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Interfaces with Other Disciplines

Fuzzy portfolio model with different investor risk attitudes

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ABSTRACT

We propose a fuzzy portfolio model designed for efficient portfolio selection with respect to uncertain or vague returns. Although many researchers have studied the fuzzy portfolio model, no researcher has yet attempted a behavioral analysis of the investor in the fuzzy portfolio model. To address this problem, we examined investor risk attitudes—risk-averse, risk-neutral, or risk-seeking behaviors—to discover an efficient method for fuzzy portfolio selection. In this study, we relied on the advantages of possibilistic mean–standard deviation models that we believed would fit the risk attitudes of investors. Thus, we developed a fuzzy portfolio model that focuses on different investor risk attitudes so that fuzzy portfolio selection for investors who possess different risk attitudes can be achieved more easily. Finally, we presented a numerical example of a portfolio selection problem to illustrate ways to address problems presented by a variety of investor risk attitudes.

1. Introduction

A majority of existing portfolio selection models are based on probability theory. Many of these portfolio selection models focus on the allocation of investors' capital into a variety of securities chosen to help investors reach their investment goals. Markowitz (1952) first proposed the mean–variance portfolio model. He also played an important role in the development of portfolio theory. A number of scholars, including Sharpe (1970), Merton (1972), Perold (1984), Pang (1980), Vörös (1986), Best and Grauer (1990), and Best and Hlouskova (2000) proposed different mathematical methods for the development of portfolio models based on probability and optimization theory.

However, many non-statistic variables exist that can be difficult to manage during applications of probabilistic methods. Two theories, the fuzzy set theory and possibility theory have been proposed by Zadeh (1965) and Dubois and Prade (1988). These theories have been used to describe fuzzy random variables that can be found in the stock market. Recently, a number of authors have turned their attention to fuzzy portfolio selection and possibilistic portfolio selection. Some researchers proposed fuzzy portfolio models in which the expected rate of return and risks were vague targets. Watada (1997), Ramaswamy (1998), and Tanaka and Guo (1999) proposed upper and lower possibility distributions. They also converted portfolio problems into quadratic programming problems. Tanaka et al. (2000) extended traditional probability measures into fuzzy possibilities by the application of a fuzzy weighted average vector and a covariance matrix. Some interval analyses addressed the problems of imprecision and uncertainty found in portfolio selection and investment decision problems (Inuiguchi and Tanino, 2000; Lai et al., 2002; León et al., 2002; Chen and Tan, 2009). Carlsson et al. (2002) proved that feasible solutions form a convex polytope that contains all optional solutions for portfolio selection problems. Zhang and Nie (2003, 2004) proposed the concept of lower and upper-possibilistic variances and covariances of fuzzy numbers in fuzzy portfolio analysis. They then stated that the admissible efficient portfolio model can be proposed based on the assumption that the expected return and risk of assets contain admissible errors. Giove et al. (2006) formulated a minimax regret portfolio problem in which the prices of securities were considered interval variables. This transformed the initial interval problem into a set of optimization problems for the portfolio. Lacagnina and Pecorella (2006) extended mean–variance models to find optimal investment strategies based on complicated financial situations. Zhang (2007) solved portfolio selection problems for bounded assets by the transformation of the mean–standard deviation model to a linear programming model based on possibility distributions. Huang (2008a) proposed two fuzzy-mean semi-variance models to address the situation in which investors attempt to obtain high returns by the avoidance of risk. Huang eliminated the asymmetry degree of return distributions. Huang then designed a hybrid intelligent algorithm to improve the effectiveness of the portfolio model (Huang, 2008b). Li et al. (2010b) proposed the concept of skewness, which they defined as the third central moment. They then extended the fuzzy mean–variance model to a mean–variance–skewness model. Li and Xu (2007, 2009) developed a possibilistic portfolio selection model that employed interval center values that were converted into a nonlinear goal programming problem. They obtained a satisfactory solution by the use of a genetic algorithm.

