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An Unified Approach to Missile Guidance Laws: A 3D Extension

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Abstract

Since the proportional navigation guidance law was first introduced, many of the researchers had proposed different methodologies to investigate the corresponding performances of all the existing guidance laws. In 1997, Yang and Yang introduced an unified approach, in that paper the authors found that under the proposed framework, all the existing guidance laws, namely TPN, IPN and PPN, are indeed special cases of the mentioned general guidance law. But their results were restricted to two dimensional space. In this paper, the author not only extends the results to three dimensional space, but also to GTPN, GIPN and PPN. Unlike conventional researchers, a modified polar coordinate (MPC) is adopted. It is shown that with the property of this modified polar coordinate, the number of differential equations required to describe the relative dynamics can be reduced from six to three, and all the terms of differential equations involve only products and additions of variables. For all the mentioned guidance laws in this paper, to describe the capture area we need only two transformed variables, while the third variable is required to provide the condition of finite turn rate.

1 Introduction

Since the proportional navigation guidance law was first introduced, many of the researchers have proposed different methodologies to investigate the corresponding performances of all the existing guidance laws, for example, TPN [1,2], IPN [3], PPN [4,5]. But unified approach did not exist until 1997. In 1997, Yang and Yang [6] introduced an unified approach, in that paper the authors found that under the proposed framework, all the existing guidance laws are indeed special cases of the mentioned general guidance law. But their results were restricted to two dimensional space.

Although there were some results for the case of there dimensional space [7–9], again, the approaches used were distinct. In this paper, the author not only extends the results to three dimensional space, but also to the general form of TPN and IPN, namely, GTPN, and GIPN. Unlike conventional researchers, a modified polar coordinate (MPC) is adopted. It is shown that with the property of this modified polar coordinate, the number of differential equations required to fully describe the relative dynamics between missile and target can be reduced from six to three, and all the terms involved in the differential equations contain only products and additions of variables. In addition, using the relative velocities, which were adopted by Yang and Yang [6], as nonlinear transformation variables reveals that for the GTPN and GIPN all the results found for two dimensional space can be applied directly to the case of nonmaneuvering target or maneuvering but the target's exact acceleration can be obtained. However, for the PPN guidance law, the results obtained from two dimensional space can only be applied to nonmaneuvering target.

In section 2, the 3D relative dynamics between missile and target are derived in MPC. The number of dynamic equations are reduced from six to three. And these three differential equations also shed the light on why the general guidance laws introduced in section 3 takes its given form, which was never explained in existing literatures. Then in section 4, using the relative closing and tangent velocities transforms the first two differential equations into another form which is used to derive the capture area and the condition for finite turn rate for all the three guidance laws.

2 Dynamic Equations in the Modified Polar Coordinate

Let the relative position vector (line of sight),r, of target and missile be defined as (see Figure 1)

$$r = r_T - r_M = \rho e_r, \tag{2.1}$$

where r is the line of sight (LOS) vector from missile platform to target, r_T and r_M are the position vectors of target and missile in an inertial coordinate $OXYZ$ respectively, ρ is the Length of LOS vector, i.e., the range between target and missile, and e_r is the unit vector in the direction of the LOS. Then the relative velocity and acceleration can be written as

$$\frac{d}{dt}r = v_r = \dot{\rho} e_r + \rho \dot{e}_r = v_T - v_M, \tag{2.2}$$

$$\frac{d}{dt}v_r = \ddot{\rho} e_r + 2\dot{\rho}\dot{e}_r + \rho \ddot{e}_r = a_T - a_M, \tag{2.3}$$

in which v_r is the relative velocity of target relative to missile, $\dot{e}_r \triangleq \frac{d}{dt}e_r$ is the time derivative of the unit LOS vector, v_T, v_M and a_T, a_M are the missile and target velocity and acceleration vectors respectively. Note that

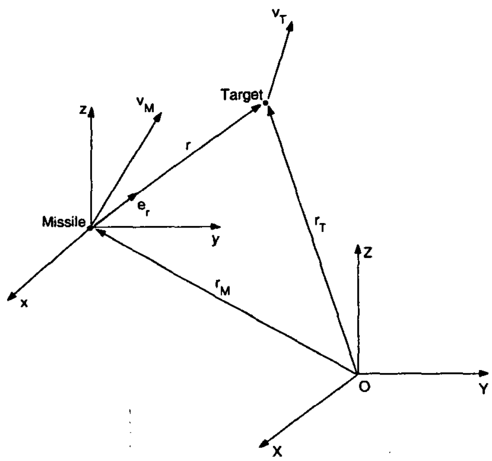

Figure 1: Engagement geometry

$v_T - v_M$ has no component along the direction $e_r \times \dot{e}_r$, that is, $(v_T - v_M)^T(e_r \times \dot{e}_r) = 0$.

