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Seller's optimal credit period and replenishment time in a supply chain with up-stream and down-stream trade credits

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Abstract In practice, a supplier often offers its retailers a permissible delay period M to settle their unpaid accounts. Likewise, a retailer in turn offers another trade credit period N to its customers. The benefits of trade credit are not only to attract new buyers who consider it a type of price reduction, but also to provide a competitive strategy other than introduce permanent price reductions. On the other hand, the policy of granting credit terms adds an additional cost to the seller as well as an additional dimension of default risk. In this paper, we first incorporate the fact that trade credit has a positive impact on demand but negative impacts on costs and default risks to establish an economic order quantity model for the seller in a supply chain with up-stream and down-stream trade credits. Then we derive the necessary and sufficient conditions to obtain the optimal replenishment time and credit period for the seller. Finally, we use some numerical examples to illustrate the theoretical results.

Keywords Inventory · Permissible delay · Trade credits · Seller · Finance

1 Introduction

In the classical economic order quantity (EOQ) model, it is assumed that the retailer must pay for the items as soon as receiving them. In practice, a supplier frequently offers its retailers a delay of payment up to M periods (i.e., an up-stream trade credit). Usually, there is no interest if the outstanding amount is paid within the permissible delay period. However, if the payment is not paid in full by the end of the permissible delay period, then interest is charged on the outstanding amount. Similarly, a retailer in turn often offers another credit period

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of N to its customers (i.e., a down-stream trade credit). The permissible delay in payment produces two benefits to the seller: (1) it attracts new buyers who consider it to be a type of price reduction, and (2) it may be applied as an alternative to price discount because it does not provoke competitors to reduce their prices and thus introduce lasting price reductions. On the other hand, the policy of granting credit terms adds not only an additional cost but also an additional dimension of default risk to the seller.

By assuming a retailer's selling price per unit is equal to its purchase cost per unit, Goyal (1985) first developed an EOQ model for a retailer when a supplier offers a fixed permissible delay period. Although Dave (1985) commented Goyal's model by addressing the fact that a retailer's unit selling price is necessarily higher than its unit purchase cost, his viewpoint did not draw much attention to the researchers in this field until Teng (2002). Shah (1993) considered a stochastic inventory model when delays in payments are permissible. Aggarwal and Jaggi (1995) extended Goyal's model to consider deteriorating items. Jamal et al. (1997) further generalized Aggarwal and Jaggi's model to allow for shortages. Hwang and Shinn (1997) added the pricing strategy to the model by Goyal (1985), and developed the optimal price and lot sizing for a retailer under the condition of permissible delay in payments. Chung (1998) developed an alternative approach to determine the economic order quantity under the condition of trade credit. By assuming a retailer's unit selling price is higher than its purchase cost per unit, Teng (2002) provided an alternative conclusion from Goyal (1985) and proved that it makes economic sense for a well-established buyer to order less quantity and take the benefits of the permissible delay more frequently. Chang et al. (2003) developed an EOQ model for deteriorating items under supplier credits linked to ordering quantity. Chung and Huang (2003) developed an economic production quantity model (EPQ) for a retailer when the supplier offers a permissible delay in payments. Huang (2003) extended Goyal's model to develop an EOQ model in which the supplier offers the retailer a permissible delay period$M$, and the retailer in turn provides a trade credit period N (with $N \leq M$) to his/her customers. Recently, Teng and Goyal (2007) complemented the shortcoming of Huang's model and proposed a generalized formulation. Many related articles can be found in Chang and Teng (2004), Chung and Liao (2004), Goyal et al. (2007), Huang (2004, 2007), Huang and Hsu (2008), Liao et al. (2000), Ouyang et al. (2005), Ouyang et al. (2006), Shinn and Hwang (2003), Teng and Chang (2009) and Teng et al. (2005, 2006, 2007).

In all these articles described above, the EOQ/EPQ inventory models are studied only from the perspective of the buyer whereas in practice the length of the credit period is set by the seller. So far, how to determine the optimal length of the credit period for the seller has received very little attention by the researchers. Abad and Jaggi (2003) determined the seller's and the buyer's policies under non-cooperative as well as cooperative relationships. However, in their model, the demand rate was not affected by offering a permissible delay. Lately, Jaggi et al. (2008) developed the optimal credit as well as replenishment policy jointly for the vendor, but did not implement the fact that granting credit terms adds not only an additional cost but also an additional dimension of default risk to the seller. Some other related papers in Supply Chain and Finance can be seen in Jaruphongsa and Lee (2008), Migdalas et al. (2004), Pardalos and Tsitsiringos (2002) and Sharma (2009).