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They then proposed a new mean–variance portfolio selection model in a hybrid uncertain environment by the definition of λ-mean–variance efficient frontiers. They concluded that the proposed model can provide results that are more flexible. Xu et al. (2011) proposed a λ mean-hybrid entropy model that provides the investor with a tradeoff frontier between security return and risk after solution of the bi-objective functions. Li et al. (2009) proposed a hybrid intelligent algorithm that they solved by the use of a genetic algorithm. This implied that the hybrid intelligent algorithm is robust and more effective in fuzzy portfolio selection. Li et al. (2010a) revised the fuzzy chance-constrained portfolio selection model and added a risk constraint to the fuzzy chance-constrained portfolio selection model. Li et al. (2012) proposed an expected regret minimization model. They proved that this model is advantageous when employed to obtain distributive investment and the reduction of investor regret.

Although many researchers have devoted themselves to the study of fuzzy portfolio models, no researcher has yet attempted an analysis of investor risk behavior during fuzzy portfolio selection. The utility theory assumes that investors who possess extremely rational attitudes are risk averse. However, this is not always true because of systematic biases that exist in human psychology. The first bias arises from investors' tendencies towards overconfidence. The second bias arises from investors' desires to avoid regret. In addition, Kahneman and Tversky (1979) proposed the concept of the value function. This concept differs from the utility function because its reference point is determined by the subjective impressions of individuals. To achieve return levels below the reference point, investors tend to display risk-seeking behavior. In contrast, to achieve return levels above this reference point, investors tend to display risk-averse behavior. When we study theories of the utility or value functions, we discover that investors cannot solely be described as risk averse. They can also be described as risk seeking. Therefore, we need to develop efficient portfolios based on these different investor risk attitudes. To address the problem stated above, we relied on the advantages provided by possibilistic mean–standard deviation models to examine different investor risk attitudes. In so doing, we developed a fuzzy portfolio model that addresses the different investor risk attitudes that occur in behavioral finance.

We organized this paper in the following manner. In Section 2, we briefly introduce fuzzy numbers, the concepts of lower and upper possibilistic means, variances, and the covariance of fuzzy numbers. In Section 3, we describe investor risk attitudes in the fuzzy portfolio model. In Section 4, we provide an illustrated example that employs the proposed model. Finally, in Section 5, we present our conclusions and suggestions for future research.

2. Lower and upper possibilistic means and variances

Let $\widetilde{A}$ be a fuzzy number with a normal, convex, and continuous membership function. Carlsson and Fuller (2001) defined the lower and upper possibilistic mean values of $\widetilde{A}$ with levels, as shown in the following equation:

$$\widetilde{A}^{\alpha} = [a_1(\alpha), a_2(\alpha)](\alpha > 0). \tag{1}$$

Then, the lower possibilistic mean value can be defined as:

$$M_*(\widetilde{A}) = \frac{\int_0^1 Pos[\widetilde{A} \leqslant a_1(\alpha)]a_1(\alpha)d\alpha}{\int_0^1 Pos[\widetilde{A} \leqslant a_1(\alpha)]d\alpha} = 2\int_0^1 \alpha \cdot a_1(\alpha)d\alpha. \tag{2}$$

Further, the upper possibilistic mean value can be defined as:

$$M^*(\widetilde{A}) = \frac{\int_0^1 Pos[\widetilde{A} \geqslant a_2(\alpha)]a_2(\alpha)d\alpha}{\int_0^1 Pos[\widetilde{A} \geqslant a_2(\alpha)]d\alpha} = 2\int_0^1 \alpha \cdot a_2(\alpha)d\alpha, \tag{3}$$

where Pos denotes the possibility. Let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy numbers. Then, we can derive the following (Carlsson and Fuller, 2001):

$$M_*(\widetilde{A} + \widetilde{B}) = M_*(\widetilde{A}) + M_*(\widetilde{B}), \tag{4}$$

$$M^*(\widetilde{A} + \widetilde{B}) = M^*(\widetilde{A}) + M^*(\widetilde{B}). \tag{5}$$

Thus, the possibilistic mean value of $\widetilde{A} + \widetilde{B}$ can be obtained as:

$$M(\widetilde{A} + \widetilde{B}) = (1/2)[M_*(\widetilde{A} + \widetilde{B}) + M^*(\widetilde{A} + \widetilde{B})]. \tag{6}$$