The components of the state vector of the modified polar coordinate system are defined as follows [10]:

$$x_P = \begin{bmatrix} x_{P1} \\ x_{P2} \\ x_{P3} \\ x_{P4} \end{bmatrix} \triangleq \begin{bmatrix} e_r \\ \dot{e}_r \\ \frac{1}{\rho} \\ \frac{1}{\rho}\frac{d\rho}{dt} \end{bmatrix}. \tag{2.4}$$

The state dynamics, are given by differentiating each component of state vector x_P:

$$\frac{d}{dt}x_P = f(x_P) + g(x_P)(a_T - a_M), \tag{2.5}$$

where

$$f(x_P) = \begin{bmatrix} x_{P2} \\ -2x_{P4}x_{P2} - (x_{P2}^T x_{P2})x_{P1} \\ -x_{P3}x_{P4} \\ (x_{P2}^T x_{P2}) - x_{P4}^2 \end{bmatrix}, \tag{2.6a}$$

$$g(x_P) = \begin{bmatrix} 0_{3\times 3} \\ x_{P3}(I_3 - x_{P1}x_{P1}^T) \\ 0_{1\times 3} \\ x_{P3}x_{P1}^T \end{bmatrix}, \tag{2.6b}$$

in which $x_{P2}^T x_{P2}$ represents the inner product of vectors x_{P2} and x_{P2}, $x_{P1}x_{P1}^T$ is the matrix formed by the product of vectors x_{P1} and x_{P1}^T, $0_{1\times 3}, 0_{3\times 3}$ are defined as 1×3 and 3×3 zero matrices, and I_3 denotes the 3×3 identity matrix. To describe the relative dynamics between missile and target in a 3-Dimensional space, six states are required. However, we utilized eight states instead, hence, there are two constraints on the above given states, namely,

$$x_{P1}^T x_{P1} = 1, x_{P1}^T x_{P2} = 0. \tag{2.7}$$

Note that x_{P2} is not a unit vector in general, indeed its magnitude is equal to the magnitude of the angular velocity of line of sight. By assuming that the angular velocity of line of sight, Ω, is orthogonal to the line of sight, we have

$$x_{P1} \times x_{P2} = x_{P1} \times (\Omega \times x_{P1}) = \Omega, \tag{2.8}$$

it follows that

$$(x_{P1} \times x_{P2})^T(x_{P1} \times x_{P2}) = x_{P2}^T x_{P2} = \Omega^T\Omega, \tag{2.9}$$

$$x_{P1} \times \Omega = -x_{P2}, \tag{2.10}$$

$$x_{P2} \times \Omega = (\Omega^T\Omega)x_{P1}, \tag{2.11}$$

in the above equations $x_{Pi} \times x_{Pj}$ denotes the cross product of vectors x_{Pi} and x_{Pj}. For convenience, let us define the following unit vectors

$$e_t \triangleq \frac{x_{P2}}{\sqrt{\Omega^T\Omega}}, \; e_\Omega \triangleq \frac{\Omega}{\sqrt{\Omega^T\Omega}}, \tag{2.12}$$

apparently e_t and e_Ω are the unit vectors in the direction of $\dot{e}_r$ and Ω respectively. By taking this orthogonal coordinate system (e_r, e_t, e_Ω), the analysis of guidance law in this paper becomes easy.

If we express a_T and a_M in this (e_r, e_t, e_Ω) coordinate system as

$$a_T \triangleq a_{T1}e_r + a_{T2}e_t + a_{T\Omega}e_\Omega, \tag{2.13}$$

$$a_M \triangleq a_{M1}e_r + a_{M2}e_t + a_{M\Omega}e_\Omega, \tag{2.14}$$

and after applying the constraints (2.7), we then have the following three coupled scalar differential equations:

$$\frac{d}{dt}x_{P4} = \Omega^T\Omega - x_{P4}^2 + x_{P3}(a_{T1} - a_{M1}), \tag{2.15a}$$

$$\frac{d}{dt}\sqrt{\Omega^T\Omega} = -2x_{P4}\sqrt{\Omega^T\Omega} + x_{P3}(a_{T2} - a_{M2}), \tag{2.15b}$$

$$\frac{d}{dt}x_{P3} = -x_{P3}x_{P4}. \tag{2.15c}$$

In addition, the dynamic equation for the angular velocity of line of sight Ω satisfies

$$\frac{d}{dt}\Omega = x_{P3}\left[(a_{T2} - a_{M2})e_\Omega - (a_{T\Omega} - a_{M\Omega})e_t\right] -2x_{P4}\Omega. \tag{2.16}$$

