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Domain wall space-times with a cosmological constant

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We solve the vacuum Einstein's field equations with the cosmological constant in space-times admitting three-parameter group of isometries with two-dimensional spacelike orbits. The general exact solutions, which are represented in the advanced and retarded null coordinates, have two arbitrary functions due to the freedom of choosing null coordinates. In the thin-wall approximation, the Israel's junction conditions yield one constraint equation on these two functions in spherical, planar, and hyperbolic domain wall space-times with reflection symmetry. The remain freedom of choosing coordinates is completely fixed by requiring that, when the surface energy density σ_0 of domain walls vanishes, the metric solutions will return to some well-known solutions. It leads us to find a planar domain wall solution, which is conformally flat, in the de Sitter universe.

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I. INTRODUCTION

The inflationary universe was originally proposed to solve horizon and flatness problems in the hot big-bang cosmological model [1]. From the discovery of the cosmic microwave background (CMB) anisotropy, the idea of cosmological inflation becomes more convincing, and it serves as the initial conditions for the subsequent hot big bang. Most popular inflation models are described by scalar fields, called inflaton fields, with their effective potentials [2]. Although there still lacks a fundamental theory to explain the origin of inflation, some effective theories reveal that inflation may naturally happen due to the spontaneous symmetry breaking and phase transitions in the early Universe [3].

When a phase transition occurred in the early Universe, various types of topological defects, which are classified by homotopy groups, can form in the vacuum manifold, and this phenomenon is known as the Kibble mechanism [4]. Therefore, domain walls, which are a particular type of the topological defects, correspond to vacuumlike hypersurfaces interpolating between separate vacua. Besides the Kibble mechanism, domain walls can also form by the quantum tunneling process of false vacuum decay, i.e., bubble nucleation [5], or quantum production of topological defects in de Sitter space [6]. (See [7] for a review of domain walls.) Reference [6] has shown that the topological defects can be continuously formed during inflation and still be present after inflation with appreciable densities. Hence, it motivates us to study the gravitational effects of domain walls in the de Sitter universe.

Since we are only interested in the macroscopic effects of domain walls, it is sufficient to study the domain wall space-times in the thin-wall approximation, where the wall is regarded as infinitely thin, with δ-function singularity in the energy-momentum tensor. Therefore, gravitational effects of domain walls are described by Einstein's field equations off the wall together with Israel's junction conditions [8]. As far as we know, domain wall solutions have been studied based on two different approaches. The first approach starts from exact solutions of Einstein's field equations off the wall in the specific coordinates, and then the wall's motion in the same coordinates is described by Israel junction conditions. The second approach is to introduce the comoving coordinates, where the wall is placed at a particular constant coordinate variable, say, $z = 0$, and then the exact solutions of Einstein's field equations off the wall are obtained in the comoving coordinates.

Cvetič *et al.* [9] studied the local and global properties of domain wall space-time with the cosmological constant Λ in the comoving coordinates. They found domain wall solutions based on three assumptions. The first one assumed that the two-dimensional spatial sections V_2 of space-times "parallel" to the wall are homogeneous and isotropic. It corresponds to space-times admitting a three-parameter group of isometries with two-dimensional spacelike orbits, i.e. V_2 are two-dimensional spheres, planes, or hyperboloids [10,11]. The second one assumed that the space-time section orthogonal to the wall, say, (t, z)-plane, is static. It means that the metric components g_{tt} and g_{zz} are t-independent. The third assumption required that the directions parallel to the wall are boost invariant, i.e., extrinsic curvature of constant-z hypersurfaces is boost invariant. It yields that the metric function intrinsic to V_2 are separable. It seems to us that the second

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and third assumptions are not satisfactory since they may eliminate some interesting domain wall solutions. For example, our planar domain wall solutions in the de Sitter universe, which are obtained in Sec. IV B [see (66)], are conformally flat and will return to the metric of the de Sitter universe when the surface energy density of the domain wall vanishes. Solution (66) cannot be found in [9] since it does not satisfy the second and third assumptions.

In this paper, we release the last two assumptions in [9] and find the general solutions of Einstein's field equations with Λ in the double-null coordinates. Solving Einstein's equations in null coordinates has been used to find Schwarzschild and Reissner-Nordström solutions in [12]. Since the nondegenerate general solutions contain two arbitrary functions $F(v)$ and $G(u)$ due to the freedom of the null-coordinate choices, the Israel's junction conditions yield one constraint equation on these two functions, where the domain wall is placed at a constant-z hypersurface.

From the generalized Birkhoff theorem [10,11], one can find a coordinate transformation [see Eq. (35)] to make our nondegenerate solutions become

$$g = -U(R)dT \otimes dT + \frac{1}{U(R)} dR \otimes dR + R^2 dV_2, \quad (1)$$

where $U(R) = K - 2M/R - (\Lambda/3)R^2$ and $dV_2 = (1 - Kr^2)^{-1} dr \otimes dr + r^2 d\phi \otimes d\phi$. It is clear that the metric (1) is static in the certain range of coordinates. Moreover, it turns out that the domain wall, which is originally sitting at a constant-z hypersurface, becomes moving in the coordinates (T, R, r, ϕ). So one may expect that a domain wall solution in the comoving coordinates is locally equivalent to a moving wall embeded in a space-time with metric (1).[1] Here, we call these two different approaches the "comoving-coordinate" approach and the "moving wall" approach. The equivalence of these two approaches has been demonstrated by Bowcock *et al.* [13],[2] so there exist coordinate transformations between these two approaches. Having shown this equivalence, Bowcock *et al.* solved Israel's junction conditions to find the most general brane-universe solutions based on the moving-wall approach. In the study of brane cosmologies, it is more suitable to use the moving-wall approach.

