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# An economic manufacturing quantity model for a two-stage assembly system with imperfect processes and variable production rate ☆

Horng-Jinh Chang [a], Rung-Hung Su [b], Chih-Te Yang [c,*], Ming-Wei Weng [d]

[a] Graduate Institute of Management Sciences, Tamkang University, Tamsui, New Taipei, Taiwan
[b] Gerontechnology Research Center, Yuan Ze University, Taoyuan, Taiwan
[c] Department of Industrial Management, Ching Yun University, Taoyuan, Taiwan
[d] Center for Business Education Innovation, Feng Chia University, Taichung, Taiwan

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ABSTRACT

This article considers a two-stage assembly system with imperfect processes. The former is an automatic stage in which the required components are manufactured. The latter is a manual stage which deals with taking the components to assemble the end product. In addition, the component processes are independent of each other, and the assembly rate is variable. Shortage is allowed, and the unsatisfied demand is completely backlogged. Then, we formulate the proposed problem as a cost minimization model where the assembly rate and the production run time of each component process are decision variables. An algorithm for the computations of the optimal solutions under the constraint of assembly rate is also provided. Finally, a numerical example and sensitivity analysis are carried out to illustrate the model.


## 1. Introduction

Over the past few decades of research on economic manufacturing quantity (EMQ) model, the scads of issues have appeared. The traditional EMQ model is developed based on the single item and simple (single-stage) production system. For example, Jamal, Sarker, and Mondal (2004) dealt with the optimum batch quantity in a single-stage production system in which rework is done under two different operational policies to minimize the total system cost. Cárdenas-Barrón (2007) presented the correct solutions to the two numerical examples presented by Jamal et al. (2004). Furthermore, Cárdenas-Barrón (2008) considered a simple derivation of the two inventory policies proposed by Jamal et al. (2004). In 2009, Cárdenas-Barrón (2009b) further developed an economic production quantity (EPQ) model with planned backorders for a single product at a single-stage manufacturing system that generates imperfect quality products, and all these defective products are reworked in the same cycle.

However, in the present industrial settings, the end product is manufactured through multi-stage production system, in which the raw material is transformed into the end product in a series of processing stages such as forming, cutting, grinding, assembling, polishing, and painting. Besides, since the production rates of all stages are not exactly the same, the semi-finished products are going to be accumulated between the stages under no starveling stage. Therefore, the holding cost of semi-finished products should be taken into account. Note that the two-stage system can be also used to approximate more complex multi-stage system. Early theorization of single product in a multi-stage perfect system is discussed by Taha and Skeith (1970). After, several authors have developed various extensions for the multi-stage system in the literatures. Szendrovits (1975) considered that the manufacturing cycle time as a function of the lot size in the EPQ model. Kumar and Vrat (1979) developed a stochastic model to determine the optimum level of inventory at every stage of production. Karimi (1992) determined the optimal stationary, cyclic schedules for minimizing the sum of set-up and inventory costs. Kim (1999) developed various lot sizing and inventory batching (i.e., operation-unit batching (OUB) and unit–unit batching (UUB)) models under different system characteristics and lot sizing and inventory policies.

Assembly process is a practical production system frequently occurred in the manufacturing industry, and is one of the two-stage (multi-stage) production systems. The former stage is to manufacture required components. In the latter stage, the end product is assembled from these components. Some researchers have studied the multi-stage assembly system. Crowston, Wagner, and Williams (1973) considered the optimal lot size problem for multi-stage assembly systems where each facility may have many predecessors but only a single successor. Schwarz and Schrage

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* Corresponding author.
*E-mail addresses:* chj@mail.tku.edu.tw (H.-J. Chang), runghung316@gmail.com (R.-H. Su), ctyang@cyu.edu.tw (C.-T. Yang), mingweiweng@gmail.com (M.-W. Weng).

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(1975) proposed optimal and near optimal polices for multi-stage assembly systems under continuous review with constant demand over an infinite planning horizon. Schmidt and Nahmias (1985) considered that an end product is assembled from two components. Besides, the end product has final demand, which is assumed to be random. Dellaert, De Kok, and Wang (2000) analyzed two alternative strategies for the production control of an assembly system. Dellaert and De Kok (2004) studied a simple multi-stage assembly system, with stationary stochastic demand for a single final product that is made-to-stock.

In many production systems, the defective good is a realistic phenomenon due to deterioration of machine. However, all researches above assumed that the production facility is perfect and all products are good quality. In fact, the product quality is not always perfect and usually depends on the state of the production process. Some studies have pointed out that the unreliable production facility in the multi-stage production system. Chakraborty and Rao (1988) determined the optimal batch quantity in a multi-stage production system considering the rework of defective items. Giri, Yun, and Dohi (2005a) considered an unreliable two-stage lot sizing problem in which the failure-prone machine at the first stage produces semi-finished products in batches that are transferred continuously to the next stage where the failure-free machine produces the finished products in batches. Darwish and Ben-Daya (2007) investigated the effect of imperfect production processes involving variable the frequency of preventive maintenance. Sarker, Jamal, and Mondal (2008) developed models for an optimal batch quantity for a multi-stage production system that allows rework of defective items under two operational policies reworking defectives within the same cycle and after $N$ cycles. Cárdenas-Barrón (2009a) further corrected the mathematical expressions presented by Sarker et al. (2008). Pearn, Su, Weng, and Hsu (2011) considered a two-stage production system, in which the former and the latter are automatic process and manual process, respectively. Besides, the capital investment in process quality is taken into account. Unfortunately, they did not consider for the multi-stage assembly system.

On the other hands, most researches usually assume that the production rate is predetermined and inflexible. In real life, the manufacturers often shift the production rate for achieving the cost-effective production. Recently, some articles developed various unit production costs which are the convex functions of the production rate, and the production rate is regarded as a decision variable (Eiamkanchanalai & Banerjee, 1999; Giri, Yun, & Dohi, 2005b; Khouja & Mehrez, 1994; Larsen, 1997). Besides, some authors also proposed various settings for the production rate, for example, the production rate varies with time (Balkhi & Benldlerouf, 1996), or on-hand inventory level (Bhunia & Maiti, 1997; Su & Lin, 2001). Ben-Daya, Hariga, and Khursheed (2008) further considered the shifting production rate in EPQ model.

In this paper, we probe an assembly process into the two-stage model proposed by Pearn et al. (2011). The former is an automatic stage in which the required components are manufactured. The component processes start at the same time and are independent each other. The latter is a manual stage which deals with assembling the components to the end product. To the best of our knowledge, production rate of automatic process is always higher than manual process. Under this situation, the components are going to be accumulated between two stages. Most of manufacturing industries correspond to automatic-manual (two-stage) assembly system such as computer, semiconductor, TFT–LCD, automobile, cell phone, and food industries. Besides, the production rates of the components are different, and the assembly rate is variable and can be controlled by modulating manpower. Note that Pearn et al. (2011) considered that the production rates in two stages are invariable. Then, we formulate the proposed problem as a cost minimization model where the assembly rate and the production run time of each component process are decision variables. We also prove that the optimal solution not only exists but also is unique. Finally, a numerical example is presented to demonstrate the theoretical results and the solution procedure, and then the sensitivity analysis of the optimal solution with respect to major parameters is also carried out.