3 Guidance Laws

Observe equations (2.15) and (2.16) we know that the acceleration components $a_{M\Omega}, a_{T\Omega}$ do not have any effect on capturing the target but do change the direction of Ω. Furthermore, they influence system observability [11]. The above observation explains why most of the existing guidance laws are in the form of [6]

$$a_M = v_g \times \Omega, \tag{3.1}$$

where the guidance reference vector v_g is defined as

$$v_g \triangleq v_{g1}e_r + v_{g2}e_t + v_{g\Omega}e_\Omega. \tag{3.2}$$

Hence

$$a_{M1} = v_{g2}\sqrt{\Omega^T\Omega},\ a_{M2} = -v_{g1}\sqrt{\Omega^T\Omega},\ a_{M\Omega} = 0, \tag{3.3}$$

and equations (2.15) can be further written as

$$\frac{d}{dt}x_{P4} = \Omega^T\Omega - x_{P4}^2 + x_{P3}(a_{T1} - v_{g2}\sqrt{\Omega^T\Omega}), \tag{3.4a}$$

$$\frac{d}{dt}\sqrt{\Omega^T\Omega} = -2x_{P4}\sqrt{\Omega^T\Omega} + x_{P3}(a_{T2} + v_{g1}\sqrt{\Omega^T\Omega}), \tag{3.4b}$$

$$\frac{d}{dt}x_{P3} = -x_{P3}x_{P4}. \tag{3.4c}$$

For convenience, let us define the velocity vector of missile and target as

$$v_M \triangleq V_M e_{vM} = v_{M1}e_r + v_{M2}e_t + v_{M\Omega}e_\Omega, \tag{3.5}$$

$$v_T \triangleq V_T e_{vT} = v_{T1}e_r + v_{T2}e_t + v_{T\Omega}e_\Omega, \tag{3.6}$$

where $V_M = \sqrt{v_M^T v_M}, V_T = \sqrt{v_T^T v_T}$, and e_{vM}, e_{vT} are the unit vectors in the direction of v_M, v_T, respectively.

Table 1: Expressions of v_{g1}, v_{g2} and $v_{g\Omega}$ for existing guidance laws

	v_{g1}	v_{g2}	$v_{g\Omega}$
TPN	$\beta\frac{x_{P4}(t_0)}{x_{P3}(t_0)}$	0	0
RTPN	$\beta\frac{x_{P4}(t)}{x_{P3}(t)}$	0	0
GTPN	$\beta\frac{x_{P4}(t_0)}{x_{P3}(t_0)}$	$\alpha\frac{\sqrt{\Omega^T(t_0)\Omega(t_0)}}{x_{P3}(t_0)}$	0
GIPN	$\beta\frac{x_{P4}(t)}{x_{P3}(t)}$	$\alpha\frac{\sqrt{\Omega^T(t)\Omega(t)}}{x_{P3}(t)}$	0
PPN	$-\beta v_{M1}(t)$	$-\beta v_{M2}(t)$	$-\beta v_{M\Omega}(t)$
OPN	$v_{1opt}(t)$	$v_{2opt}(t)$	0

Table 1 shows the guidance reference vectors adopted by some of the existing guidance laws [6, 11], in which α, β are navigation constants (or variables). In most of the literatures $\alpha = \beta =$ constant, however, it is not necessarily to be held [11]. Note that for the case $\beta = \alpha =$ constant, then

$$v_g = \beta(v_T - v_M) = \beta v_r, \text{ for IPN}, \tag{3.7}$$

$$= \beta v_r(t_0), \text{ for TPN}, \tag{3.8}$$

$$= -\beta v_M, \text{ for PPN}. \tag{3.9}$$

In this paper, we adopt the definition of "capture" of target given in [9].

Definition 3.1. The capture of target by missile is characterized by a finite final time t_f at which the range $\rho(t_f)$ is equal to zero. This can be formulated as

$$\exists t_f < \infty \text{ such that } \rho(t_f) = 0 \text{ or } x_{P3}(t_f) = \infty. \tag{3.10}$$

To avoid the turn rate of the line of sight, i.e. $\|\dot{e}_r\|$, from infinite, we also require that

$$\sqrt{\Omega^T(t_f)\Omega(t_f)} \text{ is finite}. \tag{3.11}$$

Definition 3.2. The time at which missile is launched is denoted by t_0 and is called the initial time.

In the following sections we assume that all the required states, x_P, can be measured.