In this paper, we solve Israel's junction conditions and find the general domain-wall solutions in the comoving-coordinate approach. We will show that the domain-wall solutions in the comoving-coordinate approach are more useful for studying gravitational perturbations and quantum fluctuations than the moving-wall approach, which is normally adopted to study the dynamics of the brane-universe. It is known that a proper coordinate choice can largely simplify physical problems and equations. For example, when one studies metric perturbations in a specific background space-time, a proper choice of background coordinates is important since it may largely simplify perturbed equations or make these equations solvable. Moreover, when we study quantum fluctuations in curved space-time, the coordinate choices become significant since there is no coordinate-invariant definition of the vacuum state, i.e., the vacuum state is observer dependent [14]. In the comoving-coordinate approach, the Israel's junction conditions only fix one freedom of double-null coordinate choices, so the remaining freedom can be used to further simplify our domain-wall solutions or to avoid coordinate singularities that appear in metric solutions. However, in the moving-wall approach, the metric solutions outside the wall are fixed in the static form, i.e., Eq. (1), which has coordinate singularities at some finite values of R, and the Israel's junction conditions will completely determined the trajectories of walls. In this work, we propose a reasonable way to fix the remaining coordinate freedom by requiring that, when the surface energy density σ_0 of domain walls vanishes, the metric solutions will return to some well-known solutions. For example, when $\sigma_0 = 0$, planar domain wall solutions with $M = 0$ and $\Lambda > 0$ [see (66)] will return to the de Sitter universe in conformal-time coordinates, where the quantum fluctuations have been largely studied. So the domain-wall effects on primordial quantum fluctuations in the early Universe may clearly be seen by studying quantum fluctuations in solution (66).

The plan of this paper is as follows. In Sec. II, we briefly review the thin-wall approximation and Israel's formalism in the covariant approach. Sec. III presents general solutions of Einstein's field equations with Λ in space-times admitting a three-parameter group of isometries with two-dimensional spacelike orbits. The general nondegenerate solutions have $F(v)$, $G(u)$ and three parameters: two-dimensional constant curvature K, gravitational mass M, and Λ. We also show that the solutions will return to some well-known exact solutions in specific coordinates by choosing F and G. In Sec. IV, we limit our discussion in space-time having reflection symmetry with respect to the wall, and the Israel's junction conditions become an algebraic equation for the two functions F and G. Sec. IV A discusses spherical and hyperbolic domain walls with $M = 0$ in the de Sitter universe. In Sec. IV B, we present a planar domain-wall solution, which is conformally flat, in the de Sitter universe. When the surface energy of the domain wall vanishes, the solution returns to the de Sitter metric in the conformal time. We conclude in Sec. V.

We use the units $\hbar = c = 1$, and the metric signature is $(- + + +)$. The Latin indices $a, b, \cdots$ refer to coordinate indices and the Greek indices $\alpha, \beta, \gamma \cdots$ refer to orthonormal frame indices. g and ∇ denote the metric tensor and Levi-Civita connection, respectively.

[1] It is possible that the solutions obtained in these two coordinate systems may have different global structures of space-time.

[2] In [13], these two different approaches are called the "brane-based" approach and the "bulk-based" approach.

II. THE THIN-WALL APPROXIMATION

In the thin-wall approximation, the thickness ε of a thin wall is taken to be zero, so the infinitely thin wall becomes a three-dimensional timelike, null or spacelike hypersurface Σ in four-dimensional space-times, and its associated stress-energy tensor $T^a{}_b$ of the space-times has a δ-function singularity on Σ. Here, we will assume Σ to be a three-dimensional timelike hypersurface for our current interest. From Einstein's field equations and the singular property of $T^a{}_b$, it turns out that the extrinsic curvature π_{ab} of Σ has a jump discontinuity across Σ, and its discontinuity is naturally related to surface stress-energy distribution of matter fields on Σ [8,15].

A. Israel's junction conditions

Since Σ is a three-dimensional timelike hypersurface, one may first introduce its unit spacelike normal n satisfying $g(n, n) = 1$, and the intrinsic metric h of Σ is then given by

$$h = g - \tilde{n} \otimes \tilde{n},$$

where $\tilde{n} = g(n, -)$ is the metric dual of n. So the extrinsic curvature π_{ab} of Σ is defined by

$$\pi_{ab} = \frac{1}{2}(\mathcal{L}_n \bar{h})_{ab}|_{\Sigma}, \tag{2}$$

where $\mathcal{L}_n$ denotes the Lie derivative along the unit normal n, and $\bar{h}$ is any extension of h to a neighborhood of Σ. Because of the discontinuity of π_{ab} across Σ, it is convenient to introduce the notation $\pi_{ab}|_{\pm}$, where the subscripts $\pm$ refer to values just off the surface on the side determined by the direction of $\pm n$. In [8], it showed that

$$\gamma_{ab} \equiv \pi_{ab}|_{+} - \pi_{ab}|_{-} = -\kappa(S_{ab} - \frac{1}{2}h_{ab}S_c{}^c) \tag{3}$$

in the thin-wall limit $\varepsilon \to 0$, where $\kappa = 8\pi G_N$ and $S_{ab} \equiv \lim_{\varepsilon\to 0}\int_0^{\varepsilon} dl T_{ab}$ denotes the surface stress-energy tensor. Here l is the proper distance through Σ in the direction of n. Equation (3) is normally called Israel's junction conditions. It is worth pointing out that Eq. (3) is also valid when we include the cosmological constant Λ in Einstein's field equations. One may further impose that the space-time geometry be reflection symmetric with respect to Σ, which will be considered in Sec. IV, and it yields $\pi_{ab}|_{+} = -\pi_{ab}|_{-}$.