## 2. Notation and assumptions

### 2.1. Notation

To develop the mathematical model of the two-stage assembly system with imperfect processes, the notation adopted in this paper includes as follows.

*System parameters*

- $n$ Number of required components in automatic stage
- $p_i$ Production rate of the component $i$ in units per unit time, where $i = 1, 2, \ldots, n$, and $p_1 > p_2 > \ldots > p_n$
- $D$ Demand rate in units per unit time
- $k$ Setup cost per cycle
- $h_i$ Holding cost for a component $i$ per unit time, where $i = 1, 2, \ldots, n$
- $h_e$ Holding cost for an end product per unit time
- $s$ Shortage cost for an end product per unit time
- $\theta_i$ Defective rate of component $i$ in automatic stage, where $i = 1, 2, \ldots, n$
- $\theta_e$ Defective rate of end product in manual stage
- $r_i$ Rework cost for a defective component $i$, where $i = 1, 2, \ldots, n$
- $r_e$ Rework cost for a defective end product
- $t_{id}$ Time period when inventory of the component $i$ depletes, where $i = 1, 2, \ldots, n$
- $t_{ed}$ Time period when inventory of the end product depletes
- $t_b$ Time period when backorder is replenished
- $T$ Length of cycle time
- $Z_i$ Maximum inventory level of the component $i$, where $i = 1, 2, \ldots, n$
- $Z_e$ Maximum inventory level of the end product
- $Z_b$ Maximum backorder level of the end product

*Decision variables*

- $p_e$ Assembly rate of the end product in units per unit time
- $t_0$ Time period when there is no production and shortage occurs
- $t_i$ Production run time of the component $i$, where $i = 1, 2, \ldots, n$

### 2.2. Assumptions

In addition, the following assumptions are used throughout this paper:

(1) The production cycle repeats infinitely.
(2) The production system is separated into two stages, automatic stage (Stage 1) and manual stage (Stage 2). This system is depicted in Fig. 1. From Fig. 1, the required components are manufactured by each machine in Stage 1. Then these components are transported from each warehouse to the assembly line. Finally, the end products are assembled from required components in Stage 2. Note that each process work has own production line and machines. Therefore, the production processes of two stages are independent each other. In addition, all of components and end

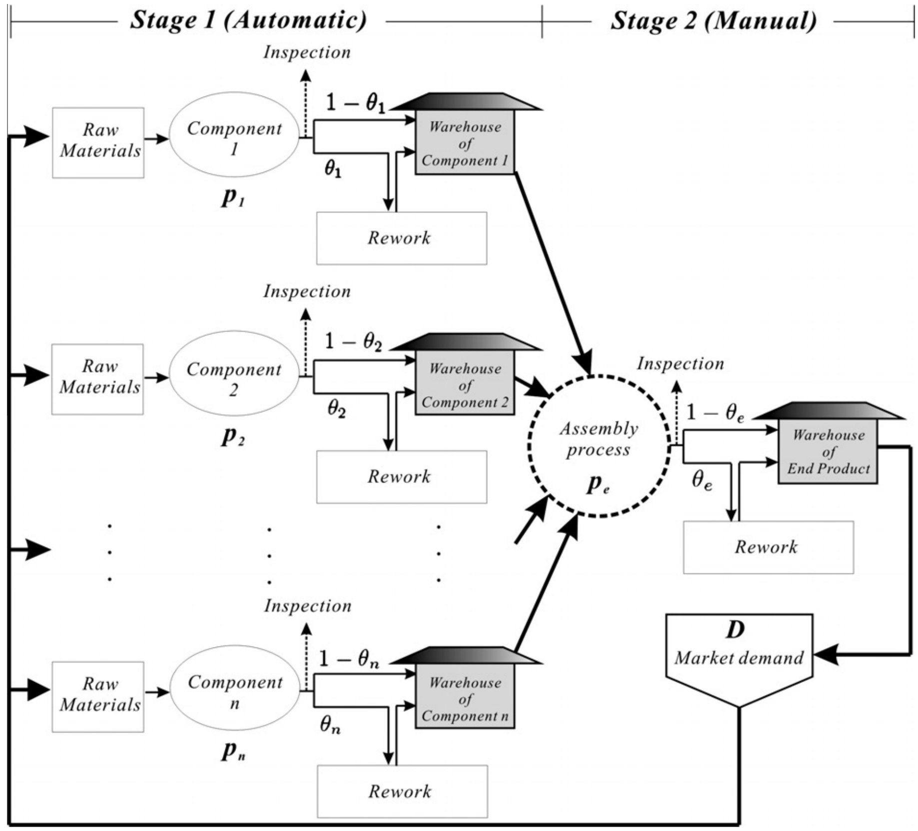


**Fig. 1.** Two-stage assembly system.

products must be inspected for finding the defective items before putting these into warehouses.

(3) For the sake of simplicity, we assume that the process quality of two stages is independent, and the inspection time is so short that it can be neglected. Moreover, the rework time of defective items is neglected in this paper.

(4) The production rate of the former stage must always be greater than the latter stage (or demand rate) due to the basic assumption of the EMQ model. Therefore, in order to avoid starvation in that stage due to lack of input from the previous stage, the minimal production rate, assembly rate, and demand rate should satisfy the condition $p_n > p_e > D$.

(5) Based on Giri et al. (2005b), the production cost for an end product consists of the following three elements: (a) the material and manufacturing costs of required components in Stage 1, $\beta_0 \geqslant 0$; (b) the labor cost for taking the components to assemble an end product in Stage 2, $\beta_1/p_e$, where $\beta_1 \geqslant 0$ is the labor cost per unit time; (c) the manpower cost for increasing assembly rate in Stage 2, $\beta_2 p_e$, where $\beta_2 \geqslant 0$ is the marginal cost of assembly rate.

## 3. Mathematical formulation

Under the notation and assumptions in the previous section, the graphic representation of inventory level can be shown as in Fig. 2. Referring to Fig. 2, we have the following results:

I. *The production run time of the component i and the cycle time:*

Since the model is completely backlogged and perfect rework process, the production quantities of all components are equal to the demand in a cycle (i.e, $p_i t_i = p_n t_n = DT$). Therefore, $t_i$ and $T$ can be expressed as

$$t_i = \frac{p_n t_n}{p_i}, \tag{1}$$

where, $i = 1, 2, \ldots, n$, and

$$T = \frac{p_n t_n}{D}, \tag{2}$$

respectively.