4 Nonlinear Transformation Using Relative Velocities

In this section we adopt the relative velocity components in the direction of e_r, e_t [6], and x_{P3} as the intermediate variables u, v, and w then we have

$$u \triangleq \frac{x_{P4}}{x_{P3}}, v \triangleq \frac{\sqrt{\Omega^T\Omega}}{x_{P3}}, w \triangleq x_{P3}, \tag{4.1}$$

and the relative dynamics can be characterized by

$$\frac{du}{dt} = v^2 w + (a_{T1} - a_{M1}),\ u(t_0) = u_0, \tag{4.2a}$$

$$\frac{dv}{dt} = -uvw + (a_{T2} - a_{M2}),\ v(t_0) = v_0, \tag{4.2b}$$

$$\frac{dw}{dt} = -uw^2,\ w(t_0) = w_0, \tag{4.2c}$$

with the initial conditions

$$u_0 \triangleq \frac{x_{P4}(t_0)}{x_{P3}(t_0)}, v_0 \triangleq \frac{\sqrt{\Omega^T(t_0)\Omega(t_0)}}{x_{P3}(t_0)}, w_0 \triangleq x_{P3}(t_0).$$

To have a direct comparison with the results in [6], assume that

$$a_T = 0 \text{ or } a_M = a_T + v_g \times \Omega,$$

hence we have

$$a_{M1} = a_{T1} + v_{g2}\sqrt{\Omega^T\Omega},\ a_{M2} = a_{T2} - v_{g1}\sqrt{\Omega^T\Omega},$$
$$a_{M\Omega} = a_{T\Omega}. \tag{4.3}$$

Equations (4.2) can be further transformed into

$$\frac{du}{d\tau} = v^2 - v_{g2}v,\ u(\tau_0) = u_0, \tag{4.4a}$$

$$\frac{dv}{d\tau} = -uv + v_{g1}v,\ v(\tau_0) = v_0, \tag{4.4b}$$

$$\frac{dw}{d\tau} = -uw,\ w(\tau_0) = w_0, \tag{4.4c}$$

where the independent variable τ is defined as $d\tau \triangleq w dt$. Note that the phase portraits on the (u, v)-plane in the "t" domain are the same as those in the "τ" domain. Moreover, $v = 0$ is the "equilibrium line" on the (u, v) plane in τ domain.

The capture conditions (3.10) and (3.11) imply that at final time t_f, $v(t_f) = 0$. In addition, it is obvious that at final time, we have a finite closing velocity $\dot{\rho}(t_f) < 0$. Henceforth, we define the capture region as:

Definition 4.1. The capture region is the region on the (u, v)-plane such that whenever the initial states (u_0, v_0) are started inside this region, the state trajectories will lead to $(u_f, 0)$ at t_f or τ_f, where u_f is a negative finite number, and the turn rate $\sqrt{\Omega^T(t_f)\Omega(t_f)}$ remains finite as well.

Although the conditions given in definition 4.1 are necessary conditions of definition 3.1, however, for the guidance laws considered in this paper, the corresponding given conditions turn out also to be sufficient.

Comparing the first two equations of (4.4) with the equations (5a,b) in [6] reveals that both of the equations indeed are the same, which in turn indicates that equations (4.4) are the three dimensional extension of [6]. Hence, most of the results given in [6] can be applied directly. While the conclusion regarding PPN guidance needs some modification as will be seen in the following consecutive subsection. For comparison, the following three cases are considered:

4.1 Case 1: GIPN

The differential equations for guidance law using GIPN can be formulated as

$$\frac{du}{d\tau} = -(\alpha - 1)v^2, \ u(\tau_0) = u_0, \tag{4.5a}$$

$$\frac{dv}{d\tau} = (\beta - 1)uv, \ v(\tau_0) = v_0, \tag{4.5b}$$

$$\frac{dw}{d\tau} = -uw, \ w(\tau_0) = w_0. \tag{4.5c}$$

It is easy to show that for constant α, β the phase portraits on the (u, v) plane satisfy

$$(\beta - 1)u^2 + (\alpha - 1)v^2 = (\beta - 1)u_0^2 + (\alpha - 1)v_0^2, \tag{4.6}$$

which is an ellipse or hyperbola centered at $(0, 0)$, and the phase portraits on the (v, w) plane satisfy

$$v(\tau)w^{(\beta-1)}(\tau) = v_0 w_0^{(\beta-1)}, \quad \text{or}$$
$$\sqrt{\Omega^T(\tau)\Omega(\tau)}\rho^{(2-\beta)}(\tau) = \sqrt{\Omega^T(t_0)\Omega(t_0)}\rho^{(2-\beta)}(t_0). \tag{4.7}$$

Apparently, to have finite turn rate at final time t_f, $\beta > 2$ is required. The capture area is determined by [11]:

1. For navigation variables $\alpha > 1, \beta > 2$:

   any initial condition except $u_0 \geq 0$ and $v_0 = 0$. (4.8)