Zhang and Nie (2003) introduced the lower and upper possibilistic variances and possibilistic covariances of fuzzy numbers that correspond to the lower and upper possibilistic means. The lower and upper possibilistic variances of fuzzy number $\widetilde{A}$, with $\widetilde{A}^{\alpha} = [a_1(\alpha), a_2(\alpha)](\alpha > 0)$, are defined in Eqs. (7) and (8), respectively:

$$\begin{aligned} Var_*(\widetilde{A}) &= \frac{\int_0^1 Pos[\widetilde{A} \leqslant a_1(\alpha)][M_*(\widetilde{A}) - a_1(\alpha)]^2 d\alpha}{\int_0^1 Pos[\widetilde{A} \leqslant a_1(\alpha)]d\alpha} \\ &= 2\int_0^1 \alpha \cdot [M_*(\widetilde{A}) - a_1(\alpha)]^2 d\alpha, \end{aligned} \tag{7}$$

$$\begin{aligned} Var^*(\widetilde{A}) &= \frac{\int_0^1 Pos[\widetilde{A} \geqslant a_2(\alpha)][M^*(\widetilde{A}) - a_2(\alpha)]^2 d\alpha}{\int_0^1 Pos[\widetilde{A} \geqslant a_2(\alpha)]d\alpha} \\ &= 2\int_0^1 \alpha \cdot [M^*(\widetilde{A}) - a_2(\alpha)]^2 d\alpha. \end{aligned} \tag{8}$$

Then, the possibilistic standard deviation value of $\widetilde{A}$, $SD\ (\widetilde{A})$, can be defined as:

$$SD(\widetilde{A}) = 1/2\{[Var_*(\widetilde{A})]^{0.5} + [Var^*(\widetilde{A})]^{0.5}\}. \tag{9}$$

The lower and upper possibilistic covariances between fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are defined as:

$$Cov_*(\widetilde{A}, \widetilde{B}) = 2\int_0^1 \alpha \cdot [M_*(\widetilde{A}) - a_1(\alpha)][M_*(\widetilde{B}) - b_1(\alpha)]d\alpha, \tag{10}$$

$$Cov^*(\widetilde{A}, \widetilde{B}) = 2\int_0^1 \alpha \cdot [M^*(\widetilde{A}) - a_2(\alpha)][M^*(\widetilde{B}) - b_2(\alpha)]d\alpha, \tag{11}$$

respectively, where $\widetilde{A}^{\alpha} = [a_1(\alpha), a_2(\alpha)]$ and $\widetilde{B}^{\alpha} = [b_1(\alpha), b_2(\alpha)]\ \forall \alpha \in [0, 1]$. If and are any numbers, then the lower and upper possibilistic variances of the fuzzy number $\rho\widetilde{A} + \mu\widetilde{B}$ are derived as follows:

$$Var_*(\rho\widetilde{A} + \mu\widetilde{B}) = 2\int_0^1 \alpha \cdot [M_*(\rho\widetilde{A} + \mu\widetilde{B}) - (\rho a_1(\alpha) + \mu b_1(\alpha))]^2 d\alpha, \tag{12}$$

$$Var^*(\rho\widetilde{A} + \mu\widetilde{B}) = 2\int_0^1 \alpha \cdot [M^*(\rho\widetilde{A} + \mu\widetilde{B}) - (\rho a_2(\alpha) + \mu b_2(\alpha))]^2 d\alpha. \tag{13}$$

3. A fuzzy portfolio model that focuses on different investor risk attitudes

In order to address possible uncertainty, Tanaka and Guo (1999) introduced non-probabilistic ways to solve problems of vagueness and ambiguity in the stock market that are associated with the portfolio selection problem. Although many researchers have studied the problems inherent in fuzzy portfolio selection, at present, no researcher has clearly discussed the possible efficient solutions inherent in fuzzy portfolio selection with respect to different risk attitudes. Decision makers respond to similar risk events with a variety of emotional reactions. Therefore, Von Neumann and Morgenstern (Fishburn, 1989) proposed the expected utility theory in the face of risk based on the strict logic of an individual's behavioral assumptions. An individual's assumptions are the guiding principles employed during the decision-making process in the face of risk. An individual's risk attitudes can be summarized as risk averse, risk neutral, and risk seeking. Individuals who are afraid of risk or are sensitive to risk can be considered risk averse. The utility function of risk-averse behavior is concave. Alternatively, a convex utility function indicates the behavior of a risk-seeking investor who is eager to engage in a challenging task. Finally, a risk-neutral decision maker has only mild concern about risk. Thus, the utility function of risk-neutral behavior is linear. In general, a risk-seeking decision maker tends to find and weigh positive outcomes. In so doing, he or she may overestimate the utility of a gain in relation to the utility of a loss that carries a lower risk perception. In contrast, a risk-averse decision maker will give more weight to negative outcomes. This leads to a heightened perception of risk. Further, in a fuzzy or uncertain environment, a risk-seeking investor will positively weigh the fuzzy return rate with an overestimated membership degree. A risk-averse investor will negatively weigh the fuzzy return rate with an underestimated membership degree. Theoretically, based on a fuzzy portfolio with n risky assets, the fuzzy return rate of asset j with different risk attitudes can be defined as a triangular fuzzy number or as $\tilde{r}_j = (r_j, c_j, d_j),\ j = 1, \ldots, n$, where r_j is its central value, c_j and d_j are its left and right spread values, respectively. Then, the membership function of the fuzzy return of asset j with different risk attitudes can be defined as:

$$u_{\tilde{r}_j}(x) = \begin{cases} 1 - \left(\frac{r_j - x}{c_j}\right)^k, & r_j - c_j \leqslant x \leqslant r_j, \\ 1, & x = r_j, \\ 1 - \left(\frac{x - r_j}{d_j}\right)^k, & r_j \leqslant x \leqslant r_j + d_j, \end{cases} \tag{14}$$

where k is an adaptive value for the risk attitude. In addition, the adaptive value k is defined as a positive value because the membership degree of the fuzzy return of asset j should be positive.

Clearly, the second derivative of $u_{\tilde{r}_j}(x),\ j = 1, \ldots, n$, can be obtained as follows:

$$[u_{\tilde{r}_j}(x)]'' = \begin{cases} -k(k-1)\left(\frac{r_j - x}{c_j}\right)^{k-2}, & r_j - c_j \leqslant x \leqslant r_j, \\ -k(k-1)\left(\frac{x - r_j}{d_j}\right)^{k-2}, & r_j \leqslant x \leqslant r_j + d_j. \end{cases} \tag{15}$$

If we observe the second derivative in Eq. (15), decision makers who are afraid of risk or are sensitive to risk can be considered risk averse. The membership function $u_{\tilde{r}_j}(x)$ of risk-averse behavior is convex with $k < 1$, the membership function of risk-neutral behavior is linear with $k = 1$, and the membership function of risk-seeking behavior is convex with $k > 1$. This indicates that the decision maker is eager to engage in a challenging investment. Without losing generality, it is reasonable to set $k = 0.5$ when investors are risk averse, $k = 1$ when they are risk neutral, and $k = 2$ when they are risk seeking. For different investors with different risk attitudes in the fuzzy portfolio model, based on Eqs. (2) and (3) (Carlsson and Fuller, 2001), the lower and upper possibilistic mean values of fuzzy returns $\tilde{r}_j,\ j = 1, 2, \ldots, n$, which are defined as $M_*(\tilde{r}_j)$ and $M^*(\tilde{r}_j)$, are derived as follows:

$$M_*(\tilde{r}_j) = 2\int_0^1 \alpha \times r_{j1}(\alpha)d\alpha = r_j - \frac{2k^2}{(2k+1)(k+1)}c_j, \quad \forall k > 0, \tag{16}$$

$$M^*(\tilde{r}_j) = 2\int_0^1 \alpha \times r_{j2}(\alpha)d\alpha = r_j + \frac{2k^2}{(2k+1)(k+1)}d_j, \quad \forall k > 0, \tag{17}$$

where $r_{j1}(\alpha) = r_j - (1-\alpha)^{\frac{1}{k}}c_j$ and $r_{j2}(\alpha) = r_j + (1-\alpha)^{\frac{1}{k}}d_j$ are the lower and upper bounds of the fuzzy returns $\tilde{r}_j, j = 1, 2, \ldots, n$, under their α-level cut. Then, we can obtain the crisp possibilistic mean value $M(\tilde{r}_j)$ of the fuzzy return for security j as follows:

$$M(\tilde{r}_j) = \frac{M_*(\tilde{r}_j) + M^*(\tilde{r}_j)}{2} = r_j + \frac{k^2}{(2k+1)(k+1)}(d_j - c_j), \quad \forall k > 0. \tag{18}$$