B. The surface stress-energy tensor

The surface stress-energy tensor $S = S_{ab}dx^a \otimes dx^b$ is usually assumed to have the following perfect-fluid form,

$$S = (\sigma - \tau)\tilde{u} \otimes \tilde{u} - \tau h, \tag{4}$$

where σ and τ denote surface-energy density and tension of Σ, respectively. $\tilde{u} = h(u, -)$ is the intrinsic metric dual of the unit timelike vector field u, which lies within Σ. It is known that dynamics of S can be determined by Eq. (3), Einstein's field equations, and the Gauss-Codazzi equations in the thin-wall approximation [8,15]. In particular, one of the Gauss-Codazzi equations yields the conservation equation of S, which is

$$D \cdot S = (\sigma - \tau)D_u\tilde{u} + \tilde{u}D \cdot [(\sigma - \tau)u] - d\tau = 0, \tag{5}$$

where D is the three-dimensional intrinsic covariant derivative on Σ satisfying $Dh = 0$ and the torsion-free condition. $D\cdot$ denotes the divergence. In terms of coordinate components, it is easy to show that, for any tensor field T on Σ,

$$D_a T^{b\ldots c}{}_{d\ldots e} = h_a{}^p h_q{}^b \cdots h_r{}^c h_d{}^s \cdots h_e{}^t \nabla_p \bar{T}^{q\ldots r}{}_{s\ldots t}, \tag{6}$$

where $\bar{T}$ is any extension of T to a neighborhood of Σ.

For dust walls, i.e., $\tau = 0$, Eq. (5) yields that the world lines of u are geodesics and the surface energy density is conserved, which is $D \cdot (\sigma u) = 0$. For domain walls, i.e., $\tau = \sigma$, we simply obtain $d\sigma = 0$, which means that $\sigma = \sigma_0$ is a constant on Σ. In the following discussion (see Sec. IV), we will concentrate only on domain-wall space-times with reflection symmetry, so Eq. (3) becomes

$$\gamma_{ab} = 2\pi_{ab}|_{+} = -2\pi_{ab}|_{-} = -\frac{\kappa\sigma_0}{2}h_{ab}. \tag{7}$$

Therefore, the reflection symmetric domain-wall space-time will be described by vacuum solutions of Einstein's field equations with the cosmological constant off Σ and Eq. (7).

III. SPACE-TIMES ADMITTING A THREE-PARAMETER GROUP OF ISOMETRIES

It is still a great challenge for mathematicians and physicists to find an exact solution of Einstein's field equations without assuming any symmetry of space-time. Most of the well-known exact solutions are found in spaces of high symmetry, so the group of isometries becomes a useful method of classification and also for finding exact solutions [10]. In Sec. IV, we will consider that the domain wall is homogeneous and isotropic in its two space dimensions, so it leads us to assume that the four-dimensional space-time geometry induced by the domain-wall source has the same symmetric property, i.e., two-dimensional spatial sections V_2 parallel to the wall are homogeneous and isotropic [9]. This assumption of space-time symmetry may correspond to space-times admitting a three-parameter group of isometries with two-dimensional spacelike orbits, denoted by $G_3(2, s)$. In this section, we shall solve the vacuum Einstein's field equations with the cosmological constant off Σ under $G_3(2, s)$.

It is known that an n-dimensional Riemannian space V_n admitting G_q, where $q = n(n + 1)/2$, is a space of constant curvature [10]. Hence the orbits V_2 of $G_3(2, s)$ must have constant Gaussian curvature K, and correspond to two-dimensional spheres ($K > 0$), planes ($K = 0$), or hyperboloids ($K < 0$). Moreover, it is always possible to introduce coordinates (t, z, r, ϕ) such that the metric tensor g with $G_3(2, s)$ has the form [11,16]

$$g = e^{2\nu(t,z)}(-dt \otimes dt + dz \otimes dz) + e^{2\lambda(t,z)} dV_2, \qquad (8)$$

where $dV_2 = (1 - Kr^2)^{-1} dr \otimes dr + r^2 d\phi \otimes d\phi$. By rescaling $e^{2\lambda}$, one can normalize the constant curvature to be $K = +1, 0, -1$. It is clear that the two space dimensions of Σ are placed at $t, z =$ constant. In [9], its metric ansatz yields

$$g = e^{2\nu(z)}(-dt \otimes dt + dz \otimes dz) + e^{2[\alpha(t)+\beta(z)]} dV_2, \qquad (9)$$

which is only a special form of the metric (8). We shall stress again that the general exact solutions of $\nu(t, z)$ and $\lambda(t, z)$ obtained in this section will only be valid off Σ.

