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Exact Solution of a Deterministic Sandpile Model in One Dimension

S.-C. Lee, N. Y. Liang, and W.-J. Tzeng

Institute of Physics, Academia Sinica Taipei, Taiwan 11529, Republic of China

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We present an exact solution of a one-dimensional sandpile model for which sand is dropped along the wall and $N=2$ grains of sand fall over the neighboring downhill sites when the critical slope is exceeded. The slopes of N consecutive sites organize into a local state. The time evolution of the local states along the spatial direction shows a natural tree structure. As a result, various multifractals can be identified. The spatial two-point correlation function decreases exponentially with a correlation length of the order of the lattice spacing.

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Sandpile models originally proposed by Bak, Tang, and Wiesenfeld [1] are perhaps the simplest models that capture the essence of many nonlinear dissipative systems. As such, they have attracted the attention of many physicists. In particular, in searching for scaling and universality, Kadanoff, Nagel, Wu, and Zhou [2] have studied many models in one and two dimensions numerically. We should try to adhere to their convention and terminology. Hence, the model we discuss is the "limited," "nonlocal" model with $N=2$ in one dimension. In most of the investigations on the sandpile models so far, sand is dropped randomly on the various sites. It is important, however, to distinguish the effect of this "random perturbation" from a potential stochasticity that may arise as a result of the intrinsic nonlinear dynamics. In this respect, we should mention that Wiesenfeld, Theiler, and McNamara [3] had investigated numerically a two-dimensional "Abelian" model [4] in which sand is added only at the center. They concluded that randomness in the sand drops is not an essential element in order for a system to show self-organized critical behavior.

Consider a one-dimensional lattice and let $i=0,\dots,L$ label the lattice sites. The number of sand grains at site i is denoted by $h(i)$. We should focus on the slope variable $\sigma(i)=h(i)-h(i+1)$ and interpret $\sigma(i)$ to be the number of "particles" at site i. The dynamics can then be described in terms of the creation and annihilation operators $a_i^\dagger$ and a_i. Specifically, dropping a sand grain at site i, which causes the change $h(i)\to h(i)+1$ in the height variables and the changes $\sigma(i)\to\sigma(i)+1$, $\sigma(i-1)\to\sigma(i-1)-1$ in the slope variables, is given by the "kinetic term"

$$a_i^\dagger a_{i-1}\,, \tag{1}$$

while an avalanche triggered at site i, which causes the changes $h(i)\to h(i)-N$, $h(i+1)\to h(i+1)+1,\dots,$ $h(i+N)\to h(i+N)+1$ in the height variables when $h(i)-h(i+1)>N$ and the changes $\sigma(i-1)\to\sigma(i-1)+N$, $\sigma(i)\to\sigma(i)-N-1$, $\sigma(i+N)\to\sigma(i+N)+1$ in the slope variables when $\sigma(i)>N$, is described by the "interaction term"

$$(a_{i-1}^\dagger)^N a_{i+N}^\dagger a_i^{N+1}\theta(\sigma(i)-N)\,, \tag{2}$$

where $\theta(x)=1$ for $x>0$ and vanishes otherwise. We shall investigate the "strong-coupling limit," i.e., the hopping term (1) will be treated as a perturbation and for each hopping the interaction term (2) will be applied any number of times until further application of it no longer produces an effect. The boundary conditions are such that particles can leave the system from the left edge but not from the right edge. Thus if a particle tries to hop beyond site L, it lands on site L [5]. In general, one may consider the hopping term

$$\sum_i p_i a_i^\dagger a_{i-1},\quad 0\le p_i\le 1,\quad \sum p_i=1\,, \tag{3}$$