2. For navigation variables $\alpha_{min} \leq \alpha \leq 1, \beta > 2$:

   $$u_0 < 0 \text{ and } u_0^2 + (\alpha_{min} - 1)v_0^2 > 0. \tag{4.9}$$

3. For constant α, β such that $\alpha < 1, \beta > 2$:

   $$u_0 < 0 \text{ and } (\beta - 1)u_0^2 + (\alpha - 1)v_0^2 > 0. \tag{4.10}$$

4.2 Case 2: GTPN

For the guidance law uses GTPN, the relative dynamics can be described by

$$\frac{du}{d\tau} = v^2 - \alpha v_0 v, \ u(\tau_0) = u_0, \tag{4.11a}$$

$$\frac{dv}{d\tau} = -uv + \beta u_0 v, \ v(\tau_0) = v_0, \tag{4.11b}$$

$$\frac{dw}{d\tau} = -uw, \ w(\tau_0) = w_0. \tag{4.11c}$$

The state trajectories on the (u, v) plane can be easily shown in the following form

$$\begin{bmatrix} u(\theta) \\ v(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta - \theta_0) & \sin(\theta - \theta_0) \\ -\sin(\theta - \theta_0) & \cos(\theta - \theta_0) \end{bmatrix} \begin{bmatrix} -(\beta - 1)u_0 \\ -(\alpha - 1)v_0 \end{bmatrix} + \begin{bmatrix} \beta u_0 \\ \alpha v_0 \end{bmatrix} \tag{4.12}$$

where θ is defined by $d\theta \triangleq v d\tau$, and $u(\theta_0) = u_0, v(\theta_0) = v_0$. It is easy to see that on the (u, v) plane, the phase portraits satisfy

$$(u - \beta u_0)^2 + (v - \alpha v_0)^2 = r_{GTPN}^2, \tag{4.13}$$

which is a circle centered at $(\beta u_0, \alpha v_0)$ with radius $r_{GTPN} = \sqrt{(\beta - 1)^2 u_0^2 + (\alpha - 1)^2 v_0^2}$ (see Figure 2). Note that $\frac{d}{d\tau}\theta = v > 0$, for all $v_0 > 0$, and the trajectory of $(u(\theta), v(\theta))$ will move in a clock-wise direction.

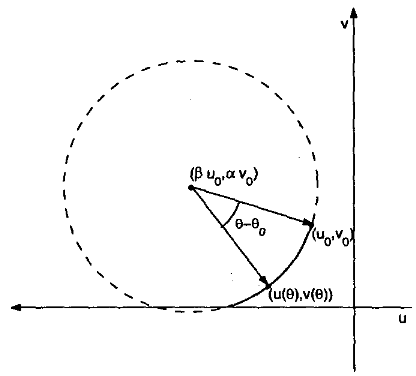

Figure 2: Phase portraits on (u, v) plane for GTPN guidance law with $\alpha > 1, \beta > 2$.

For constant β, α, to have $u(\tau_f) < 0, v(\tau_f) = 0$, the region is determined by:

$$\alpha^2 v_0^2 < r_0^2 < \beta^2 u_0^2, \text{ and } u_0 < 0. \tag{4.14}$$

On the (v, w) plane, it can be shown that

$$\frac{w(\tau)}{w(\tau_0)} = e^{\int_{v(\tau_0)}^{v(\tau)} \left[1 + \frac{\beta u_0}{\sqrt{r_0^2 - (v - \alpha v_0)^2}}\right] \frac{dv}{v}}, \tag{4.15a}$$

$$v(\tau)w(\tau) = v(\tau_0)w(\tau_0)e^{\int_{\tau_0}^{\tau} (\beta u_0 - 2u) d\tau}. \tag{4.15b}$$

or, equivalently,

$$\rho(\tau) = \rho(\tau_0)e^{-\int_{v(\tau_0)}^{v(\tau)} \left[1 + \frac{\beta u_0}{\sqrt{r_0^2 - (v - \alpha v_0)^2}}\right] \frac{dv}{v}},$$
$$\sqrt{\Omega^T(\tau)\Omega(\tau)} = \sqrt{\Omega^T(\tau_0)\Omega(\tau_0)}e^{\int_{\tau_0}^{\tau} (\beta u_0 - 2u) d\tau}.$$

It is easy to show that if $\beta > 2, u_0 < 0$, the minimum value of u occurs at u_0 or $u(\tau_f)$. In addition, if

$$(\frac{3}{4}\beta^2 - 2\beta + 1)u_0^2 + (1 - 2\alpha)v_0^2 > 0, \tag{4.16}$$

then we have $\beta u_0 - 2u < 0$, for $\tau > \tau_0$, which in turn results in finite turn rate at final time τ_f,

$$\sqrt{\Omega^T(\tau_f)\Omega(\tau_f)} < \sqrt{\Omega^T(\tau_0)\Omega(\tau_0)}, \tag{4.17}$$

and $\rho(\tau_f) = 0$. Note that condition (4.16) will be satisfied automatically if $\beta > 2, \alpha \leq \frac{1}{2}$.

