, we first complement an inappropriate mathematical error on the total cost in the previously published paper by Chung and Wee [2007, 'Optimal the Economic Lot Size of a Three-stage Supply Chain With Backlogging Derived Without Derivatives', *European Journal of Operational Research*, 183, 933–943] related to buyer–distributor–vendor three-stage supply chain with backlogging derived without derivatives. Then, an arithmetic–geometric inequality method is proposed not only to simplify the algebraic method of completing prefect squares, but also to complement their shortcomings. In addition, we provide a closed-form solution to integral number of deliveries for the distributor and the vendor without using complex derivatives. Furthermore, our method can solve many cases in which their method cannot, because they did not consider that a squared root of a negative number does not exist. Finally, we use some numerical examples to show that our proposed optimal solution is cheaper to operate than theirs.

Keywords: supply chain management; inventory; arithmetic–geometric mean; backlogging

1. Introduction

The integrated production–inventory models using differential calculus to derive an optimal solution for the inventory model with multi-variable problems are prevalent in operational research studies. However, students who are unfamiliar with calculus may not be capable of understanding the solution procedure easily. Consequently, few researchers focused on the easy solution methods for the individual/integrated inventory system. Grubbström and Erdem (1999) developed a new approach to derive an economic order quantity (EOQ) policy with backlogging without derivatives. Cárdenas-Barrón (2001) developed a single-level economic production quantities (EPQ) model with shortage without derivatives. Yang and Wee (2002) derived an economic lot size of the integrated vendor–buyer inventory system without derivatives. Under a different economic issue, Wee, Chung, and Yang (2003) later developed an economic ordering quantity model with temporary sale price without using derivatives. Zanoni and Grubbström (2004) used the approach of Grubbström and Erdem (1999) to develop an analytic formulation. However, the integrated multiple-stage production–inventory system with backlogging is neglected in the inventory model development.

Recently, Chung and Wee (2007) established a three-stage integrated production–inventory supply chain system as follows. If the buyer's stock is depleted to the maximum allowed amount of shortages B, then the order quantity of q is replenished on time by the distributor. The distributor periodically delivers q items to the buyer, and orders Mq units from the vendor with an integral M. Finally, the vendor periodically delivers n times of Mq units to the distributor's warehouse, and produces a lot-size of $Q = n(Mq)$ units with an integral n. Then, they used the simple algebraic method of completing perfect squares (CPS) to obtain the optimal solution for this integrated three-stage supply chain production–inventory problem with four decision variables, B, q, M and n. This article revisits the model by Chung and Wee (2007) and Yang and Wee (2002) to analyse a multi-stage supply chain inventory problem, and provides the following contributions beyond Chung and Wee (2007): (1) we first point out an inappropriate mathematical error on the shortage cost, (2) we use a simple-to-use arithmetic–geometric inequality (AGI) approach to

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solve the problem, (3) we establish a closed-form solution to integral number of deliveries for the vendor and the distributor, (4) we completely discuss the boundary conditions when the optimal number of distributor's deliveries $M^* = 1$, and when the optimal number of vendor's deliveries $n^* = 1$, (5) we can solve many cases in which their method cannot, as shown in Examples 2, 3, 4, 9 and 10 and (6) we use the same numerical example to show our proposed optimal solution is cheaper to operate than that in Chung and Wee (2007).

As we know, the arithmetic mean is always greater than or equal to the geometric mean. In short, for any two real positive numbers, say a and b, we have

$$\frac{a+b}{2} \geq \sqrt{ab} \tag{1}$$

The equality equation holds only if $a = b$, as shown in Teng (2009), Teng and Goyal (2009), Cárdenas-Barrón (2010a, b), Cárdenas-Barrón, Wee, and Blos (2011) and Teng, Cárdenas-Barrón, and Rou (2011). The proposed method seems to be easier-to-use than the standard operation process of calculus.

2. Parameters, decision variables and assumptions

For simplicity, we use the same notation as those in Chung and Wee (2007). However, we define the parameters first and then the decision variables.