but we shall choose $p_0=1$ in the following, so that particles are injected one by one from the left edge corresponding to sand being dropped one by one at site $i=0$. We call each particle injection a time step. Starting from any given configuration, the system will reach a "limit cycle" after a finite number of time steps. The period of the limit cycle is N^L. The states in the limit cycle are characterized by two conditions: (1) For any N consecutive sites, there is at least one site i such that $\sigma(i)=N$. (2) Let $\sigma(L)=k$; then $k\geq 1$ and there exists at least one site i such that $\sigma(i)=N$ for $L-k\leq i<L$. Thus the system organizes into intervals of N consecutive sites. For $N=2$, we assume $L=2K$ for simplicity so that there are K intervals. By condition (1) above, there are five allowed states for each interval, namely, $u_1=(02)$, $u_2=(12)$, $w=(22)$, $v_1=(21)$, and $v_2=(20)$ where the integers in the brackets give the particle number of the two consecutive sites. A state in the limit cycle is described by a sequence $[z_1z_2\cdots z_K]$, where $z_i\in\{u_1,u_2,w,v_1,v_2\}$, with the constraints that (1) $z_i=u_1$ or u_2 implies $z_{i-1}=u_1$, u_2, or w, (2) $z_i=v_1$ or v_2 implies $z_{i+1}=v_1$, v_2, or w, and (3) $z_K\neq v_2$. We shall consider sequences of lengths other than K. An "allowed sequence" is one such that conditions (1) and (2) are satisfied. In particular, u_i, w, and v_i may be regarded as allowed sequences of length one. It is easy to show that the number of allowed sequences $[z_1z_2\cdots z_k]$ of length k is n_{k-1} if $z_k=u_1,u_2$ or $z_1=v_1,v_2$ and is m_{k-1} if $z_k=w,v_1,v_2$ or $z_1=u_1,u_2,w$, where

$$n_k=\tfrac{1}{3}(2^{2k+1}+1),\quad m_k=\tfrac{1}{3}(2^{2k+2}-1).\tag{4}$$

To describe time evolution of the system, we introduce integer-valued "time functions" t_j defined on the allowed sequences. For sequences of length one, we have

$$\begin{aligned}&t_j(u_1)=0,\quad t_j(u_2)=n_j,\quad t_j(w)=2n_j,\\&t_j(v_1)=n_{j+1},\quad t_j(v_2)=n_{j+1}+m_j.\end{aligned}\tag{5}$$

In general,

$$\begin{aligned}&t_0([z_1z_2\cdots z_k])=\sum_{j=1}^{k}t_{j-1}(u_j),\\&t_l([z_1z_2\cdots z_k])=\sum_{j=1}^{k}t_{j+l-1}(u_j).\end{aligned}\tag{6}$$

It can be shown that $t_0([z_1z_2\cdots z_K])$ provides time ordering for the states in the limit cycle. For a given allowed sequence $[z_1z_2\cdots z_k]$, we denote its "next sequence" by $[z_1z_2\cdots z_k]+1$ which is defined by the relation

$$t_0([z_1z_2\cdots z_k]+1)=t_0([z_1z_2\cdots z_k])+1.\tag{7}$$

Then we have

$$\begin{aligned}&[u_i\cdots]+1=[u_i+1\cdots],\\&[ww\cdots]+1=[v_1w\cdots],\\&[wv_i\cdots]+1=[v_1v_i\cdots],\\&[wu_i\cdots]+1=[u_1u_i+1\cdots],\\&[v_1\cdots]+1=[v_2+1\cdots],\\&[v_2\cdots v_2[\cdots]]+1=[u_1\cdots u_1[\cdots]+1].\end{aligned}\tag{8}$$

In the last relation of Eq. (8), the number of consecutive v_2's is the same as that of consecutive u_1's. It is instructive to note the relation

$$t_0([z_1z_2\cdots z_l])=t_0([z_1z_2\cdots z_k])+t_k([z_{k+1}\cdots z_l]),\tag{9}$$

which in fact describes the "scaling" property of the time evolution. In plain words, the time evolution of a state at the kth interval is given as follows. Each state of u_1 or u_2 will last n_{k-1} times steps and that of v_1, v_2, or w will last m_{k-1} times steps. If we ignore this repetition of a given state, the time evolution of the kth interval is described by the time evolution of an allowed sequence of length $K-k+1$. Consider the Kth interval. The four states u_1, u_2, w, and v_1 appear in this order and each repeats n_{K-1}, n_{K-1}, m_{K-1}, and m_{K-1} time steps, respectively. At the $(K-1)$th interval, there are sixteen time intervals corresponding to the sixteen allowed sequences $[z_{K-1}z_K]$ with $z_K\neq v_2$. The time ordering of the sixteen intervals is determined by $t_0([z_{K-1}z_K])$. Thus we see that, in general, the state $[z_1z_2\cdots z_K]$ in the limit cycle corresponds to a path in a tree. It is now possible to identify a multifractal along the time axis. We rescale the time variable so that each time step is of length $\delta=2^{-L}$ and the duration for a limit cycle fits into the time interval [0,1]. Consider the states $[z_1z_2\cdots z_K]$ in the limit cycle which contain N_1 u_1's, N_2 u_2's, N_0 w's, $\tilde N_1$ v_1's, and $\tilde N_2$ v_2's. The corresponding time steps for such states form a fractal in the interval [0,1] with the fractal dimension