We first introduce the Einstein's three-forms [17]

$$G_\mu = R_{\alpha\beta} \wedge *(e^\alpha \wedge e^\beta \wedge e_\mu), \qquad (10)$$

where $R_{\alpha\beta}$ are curvature two-forms defined in terms of Levi-Civita connection ∇, e^α are orthonormal coframes, and $*$ denotes the Hodge map associated with g. So Einstein's equations with Λ are

$$G_\mu = -2\kappa\tau_\mu + 2\Lambda * e_\mu, \qquad (11)$$

where τ_μ are stress-energy three-forms of matter fields. Since we are only interested in vacuum solutions of Eq. (11), τ_μ will be assumed to vanish in the following calculation. Substituting the metric (8) into vacuum Eq. (11) yields

$$\dot\lambda\lambda' - \dot\nu\lambda' + \dot\lambda' - \nu'\dot\lambda = 0, \qquad (12)$$

$$e^{-2\nu}(\dot\lambda^2 - 3\lambda'^2 - 2\lambda'' + 2\dot\lambda\,\dot\nu + 2\lambda'\nu') + e^{-2\lambda}K = \Lambda, \qquad (13)$$

$$e^{-2\nu}(3\dot\lambda^2 - \lambda'^2 + 2\ddot\lambda - 2\dot\lambda\,\dot\nu - 2\lambda'\nu') + e^{-2\lambda}K = \Lambda, \qquad (14)$$

$$e^{-2\nu}(\ddot\nu - \nu'' + \ddot\lambda - \lambda'' + \dot\lambda^2 - \lambda'^2) = \Lambda, \qquad (15)$$

where dots and primes here and in the following denote the differentiation with respect to t and z, respectively. These four nonlinear partial differential equations, Eqs. (12)–(15) , are difficult to find an analytic nontrivial solution for due to their highly coupling nature. However, they can be largely simplified by introducing advanced and retarded null coordinates u, v, defined by $u = \frac{1}{2}(t + z)$, $v = \frac{1}{2} \times (t - z)$. A similar procedure has been used to find Schwarzschild and Reissner-Nordström solutions and their maximally analytic extensions [12].

It is convenient to set

$$A(t, z) = e^{2\nu}, \qquad B^2(t, z) = e^{2\lambda}, \qquad (16)$$

where $A > 0$, so the metric (8) becomes

$$g = A(t, z)(-dt \otimes dt + dz \otimes dz) + B^2(t, z) dV_2. \qquad (17)$$

By transforming (t, z) coordinate variables to the null coordinates (u, v), Eqs. (12)–(15) with some linear combination yield

$$AB_{,uu} - A_{,u}B_{,u} = 0, \qquad (18)$$

$$AB_{,vv} - A_{,v}B_{,v} = 0, \qquad (19)$$

$$BB_{,uv} + B_{,u}B_{,v} + AK = \Lambda AB^2, \qquad (20)$$

$$B(\ln A)_{,uv} + 2B_{,uv} = 2\Lambda AB, \qquad (21)$$

where a subscript comma denotes partial differentiation with respect to the coordinates following it. It turns out that Eqs. (18)–(21) become much simpler and solvable. If we put $K = 1$ and $\Lambda = 0$, Eqs. (18)–(21) agree with Eqs. (31)–(34) in Sec. 17 of [12].[3]

From Eqs. (18) and (19), we observe that there exist degenerate nontrivial solutions in the case of $B_{,u}$ or $B_{,v}$ vanishing. Hence it is better to study the degenerate and nondegenerate solutions separately. In the special case of $K = \Lambda = 0$, i.e., plane symmetry, the degenerate and nondegenerate solutions have been studied in [15,18], where [15] called them class-I and class-II solutions, respectively.

A. The degenerate case: $B_{,u} = 0$ or $B_{,v} = 0$ (but not both)

Since Eqs. (18)–(21) are invariant under switching the coordinate variables u and v, it is necessary only to study either $B_{,u} = 0$ or $B_{,v} = 0$. Suppose $B_{,u} = 0$ and $B_{,v} \neq 0$, i.e., $B = B(v)$. Then Eq. (18) is trivial satisfying and Eq. (20) yields

$$B^2 = \frac{K}{\Lambda} = \text{const.} \geqslant 0, \qquad (22)$$

where $K, \Lambda \neq 0$. It is clear that both $B_{,u}$ and $B_{,v}$ vanish, which gives a trivial solution. So the case of $K, \Lambda \neq 0$ does not give a degenerate nontrivial solution, and we will not proceed further with our discussion in this case.

For $K = 0$, i.e., plane symmetry, Eq. (20) yields $\Lambda = 0$ since A and B cannot vanish. It means that the nontrivial degenerate solutions with plane symmetry cannot allow the nonvanishing cosmological constant. Then we expect to recover the class-I solutions in [15]. One may first solve Eq. (19) to yield $A = G(u)B_{,v}$, where G is an arbitrary function of u, and the Eq. (21) becomes trivial satisfying. So this solution does return to class-I solutions of [15]. It is worth mentioning that the planar domain-wall solutions obtained in [15,19] have the class-I solutions outside the wall.

[3] We point out that the metric signature $(+, -, -, -)$ used in [12] is different from ours.

B. The nondegenerate case: $B_{,u} \neq 0$ and $B_{,v} \neq 0$

In this case, Eqs. (18) and (19) yield

$$A(u, v) = F(v)B_{,u}, \tag{23}$$

$$A(u, v) = G(u)B_{,v}, \tag{24}$$

respectively, where $F(v) \neq 0$ and $G(u) \neq 0$ are arbitrary functions of their arguments. Substituting Eq. (23) into Eq. (20) gives

$$\left(BB_{,v} + KF(v)B - \frac{\Lambda}{3}F(v)B^3\right)_{,u} = 0. \tag{25}$$

Hence, from Eq. (25), we obtain

$$B_{,v} = -KF(v) + \frac{H(v)}{B} + \frac{\Lambda}{3}F(v)B^2, \tag{26}$$

where $H(v)$ is an arbitrary function of v. A similar result can be obtained by substituting Eq. (24) into Eq. (20), which yields

$$B_{,u} = -KG(u) + \frac{J(u)}{B} + \frac{\Lambda}{3}G(u)B^2, \tag{27}$$

where $J(u)$ is an arbitrary function of u. Since Eqs. (26) and (27) possess some symmetry between u and v, we multiply Eqs. (26) and (27), and use Eqs. (23) and (24), which gives