By contrast to most researchers focusing only from the perspective of the buyer, we will establish an EOQ model for a seller to obtain its optimal credit period and replenishment time by incorporating the facts that (1) the credit period has a positive impact on demand but negative impacts on costs and default risks, (2) both the supplier and the retailer often offer trade credits to their buyers, and (3) the longer the credit period, the higher the default risk as well as the cost. In fact, the proposed model is an extension of Teng and Chang (2009) by considering both up-stream and down-stream trade credits. We also establish the necessary and sufficient conditions for finding the optimal solution, and characterize the impact of various parameters on the optimal solution. Finally, we provide some numerical examples to illustrate the theoretical results.

2 Notation and assumptions

The following notation and assumptions are used in the paper.

2.1 Notation

A	The retailer's ordering cost per order
c	The retailer's purchasing cost per unit
p	The retailer's selling price per unit, with $p > c$
h	The retailer's unit holding cost per year excluding interest charge
I_e	The retailer's interest earned per dollar per year
I_c	The retailer's interest charged per dollar per year
M	The up-stream trade credit period in years offered by the supplier
N	The down-stream trade credit period in years offered by the retailer (a decision variable)
$D(N)$	The retailer's annual demand rate which is a function of N
T	The retailer's replenishment cycle time in years (a decision variable)
Q	The retailer's order quantity
$TP(N, T)$	The retailer's annual expected total profit, which is a function of N and T

2.2 Assumptions

The following assumptions are made to establish the mathematical inventory model.

1. In today's global competition, many retailers have no pricing power.Therefore, in today's global competition and low inflation environment, we may assume without loss of generality (WLOG) that the selling price is constant within a year.
2. As stated in Jaggi et al. (2008), "it is observed that credit period offered by the retailer to its customers has a positive impact on demand of an item." For simplicity, we assume that the buyer's demand rate $D(N)$ with the retailer's trade credit of N periods is given by
$$D(N) = Ke^{aN}, \tag{1}$$
where K and a are positive constants.
3. Since the longer the credit period to the buyer, the higher the default risk, we may assume WLOG that the rate of default risk given the credit period N is
$$F(N) = 1 - e^{-bN}, \tag{2}$$
where b is a positive constant.
4. The retailer would settle the account at time M and pay for the interest charges on items in stock with rate I_c over the interval $[M, T]$ when $T \geq M$. Alternatively, the retailer settles the account at time M and is not required to pay any interest charge for items in stock during the whole cycle when $T \leq M$. On the other hand, the retailer can accumulate revenue and earn interest during the period from N to M (when $M > N$) with rate I_e under the trade credit conditions.

Fig. 1 $N \leq M$ and $M \leq T + N$

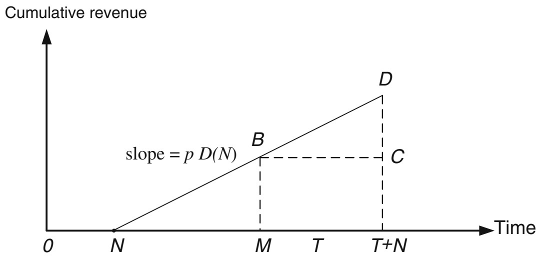
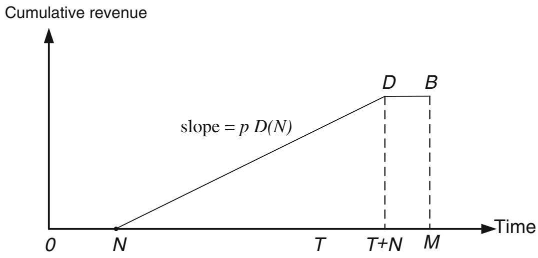


Fig. 2 $N \leq M$ and $T + N \leq M$



5. Lead time is negligible.
6. Shortages are not allowed to occur.

3 Mathematical formulation of the model

From the values of N and M, we have two potential cases: (1)$N \leq M$, and (2)$N \geq M$.