II. *The maximum inventory level of the component i and the maximum backorder level:*

$$Z_i = (p_i - p_e)t_i, \tag{3}$$

where $i = 1, 2, \ldots, n$, and

$$Z_b = Dt_0 = (p_e - D)t_b. \tag{4}$$

III. *The time period when inventory of the component i depletes*:

$$t_{id} = \frac{Z_i}{p_e} = \frac{(p_i - p_e)t_i}{p_e} = \left(\frac{p_i}{p_e} - 1\right)t_i, \tag{5}$$

where $i = 1, 2, \ldots, n$.

IV. *The maximum inventory level of the end product:*

$$\begin{aligned} Z_e &= (p_e - D)(t_n + t_{nd} - t_b) \\ &= (p_e - D)\left[t_n + \left(\frac{p_n}{p_e} - 1\right)t_n - \frac{Z_b}{p_e - D}\right] \quad \text{(from Eq.(4))} \\ &= (p_e - D)\left(\frac{p_n}{p_e}t_n - \frac{D}{p_e - D}t_0\right). \end{aligned} \tag{6}$$

Based on the above results, the total cost per cycle consists of the following six elements:

1. *Setup cost:*

The setup cost per cycle (denoted by $C_s$) is

$$C_s = k. \tag{7}$$

2. *Holding cost of end product:*

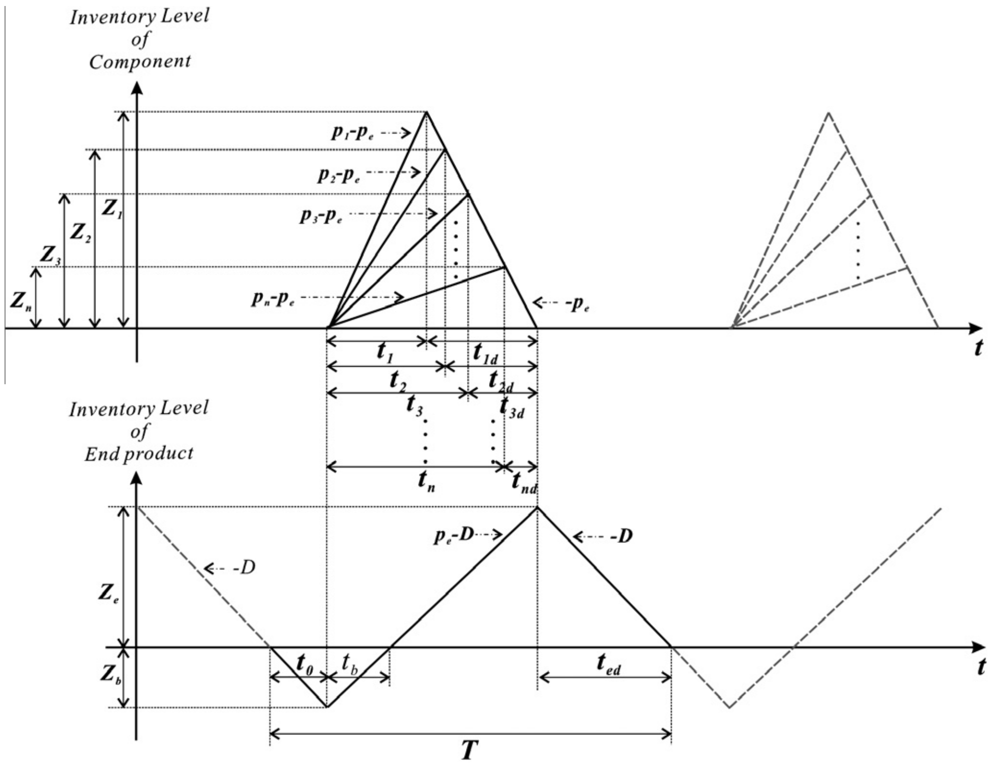


**Fig. 2.** The graph of inventory level during time period $[0, T]$.

The holding cost of end product per cycle (denoted by $HC_e$) is given by

$$HC_e = \frac{h_e Z_e (T - t_0 - t_b)}{2} = \frac{h_e D(p_e - D)}{2p_e}\left(\frac{p_n}{D}t_n - \frac{p_e t_0}{p_e - D}\right)^2. \quad \text{(from Eqs.(1), (2), and(6))} \tag{8}$$

3. *Holding cost of all components:*
The holding cost of all components per cycle (denoted by $HC_c$) is given by

$$HC_c = \sum_{i=1}^{n}\frac{h_i Z_i (t_i + t_{id})}{2} = \frac{p_n^2 t_n^2}{2}\left[\sum_{i=1}^{n} h_i\left(\frac{1}{p_e} - \frac{1}{p_i}\right)\right]. \quad \text{(from Eqs.(1), (3), and(5))} \tag{9}$$

4. *Shortage cost:*
The shortage cost per cycle (denoted by $SC$) is given by

$$SC = \frac{sZ_b(t_0 + t_b)}{2} = \frac{sp_e D}{2(p_e - D)}t_0^2 \quad \text{(from Eq.(4))} \tag{10}$$

5. *Rework costs:*
The rework costs for defective end product and all components per cycle (denoted by $RC$) is given by

$$RC = \left(r_e\theta_e + \sum_{i=1}^{n} r_i\theta_i\right)DT = \left(r_e\theta_e + \sum_{i=1}^{n} r_i\theta_i\right)p_n t_n. \quad \text{(from Eq.(2))} \tag{11}$$

6. *Production cost:*
Based on Giri et al. (2005b), the production cost per cycle (denoted by $PC$) is

$$PC = \left(\beta_0 + \frac{\beta_1}{p_e} + \beta_2 p_e\right)p_n t_n. \tag{12}$$

Therefore, the total cost per unit time (denoted by $AC(t_0, t_n, p_e)$) is given by

$$\begin{aligned} AC(t_0, t_n, p_e) &= \frac{1}{T} \times \{HC_e + HC_c + SC + RC + PC + C_s\} \\ &= \frac{D}{p_n t_n} \times \left\{\frac{p_n^2 t_n^2}{2}\left[\sum_{i=1}^{n} h_i\left(\frac{1}{p_e} - \frac{1}{p_i}\right)\right] \right. \\ &\quad + \frac{h_e D(p_e - D)\left(\frac{p_n t_n}{D} - \frac{p_e t_0}{p_e - D}\right)^2}{2p_e} + \frac{sp_e D t_0^2}{2(p_e - D)} \\ &\quad \left. + \left[\left(r_e\theta_e + \sum_{i=1}^{n} r_i\theta_i\right) + \left(\beta_0 + \frac{\beta_1}{p_e} + \beta_2 p_e\right)\right]p_n t_n + k\right\}. \end{aligned} \tag{13}$$

**Remark 1.** If the end product is assembled from only one required component, i.e., $n = 1$, and the assembly rate is invariable, i.e., $\beta_0 = \beta_1 = \beta_2 = 0$, the objective function will be reduced to Pearn et al. (2011) without capital investment in process quality. The decision variables are production run time of all components, $(t_1, t_2, \ldots, t_n)$, shortage run time $(t_0)$, and assembly rate $(p_e)$, and these variables are independent each other. Note that $t_1, t_2, \ldots,$ and $t_{n-1}$ are the function of $t_n$ from Eq. (1). Therefore, our problem is to minimize the total cost per unit time, $AC(t_0, t_n, p_e)$, by simultaneously optimizing $t_0$, $t_n$, and $p_e$, constrained on $t_0 > 0$, $t_n > 0$, and $D < p_e < p_n$.