$$B_{,u}B_{,v} = -KA + \frac{J(u)}{B}B_{,v} + \frac{\Lambda}{3}AB^2 \tag{28}$$

$$= -KA + \frac{H(v)}{B}B_{,u} + \frac{\Lambda}{3}AB^2. \tag{29}$$

If $\{J(u), H(v)\} \neq 0$, Eqs. (28) and (29) yield

$$\frac{J(u)}{H(v)} = \frac{B_{,u}}{B_{,v}} = \frac{G(u)}{F(v)}. \tag{30}$$

So we finally obtain

$$\frac{J(u)}{G(u)} = \frac{H(v)}{F(v)} = 2M, \tag{31}$$

where M is a constant. By substituting Eqs. (23) and (27) into Eq. (21) and using Eqs. (20) and (31), one can easily verify that Eq. (21) is automatically satisfying. Therefore, the general exact solution of Eqs. (18)–(21) yields

$$A = -F(v)G(u)\left(K - \frac{2M}{B} - \frac{\Lambda}{3}B^2\right), \tag{32}$$

with $B(u, v)$ satisfying

$$dB = -\left(K - \frac{2M}{B} - \frac{\Lambda}{3}B^2\right)(G(u)du + F(v)dv). \tag{33}$$

Also, the metric Eq. (17) becomes

$$g = 4F(v)G(u)\left(K - \frac{2M}{B} - \frac{\Lambda}{3}B^2\right)du \otimes dv + B^2 dV_2. \tag{34}$$

From the metric (34), it is clear that the existence of the two arbitrary functions $F(v)$ and $G(u)$ is due to the freedom of choosing null coordinates u, v. Hence, the different choices of $F(v)$ and $G(u)$ may correspond to the metric g in different null coordinates. According to the generalized Birkhoff theorem [10,11], the metric (34) can actually be put into a static form [see Eq. (1)] by introducing coordinates:

$$T = \int F(v)dv + \int G(u)du, \qquad R = B. \tag{35}$$

In the following, we show that the metric (34) will return to some well-known solutions in various different coordinates by choosing three parameters K, Λ, M, and two functions $F(v)$, $G(u)$.

In the case of $K = 1$ and $\Lambda = 0$, Eqs. (32) and (33) yield the Schwarzschild solution obtained in [12], and M will be regarded as gravitational mass. Moreover, by setting

$$F(v) = \frac{2M}{v} \quad \text{and} \quad G(u) = \frac{2M}{u}, \tag{36}$$

we obtain the Schwarzschild solution in the Kruskal coordinates, which is the maximal extension of the Schwarzschild solution [12,20]. If one further sets $v/u = e^{t/2M}$, the metric (34) will be transited to Schwarzschild coordinates (t, R, θ, ϕ), which yields

$$g = -h(R)dt \otimes dt + h(R)^{-1}dR \otimes dR + R^2 d\Omega_2, \tag{37}$$

where $B \equiv R$ and $h(R) = 1 - 2M/R$. $d\Omega_2 \equiv d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$ denotes surface element of two-spheres. It is clear that the coordinates (t, R) cover only the regions of both u, v being positive and negative in the (u, v)-plane.

In the case of $K = \Lambda = 0$, i.e., plane symmetry with vanishing cosmological constant, it is not difficult to verify that Eqs. (32) and (33) give the class-II solutions of [15].[4]

In the case of $K = 1$, $M = 0$ and $\Lambda > 0$, by setting

$$F(v) = \sqrt{\frac{3}{4\Lambda}}\frac{1}{v} \quad \text{and} \quad G(u) = \sqrt{\frac{3}{4\Lambda}}\frac{1}{u}, \tag{38}$$

we obtain the de Sitter metric in the Kruskal coordinates:

$$g = -\frac{12}{\Lambda}\frac{1}{(uv - 1)^2}du \otimes dv + \frac{3}{\Lambda}\frac{(uv + 1)^2}{(uv - 1)^2}d\Omega_2. \tag{39}$$

Its global properties have been discussed in [21].

Finally, we consider $K = M = 0$, which will be useful for cosmological models. Since $K = M = 0$, Eq. (33) gives

[4]It should be noted that the two arbitrary functions F and G in [15] are different from ours.

$$-\frac{3}{\Lambda}\frac{1}{B} + c = \int G(u)du + \int F(v)dv, \qquad (40)$$

where c is a constant of integration, and the metric (34) becomes

$$g = B^2\Big(-\frac{4\Lambda}{3}F(v)G(u)du \otimes dv + dX_2\Big), \qquad (41)$$

where $dX_2 = dx \otimes dx + dy \otimes dy$. It is interesting to discuss the sign of F and G in the cases of Λ being positive or negative. For $\Lambda > 0$, the metric (41) indicates that $FG > 0$. On the other hand, the metric (41) yields $FG < 0$ in the case of $\Lambda < 0$. Since we are interested in the de Sitter universe, we will consider only $\Lambda > 0$.

It is clear that the metric (41) is conformally flat if we simply choose

$$F(v) = -\sqrt{\frac{3}{\Lambda}}, \qquad G(u) = -\sqrt{\frac{3}{\Lambda}}, \qquad (42)$$

for $\Lambda > 0$, and Eq. (40) yields

$$B = \sqrt{\frac{3}{\Lambda}}\frac{1}{(u + v - c)}. \qquad (43)$$

Furthermore, the metric (41) becomes

$$g = \frac{1}{\frac{\Lambda}{3}\eta^2}(-d\eta \otimes d\eta + dz \otimes dz + dX_2), \qquad (44)$$

where $\eta = t + c$. The metric (44) is a well-known solution of a flat expanding de Sitter universe in the conformal time [22]. One can also realize that the metric (44) is a background metric for describing a slow-roll inflation in the early Universe [2].