According to the above Eqs. (16)–(18), the combination of the possibilistic mean value of the expected fuzzy returns $M\left(\sum_{j=1}^n \tilde{r}_j x_j\right)$ associated with the portfolio selection can be obtained as follows:

$$\begin{aligned} M\left(\sum_{j=1}^n \tilde{r}_j x_j\right) &= \sum_{j=1}^n x_j M(\tilde{r}_j) \\ &= \sum_{j=1}^n \left[r_j x_j + \frac{k^2}{(2k+1)(k+1)}(d_j - c_j)x_j\right], \quad \forall k > 0. \end{aligned} \tag{19}$$

The possibilistic variance for the expected fuzzy return with different risk attitudes can be obtained as follows:

$$\begin{aligned} Var(\tilde{r}_j x_j) &= \frac{x_j^2}{2}\int_0^1 \alpha(r_{j2}(\alpha) - r_{j1}(\alpha))^2 d\alpha \\ &= \frac{x_j^2}{2}\int_0^1 \alpha(1-\alpha)^{\frac{2}{k}}(d_j + c_j)^2 d\alpha \\ &= \frac{k^2}{4(k+1)(k+2)}x_j^2(d_j + c_j)^2, \quad \forall k > 0 \end{aligned} \tag{20}$$

and the covariance between fuzzy return rate $\tilde{r}_i$ and $\tilde{r}_j, \forall i \neq j$, is derived as follows:

$$\begin{aligned} Cov(\tilde{r}_i, \tilde{r}_j) &= \frac{1}{2}\int_0^1 \alpha[r_{j2}(\alpha) - r_{j1}(\alpha)][r_{i2}(\alpha) - r_{i1}(\alpha)]d\alpha \\ &= \frac{1}{2}\int_0^1 \alpha[(1-\alpha)^{\frac{1}{k}}(d_j + c_j)][(1-\alpha)^{\frac{1}{k}}(d_i + c_i)]d\alpha \\ &= \frac{1}{2}(d_i + c_i)(d_j + c_j)\int_0^1 \alpha(1-\alpha)^{\frac{2}{k}}d\alpha \\ &= \frac{k^2}{4(k+1)(k+2)}(d_i + c_i)(d_j + c_j), \quad \forall k > 0. \end{aligned} \tag{21}$$

Therefore, the variance of the expected fuzzy return in the fuzzy portfolio model for n risk securities can be obtained as follows:

$$\begin{aligned} Var\left(\sum_{j=1}^n \tilde{r}_j x_j\right) &= \sum_{j=1}^n x_i^2 Var(\tilde{r}_i) + \sum_{j=1}^n 2|x_i x_j| Cov(\phi(x_i)\tilde{r}_i, \phi(x_j)\tilde{r}_j) \\ &= \frac{k^2}{4(k+1)(k+2)}\sum_{j=1}^n x_i^2(d_j + c_j)^2 + 2 \\ &\quad \times \frac{k^2}{4(k+1)(k+2)}\sum_{i \neq j}^n x_i x_j (d_i + c_i)(d_j + c_j) \\ &= \frac{k^2}{4(k+1)(k+2)}\left[\sum_{j=1}^n x_j(d_j + c_j)\right]^2, \quad \forall k > 0, \end{aligned} \tag{22}$$

where $\phi(x_i)$ is a sign function defined as follows.

$$\phi(x_i) = \begin{cases} 1 & \text{if } x_i > 0, \\ 0 & \text{if } x_i = 0, \\ -1 & \text{if } x_i < 0. \end{cases} \tag{23}$$

Analogous to Markowitz's mean–variance methodology, the possibilistic mean value represents the invested return of the portfolio. This is also the objective function that will be maximized. The possibilistic standard deviation represents the risk for the portfolio that will be constrained by the upper bound of the desired values of the investor. From this perspective, the possibilistic mean–standard deviation model of portfolio selection with fuzzy return rate and fuzzy proportion can be obtained as follows:

$$\begin{aligned} &\text{Max} \quad \sum_{j=1}^{n}\left[r_j x_j + \frac{k^2}{(2k+1)(k+1)}(d_j - c_j)x_j\right],\\ &\text{s.t.} \quad \frac{k}{2\sqrt{(k+1)(k+2)}}\left[\sum_{j=1}^{n} x_j(d_j + c_j)\right] \leqslant \sigma,\\ &\qquad \sum_{j=1}^{n} x_j \leqslant 1,\\ &\qquad l_j \leqslant x_j \leqslant u_j, \quad \forall j = 1,2,\ldots,n, \end{aligned} \tag{24}$$

where l_j and u_j are the lower and upper bounds on the invested proportion x_j, $j = 1,2,\ldots,n$, respectively.