4.3 Case 3: PPN

For PPN guidance law, the missile's velocity is adopted as the guidance reference vector, and consider nonmaneuvering target ($a_T = 0$), hence,

$$\begin{aligned} a_M &= -\beta V_M e_{vM} \times \Omega, \\ &= -\beta V_M \sqrt{\Omega^T\Omega}\left[(e_{vM}^T e_t)e_r - (e_{vM}^T e_r)e_t\right] \end{aligned} \quad (4.18)$$

and

$$\frac{du}{d\tau} = v^2 + \beta V_M (e_{vM}^T e_t)v, \; u(\tau_0) = u_0, \quad (4.19a)$$

$$\frac{dv}{d\tau} = -uv - \beta V_M (e_{vM}^T e_r)v, \; v(\tau_0) = v_0, \quad (4.19b)$$

$$\frac{dw}{d\tau} = -uw, \; w(\tau_0) = w_0. \quad (4.19c)$$

Next, it is easy to see that

$$a_M = \dot{v}_M = \dot{V}_M e_{vM} + V_M \dot{e}_{vM}. \quad (4.20)$$

However, without considering aerodynamic drag, comparing equations (4.18) and (4.20) yields

$$\dot{V}_M = 0, \quad (4.21)$$

$$\begin{aligned} V_M \dot{e}_{vM} &= -\beta V_M e_{vM} \times \Omega, \\ &= -\beta V_M \sqrt{\Omega^T\Omega}\left[(e_{vM}^T e_t)e_r - (e_{vM}^T e_r)e_t\right] \end{aligned} \quad (4.22)$$

It can be shown that the direction cosines $e_{vM}^T e_r, e_{vM}^T e_t$ and $e_{vM}^T e_\Omega$ satisfy

$$\frac{d}{d\tau}(e_{vM}^T e_r) = -(\beta - 1)(e_{vM}^T e_t)v, \quad (4.23a)$$

$$\frac{d}{d\tau}(e_{vM}^T e_t) = (\beta - 1)(e_{vM}^T e_r)v, \quad (4.23b)$$

$$\frac{d}{d\tau}(e_{vM}^T e_\Omega) = 0, \quad (4.23c)$$

with initial conditions $(e_{vM}^T e_r)(\tau_0) = (e_{vM}^T e_r)_0, (e_{vM}^T e_t)(\tau_0) = (e_{vM}^T e_t)_0, (e_{vM}^T e_\Omega)(\tau_0) = (e_{vM}^T e_\Omega)_0$. and the following unity length constraint as well

$$1 = (e_{vM}^T e_r)^2 + (e_{vM}^T e_t)^2 + (e_{vM}^T e_\Omega)^2. \quad (4.24)$$

Observing equations (4.23) reveals that $(e_{vM}^T e_r)$ and $(e_{vM}^T e_t)$ can be solved independently from (4.19),

$$\begin{bmatrix}(e_{vM}^T e_r)(\theta) \\ (e_{vM}^T e_t)(\theta)\end{bmatrix} = \begin{bmatrix}\cos((\beta-1)(\theta-\theta_0)) & -\sin((\beta-1)(\theta-\theta_0)) \\ \sin((\beta-1)(\theta-\theta_0)) & \cos((\beta-1)(\theta-\theta_0))\end{bmatrix} \cdot \begin{bmatrix}(e_{vM}^T e_r)_0 \\ (e_{vM}^T e_t)_0\end{bmatrix}, \quad (4.25)$$

where θ is defined by $d\theta \triangleq v d\tau$. After some mathematical manipulations, it can be shown that on the (u, v) plane we have

$$\begin{aligned} \begin{bmatrix}u(\theta) \\ v(\theta)\end{bmatrix} &= \begin{bmatrix}\cos(\theta-\theta_0) & \sin(\theta-\theta_0) \\ -\sin(\theta-\theta_0) & \cos(\theta-\theta_0)\end{bmatrix} \begin{bmatrix}u_0 + V_M(e_{vM}^T e_r)_0 \\ v_0 + V_M(e_{vM}^T e_t)_0\end{bmatrix} \\ &+ \begin{bmatrix}-\cos((\beta-1)(\theta-\theta_0)) & \sin((\beta-1)(\theta-\theta_0)) \\ -\sin((\beta-1)(\theta-\theta_0)) & -\cos((\beta-1)(\theta-\theta_0))\end{bmatrix} \cdot \begin{bmatrix}V_M(e_{vM}^T e_r)_0 \\ V_M(e_{vM}^T e_t)_0\end{bmatrix} \\ &= \begin{bmatrix}\cos(\theta-\theta_0) & \sin(\theta-\theta_0) \\ -\sin(\theta-\theta_0) & \cos(\theta-\theta_0)\end{bmatrix} \begin{bmatrix}V_T(e_{vT}^T e_r)_0 \\ V_T(e_{vT}^T e_t)_0\end{bmatrix} \\ &- \begin{bmatrix}\cos((\beta-1)(\theta-\theta_0)) & -\sin((\beta-1)(\theta-\theta_0)) \\ \sin((\beta-1)(\theta-\theta_0)) & \cos((\beta-1)(\theta-\theta_0))\end{bmatrix} \cdot \begin{bmatrix}V_M(e_{vM}^T e_r)_0 \\ V_M(e_{vM}^T e_t)_0\end{bmatrix}, \end{aligned} \quad (4.26)$$