It is known that the metric solution (34) is only valid off the Σ. Hence, a general solution of domain-wall space-time should also need to satisfy the Israel's junction conditions. As we mentioned in Sec. I, the study of the domain-wall space-times can be separated into two different approaches mainly due to the different choices of coordinates, namely, the moving-wall approach and the comoving-coordinate approach. The moving-wall approach is usually used to study the two space-times, M_+ and M_- with a common moving boundary Σ [13,23,24]. In this approach, the metric solutions of M_+ and M_- have the static form (1), and it turns out that Σ is moving in this coordinate system. Therefore, the Israel's junction conditions become equations of motion for Σ.

The comoving-coordinate approach is to introduce the comoving coordinates of the wall system, i.e., the rest frame of the wall Σ [9]. In the comoving coordinates, the wall Σ is normally placed at a constant z-coordinate position, say, $z = 0$, so Israel's junction conditions serve as boundary conditions of the metric solutions at $z = 0$. In Sec. IV, we will adopt the second approach by considering a domain wall sitting at $z = 0$ with space-time being reflection symmetry. It turns out that the Israel's junction condition, i.e., Eq. (7), yields some constraints on $F(v)$ and $G(u)$ at $z = 0$, so $F(v)$ and $G(u)$ cannot be arbitrarily chosen.

IV. DOMAIN-WALL SPACE-TIMES

In this section, we limit our discussion on reflection-symmetric domain-wall space-times, so Israel's junction conditions yield Eq. (7) and constant surface energy density $\sigma = \tau = \sigma_0$. Since the metric solution (34) has a freedom of choosing double-null coordinates (u, v), we may simply assume that Σ is placed at $z \equiv u - v = 0$. According to the reflection symmetry, it only needs to study the metric (34) at $z > 0$. Hence, Eq. (7) may be considered as the boundary conditions of the metric (34) at $z = 0$.

Since Σ is placed at $z = 0$, the unit normal n and unit timelike vector u in the coordinates (t, z, r, ϕ) become

$$n = \frac{1}{\sqrt{A(t, z)}}\partial_z, \qquad u = \frac{1}{\sqrt{A(t, z)}}\partial_t, \qquad (45)$$

where ∂_t and ∂_z denote the coordinate basis, and the intrinsic metric h of Σ yields

$$h = -A|_{z=0}dt \otimes dt + B^2|_{z=0}dV_2. \qquad (46)$$

Substituting (46) into Eq. (7) and using Eq. (2) gives two nonvanishing equations

$$A'|_+ = -\frac{\kappa\sigma_0}{2}A^{3/2}|_{z=0}, \qquad (47)$$

$$B'|_+ = -\frac{\kappa\sigma_0}{4}\sqrt{A}B|_{z=0}. \qquad (48)$$

It is clear that Eqs. (47) and (48) will give $F(v)$ and $G(u)$ further constraints, so $F(v)$ and $G(u)$ cannot be arbitrary functions.

By substituting the solution (32) into Eqs. (47) and (48) and using (33), a tedious but straightforward calculation yields

$$\Big(-\frac{F_{,v}}{F} + \frac{G_{,u}}{G} + \frac{L_{,u} - L_{,v}}{L}\Big)\Big|_{z=0} = -\kappa\sigma_0\sqrt{-FGL}\Big|_{z=0}, \qquad (49)$$

$$L(F - G)|_{z=0} = -\frac{\kappa\sigma_0}{2}(B\sqrt{-FGL})|_{z=0}, \qquad (50)$$

where

$$L(u, v) := K - \frac{2M}{B} - \frac{\Lambda}{3}B^2. \qquad (51)$$

However, Eqs. (49) and (50) actually are not independent. To verify this, one may first differentiate Eq. (50) with respect to t. Then, by using Eqs. (33) and (50), one can show that Eq. (49) is implied by Eq. (50) if $(F + G)|_{z=0} \neq 0$, which is assumed in the following discussion. It is interesting to note that, when $(F-G)|_{z=0}=0$, Eq. (50) yields $\sigma_0=0$. We then obtain an important result that, if the domain wall exists, i.e., $\sigma_0\neq 0$, in the space-time, one cannot choose $F(t)=G(t)$. It turns out that Eqs. (36), (38), and (42), are not valid choices in the domain-wall space-time. We shall mention that Eq. (50) cannot uniquely determine the $F(v)$ and $G(u)$, so one still has freedom of choosing these two functions.

A. Spherical and hyperbolic domain walls

In this subsection, we study spherical and hyperbolic domain walls in the de Sitter universe, i.e., $K=\pm 1$, $M=0$, and $\Lambda>0$. It is convenient to introduce

$$\mathcal{F}(v)=\int F(v)dv, \qquad \mathcal{G}(u)=\int G(u)du, \tag{52}$$

and, in the case of $K=1$, Eq. (33) yields

$$B=\begin{cases}-\sqrt{\frac{3}{\Lambda}}\coth\Big[\sqrt{\frac{\Lambda}{3}}(\mathcal{F}+\mathcal{G})\Big] & \text{for } B^2>\frac{3}{\Lambda},\\ -\sqrt{\frac{3}{\Lambda}}\tanh\Big[\sqrt{\frac{\Lambda}{3}}(\mathcal{F}+\mathcal{G})\Big] & \text{for } B^2<\frac{3}{\Lambda}.\end{cases} \tag{53}$$