The detail solution procedure is shown as follows. The necessary conditions for the total cost per unit time $AC(t_0, t_n, p_e)$ in Eq. (13) to be minimized are $\partial AC(t_0, t_n, p_e)/\partial t_0 = 0$, $\partial AC(t_0, t_n, p_e)/\partial t_n = 0$, and $\partial AC(t_0, t_n, p_e)/\partial p_e = 0$, simultaneously. That is,

$$\frac{\partial AC(t_0, t_n, p_e)}{\partial t_0} = \frac{D}{p_n t_n}\left[-h_e p_n t_n + \frac{p_e D(h_e + s)t_0}{p_e - D}\right] = 0, \tag{14}$$

**Table 1**
Iterations to find the optimal solutions.

| $j$ | $p_{ej}$ | $t_0^*$ | $t_3^*$ | $\mathrm{LP}_n$ | $p_{ej+1}$ | $p_{ej+1}-p_{ej}$ |
|---|---|---|---|---|---|---|
| 1 | 400.000 | 0.27285 | 1.84173 | 0.74059 | 370.556 | −29.444 |
| 2 | 370.556 | 0.18942 | 1.67872 | 0.83192 | 359.490 | −11.066 |
| 3 | 359.49 | 0.15909 | 1.62231 | 0.85104 | 355.104 | −4.386 |
| 4 | 355.104 | 0.14718 | 1.60058 | 0.85652 | 353.324 | −1.780 |
| 5 | 353.324 | 0.14237 | 1.59186 | 0.85841 | 352.595 | −0.729 |
| 6 | 352.595 | 0.14040 | 1.58831 | 0.85913 | 352.295 | −0.300 |
| 7 | 352.295 | 0.13959 | 1.58685 | 0.85942 | 352.171 | −0.124 |
| 8 | 352.171 | 0.13925 | 1.58625 | 0.85953 | 352.120 | −0.051 |
| 9 | 352.12 | 0.13912 | 1.58600 | 0.85958 | 352.099 | −0.021 |
| 10 | 352.099 | 0.13906 | 1.58590 | 0.85960 | 352.091 | −0.008 |
| 11 | 352.091 | 0.13904 | 1.58586 | 0.85961 | 352.087 | −0.004 |
| 12 | 352.087 | 0.13903 | 1.58584 | 0.85961 | 352.086 | −0.001 |
| 13 | 352.086 | 0.13902 | 1.58583 | 0.85961 | 352.085 | −0.001 |
| 14 | 352.085 | 0.13902 | 1.58583 | 0.85961 | 352.085 | 0.000 |

*Sol*: $p_e^* = 352.085$, $t_0^* = 0.13902$, $t_3^* = 1.58583$, $t_1^* = 1.05722$, $t_2^* = 1.26866$

$$\frac{\partial AC(t_0,t_n,p_e)}{\partial t_n} = \frac{D}{p_n t_n^2}\left\{\frac{p_n^2 t_n^2}{2}\left[\sum_{i=1}^{n} h_i\left(\frac{1}{p_e}-\frac{1}{p_i}\right)\right] - \frac{s p_e D t_0^2}{2(p_e-D)} - k + \frac{h_e D(p_e-D)}{2p_e} \times \left[\left(\frac{p_n t_n}{D}\right)^2 - \left(\frac{p_e t_0}{p_e-D}\right)^2\right]\right\} = 0, \tag{15}$$

and

$$\frac{\partial AC(t_0,t_n,p_e)}{\partial p_e} = \frac{D}{p_n t_n}\left\{\frac{p_n^2 t_n^2}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e+s)D^2 t_0^2}{2(p_e-D)^2} + \left(\beta_2 - \frac{\beta_1}{p_e^2}\right)p_n t_n\right\} = 0. \tag{16}$$

It is not easy to find the closed-form solution of $(t_0, t_n, p_e)$ from Eqs. (14)–(16). Besides, due to the high-power expression of the exponential function, the convex property of the total cost per unit time cannot be proved by using the Hessian matrix. Instead, we solve the problem by using the following search procedure. First, we prove that for any given $p_e \in (D, p_n)$, the optimal solution of $(t_0, t_n)$ (say $(t_0^*, t_n^*)$) not only exists but also is unique (the proof is shown as in Appendix A).

Next, we study the optimal assembly rate also exists and is unique under the solution $(t_0^*, t_n^*)$. For given $t_0^*$ and $t_n^*$, the first-order necessary condition for $AC(p_e|t_0^*, t_n^*)$ to be minimum is

$$\frac{\mathrm{d}AC(p_e|t_0^*,t_n^*)}{\mathrm{d}p_e} = \frac{D}{p_n t_n^*}\left\{\frac{p_n^2 t_n^{*2}}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e+s)D^2 t_0^{*2}}{2(p_e-D)^2} + \left(\beta_2 - \frac{\beta_1}{p_e^2}\right)p_n t_n^*\right\} = 0. \tag{17}$$

From Eq. (17), we have

$$\frac{p_n^2 t_n^{*2}}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e+s)D^2 t_0^{*2}}{2(p_e-D)^2} + \left(\beta_2 - \frac{\beta_1}{p_e^2}\right)p_n t_n^* = 0. \tag{18}$$

Let $L(p_e)$ denote the left hand side of Eq. (18), i.e.,

$$L(p_e) = \frac{p_n^2 t_n^{*2}}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e+s)D^2 t_0^{*2}}{2(p_e-D)^2} + \left(\beta_2 - \frac{\beta_1}{p_e^2}\right)p_n t_n^*. \tag{19}$$

We first rewrite Eq. (18) and have

$$\frac{p_n^2 t_n^{*2}}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e+s)D^2 t_0^{*2}}{2(p_e-D)^2} - \frac{\beta_1 p_n t_n^*}{p_e^2} = -\beta_2 p_n t_n^*.$$

Because the right hand side $-\beta_2 p_n t_n^* < 0$, then we have $\Delta < 0$, where