This implies that the direction cosines of target on the (u, v) plane satisfy

$$\begin{bmatrix}(e_{vT}^T e_r)(\theta) \\ (e_{vT}^T e_t)(\theta)\end{bmatrix} = \begin{bmatrix}\cos(\theta-\theta_0) & \sin(\theta-\theta_0) \\ -\sin((\theta-\theta_0) & \cos(\theta-\theta_0)\end{bmatrix} \begin{bmatrix}(e_{vT}^T e_r)_0 \\ (e_{vT}^T e_t)_0\end{bmatrix}, \quad (4.27)$$

where $(e_{vT}^T e_r)_0 \triangleq (e_{vT}^T e_r)(t_0), (e_{vT}^T e_t)_0 \triangleq (e_{vT}^T e_t)(t_0)$. It follows that the trajectory on the (u, v) plane can be realized as a moving circle centered at $(V_T(e_{vT}^T e_r), V_T(e_{vT}^T e_t))$ with constant radius $r_{PPN} = \sqrt{V_M^2 - V_T^2(e_{vT}^T e_\Omega)_0^2}$ (see Figure 3), that is,

$$\left[u - V_T(e_{vT}^T e_r)\right]^2 + \left[v - V_T(e_{vT}^T e_t)\right]^2 = r_{PPN}^2. \quad (4.28)$$

Here we have applied the facts that $v_T - v_M$ has no component in the e_Ω direction, that is,

$$0 = V_M e_{vM}^T e_\Omega - V_T e_{vT}^T e_\Omega, \quad (4.29)$$

and $e_{vT}^T e_\Omega$ is a constant. Recall that due to the definition of v (equation (4.1)), we have $v(\theta_0) \geq 0$. At final time, the corresponding θ_f is determined by letting $u(\theta_f) < 0, v(\theta_f) = 0$, that is,

$$\begin{aligned} 0 &> \cos(\theta_f - \theta_0)V_T(e_{vT}^T e_r)_0 + \sin(\theta_f - \theta_0)V_T(e_{vT}^T e_t)_0 \\ &\quad - \cos((\beta-1)(\theta_f - \theta_0))V_M(e_{vM}^T e_r)_0 \\ &\quad + \sin((\beta-1)(\theta_f - \theta_0))V_M(e_{vM}^T e_t)_0, \\ 0 &= -\sin(\theta_f - \theta_0)V_T(e_{vT}^T e_r)_0 + \cos(\theta_f - \theta_0)V_T(e_{vT}^T e_t)_0 \\ &\quad - \sin((\beta-1)(\theta_f - \theta_0))V_M(e_{vM}^T e_r)_0 \\ &\quad - \cos((\beta-1)(\theta_f - \theta_0))V_M(e_{vM}^T e_t)_0, \end{aligned}$$

or,

$$\begin{aligned} 0 &> -V_M\sqrt{(e_{vM}^T e_r)_0^2 + (e_{vM}^T e_t)_0^2}\cos[\theta_{vM0} + (\beta-1)(\theta_f - \theta_0)] \\ &\quad + V_T\sqrt{(e_{vT}^T e_r)_0^2 + (e_{vT}^T e_t)_0^2}\cos(\theta_{vT0} - \theta_f + \theta_0), \end{aligned} \quad (4.30a)$$

$$\begin{aligned} 0 &= -V_M\sqrt{(e_{vM}^T e_r)_0^2 + (e_{vM}^T e_t)_0^2}\sin[\theta_{vM0} + (\beta-1)(\theta_f - \theta_0)] \\ &\quad + V_T\sqrt{(e_{vT}^T e_r)_0^2 + (e_{vT}^T e_t)_0^2}\sin(\theta_{vT0} - \theta_f + \theta_0), \end{aligned} \quad (4.30b)$$

where θ_{vT0} and θ_{vM0} are defined by

$$\cos\theta_{vT0} = \frac{(e_{vT}^T e_r)_0}{\sqrt{(e_{vT}^T e_r)_0^2 + (e_{vT}^T e_t)_0^2}},$$

$$\cos\theta_{vM0} = \frac{(e_{vM}^T e_r)_0}{\sqrt{(e_{vM}^T e_r)_0^2 + (e_{vM}^T e_t)_0^2}},$$