2.1. *Parameters*

- d demand rate per year
- p production rate per year, with $p > d$
- ρ production time in years, $\rho = d/p$,
- H_v holding cost per unit per year for the vendor
- H_d holding cost per unit per year for the distributor
- H_b holding cost per unit per year for the buyer
- C_v vendor's setup cost per production cycle
- C_d distributor's ordering cost per order
- C_b buyer's ordering cost per order
- b backlogging cost per unit per year for the buyer

2.2. *Decision variables*

- q buyer's order quantity periodically delivered from the distributor to the buyer
- B buyer's maximum allowed backlogging quantity
- n number of deliveries per production cycle from the vendor to the distributor, with $n \geq 1$
- M number of deliveries per replenishment cycle from the distributor to the buyer, with $M \geq 1$
- Q vendor's production lot size per production cycle with $Q = nMq$

2.3. *Assumptions*

The assumptions proposed here are similar to those in Chung and Wee (2007).

(1) Production rate and demand rate are constant and known.
(2) Shortages are allowed and completely backlogged for the buyer.
(3) Replenishment is instantaneous.
(4) Quantity discount, trade credit, pricing and advertising, defective and deteriorating items are not considered.
(5) The delivery policy is a just-in-time multiple deliveries for the integrated inventory system (Chung and Wee 2007).

3. Closed-form solution by using AGI and CPS

The annual total cost (TC) of the integrated system is shown in (2) of Chung and Wee (2007). However, during the shortage period, we only pay the shortage cost, and not both the shortage and inventory costs at the same time. Hence, the annual shortage cost, which is the last term of (2), is $\frac{bB^2}{2q}$, not $\frac{B^2}{2q}(b + H_b)$. Notice that Chung (2010) published a closed related paper, and provided some comments on the economic lot size of a three-stage supply chain with backordering derived without derivatives. However, he made the same inappropriate mathematical error on the TC and solved the problem by calculus. Consequently, our optimal solution is always cheaper to operate than his. Therefore, the revised total annual cost is as follows:

$$\begin{aligned} TC(B, q, M, n) = {} & \frac{dC_v}{Q} + \frac{QH_v}{2n}[(n-1)(1-\rho) + \rho] \\ & + \frac{(M-1)q}{2}H_d + \frac{ndC_d}{Q} + \frac{dC_b}{q} \\ & + \frac{(q-B)^2 H_b}{2q} + \frac{bB^2}{2q} \end{aligned} \tag{2}$$

$$= \frac{b+H_b}{2q}\left(B - \frac{H_b q}{b+H_b}\right)^2 + \frac{d}{q}W + \frac{q}{2}Y, \quad (3)$$

where

$$W = \left[C_b + \frac{1}{M}\left(C_d + \frac{C_v}{n}\right)\right] > 0 \quad (4)$$

and

$$Y = MH_v[(n-1)(1-\rho)+\rho] + MH_d + \frac{bH_b}{b+H_b} - H_d > 0. \quad (5)$$

It is clear from (3) that the optimal buyer's maximum allowed shortage amount is

$$B^* = \frac{H_b q}{b+H_b}. \quad (6)$$

Notice that B^* in (6) is significantly different from the optimal solution in (6) of Chung and Wee (2007) as

$$B^* = \frac{H_b q}{b+2H_b}.$$

Likewise, the other optimal solutions such as q^*, M^*, n^* and TC^* obtained by our model are completely different from those in Chung and Wee (2007). Substituting (6) into (3), the total annual cost is simplified to

$$\mathrm{TC}(q, M, n) = TC(B^*, q, M, n) = \frac{d}{q}W + \frac{q}{2}Y. \quad (7)$$

Using the AGI and (7), we set

$$\frac{d}{q}\left[C_b + \frac{1}{M}\left(C_d + \frac{C_v}{n}\right)\right] = \frac{q}{2}\left\{MH_v[(n-1)(1-\rho)+\rho]. + MH_d + \frac{bH_b}{b+H_b} - H_d\right\}$$

to solve for q, and obtain the optimal buyer's order quantity as

$$q^* = \sqrt{\frac{2dW}{Y}} = \sqrt{\left\{\begin{array}{c} 2d\left[C_b + \frac{1}{M}\left(C_d + \frac{C_v}{n}\right)\right]/ \\ \{MH_v[(n-1)(1-\rho)+\rho] \\ + MH_d + \frac{bH_b}{b+H_b} - H_d\}. \end{array}\right\}} \quad (8)$$