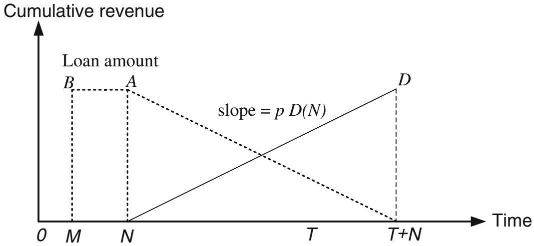
Case 1. $N \leq M$ Based on the values of M (i.e., the time at which the retailer must pay the supplier to avoid interest charge) and $T + N$ (i.e., the time at which the retailer receives the payment from the last customer), we have two possible sub-cases. If $T + N > M$, then the retailer pays off all units sold by $M - N$ at time M, keeps the profits, and starts paying for the interest charges on the items sold after $M - N$, which is shown in Fig. 1. Otherwise (i.e., if $T + N \leq M$), the retailer will receive the total revenue at time $T + N$, and will pay off the total purchase cost at time M. The graphical representation of this case is shown in Fig. 2. Now, let us discuss the detailed formulation in each sub-case.

Sub-case 1–1: $M \leq T + N$ In this sub-case, the retailer can not payoff the supplier by M because the supplier credit period M is shorter than the customer last payment time $T + N$. As a result, the retailer must finance all items sold after time $M - N$ at an interest charged I_c per dollar per year. As a result, the interest charged per cycle is $(c/p)I_c$ times the area of the triangle BCD shown in Fig. 1. Notice that (1) the vertical axis in Figs. 1, 2, 3 represents the cumulative revenue, not cumulative sale volume, and (2) the slope of the increasing line in Figs. 1, 2, and 3 is $pD(N)$. Therefore, the interest charged per year is given by

$$\frac{cI_c D(N)}{2T}[T + N - M]^2. \tag{3}$$

On the other hand, the retailer starts selling products at time 0, but getting the money at time N. Consequently, the retailer accumulates revenue in an account that earns I_e per dollar

Fig. 3 $N \geq M$



per year starting from N through M. Therefore, the interest earned per cycle is I_e multiplied by the area of the triangle *NMB* as shown in Fig. 1. Hence, the interest earned per year is

$$\frac{pI_eD(N)(M-N)^2}{2T}. \tag{4}$$

Since the annual expected revenue is $pD\,(N)e^{-bN}$, the annual cost is $cD(N)$, the annual ordering cost is A/T, and the annual holding cost excluding interest charges is $hD(N)T/2$, we obtain the annual expected total profit for the retailer as

$$TP_1\,(N,T) = pD\,(N)\,e^{-bN} - cD(N) - \frac{A}{T} - \frac{hD(N)T}{2} - \frac{cI_cD(N)}{2T}[T+N-M]^2 + \frac{pI_eD(N)(M-N)^2}{2T} \tag{5}$$

Sub-case 1–2: $M > T+N$ In this sub-case, the retailer receives the total revenue at time $T+N$, and is able to pay the supplier the total purchase cost at time M. Consequently, there is no interest charge while the interest earned per cycle is I_e multiplied by the area of the trapezoid on the interval $[N, M]$ as shown in Fig. 2. As a result, the annual interest earned is

$$\frac{pI_eD(N)T^2}{2T} + \frac{pI_eD(N)T(M-T-N)}{T} = pI_eD(N)(M-N) - \frac{pI_eD(N)T}{2} \tag{6}$$

Hence, the annual expected total profit is

$$TP_2(N,T) = pD(N)e^{-bN} - cD(N) - \frac{A}{T} - \frac{hD(N)T}{2} + pI_eD(N)(M-N) - \frac{pI_eD(N)T}{2} \tag{7}$$

Case 2. $N \geq M$

Since $N \geq M$, there is no interest earned for the retailer. In addition, the retailer must finance all items ordered at time M at an interest charged I_c per dollar per year, and start to payoff the loan after time N. Hence, the interest charged per cycle is $(c/p)I_c$ multiplied by the area of the trapezoid on the interval $[M, T+N]$, as shown in Fig. 3. Therefore, the interest charged per year is given by

$$\frac{cI_cD(N)}{2}[2(N-M)+T]. \tag{8}$$

Hence, the annual expected total profit is

$$TP_3(N,T) = pD(N)e^{-bN} - cD\,(N) - \frac{A}{T} - \frac{hD(N)T}{2} - \frac{cI_cD(N)}{2}[2(N-M)+T] \tag{9}$$