Equation (4.30) implies that

$$V_M\sqrt{(e_{vM}^T e_r)_0^2 + (e_{vM}^T e_t)_0^2} > V_T\sqrt{(e_{vT}^T e_r)_0^2 + (e_{vT}^T e_t)_0^2} \quad (4.31a)$$

$$\beta > 2. \quad (4.31b)$$

is a sufficient condition to guarantee the satisfaction of capture condition for any initial engagement geometry. However, due to the fact explained by equation (4.29), condition (4.31) is equivalent to

$$V_M > V_T \text{ and } \beta > 2. \quad (4.32)$$

Next, on the (v, w) plane, it can be shown that

$$\begin{aligned} v(\tau)w(\tau) &= v_0 w_0 e^{\int_{\tau_0}^{\tau}[-2u - \beta V_M(e_{vM}^T e_r)]d\tau}, \\ &= v_0 w_0 e^{\int_{\tau_0}^{\tau}[-2V_T(e_{vT}^T e_r) + (2-\beta)V_M(e_{vM}^T e_r)]d\tau} \end{aligned} \quad (4.33)$$

$$\begin{aligned} w(\tau) &= w_0 e^{-\int_{\tau_0}^{\tau} u d\tau}, \\ &= w_0 e^{-\int_{\theta_0}^{\theta} \frac{u}{v} d\theta}, \end{aligned} \quad (4.34)$$

which in turn indicates that if β is chosen such that

$$\int_{\tau_0}^{\tau_f} [-2V_T(e_{vT}^T e_r) + (2-\beta)V_M(e_{vM}^T e_r)]d\tau < 0, \quad (4.35)$$

then the finite turn rate constraint can be retained. It can be seen from Figure 3 that if

$$V_M > V_T, \beta > 2 \text{ and } (e_{vT}^T e_r)_0 > 0, (e_{vM}^T e_r)_0 > 0, \quad (4.36)$$

then inequality (4.35) can be satisfied. In addition it can be shown that for $0 < \Delta \ll 1$

$$\frac{u(\theta_f - \theta_0 - \Delta)}{v(\theta_f - \theta_0 - \Delta)} \approx \frac{u_f}{\Delta[u_f + \beta V_M(e_{vM}^T e_r)(\theta_f)]}, \quad (4.37)$$

which in turn renders $\rho(\tau_f) = 0$.

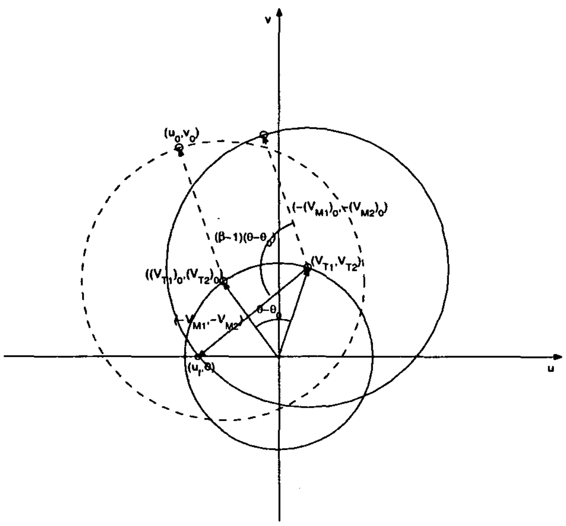

Figure 3: Velocity diagram on (u, v) plane for PPN guidance law.

5 Conclusion

In this paper, the author adopted the modified polar coordinate to describe the relative dynamics between target and missile. It is shown that with the property of this modified polar coordinate, the number of differential equations required to describe the relative dynamics can be reduced from six to three, and all the terms of differential equations involve only products and additions of variables. Then utilized relative velocities as the transformation variable, all the results derived by Yang and Yang in [6] can be recovered. More specifically, for the case of GTPN and GIPN, the results found for two dimensional space can be applied directly to the case of nonmaneuvering target or maneuvering target but with measurable exact acceleration. However, for the PPN guidance law, the results obtained from two dimensional space can only be applied to nonmaneuvering target. For all the mentioned guidance laws in this paper, to describe the capture area we need only two transformed variables, while the third variable is required to provide the information of finite turn rate. The proposed mathematical model in this paper not only provides an intuitive explanation on the reason why most of the existing guidance appears in their current forms, but also paves the way to analyze the capture area and performance for even more general guidance laws with more realistic situations, e.g., the maneuverabilities (accelerations) of missiles and targets are bounded.

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