As a result, the TC per year is reduced to

$$\mathrm{TC}(M, n) = \mathrm{TC}(B^*, q^*, M, n) = \sqrt{2dWY} \quad (9)$$

Since d is a constant, we know that the optimal solution that minimises (9) is equivalent to the optimal solution that minimises WY. After re-arranging the terms in WY, we have

$$\begin{aligned} WY(M,n) &= MC_b\{H_v[(n-1)(1-\rho)+\rho] + H_d\} \\ &+ \frac{C_d + \frac{C_v}{n}}{M}\left[\frac{bH_b}{b+H_b} - H_d\right] \\ &+ \left(C_d + \frac{C_v}{n}\right)\{H_v[(n-1)(1-\rho)+\rho] + H_d\} \\ &+ C_b\left[\frac{bH_b}{b+H_b} - H_d\right]. \end{aligned} \quad (10)$$

If $\frac{bH_b}{b+H_b} - H_d \le 0$, it is obvious from the first two terms of (10) that $WY(M,n)$ reaches its global minimum value at $M^* = 1$. Consequently, $WY(M,n)$ is simplified to

$$\begin{aligned} WY(1,n) = WY(n) &= (C_b + C_d)H_v(1-\rho)n \\ &+ \frac{C_v}{n}\left[\frac{bH_b}{b+H_b} - H_v(1-2\rho)\right] \\ &+ (C_b + C_d)\left[\frac{bH_b}{b+H_b} - H_v(1-2\rho)\right] \\ &+ C_v H_v(1-\rho). \end{aligned} \quad (11)$$

Similarly, if $\frac{bH_b}{b+H_b} - H_v(1-2\rho) \le 0$, it is clear from (11) that $WY(n)$ achieves its global minimum value at $n^* = 1$. As a result, we have the following theorem.

Theorem 1: *If* $\frac{bH_b}{b+H_b} - H_d \le 0$ *and* $\frac{bH_b}{b+H_b} - H_v(1-2\rho) \le 0$, *the optimal solution is as follows*:

$$M^* = n^* = 1,$$

$$q^* = \sqrt{\frac{2d(C_b + C_d + C_v)}{H_v\rho + \frac{bH_b}{b+H_b}}},$$

$$B^* = \frac{H_b}{b+H_b}\sqrt{\frac{2d(C_b + C_d + C_v)}{H_v\rho + \frac{bH_b}{b+H_b}}}$$

and

$$\mathrm{TC}^* = \sqrt{2d(C_b + C_d + C_v)\left[H_v\rho + \frac{bH_b}{b+H_b}\right]}. \quad (12)$$

Proof: Substituting $M^* = n^* = 1$ into (8), (6) and (9), we obtain the above results, respectively.

For the case in which $\frac{bH_b}{b+H_b} - H_v(1-2\rho) > 0$, to find the optimal n^*, let the first two terms of (11) be

$$\begin{aligned} X(n) &= (C_b + C_d)H_v(1-\rho)n \\ &+ \frac{C_v}{n}\left[\frac{bH_b}{b+H_b} - H_v(1-2\rho)\right] \\ &= A_1 n + \frac{A_2}{n}, \end{aligned} \quad (13)$$

where $A_1 = (C_b + C_d)H_v(1-\rho) > 0$ and

$$A_2 = C_v\left[\tfrac{bH_b}{b+H_b} - H_v(1-2\rho)\right] > 0.$$

Since $X(n)$ is strictly convex, in order to minimise $X(n)$, it is necessary to find the smallest positive integer n^* such that $X(n+1) - X(n) \geq 0$. From (13), simplifying $X(n+1) - X(n) \geq 0$, and solving the quadratic equation, we have

$$\begin{aligned} n^* &= \text{the smallest integer } n \text{ such that } n^2 + n - A_2/A_1 \geq 0 \\ &= \text{the smallest integer which is greater or equal to} \\ &\quad -0.5 + \sqrt{0.25 + A_2/A_1} \end{aligned} \tag{14}$$

Neither A_1 nor A_2 is a function of M. Hence, n^* can be derived from (14) without knowing M. Note that the above method to obtain n^* is similar to the method of obtaining an integral order quantity in García-Laguna, San-José, Cárdenas-Barrón, and Sicilia et al. (2010). Similar to Theorem 1, one has the following result.