For different risk attitudes, without losing generality, we can reasonably set $k = 0.5$ when investors are risk averse and obtain fuzzy portfolio model (25) for the solution of the efficient portfolio problem. Similarly, if we set $k = 1$ for risk-neutral behavior and $k = 2$ for risk-seeking behavior, then we can obtain fuzzy portfolio models (26) and (27) to obtain the efficient portfolios for risk neutral and risk-seeking behaviors, respectively. When we compare models (25)–(27), we can see that the right-hand-side values in their constraints vary because of different investor risk attitudes. For example, if a satisfactory level value σ for investment risk is assigned to models (25)–(27), then model (25) will have the largest right-hand-side value among all models. Therefore, if the optimal solution for the fuzzy portfolio models exists at a minimum feasible region, then model (25) could be solved at the lowest investment risk value σ. This means that a risk-averse investor will prefer to adopt the efficient portfolio at the lowest investment risk. In contrast with model (25), model (27) should be solved with a larger value of investment risk σ because its multiplier is $2\sqrt{3}$ that is smaller than the multiplier $2\sqrt{15}$ of model (25). Thus, a risk-seeking investor will prefer to maintain the portfolio with a higher investment risk. Besides, we know that different investment risk values σ will derive different solutions for portfolios based on these models. In the final analysis, each model from (25)–(27) has its own reference point that can be derived from the given upper bound of investment risk σ. If the given value σ is smaller than the reference point, then the proposed model can solve the efficient portfolio problem based on the risk attitude of the investor. Otherwise, the proposed model will be solved and we will obtain an efficient portfolio that is the same as that of the reference point's solution. This occurs because the given risk value σ does not bring any change in the feasible region in the corresponding fuzzy portfolio model.

$$\begin{aligned} &\text{Max} \quad \sum_{j=1}^{n}\left[r_j x_j + \frac{1}{14}(d_j - c_j)x_j\right],\\ &\text{s.t.} \quad \left[\sum_{j=1}^{n} x_j(d_j + c_j)\right] \leqslant 2\sqrt{15}\sigma,\\ &\qquad \sum_{j=1}^{n} x_j \leqslant 1,\\ &\qquad l_j \leqslant x_j \leqslant u_j, \quad \forall j = 1,2,\ldots,n, \end{aligned} \tag{25}$$

$$\begin{aligned} &\text{Max} \quad \sum_{j=1}^{n}\left[r_j x_j + \frac{1}{6}(d_j - c_j)x_j\right],\\ &\text{s.t.} \quad \left[\sum_{j=1}^{n} x_j(d_j + c_j)\right] \leqslant 2\sqrt{6}\sigma,\\ &\qquad \sum_{j=1}^{n} x_j \leqslant 1,\\ &\qquad l_j \leqslant x_j \leqslant u_j, \quad \forall j = 1,2,\ldots,n, \end{aligned} \tag{26}$$

$$\begin{aligned} &\text{Max} \quad \sum_{j=1}^{n}\left[r_j x_j + \frac{4}{15}(d_j - c_j)x_j\right],\\ &\text{s.t.} \quad \left[\sum_{j=1}^{n} x_j(d_j + c_j)\right] \leqslant 2\sqrt{3}\sigma,\\ &\qquad \sum_{j=1}^{n} x_j \leqslant 1,\\ &\qquad l_j \leqslant x_j \leqslant u_j, \quad \forall j = 1,2,\ldots,n. \end{aligned} \tag{27}$$