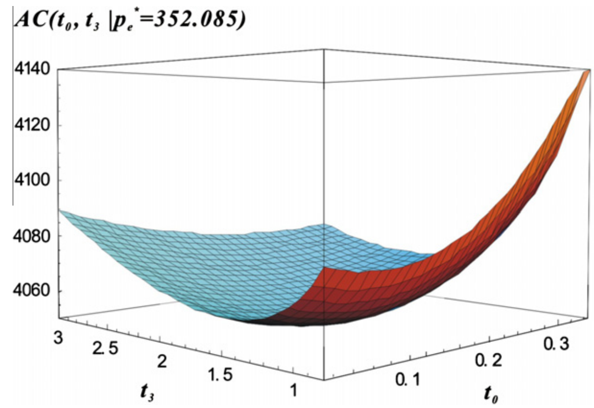


**Fig. 3.** The total cost per unit time, $AC(t_0, t_3|p_e^* = 352.085)$.

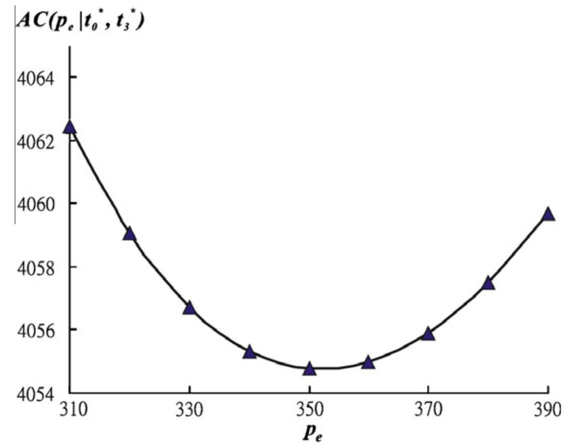


**Fig. 4.** Graphical representation of $AC(p_e|t_0^*, t_n^*)$.

$$\Delta \equiv \frac{p_n^2 t_n^{*2}}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h+s)D^2 t_0^{*2}}{2(p_e-D)^2} - \frac{\beta_1 p_n t_n^*}{p_e^2},$$

Next, taking the first-order derivative of $L(p_e)$ with respect to $p_e$, we obtain

$$\begin{aligned}\frac{\mathrm{d}L(p_e)}{\mathrm{d}p_e} &= -\frac{p_n^2 t_n^{*2}}{p_e^3}\left(h_e - \sum_{i=1}^{n} h_i\right) + \frac{(h_e+s)D^2 t_0^{*2}}{(p_e-D)^3} + \frac{2\beta_1 p_n t_n^*}{p_e^3}\\ &= -\frac{2}{p_e}\left\{\frac{p_n^2 t_n^{*2}}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e+s)D^2 p_e t_0^{*2}}{2(p_e-D)^3} - \frac{\beta_1 p_n t_n^*}{p_e^2}\right\}\\ &= -\frac{2}{p_e}\left\{\frac{p_n^2 t_n^{*2}}{2p_e^2}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e+s)D^2 t_0^{*2}}{2(p_e-D)^2} \times \left(1 + \frac{D}{p_e-D}\right) - \frac{\beta_1 p_n t_n^*}{p_e^2}\right\}\\ &= -\frac{2}{p_e}\Delta + \frac{(h_e+s)D^3 t_0^{*2}}{p_e(p_e-D)^3} > 0.\end{aligned}$$

Therefore, $L(p_e)$ is a strictly increasing function of $p_e \in (D, p_n)$. Furthermore, calculating the following limit:

$$\lim_{p_e \to D^+} L(p_e) = -\infty < 0. \tag{20}$$

According to the above, it is clear that when the stock of end product increases, the total cost per unit time under the solution $(t_0^*, t_n^*), AC(p_e|t_0^*, t_n^*)$, decreases. It implies that although the holding cost of end product increases, the holding cost of all components decreases. Next, we also calculate the following limit and define as $LP_n$.

**Table 2**
Effect of changes in various parameters of the inventory model.

| Parameter | % Change | Optimal solutions | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $t_0^*$ | $t_1^*$ | $t_2^*$ | $t_3^*$ | $T^*$ | $p_e^*$ | $AC$ |
| $h_1$ | 50 | 0.13842 | 1.00830 | 1.20996 | 1.51246 | 2.01661 | 354.795 | 4060.15 |
| | 25 | 0.13870 | 1.03170 | 1.23804 | 1.54755 | 2.06340 | 353.458 | 4057.49 |
| | −25 | 0.13938 | 1.08520 | 1.30224 | 1.62779 | 2.17039 | 350.671 | 4051.91 |
| | −50 | 0.13980 | 1.11605 | 1.33926 | 1.67407 | 2.23209 | 349.210 | 4048.96 |
| $h_2$ | 50 | 0.14535 | 0.99950 | 1.19940 | 1.49925 | 1.99900 | 358.681 | 4062.27 |
| | 25 | 0.14228 | 1.02616 | 1.23140 | 1.53925 | 2.05233 | 355.441 | 4058.62 |
| | −25 | 0.13551 | 1.09377 | 1.31252 | 1.64065 | 2.18753 | 348.584 | 4050.62 |
| | −50 | 0.13162 | 1.13737 | 1.36485 | 1.70606 | 2.27474 | 344.900 | 4046.18 |
| $h_3$ | 50 | 0.16920 | 1.06165 | 1.27398 | 1.59247 | 2.12330 | 365.539 | 4058.85 |
| | 25 | 0.15353 | 1.05533 | 1.26640 | 1.58300 | 2.11066 | 358.706 | 4056.99 |
| | −25 | 0.12497 | 1.06712 | 1.28055 | 1.60069 | 2.13425 | 345.523 | 4052.10 |
| | −50 | 0.11073 | 1.08580 | 1.30296 | 1.62870 | 2.17160 | 338.879 | 4049.05 |
| $h_e$ | 50 | 0.14848 | 1.01684 | 1.22021 | 1.52526 | 2.03368 | 346.363 | 4056.96 |
| | 25 | 0.14479 | 1.03371 | 1.24045 | 1.55056 | 2.06741 | 348.867 | 4056.00 |
| | −25 | 0.12967 | 1.09224 | 1.31068 | 1.63836 | 2.18447 | 356.417 | 4053.04 |
| | −50 | 0.11362 | 1.15021 | 1.38026 | 1.72532 | 2.30043 | 362.696 | 4050.55 |
| $s$ | 50 | 0.09778 | 1.02540 | 1.23048 | 1.53810 | 2.05080 | 347.656 | 4056.47 |
| | 25 | 0.11492 | 1.03858 | 1.24629 | 1.55786 | 2.07715 | 349.557 | 4055.74 |
| | −25 | 0.17530 | 1.08561 | 1.30273 | 1.62841 | 2.17122 | 355.633 | 4053.35 |
| | −50 | 0.23607 | 1.13423 | 1.36108 | 1.70135 | 2.26847 | 361.050 | 4051.20 |
| $\theta_1$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.35 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.05 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.45 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.15 |
| $\theta_2$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.65 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.20 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.30 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4053.85 |
| $\theta_3$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.65 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.20 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.30 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4053.85 |
| $\theta_e$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.35 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.05 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.45 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.15 |
| $r_1$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.35 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.05 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.45 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.15 |
| $r_2$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.65 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.20 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.30 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4053.85 |
| $r_3$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.65 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.20 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.30 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4053.85 |
| $r_e$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.35 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4055.05 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.45 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4054.15 |
| $\beta_0$ | 50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 5554.75 |
| | 25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 4879.75 |
| | −25 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 3304.75 |
| | −50 | 0.13902 | 1.05722 | 1.26866 | 1.58583 | 2.11444 | 352.085 | 2554.75 |
| $\beta_1$ | 50 | 0.27285 | 1.22782 | 1.47338 | 1.84173 | 2.45564 | 399.999 | 4249.95 |
| | 25 | 0.24339 | 1.18860 | 1.42632 | 1.78290 | 2.37720 | 389.797 | 4155.85 |
| | −25 | 0.02653 | 0.92959 | 1.11551 | 1.39439 | 1.85919 | 309.953 | 3941.46 |
| | −50 | 0.00000 | 0.90167 | 1.08200 | 1.35251 | 1.80334 | 300.000 | 3816.91 |
| $\beta_2$ | 50 | 0.00000 | 0.90167 | 1.08200 | 1.35251 | 1.80334 | 300.000 | 4291.91 |
| | 25 | 0.03020 | 0.93352 | 1.12023 | 1.40028 | 1.86705 | 311.332 | 4178.67 |
| | −25 | 0.27285 | 1.22782 | 1.47338 | 1.84173 | 2.45564 | 399.999 | 3912.45 |
| | −50 | 0.27285 | 1.22782 | 1.47338 | 1.84173 | 2.45564 | 399.999 | 3732.45 |