Theorem 2: *If* $\frac{bH_b}{b+H_b} - H_d \leq 0$ *and* $\frac{bH_b}{b+H_b} - H_v(1-2\rho) > 0$, *then the optimal solution is as follows*

$M^* = 1$ *and* n^* *is derived from* (14)

$$q^* = \sqrt{\frac{2d\left(C_b + C_d + \frac{C_v}{n^*}\right)}{H_v[(n^*-1)(1-\rho)+\rho] + \frac{bH_b}{b+H_b}}},$$

$$B^* = \frac{H_b}{b+H_b}\sqrt{\frac{2d\left(C_b + C_d + \frac{C_v}{n^*}\right)}{H_v[(n^*-1)(1-\rho)+\rho] + \frac{bH_b}{b+H_b}}}$$

and

$$\mathrm{TC}^* = \sqrt{\left\{\begin{array}{l} 2d\left(C_b + C_d + \frac{C_v}{n^*}\right) \\ \quad \left\{H_v[(n^*-1)(1-\rho)+\rho] + \frac{bH_b}{b+H_b}\right\} \end{array}\right\}}. \tag{15}$$

Proof: It immediately follows by substituting $M^* = 1$ and n^* as in (14) into (8), (6) and (9), respectively.

We know from (13) that if $n = \sqrt{A_2/A_1}$ is an integer, then the lower bound (LB) for TC is given as

$$\mathrm{LB} = \sqrt{2d}\left\{\sqrt{C_vH_v(1-\rho)} + \sqrt{(C_b + C_d)\left[\frac{bH_b}{b+H_b} - H_v(1-2\rho)\right]}\right\}$$

Besides the case in which $\frac{bH_b}{b+H_b} - H_d \leq 0$, the case when $\frac{bH_b}{b+H_b} - H_d > 0$ should also be considered. To solve n, we re-arrange the terms in (10), and obtain as follows:

$$\begin{aligned} WY(M,n) &= (C_bM + C_d)H_v(1-\rho)n \\ &\quad + \frac{C_v}{n}\left[H_d - H_v(1-2\rho) + \frac{\frac{bH_b}{b+H_b} - H_d}{M}\right] \\ &\quad + (C_bM + C_d)[H_d - H_v(1-2\rho) \\ &\quad + \frac{\frac{bH_b}{b+H_b} - H_d}{M}\Bigg] + C_vH_v(1-\rho). \end{aligned} \tag{16}$$

For simplicity, we set

$$T \equiv H_d - H_v(1-2\rho) + \left(\frac{bH_b}{b+H_b} - H_d\right)/M$$

If $H_d - H_v(1-2\rho) \geq 0$, then $T > 0$. If $H_d - H_v(1-2\rho) + \left(\frac{bH_b}{b+H_b} - H_d\right) < 0$, then $T < 0$. When $H_d - H_v(1-2\rho) < 0$ and $H_d - H_v(1-2\rho) + \left(\frac{bH_b}{b+H_b} - H_d\right) > 0$, we know that (1) if $T \leq 0$, then $M \geq \left(\frac{bH_b}{b+H_b} - H_d\right)/[H_v(1-2\rho) - H_d]$, and (2) if $T > 0$, then $M < \left(\frac{bH_b}{b+H_b} - H_d\right)/[H_v(1-2\rho) - H_d]$. Now, we are ready to find the optimal solution to $WY(M, n)$ in (16). If $T \leq 0$, from the first two terms of (16), one can see that the optimal number of vendor's deliveries is $n^* = 1$. Substituting $n^* = 1$ into (10), we have

$$\begin{aligned} WY(M) &= \mathrm{M}C_b(H_v\rho + H_d) + \frac{C_d + C_v}{M}\left[\frac{bH_b}{b+H_b} - H_d\right] \\ &\quad + (C_d + C_v)(H_v\rho + H_d) + C_b\left[\frac{bH_b}{b+H_b} - H_d\right]. \end{aligned} \tag{17}$$