4. Numerical example

To illustrate our proposed effective means and approaches to the efficient portfolios discussed in this paper, we chose a portfolio selection example found in Zhang (2007) [28]. In this example, five selected securities were chosen to form a portfolio. On the basis of historical data and the corporations' financial reports and future information, the fuzzy return of each selected security can be estimated with the following possibility distributions: $\tilde{r}_1 = (0.073, 0.054, 0.087)$, $\tilde{r}_2 = (0.105, 0.075, 0.102)$, $\tilde{r}_3 = (0.138, 0.096, 0.123)$, $\tilde{r}_4 = (0.168, 0.126, 0.162)$, and $\tilde{r}_5 = (0.208, 0.168, 0.213)$, where the first value for the fuzzy returns is its center value, the second value is the left spread value, and the third value is the right spread value. The lower bounds of investment proportion x_j for the security j are given by $(l_1, l_2, l_3, l_4, l_5) = (0.1, 0.1, 0.1, 0.1, 0.1)$, and the upper bounds are given by $(u_1, u_2, u_3, u_4, u_5) = (0.4, 0.4, 0.4, 0.5, 0.6)$. After application of model (25) to model (27), the efficient portfolios obtained based on some different risk attitudes are shown in Tables 1–3 (risk-averse, risk-neutral, and risk-seeking, respectively).

For a risk-averse investor, the sum of the investment proportions for the securities is partial $\left(\sum_{j=1}^{5} x_j \leqslant 1\right)$ when the investment risks are smaller than or equal to 2.6% and their returns are smaller than or equal to 38.08% (see, e.g., Table 1). In addition, when the range of the invested risk is between 1.6 % and 2.6%, partial investments show that the proportion of securities 1 and 3 increase, but the proportions of securities 2, 4, and 5 remain the same. Thus, under lower investment risk (σ < 2.6%) in their partial investments, the risk-averse investors are interested in security x_1 because security x_1 has the lowest uncertainty (spread values) in its fuzzy return rate. In contrast, when the invested risk exceeds or is equal to 2.8%, the investment proportions to securities 1 and 2 are the same, but the proportions of securities 3 and 4 decrease. Moreover, the proportions of security 5 increase because of its higher return. This occurs because higher return security with higher vagueness is chosen to increase the investor's expected return in the range of increasing invested risk and then reduce his or her investment to some securities with lower rates of return. In addition, we can define the investment risk as larger than 3.2%. However, the efficient portfolio is the same. This means that the largest risk accepted by a risk-averse investor would be 3.2%.

As shown in Table 2, with respect to a risk-neutral investor, the sum of the investment proportions is full when the investment risk exceeds or is equal to 4.4%, in which the invested proportions of securities 1 and 2 are the same among different investment risks. However, the proportion of securities 3 and 4 decrease, and the proportion of security 5 increases. In addition, for a risk-neutral investor, security 1 is the most important because it is invested with the highest proportion. Security 5 is invested with increasing proportion in relation to the increasing investment because this security has higher return and higher vagueness. In contrast, when the invested risk is between 3% and 4.2%, the total proportions for the securities are partial, where the investment proportions to securities 1 and 3 increase, but securities 2, 4, and 5 remain the same. Thus, under lower risk in partial investment, the lowest

Table 1
Possibilistic efficient portfolios based on risk aversion.

σ (%)	1.6	1.8	2	2.2	2.4	2.6	2.8	3	3.2	3.4
M (%)	15.36	23.41	31.46	36.09	37.09	38.08	38.87	39.553	39.85	39.85
x_1	0.1236	0.2335	0.3434	0.4	0.4	0.4	0.4	0.4	0.4	0.4
x_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
x_3	0.1	0.1	0.1	0.1343	0.205	0.2758	0.1524	0.1	0.1	0.1
x_4	0.1	0.1	0.1	0.1	0.1	0.1	0.2476	0.1723	0.1	0.1
x_5	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2277	0.3	0.3
$\sum_{j=1}^{5} x_j$	0.5236	0.6335	0.73434	0.8343	0.905	0.9758	1	1	1	1

Table 2
Possibilistic efficient portfolios based on risk-neutral behavior.

σ (%)	3	3.2	3.4	3.6	3.8	4	4.2	4.4	4.6	4.8	5
M (%)	27.52	32.64	36.07	36.71	37.35	37.99	38.62	39.09	39.53	39.97	40.15
x_1	0.287	0.3565	0. 4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
x_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
x_3	0.1	0.1	0.1168	0.1615	0.2062	0.251	0.2957	0.1716	0.1	0.1	0.1
x_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2284	0.2478	0.1424	0.1
x_5	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1522	0.2576	0.3
$\sum_{j=1}^{5} x_j$	0.687	0.7565	0.8168	0.8615	0.9062	0.951	0.9957	1	1	1	1

Table 3
Possibilistic efficient portfolios based on risk-seeking behavior.