It is known that $B^2=\frac{3}{\Lambda}$ corresponds to cosmological horizons [21]. Substituting Eq. (53) into Eq. (50) gives

$$\frac{\dot f_-}{\sqrt{\dot f_+^2-\dot f_-^2}}=\mp\frac{\kappa\sigma_0}{4}\sqrt{\frac{3}{\Lambda}}\cosh\Big[\sqrt{\frac{\Lambda}{3}}f_+\Big] \quad \text{for} \quad B^2>\frac{3}{\Lambda},$$
$$\frac{\dot f_-}{\sqrt{\dot f_-^2-\dot f_+^2}}=-\frac{\kappa\sigma_0}{4}\sqrt{\frac{3}{\Lambda}}\sinh\Big[\sqrt{\frac{\Lambda}{3}}f_+\Big] \quad \text{for} \quad B^2<\frac{3}{\Lambda}, \tag{54}$$

where $f_+(t):=(\mathcal{F}+\mathcal{G})|_{z=0}$ and $f_-(t):=(\mathcal{F}-\mathcal{G})|_{z=0}$. It is clear that $\dot f_\pm=\frac{1}{2}(F\pm G)|_{z=0}$. The $\mp$ sign corresponds to $f_+>0$ or $f_+<0$, respectively. Equation (54) indicates that either f_+ or f_- is given; the other will be determined.

In the case of $K=-1$, a similar calculation yields

$$B=\sqrt{\frac{3}{\Lambda}}\tan\Big[\sqrt{\frac{\Lambda}{3}}(\mathcal{F}+\mathcal{G})\Big], \tag{55}$$

and

$$\frac{\dot f_-}{\sqrt{\dot f_+^2-\dot f_-^2}}=\frac{\kappa\sigma_0}{4}\sqrt{\frac{3}{\Lambda}}\sin\Big[\sqrt{\frac{\Lambda}{3}}f_+\Big], \tag{56}$$

for $\sec[\sqrt{\frac{\Lambda}{3}}(\mathcal{F}+\mathcal{G})]>0$. We expect that the choices of f_+ or f_- are related to global properties of space-times and a proper choice of f_+ or f_- may yield a simpler domain-wall solution and also avoid coordinate singularities. So far, our guiding principle of choosing f_+ or f_- is that, when $\sigma_0=0$, the domain-wall solutions should return to some well-known solutions. Here, we present an example of choosing f_+ in the case of $K=1$. A more comprehensive study on f_+, f_- and also global properties of domain-wall space-times will be present in our future work.

In the case of $K=1$, the de Sitter metric (39) in Kruskal coordinates has been presented in Sec. III. From Eq. (38), we know that $f_+=\frac{1}{2}\sqrt{\frac{3}{\Lambda}}\ln t$ for $t>0$, and then substituting f_+ into Eq. (54) yields

$$\dot f_-=\frac{-\sqrt{\frac{3}{\Lambda}}(t+1)}{2t\sqrt{\frac{\Lambda}{3}(\frac{8}{\kappa\sigma_0})^2t+(t+1)^2}}, \tag{57}$$

where we consider only $B^2>\frac{3}{\Lambda}$. We then obtain F and G, which are

$$F(v)=\sqrt{\frac{3}{4\Lambda}}\frac{1}{v}\bigg(1-\frac{(v+1)}{\sqrt{\frac{\Lambda}{3}(\frac{8}{\kappa\sigma_0})^2v+(v+1)^2}}\bigg),$$
$$G(u)=\sqrt{\frac{3}{4\Lambda}}\frac{1}{u}\bigg(1+\frac{(u+1)}{\sqrt{\frac{\Lambda}{3}(\frac{8}{\kappa\sigma_0})^2u+(u+1)^2}}\bigg). \tag{58}$$

It is clear that when $\sigma_0=0$, Eq. (58) returns to Eq. (38), and the metric should become the de Sitter metric in Kruskal coordinates.

B. Planar domain walls

In this subsection, we study the planar domain wall in the de Sitter universe, i.e., $K=M=0$ and $\Lambda>0$. In the planar domain-wall case, the Israel's junction condition (50) becomes

$$\frac{\dot f_-}{\sqrt{\dot f_+^2-\dot f_-^2}}=\pm\frac{\kappa\sigma_0}{4}\sqrt{\frac{3}{\Lambda}}, \tag{59}$$

which is much simpler. The $\pm$ sign corresponds to $B>0$ or $B<0$, respectively. In Sec. III, we have found the metric (44), which is conformally flat. It was obtained by choosing $F=G=$ constant. We observe that, if F and G are two different constants, Eq. (59) can also be satisfied. Therefore, we consider

$$F(v)=F_0, \qquad G(u)=F_0\Gamma, \tag{60}$$

where F_0 and Γ are constants. Since $FG>0$ for $\Lambda>0$, so $\Gamma>0$. If one considers $F_0<0$, then substituting Eq. (60) into (50) yields

$$\sqrt{\frac{\Lambda}{3}}(1-\Gamma)=\mp\frac{\kappa\sigma_0}{2}\sqrt{\Gamma}, \tag{61}$$

for B being positive or negative. If one further assumes $\sigma_0\geqslant 0$, which is physically reasonable, Eq. (61) yields that $\Gamma\geqslant 1$ for $B>0$, and $\Gamma\leqslant 1$ for $B<0$. The algebraic equation (61) has two real positive roots, which are

$$\Gamma = 1 + \frac{3\kappa^2\sigma_0^2}{8\Lambda} \pm \frac{\sqrt{48\kappa^2\sigma_0^2\Lambda + 9\kappa^4\sigma_0^4}}{8\Lambda}, \tag{62}$$

so the larger root of Γ corresponds to the case of $B > 0$, and the smaller root to $B < 0$.