To obtain the optimal M^*, we set the first two terms of (17) as

$$\begin{aligned} Z(M) &= MC_b(H_v\rho + H_d) + \frac{C_d + C_v}{M}\left[\frac{bH_b}{b+H_b} - H_d\right] \\ &= A_3M + \frac{A_4}{M}, \end{aligned} \tag{18}$$

where $A_3 = C_b(H_v\rho + H_d) > 0$ and $A_4 = \left[\frac{bH_b}{b+H_b} - H_d\right](C_d + C_v) > 0$. Since $Z(M)$ is strictly convex, using the same analogous argument as in (14) we can obtain the optimal M^* as follows:

$$\begin{aligned} M^* &= \text{the smallest integer } M \text{ such that} \\ &\quad M^2 + M - A_4/A_3 \geq 0 \\ &= \text{the smallest integer greater than or equal to} \\ &\quad -0.5 + \sqrt{0.25 + A_4/A_3}. \end{aligned} \tag{19}$$

Notice that if $H_d - H_v(1-2\rho) < 0$ and $H_d - H_v(1-2\rho) + \left(\frac{bH_b}{b+H_b} - H_d\right) > 0$, then the optimal M^* is the larger value between $\left(\frac{bH_b}{b+H_b} - H_d\right)/[H_v(1-2\rho) - H_d]$ and the solution to (19). Therefore, one has the following result.

Table 1. Data for 10 examples.

Example	d	p	C_b	C_v	H_b	H_v	C_d	b	H_d
1	1000	3200	25	400	5	4	40	30	4.2
2	1100	4100	5	10	2	4	8	3	12
3	2100	3200	13	21	15	2	13	10	15
4	1000	5000	21	98	90	65	169	89	15
5	9800	20,030	100	4500	600	10	600	265	69
6	210,000	260,000	13	6500	130	11	210	9	7
7	3,200,000	6,400,000	5	100	44.8	10	2	160	10
8	12,100	12,600	45	150	90	15	60	90	10
9	600	1200	12.6	21	15	2.5	12.6	10	15
10	400	1100	10	125	90	66	225	60	11

Theorem 3: *If* $\frac{bH_b}{b+H_b} - H_d > 0$ *and* $T \le 0$, *then the optimal solution is as follows*:

$$n^* = 1,\ M^* \text{ is as in (19)},$$

$$q^* = \sqrt{\frac{2d\left(C_b + \frac{C_d+C_v}{M^*}\right)}{M^*(H_v\rho + H_d) + \frac{bH_b}{b+H_b} - H_d}},$$

$$B^* = \frac{H_b}{b+H_b}\sqrt{\frac{2d\left(C_b + \frac{C_d+C_v}{M^*}\right)}{M^*(H_v\rho + H_d) + \frac{bH_b}{b+H_b} - H_d}}$$

and

$$\mathrm{TC}^* = \sqrt{\left\{\begin{array}{l} 2d\left(C_b + \frac{C_d+C_v}{M^*}\right) \\ \left[M^*(H_v\rho + H_d) + \frac{bH_b}{b+H_b} - H_d\right]. \end{array}\right\}} \qquad (20)$$

Proof: It is obvious by substituting $n^* = 1$ and M^*, as in (19), into (8), (6) and (9), respectively.