$$f(\xi_1,\xi_2,\xi_0,\tilde\xi_1,\tilde\xi_2)=\frac{1}{2\ln 2}\left[\xi_1\ln\left(1+\frac{\xi_2+\xi_0}{\xi_1}\right)+\xi_2\ln\left(1+\frac{\xi_1+\xi_0}{\xi_2}\right)+\xi_0\ln\left(1+\frac{\xi_1+\xi_2}{\xi_0}\right)+\xi_i\to\tilde\xi_i\right],\tag{10}$$

where $\xi_i=N_i/K$, $\tilde\xi_i=\tilde N_i/K$, $\xi_0=N_0/K$, and satisfy $\sum_i(\xi_i+\tilde\xi_i)+\xi_0=1$.

Denoting the time average by angular brackets $\langle\cdots\rangle$, we find

$$\langle\sigma(2i-1)\rangle=\tfrac{3}{2}-1/2^{2i},\quad\langle\sigma(2i)\rangle=\tfrac{3}{2}-1/2^{2i+1}.\tag{11}$$

For the two-point function, we get, for example,

$$\langle\sigma(2i-1)\sigma(2j-1)\rangle-\langle\sigma(2i-1)\rangle\langle\sigma(2j-1)\rangle=\frac{1}{2}\frac{1}{2^{2(j-1)}}-\frac{3}{2}\frac{1}{2^{2i}}-\frac{1}{2^{2j}}.\tag{12}$$

Neglecting the boundary effect, we see that the two-point function decreases exponentially with a correlation length $\xi = a/\ln 2$, where a is the lattice spacing.

For each injection of a particle from the left edge, a number of consecutive local states will be affected. We define the corresponding affected number of sites to be the cluster size. Then one can show that the distribution of the cluster size decreases exponentially with the characteristic length ξ. A similar result holds when we consider the distribution of the durations of avalanches if we define the duration of an avalanche to be the number of times the interaction term in Eq. (2) is applied [6].

We conclude that for one dimension, the deterministic sandpile model does not show the "self-organized criticality" in the sense of a power-law distribution for, say, the cluster size [7]. The conclusion may very well be otherwise when a random perturbation is applied [2]. We do observe an interesting tree structure in the time evolution of the system. This "ordering," however, is not reflected either in the two-point correlation function or in the distribution of the cluster size.

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[1] Per Bak, Chao Tang, and Kurt Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); Chao Tang and Per Bak, Phys. Rev. Lett. **60**, 2347 (1988).

[2] Leo P. Kadanoff, Sidney R. Nagel, Lei Wu, and Su-min Zhou, Phys. Rev. A **39**, 6524 (1989).

[3] Kurt Wiesenfeld, James Theiler, and Bruce McNamara, Phys. Rev. Lett. **65**, 949 (1990).

[4] Deepak Dhar, Phys. Rev. Lett. **64**, 1613 (1990).

[5] Recall that here the "particle" refers to slope variable and should not be confused with sand grain.

[6] The expert may wonder how this can happen since if sand is dropped only at the left edge, the average number of avalanches should be of order L. In fact, the distribution for the number of avalanches T decreases exponentially for $0 \leq T \leq \frac{1}{2}L$ and increases exponentially for $\frac{1}{2}L \leq T \leq L$. The precise statement should be the distribution of T decreases exponentially toward its minimum value which is attained when $T = \frac{1}{2}L$.

[7] However, the power spectrum for the distribution of sand grains dropping out of the right edge as time develops may very well show a $1/f$ behavior.