$$LP_n \equiv \lim_{p_e \to p_n^-} L(p_e)$$
$$= \frac{1}{2}\left(h_e - \sum_{i=1}^{n} h_i\right) t_n^{*2} - \frac{(h_e + s)D^2}{2(p_n - D)^2} t_0^{*2} + \left(\beta_2 - \frac{\beta_1}{p_n^2}\right) p_n t_n^{*}. \qquad (21)$$

$LP_n$ can be expressed whether $AC(p_e|t_0^*, t_n^*)$ still decreases as $p_e \to p_n^-$. Then we have the following result:

**Theorem 1.** For any given $(t_0^*, t_n^*)$,

(a) if $LP_n \geqslant 0$, then the solution $p_e^* \in (D, p_n)$ which minimizes $AC(p_e|t_0^*, t_n^*)$ not only exists but also is unique,
(b) if $LP_n < 0$, then the optimal value of $p_e$ is $p_e^* \to p_n$.

**Proof 1.** See the Appendix B. □

Summarizing the above results, we establish the following algorithm to obtain the optimal solution of our problem.

**Algorithm**

*Step 1*: Start with $j = 1$ and $p_{ej} \to p_n$.
*Step 2*: Put $p_{ej}$ into Eq. (A5) to obtain the corresponding value of $t_n$, i.e., $t'_n$.
*Step 3*: Put $p_{ej}$ and $t'_n$ into Eq. (A6) to obtain the corresponding value of $t_0$, i.e., $t'_0$.
*Step 4*: Put $t'_n$ and $t'_0$ into Eq. (21) to obtain $LP_n$.
*Step 5*: If $LP_n$<0, let $p_{ej}^* \to p_n$, then go to Step 8. Otherwise, go to Step 6.
*Step 6*: Put $t'_n$ and $t'_0$ into Eq. (18), then solve the optimal $p_{ej+1}$.
*Step 7*: If the difference between $p_{ej}$ and $p_{ej+1}$ is sufficiently small, set $p_e^* = p_{ej+1}$, then go to Step 8. Otherwise, set $j = j + 1$ and go back to Step 2.
*Step 8*: Calculate the corresponding values of $t_n^*, t_0^*, t_i^*$, and $T^*$ by Eqs. (A5), (A6), (1), and (2) respectively, where $i = 1, 2, \ldots, n - 1$.

## 4. Numerical example and sensitivity analysis

To illustrate the results, we consider a two-stage assembly system with three components processes ($n = 3$) in Stage 1 and single assembly process in Stage 2. Some known parameters are given as follows: $k$ = \$100/cycle, $D$ = 300/unit time, $s$ = \$0.5/unit/unit time, $\beta_0 = 10$, $\beta_1 = 500$, $\beta_2 = 0.005$,

*Component 1 process*: $p_1$ = 600 per unit time, $h_1$ = \$0.1 per unit per unit time, $\theta_1 = 0.04$, $r_1$ = \$0.1/unit.
*Component 2 process*: $p_2$ = 500/unit time, $h_2$ = \$0.2/unit/unit time, $\theta_2 = 0.03$, $r_2$ = \$0.2/unit.
*Component 3 process*: $p_3$ = 400/unit time, $h_3$ = \$0.3/unit/unit time, $\theta_3 = 0.02$, $r_3$ = \$0.3/unit.
*Assembly process*: $h_e$ = \$0.4/unit/unit time, $\theta_e = 0.01$, $r_e$ = \$0.4/unit.

Then, applying the proposed algorithm, the iterations to find the optimal replenishment policy are shown in Table 1. After 14 iterations, we have $p_e^* = 352.085, t_0^* = 0.13902, t_3^* = 1.58583, t_1^* = 1.05722, t_2^* = 1.26866$, and $T^* = 2.11444$. Then, from Eq. (13), we obtain $AC(t_0^*, t_3^*, p_e^*) = 4054.75$. The three-dimensional total cost per unit time graph as $p_e^* = 352.085$ is shown in Fig. 3. Note that we run the numerical results with distinct values of $p_e = 310(10)390$. The numerical results indicate that $AC(p_e|t_0^*, t_3^*)$ is strictly concave in $p_e$, as shown in Fig. 4. Consequently, we are sure that the local minimum obtained here indeed the global minimum solution.

Now, this numerical example is considered to study the effects of changes in the system parameters $h_1, h_2, h_3, h_e, s, \theta_1, \theta_2, \theta_3, \theta_e, r_1, r_2, r_3, r_e, \beta_0, \beta_1$, and $\beta_2$ on the optimal values of $t_0^*, t_1^*, t_2^*, t_3^*, T^*, p_e^*$, and $AC(t_0^*, t_3^*, p_e^*)$. The sensitivity analysis is performed by changing each of the major parameters by +50%, +25%, −25%, and −50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 2.