Similarly, we know from (18) that if $M = \sqrt{A_4/A_3}$ is an integer, then the LB for TC is shown as

$$\mathrm{LB} = \sqrt{2d}\left\{\sqrt{(C_d + C_v)(H_v\rho + H_d)} + \sqrt{C_b\left[\frac{bH_b}{b+H_b} - H_d\right]}\right\}.$$

Finally, let us discuss the case in which $\frac{bH_b}{b+H_b} - H_d > 0$ and $T > 0$. If $\frac{bH_b}{b+H_b} - H_d > 0$, then it is clear from (10) that $WY(M, n)$ is strictly convex in M for any given n, and vice versa. To obtain the optimal M^*, we re-arrange (10) as follows:

$$WY(M,n) = F_M + F_n + C_b\left(\frac{bH_b}{b+H_b} - H_d\right) + C_d[H_v(2\rho - 1) + H_d] + C_vH_v(1-\rho), \qquad (21)$$

where $F_n = [C_dH_v(1-\rho)]n + \frac{C_v}{n}[H_v(2\rho - 1) + H_d] \equiv A_5n + \frac{A_6}{n}$ and

$$F_M = C_b\left\{H_v[n(1-\rho) + 2\rho - 1] + H_d\right\}M + \frac{1}{M}\left(C_d + \frac{C_v}{n}\right)\left[(H_b - H_d) - \frac{H_b^2}{b+H_b}\right] \equiv A_7M + \frac{A_8}{M}.$$

It is obvious that minimising $WY(M,n)$ is equivalent in minimising $F_n + F_M$. Likewise, to find the optimal M^* that minimises $F_n + F_M$ is equivalent in finding the optimal M^* that minimises F_M. Using the same analogous argument as in (19), for any given n, we can obtain the optimal M^* as follows:

M^* = the smallest integer greater than or equal to

$$-0.5 + \sqrt{0.25 + A_8/A_7}. \qquad (22)$$

Notice that if $H_d - H_v(1-2\rho) < 0$ and $H_d - H_v(1-2\rho) + \left(\frac{bH_b}{b+H_b} - H_d\right) > 0$, then the optimal M^* is the smaller value between $\left(\frac{bH_b}{b+H_b} - H_d\right)/[H_v(1-2\rho) - H_d]$ and the solution to (22).

Since F_M is a function of both n and M, it seems intractable for us to find a simple closed-form solution to the optimal n^*. However, we propose the following algorithm to simultaneously obtain the optimal n^* and M^*. Let us use the solution that minimises F_n as the initial n_1. As a result, we have

$n_1 = 1$ if $A_6 = C_v[H_v(2\rho - 1) + H_d] \le 0$. Otherwise,

n_1 = the smallest integer greater than or equal to

$$-0.5 + \sqrt{0.25 + A_6/A_5}. \qquad (23)$$

Notice that if $A_6 > 0$, then $\sqrt{0.25 + A_6/A_5} > 0.5$. Hence, the solution obtained from (23) is always $n_1 \ge 1$.

Algorithm 1: To derive M^* and n^* simultaneously

Step 1: Set the initial values: $i = 1$, n_1 as in (23). Use (22) and (9) to compute $M(n_1)$, $M(n_1 + 1)$, $\mathrm{TC}(M(n_1), n_1)$ and $\mathrm{TC}(M(n_1+1), n_1+1)$, respectively.

Table 2. Solutions to numerical examples.