Substituting Eq. (60) into Eqs. (40) and (41) yields

$$B = -\frac{3}{\Lambda F_0(\Gamma u + v - c_0)}, \tag{63}$$

$$g = B^2\left(\frac{\Lambda\Gamma}{3} F_0^2(-dt \otimes dt + dz \otimes dz) + dX_2\right). \tag{64}$$

It is interesting to note that B should take different values of Γ for $B > 0$ and $B < 0$, so these two domain regions may need to be considered separately. Since F_0 is an arbitrary negative constant, we may choose $F_0 = -\sqrt{\frac{3}{\Lambda\Gamma}}$, and then obtain

$$g = \frac{-d\eta \otimes d\eta + dz \otimes dz + dX_2}{\frac{\Lambda\Gamma}{3}(\frac{\Gamma+1}{2}\eta + \frac{\Gamma-1}{2}z)^2} \tag{65}$$

for $z > 0$. It is obvious that when $\Gamma = 1$, i.e., $\sigma_0 = 0$, the metric (65) returns to (44), which is the de Sitter universe. Moreover, the metric (65) is conformally flat, although it has z-dependence. According to the reflection symmetry, we then obtain the domain-wall solution in the de Sitter universe:

$$g = \frac{-d\eta \otimes d\eta + dz \otimes dz + dX_2}{\frac{\Lambda\Gamma(\Gamma+1)^2}{12}(\eta + \frac{\Gamma-1}{\Gamma+1}|z|)^2}. \tag{66}$$

The metric solution (66) is important and useful for cosmologists to study the domain-wall effects in the early Universe. It is worth mentioning that the metric (66) will not return to the planar domain-wall solution obtained in [15,19] when $\Lambda = 0$, since the metric (66) outside the domain wall belongs to the nondegenerate solution. Moreover, $\Lambda = 0$ will make the metric (66) become divergent. Finally, we would like point out that, if one applies the following coordinate transformations:

$$\tilde{\eta} = \frac{\Gamma+1}{2}\eta + \frac{\Gamma-1}{2}z, \qquad \tilde{z} = \frac{\Gamma-1}{2}\eta + \frac{\Gamma+1}{2}z,$$
$$\tilde{x} = \sqrt{\Gamma}x, \qquad \tilde{y} = \sqrt{\Gamma}y, \tag{67}$$

for $z > 0$, the metric (65) also returns to (44). However, in these coordinates $(\tilde{\eta}, \tilde{z}, \tilde{x}, \tilde{y})$, the wall's location becomes

$$\tilde{z} = \left(\frac{\Gamma-1}{\Gamma+1}\right)\tilde{\eta}, \tag{68}$$

which is moving along $\tilde{z}$.

Here, we consider only the spherical, planar, and hyperbolic domain-wall space-times with $M = 0$ and $\Lambda > 0$. It might also be interesting to study more general cases, e.g., $M \neq 0$ and $\Lambda < 0$. Moreover, the global properties of these domain-wall solutions are important and will be discussed in our next work. In particular, the problems of the two-bubbles collision in the early Universe exhibit the hyperbolic symmetry on two-dimensional spatial section V_2 of space-times [24,25], so it should be important to further investigate the global properties of $K = -1$.

V. CONCLUSIONS

We have systematically studied exact solutions of vacuum Einstein's field equations with Λ in space-times admitting $G_3(2, s)$. The general solutions are classified into degenerate and nondegenerate solutions. In the degenerate case of $\{K, \Lambda\} \neq 0$, we did not find any physically interesting solutions. However, in the case of either $K = 0$ or $\Lambda = 0$, each degenerate field equation required the other to be vanishing. Therefore, we obtained the planar symmetric solutions with vanishing Λ, which are equivalent to the class-I solutions of [15]. In the nondegenerate case, the general exact solutions are obtained in the double-null coordinates, and they contain three parameters, K, M, Λ, and two arbitrary functions $F(v)$ and $G(u)$ due to the freedom of choosing null coordinates. We considered nondegenerate vacuum solutions as the domain-wall solutions outside the wall.

In the domain-wall space-times, we assumed that the domain wall Σ is homogeneous and isotropic in its two space dimensions. Since the nondegenerate solutions have freedom on choosing null coordinates, we then considered that Σ is placed at $z = 0$, and the Israel's junction conditions yield one constraint equation on $F(v)$ and $G(u)$. The remaining freedom of choosing $F(v)$ and $G(u)$ is fixed by requiring that, when the surface energy density σ_0 of domain walls vanishes, the metric solutions will return to some well-known solutions. In the derivation of Israel's junction conditions, we assume only that the space-time be reflection-symmetric without putting any restriction on the three parameters. Applying the Israel's junction conditions to the case of $M = 0$ and $\Lambda > 0$, we first discussed the cases of $K = \pm 1$. It turns out that Israel's junction conditions become a first-order ordinary differential equation for two arbitrary functions, $f_+(t)$ and $f_-(t)$. An example of choosing f_+ has been presented. In the case of $K = 0$, we obtained the planar domain-wall solution, which is conformally flat, in the de Sitter universe.

We plan to use the solution (66) to study primordial quantum fluctuation during inflation in the early Universe. Since the wall is at rest at $z = 0$ and the planar domain-wall metric is conformally flat, the quantum fluctuation of scalar fields may be solved exactly. Moreover, the study of the global properties of the domain-wall solutions obtained in this paper is in progress. Besides the study of the planar domain walls, it will also be interesting to extend our current investigation to $\{M \neq 0, \Lambda < 0\}$.

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