On the basis of the results of Table 2, the following observations can be made:

(1) When the values of parameters $h_1$, $h_2$, $h_3$, and $h_e$ increase, $t_1^*, t_2^*, t_3^*$, and $T^*$ decrease but $AC(t_0^*, t_3^*, p_e^*)$ increases. It implies that if the holding cost per unit per unit time increases, one should reduce the production run time to avoid unnecessary inventory. However, when the value of $h_3$ exceeds some limit value (i.e., $h_3 > 0.357$), the production run time and cycle time start to increase for retarding the growth of the holding cost.
(2) As the shortage cost per unit per unit time, $s$, increases, $AC(t_0^*, t_3^*, p_e^*)$ increases while $t_0^*$ decreases. It implies that one should focus on the length of the period during which shortage is allowed for reducing the shortage quantity.
(3) When the values of parameters $\theta_1$, $\theta_2$, $\theta_3$, $\theta_e$, $r_1$, $r_2$, $r_3$, $r_e$, and $\beta_0$ increase, $t_0^*, t_1^*, t_2^*, t_3^*, T^*$, and $p_e^*$ are still fixed but the minimum total cost per unit time $AC(t_0^*, t_3^*, p_e^*)$ increases. If these costs and the defective rates could be reduced effectively, the total cost per unit time will be improved.
(4) With increase in the value of parameter $\beta_1$, $AC(t_0^*, t_3^*, p_e^*)$ increases. Therefore, in order to decrease minimum total cost per unit time, one should decrease the labor cost per unit time (i.e., wage or salary). Besides, $p_e^*$ increases as $\beta_1$ increases, which implies that the assembly rate should be increased for retarding the growth of the labor cost. But, when the value of $\beta_1$ exceeds some limit value i.e., $\beta_1 > 660.846$, the assembly rate stops at $p_e^* \to p_3 = 400.000$ due to the constraint of $p_e$, $D < p_e < p_3$. Also, assembly rate stops at $p_e^* = 300.000$ when $\beta_1 < 347.816$.
(5) With increase in the value of parameter $\beta_2$, $AC(t_0^*, t_3^*, p_e^*)$ increases. Therefore, in order to decrease minimum total cost per unit time, one should decrease the marginal cost of assembly rate. In addition, $p_e^*$ decreases as $\beta_2$ increases, which implies that the assembly rate should be decreased for retarding the growth of the manpower cost. But, when the value of $\beta_2$ exceeds some limit value i.e., $\beta_2 > 0.0067$, the assembly rate stops at $p_e^* = 300.000$ due to the constraint of $p_e$. Also, assembly rate stops at $p_e^* \to p_3 = 400.000$ when $\beta_2 > 0.0040$.

## 5. Conclusion

In this paper, we amended the paper of Pearn et al. (2011) with a view to making the model more relevant and applicable in practice. We investigated a two-stage assembly system in which the $n$ required components are produced in Stage 1 (automatic process) and the end products are assembled form $n$ components in Stage 2 (manual process). In addition, we assume that the production (assembly) cost is a convex function of assembly rate. Consequently, the production run time of all components $(t_1, t_2, \ldots, t_n)$, shortage time $(t_0)$, and assembly rate $(p_e)$ are the decision variables for minimizing the total related cost per unit. The proposed model can be adopted in inventory control of production system such as automobile, semiconductor, TFT–LCD, and food industries. The necessary and sufficient conditions of the existence and uniqueness of the optimal solution are shown. Next, we provided a simple algorithm to find the optimal solution of $(t_0, t_n, p_e)$ under the constraint $D < p_e < p_n$. Furthermore, the effects of the model parameters on the optimal solutions and minimum total cost per unit time are investigated through a numerical example.

The proposed model can be extended in several ways. For instance, in real life, it may take a significant amount of time to rework (disassembling, correcting and then reassembling) for the seriously defective end products in the production assembly system. Therefore, it would be interesting to relax Assumption (3), and take the rework time into account when dealing with an imperfect assembly system. Additionally, the machine, manpower, tools, and idle costs can also be considered to extend the proposed model.

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## Appendix A

From Eqs. (14)–(16), we can obtain that

$$t_0 = \frac{h_e p_n (p_e - D)}{p_e D (h_e + s)} t_n, \tag{A1}$$

$$-k + \frac{p_n^2 t_n^2}{2}\left[\sum_{i=1}^{n} h_i\left(\frac{1}{p_e} - \frac{1}{p_i}\right)\right] = \frac{s p_e D t_0^2}{2(p_e - D)} - \frac{h_e D (p_e - D)}{2 p_e}\left[\left(\frac{p_n t_n}{D}\right)^2 - \left(\frac{p_e t_0}{p_e - D}\right)^2\right], \tag{A2}$$

and

$$\frac{p_n^2 t_n^2}{2 p_e}\left(h_e - \sum_{i=1}^{n} h_i\right) - \frac{(h_e + s) D^2 t_0^2}{2(p_e - D)} + \left(\beta_2 - \frac{\beta_1}{p_e^2}\right) p_n t_n = 0. \tag{A3}$$

From the above results, $t_0$ is a function of $t_n$. Given any $D < p_e < p_n$, we substitute $t_0$ in Eq. (A1) into Eq. (A2) and obtain

$$G t_n^2 - k = 0, \tag{A4}$$

where

$$G = \frac{p_n^2}{2}\left[\sum_{i=1}^{n} h_i\left(\frac{1}{p_e} - \frac{1}{p_i}\right)\right] + \frac{s h_e p_n^2 (p_e - D)}{2 p_e D (h_e + s)} > 0.$$

Solving Eq. (A4), we can obtain the optimal value of $t_n$ that is

$$t_n^* = \sqrt{k/G}. \tag{A5}$$

Then, we substitute $t_n^*$ in Eq. (A5) into Eq. (A1), the corresponding $t_0^*$ is determined, i.e.,

$$t_0^* = \frac{h_e p_n (p_e - D)}{p_e D (h_e + s)} \sqrt{k/G}. \tag{A6}$$

Furthermore, we also calculate that

$$\frac{\partial^2 AC(t_0, t_n | p_e)}{\partial t_0^2}\Big|_{(t_0,t_n)=(t_0^*,t_n^*)} = \frac{p_e D^2 (h_e + s)}{p_n t_n^* (p_e - D)} > 0,$$

$$\frac{\partial^2 AC(t_0, t_n | p_e)}{\partial t_n^2}\Big|_{(t_0,t_n)=(t_0^*,t_n^*)} = \frac{p_n D}{t_n^*}\left[\frac{h_e (p_e - D)}{p_e D} + \sum_{i=1}^{n} h_i\left(\frac{1}{p_e} - \frac{1}{p_i}\right)\right] > 0,$$

and

$$\frac{\partial^2 AC(t_0, t_n | p_e)}{\partial p_e^2}\Big|_{(t_0,t_n)=(t_0^*,t_n^*)} = \frac{-h_e D}{t_n^*}.$$

Therefore, the determinant of the Hessian matrix at the stationary point $(t_0^*, t_n^*)$ is

$$\det(\mathrm{H}) = \frac{p_e D^3 (h_e + s)}{t_n^{*2} (p_e - D)}\left[\frac{h_e (p_e - D)}{p_e D} + \sum_{i=1}^{n} h_i\left(\frac{1}{p_e} - \frac{1}{p_i}\right)\right] - \frac{h_e^2 D^2}{t_n^{*2}} = \frac{h_e s D^2}{t_n^{*2}} + \frac{p_e D^3 (h_e + s)}{t_n^{*2} (p_e - D)} \sum_{i=1}^{n} h_i\left(\frac{1}{p_e} - \frac{1}{p_i}\right) > 0.$$

Consequently, the stationary point $(t_0^*, t_n^*)$ is the optimal solution for given $p_e \in (D, p_n)$.