Example	Conditions	Theorem	Solution (B^*, q^*, M^*, n^*, TC^*)	Lower bound
1	$\frac{bH_b}{b+H_b}-H_d=0.085714>0$ and $T\equiv H_d-H_v\left(1-\frac{2d}{p}\right)+\left(\frac{bH_b}{b+H_b}-H_d\right)/M^*=2.785714>0$	4	(189.5887, 27.0841, 1, 3, 2092.2475)	2013.4630
2	$\frac{bH_b}{b+H_b}-H_d=-10.8\le 0$ and $\frac{bH_b}{b+H_b}-H_v(1-2\rho)=-0.65366\le 0$	1	(59.6787, 149.1967, 1, 1, 339.1496)	339.1496
3	$\frac{bH_b}{b+H_b}-H_d=-9\le 0$ and $\frac{bH_b}{b+H_b}-H_v(1-2\rho)=6.625>0$	2	(75.7854, 126.3090, 1, 3, 1097.3092)	1096.8054
4	$\frac{bH_b}{b+H_b}-H_d=29.7486>0$ and $T\equiv H_d-H_v\left(1-\frac{2d}{p}\right)+\left(\frac{bH_b}{b+H_b}-H_d\right)/M^*=-16.5628\le 0$	3	(17.6917, 35.1868, 4, 1, 4987.6728)	4984.5657
5	$\frac{bH_b}{b+H_b}-H_d=114.8150>0$ and $T\equiv H_d-H_v\left(1-\frac{2d}{p}\right)+\left(\frac{bH_b}{b+H_b}-H_d\right)/M^*=107.057>0$	4	(94.5804, 136.3534, 3, 10, 64, 684.8520)	64,666.9012
6	$\frac{bH_b}{b+H_b}-H_d=1.4173>0$ and $T\equiv H_d-H_v\left(1-\frac{2d}{p}\right)+\left(\frac{bH_b}{b+H_b}-H_d\right)/M^*=15.1865>0$	4	(2373.9618, 2538.3130, 1, 14, 113, 721.1991)	11,3624.1006
7	$\frac{bH_b}{b+H_b}-H_d=25>0$ and $T\equiv H_d-H_v\left(1-\frac{2d}{p}\right)+\left(\frac{bH_b}{b+H_b}-H_d\right)/M^*=35>0$	4	(247.4874, 1131.3709, 1, 10, 96, 166.5222)	96,166.5222
8	$\frac{bH_b}{b+H_b}-H_d=35>0$ and $T\equiv H_d-H_v\left(1-\frac{2d}{p}\right)+\left(\frac{bH_b}{b+H_b}-H_d\right)/M^*=58.8095>0$	4	(103.8201, 207.6402, 1, 12, 13, 694.3636)	13,523.4188
9	$\frac{bH_b}{b+H_b}-H_d=-9\le 0$ and $\frac{bH_b}{b+H_b}-H_v(1-2\rho)=6>0$	2	(42.59577, 70.99295, 1, 2, 603.4401)	603.4401
10	$\frac{bH_b}{b+H_b}-H_d=25>0$ and $T\equiv H_d-H_v\left(1-\frac{2d}{p}\right)+\left(\frac{bH_b}{b+H_b}-H_d\right)/M^*=-2\le 0$	3	(10.73312, 17.88854, 5, 1, 3577.7087)	3577.7087

Step 2: If $\mathrm{TC}(M(n_1), n_1) > \mathrm{TC}(M(n_1+1), n_1+1)$, then compute $\mathrm{TC}(M(n_1+2), n_1+2), \ldots,$ until TC $(M(n_1+k), n_1+k)$ such that $\mathrm{TC}(M(n_1+k-1),\ n_1+k-1) \leq \mathrm{TC}(M(n_1+k), n_1+k)$. Set $n^* = n_1+k-1$, $M^* = M(n_1+k-1)$ and stop.

Step 3: If $\mathrm{TC}(M(n_1),\ n_1) < \mathrm{TC}(M(n_1+1),\ n_1+1)$ and $n_1 = 1$, then set $n^* = 1$, $M^* = M(1)$ and stop. If $\mathrm{TC}(M(n_1), n_1) < \mathrm{TC}(M(n_1+1), n_1+1)$ and $n_1 > 1$, then compute $\mathrm{TC}(M(n_1-1), n_1-1), \ldots,$ until TC $(M(n_1-k), n_1-k)$ such that $\mathrm{TC}(M(n_1-k+1),\ n_1-k+1) \leq \mathrm{TC}(M(n_1-k), n_1-k)$. Set $n^* = n_1-k+1$, $M^* = M(n_1-k+1)$ and stop.

Step 4: If $\mathrm{TC}(M(n_1), n_1) = \mathrm{TC}(M(n_1+1), n_1+1)$, then set $n^* = n_1$, $M^* = M(n_1)$ and stop.

Consequently, we have the following theoretical result.