σ (%)	3.6	4	4.4	4.8	5.2	5.4	5.6	5.8	6	6.2	6.6	7
M (%)	16.09	23.35	30.61	36.32	37.24	37.7	38.16	38.62	39.06	39.39	40.02	40.5
x_1	0.1291	0.2274	0.3257	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
x_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
x_3	0.1	0.1	0.1	0.1154	0.1787	0.2103	0.242	0.2736	0.2834	0.183	0.1	0.1
x_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1166	0.217	0.2126	0.1
x_5	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1874	0.3
$\sum_{j=1}^{5} x_j$	0.5291	0.6274	0.7257	0.8115	0.905	0.9103	0.9242	0.9274	1	1	1	1

return rate of security 1 receives more attention for investment in these portfolios. In addition, when the invested risk grows larger than 5%, the portfolio remains the same. This means that the investment risk at 5% is the reference point that a risk-neutral investor can accept as the largest risk.

As shown in Table 3, with respect to a risk-seeking investor, when the investment risk exceeds or is equal to 6%, and the fuzzy return exceeds or is equal to 39.06%, the total of investment proportions $\sum_{j=1}^{5} x_j$ is equal to 1, where the investment proportions to securities 1 and 2 are the same, those to securities 3 and 4 decrease, and those to security 5 increase. This occurs because security 5 has a higher return rate and higher vagueness (uncertainty). Thus, a risk-seeking investor may wish to invest more in order to increase his or her expected return in the increasing investment risk. He or she may attempt to reduce his or her investment to some securities with lower rates of return (i.e., x_3 and x_4). In contrast, when investment risks are between 3.6% and 5.8%, the investments are partial $\left(\sum_{j=1}^{5} x_j < 1\right)$. The investment proportions to securities 1 and 3 increase, but the investment proportions to securities 2, 4, and 5 always remain the same. In fact, security 1 plays an important role in partial investment because it represents almost 40% of the portfolio. Security 5 becomes important in the portfolio when the invested risk is larger than 7% with a proportion of 30%. Finally, in full investment, the efficient portfolio maintains the same values when the invested risk is larger than 7%. This means that a risk-seeking investor can accept the risk at 7%.

When we compare Tables 1–3, we can see that different investor risk attitudes require different portfolio selections, and the feasible region of the risk- seeking investor in the fuzzy portfolio model is smaller than that of risk-averse and risk-neutral investors. This occurs because the risk-seeking investor is interested in investments with higher invested risk. Clearly, different investors with different risk attitudes have different efficient portfolios. Risk-averse investors may start investing at a lower risk value of 1.6%. Risk-neutral investors may start investing at 3%. Risk-seeking investors may start investing at 3.6%. In addition, with respect to full investment, risk-seeking investors can expect higher invested risk at 7% and higher expected return at 40.5%. Risk-neutral investors can expect an invested risk rate at 5% and an expected return rate at 40.15%. Risk-averse investors can expect an invested risk rate at 3.4% and an expected return rate at 39.85%.

5. Conclusions

In this paper, we proposed a fuzzy portfolio model that focuses on different risk attitudes for portfolio selection. We obtained significant results from the portfolio selection based on different investor risk attitudes. In our proposed models, with respect to the risks associated with partial investments, a risk-averse investor may prefer to invest at a smaller investment risk than risk-neutral and risk-seeking investors. With respect to the risks associated with full investment, a risk-seeking investor can tolerate higher invested risk and higher invested return than risk-neutral and risk-averse investors. It is important to note that we tested the illustrated experiments with a sensitivity analysis that employed different investment risks. This approach demonstrated that the application of these proposed portfolio models can easily derive efficient portfolios for investors that possess different risk attitudes. If the fuzzy portfolio model for different risk attitudes meets expectations, this approach will be an important development in the use of fuzzy portfolio selection. However, this study examines only a small number of specific k values for risk attitudes. The choice of different k values might affect the efficient portfolio. Therefore, future research should be performed to discover the best input values for each parameter and to determine the relationships among these parameters.

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