## Appendix B

(a) If $LP_n \geqslant 0$, we can find a unique solution $p_e^* \in (D, p_n)$ such that Eq. (18) is hold by the Intermediate Value Theorem. After assembling Eq. (18), the second-order derivative of $AC(p_e | t_0^*, t_n^*)$ with respect to $p_e$ becomes

$$\frac{d^2 AC(p_e | t_0^*, t_n^*)}{d p_e^2} = \frac{D}{p_e p_n t_n^*}\left\{2\beta_2 p_n t_n^* + \frac{(h_e + s) D^3}{(p_e - D)^3} t_0^{*2}\right\} > 0.$$

Consequently, there exists a unique optimal assembly rate $p_e^* \in (D, p_n)$ which minimizes $AC(p_e | t_0^*, t_n^*)$ .

(b) If $LP_n < 0$, then it can be obtained that $L(p_e) < 0$ for $p_e^* \in (D, p_n)$. Therefore, from Eq. (17), we obtain that $\frac{dAC(p_e|t_0^*,t_n^*)}{dp_e} = \frac{DL(p_e)}{p_n t_n^*} < 0$ for $p_e^* \in (D, p_n)$, which implies that a large value of $p_e$ causes a lower value of $AC(p_e | t_0^*, t_n^*)$. Hence the minimum value of $AC(p_e | t_0^*, t_n^*)$ occurs at the point $p_e^* \to p_n$.

## References


Balkhi, Z. T., & Benldlerouf, L. (1996). A production lot size inventory model for deteriorating items and arbitrary production and demand rates. *European Journal of Operational Research, 92*, 302–309.

Ben-Daya, M., Hariga, M., & Khursheed, S. N. (2008). Economic production quantity model with a shifting production rate. *International Transactions in Operational Research, 15*, 87–101.

Bhunia, A. K., & Maiti, M. (1997). Deterministic inventory models for variable production. *The Journal of the Operational Research Society, 48*, 221–224.

Cárdenas-Barrón, L. E. (2007). On optimal manufacturing batch size with rework process at single-stage production system. *Computers & Industrial Engineering, 53*, 196–198.

Cárdenas-Barrón, L. E. (2008). Optimal manufacturing batch size with rework in a single-stage production system – A simple derivation. *Computers & Industrial Engineering, 55*, 758–765.

Cárdenas-Barrón, L. E. (2009a). On optimal batch sizing in a multi-stage production system with rework consideration. *European Journal of Operational Research, 196*, 1238–1244.

Cárdenas-Barrón, L. E. (2009b). Economic production quantity with rework process at a single-stage manufacturing system with planned backorders. *Computers & Industrial Engineering, 57*, 1105–1113.

Chakraborty, S., & Rao, N. V. (1988). EBQ for a multistage production system considering rework. *Opsearch, 25*, 75–88.

Crowston, W. B., Wagner, M., & Williams, J. F. (1973). Economic lot size determination in multi-stage assembly systems. *Management Science, 19*, 517–527.

Darwish, M. A., & Ben-Daya, M. (2007). Effect of inspection errors and preventive maintenance on a two-stage production inventory system. *International Journal of Production Economics, 107*, 301–313.

Dellaert, N., & De Kok, T. (2004). Integrating resource and production decisions in a simple multi-stage assembly system. *International Journal of Production Economics, 90*, 281–294.

Dellaert, N. P., De Kok, A. G., & Wang, W. (2000). Push and pull strategies in multi-stage assembly systems. *Statistica Neerlandica, 54*, 175–189.

Eiamkanchanalai, S., & Banerjee, A. (1999). Production lot sizing with variable production rate and explicit idle capacity cost. *International Journal of Production Economics, 59*, 251–259.

Giri, B. C., Yun, W. Y., & Dohi, T. (2005a). Optimal lot sizing in an unreliable two-stage serial production-inventory system. *International Transactions in Operational Research, 12*, 63–82.

Giri, B. C., Yun, W. Y., & Dohi, T. (2005b). Optimal design of unreliable production-inventory systems with variable production rate. *European Journal of Operations Research, 162*, 372–386.

Jamal, A. A. M., Sarker, B. R., & Mondal, S. (2004). Optimal manufacturing batch size with rework process at single-stage production system. *Computers & Industrial Engineering, 47*, 77–89.

Karimi, I. A. (1992). Optimal cycle times in multistage serial systems with set-up and inventory costs. *Management Science, 38*, 1467–1481.

Khouja, M., & Mehrez, A. (1994). Economic production lot size model with variable production rate and imperfect quality. *The Journal of the Operational Research Society, 45*, 1405–1417.

Kim, D. S. (1999). Optimal two-stage lot sizing and inventory batching policies. *International Journal of Production Economics, 58*, 221–234.

Kumar, U., & Vrat, P. (1979). Optimization of single-item multistage stochastic production inventory systems. *Journal of Operational Research Society, 30*, 547–553.
Larsen, C. (1997). Using a variable production rate as a response mechanism in the economic production lot size model. *Journal of the Operational Research Society, 48*, 97–99.
Pearn, W. L., Su, R. H., Weng, M. W., & Hsu, C. H. (2011). Optimal production run time for two-stage production system with imperfect processes and allowable shortages. *Central European Journal of Operational Research, 19*, 533–545.
Sarker, B. R., Jamal, A. M. M., & Mondal, S. (2008). Optimal batch sizing in a multi-stage production system with rework consideration. *European Journal of Operational Research, 184*, 915–929.
Schmidt, C., & Nahmias, S. (1985). Optimal policy for a two-stage assembly system under random demand. *Operations Research, 33*, 1130–1145.
Schwarz, L. B., & Schrage, L. (1975). Optimal and system myopic policies for multi-echelon production/inventory assembly systems. *Management Science, 21*, 1285–1294.
Su, C. T., & Lin, C. W. (2001). A production inventory model which considers the dependence of production rate on demand and inventory level. *Production Planning and Control, 12*, 69–75.
Szendrovits, A. Z. (1975). Manufacturing cycle time determination for a multi-stage economic production quantity model. *Management Science, 22*, 298–308.
Taha, H. A., & Skeith, R. W. (1970). The economic lot sizes in multistage production systems. *IIE Transactions, 2*, 157–162.