Theorem 4: *If* $\frac{bH_b}{b+H_b} - H_d > 0$ *and* $T > 0$, *then the optimal solution is as follows*:

n^* and M^* are derived by Algorithm 1,

$$q^* = \sqrt{\frac{2d\left[C_b + \frac{1}{M^*}\left(C_d + \frac{C_v}{n^*}\right)\right]}{\left\{\begin{array}{c} M^* H_v[(n^*-1)(1-\rho)+\rho] \\ + M^* H_d + \frac{bH_b}{b+H_b} - H_d \end{array}\right\}}},$$

$$B^* = \frac{H_b}{b+H_b} q^* \text{ and}$$

$$TC^* = \sqrt{\left\{\begin{array}{l} 2d\left[C_b + \frac{1}{M^*}\left(C_d + \frac{C_v}{n^*}\right)\right] \\ \quad \left\{\begin{array}{c} M^* H_v[(n^*-1)(1-\rho)+\rho] \\ + M^* H_d + \frac{bH_b}{b+H_b} - H_d \end{array}\right\} \end{array}\right\}}, \tag{24}$$

which has a LB as following:

$$\mathrm{LB} = \sqrt{2d}\left\{\sqrt{C_d[H_v(2\rho-1)+H_d]} + \sqrt{C_v H_v(1-\rho)} + \sqrt{C_b\left[\frac{bH_b}{b+H_b} - H_d\right]}\right\}.$$

Proof: It immediately follows by substituting M^* and n^* into (8), (6) and (9), respectively.

Note that the LB is obtained by relaxing both n and M as continuous decision variables, instead of integral variables. For example, if we solve Example 7 by relaxing both n and M as continuous decision variables, and get both optimal n and M to be integral, then the optimal TC^* reaches its LB.

4. Numerical examples

To apply the above four theorems, we present 10 numerical examples for which the data are given in Table 1. Notice that the first example is taken from Chung and Wee (2007). The solutions for these examples are given in Table 2.

For the numerical Example 1, it is easy to see that our TC of 2092.25 is cheaper to operate than the TC of 2100.69 in Chung and Wee (2007). Furthermore, notice that we cannot solve Example 2 using the method in Chung and Wee (2007) simply because both q in (5) and M in (11) of Chung and Wee (2007) do not exist in real numbers. Similarly, Examples 3, 4, 9 and 10 cannot be solved by their method either. Notice that our optimal solution reaches its LB in Examples 2, 7, 9 and 10.

5. Conclusions

In this study, we have shown the following contributions beyond Chung and Wee (2007): (1) we have corrected an inappropriate mathematical error on the TC, (2) we have established a closed-form solution to integral number of deliveries for the vendor and the distributor, (3) we have completely discussed the boundary conditions when $M^* = 1$ and $n^* = 1$, (4) we have solved many cases in which their method cannot in Examples 2, 3, 4, 9 and 10 and (5) we have used the same numerical example to show our proposed optimal solution is cheaper to operate than that in Chung and Wee (2007).

While this research points out an inappropriate mathematical form in a previous research paper and uses an easy-to-understand analysis without derivatives to complete the solution process and mathematical proof, further investigation can be conducted in a number of areas. For instance, we may extend the integral requirement to all decision variables. Also, we could generalise the model to allow for partial backlogging (e.g. Park 1982; San-Jose, García-Laguna, and Sicilia 2009; Sana 2010), rework (e.g. Cárdenas-Barrón 2008, 2009a, b), imperfect quality products (e.g. Roy, Sana, and Chaudhuri 2011; Sana 2011), discount offer and backorders (e.g. Cárdenas-Barrón, Goyal, and Smith 2010), trade credits (e.g. Teng, Chang, Chern, and Chan 2007; Huang and Huang 2008), deteriorating items and uncertain lead time (e.g. Hou and Lin 2006; Widyadana, Cárdenas-Barrón, and Wee 2011), delay in payments (e.g. Chang and Dye 2001; Chang, Hung, and Dye 2002), a multi-stage multi-customer supply chain (e.g. Cárdenas-Barrón 2007) and others. Finally, we could consider the effects of inflation rate, defective rate and inspection rate on the EOQ.

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