4 Optimal trade credit period and replenishment time

Since $N \geq M$ has no sub-cases, let us discuss this case first. Substituting $D(N) = Ke^{aN}$ into (9), and re-arranging terms, we have

$$TP_3(N,T) = pKe^{(a-b)N} - cKe^{aN} - \frac{A}{T} - \frac{(h+cI_c)T}{2}Ke^{aN} - cI_c(N-M)Ke^{aN} \quad (10)$$

To maximize the annual expected total profit $TP_3(N,T)$ in (10), taking the first-order derivatives $TP_3(N,T)$ with respect to N and T, and setting the result to be zero, we have

$$\frac{\partial TP_3(N,T)}{\partial N} = Ke^{aN}\left\{(a-b)pe^{-bN} - ac - \frac{a(h+cI_c)T}{2} - acI_c(N-M) - cI_c\right\} = 0, \quad (11)$$

and

$$\frac{\partial TP_3(N,T)}{\partial T} = \frac{A}{T^2} - \frac{(h+cI_c)}{2}Ke^{aN} = 0, \quad (12)$$

respectively. Consequently, if $N \geq M$, then the retailer's optimal trade credit period in years is:

$$N_3^* = \frac{1}{b}\ln\frac{(a-b)p}{a[c + \frac{T}{2}(h+cI_c) + cI_c(N-M+\frac{1}{a})]}, \quad (13)$$

and the retailer's optimal replenishment cycle time is

$$T_3^* = \sqrt{\frac{2A}{D(N)(h+cI_c)}}. \quad (14)$$

For the second-order derivatives of $TP_3(N,T)$ with respect to $N$and T, it is easy to see that if $(a-b)^2p < a^2c$, then $TP_3(N,T)$ is a concave function at point (N_3^*, T_3^*). Notice that we are unable to prove that $TP_3(N,T)$ is a concave function for any given point (N,T). Consequently, (N_3^*, T_3^*) is a local minimum of $TP_3(N,T)$. For detailed proof, please see Appendix A. Notice that the right-hand side of (13) is a function of N. Hence, we do not have a closed-form solution to find the optimal credit period N_3^*. Here, we propose the following algorithm to obtain the optimal credit period. By using the fact that $TP_3(N,T)$ is a concave function at point (N_3^*, T_3^*) if $(a-b)^2p < a^2c$, we know the proposed algorithm will converge to an optimal solution.

An Algorithm for the optimal credit period and replenishment time

Step 1. Set $i = 0$ and select an initial value for N_i.
Step 2. Substitute N_i into (14) to get T_i, and then substitute N_i and T_i the right-hand side of (13) to get N_{i+1}.
Step 3. If $N_{i+1} \approx N_i$, then we set $N^* = N_{i+1}$, and substitute N^* into (14) to have T^*, and stop. Otherwise, set $i = i + 1$, and go back to Step 2.

Next, let us discuss the case in which $M > T + N$. Substituting $D(N) = Ke^{aN}$ into (7), and re-arranging terms, we get

$$TP_2(N,T) = pKe^{(a-b)N} - cKe^{aN} - \frac{A}{T} - \frac{(h+pI_e)T}{2}Ke^{aN} + pI_e(M-N)Ke^{aN}. \quad (15)$$

Again, by taking the first-order derivatives of $TP_2(N, T)$ with respect to $N$and T, and setting the results to be zero, we obtain

$$\frac{\partial TP_2(N,T)}{\partial N} = Ke^{aN}\left\{(a-b)pe^{-bN} - ac - \frac{a(h+pI_e)T}{2} + pI_e[a(M-N)-1]\right\} = 0, \tag{16}$$

and

$$\frac{\partial TP_2(N,T)}{\partial T} = \frac{A}{T^2} - \frac{1}{2}(h+pI_e)Ke^{aN} = 0, \tag{17}$$

respectively. As a result, if $M > T + N$, then the retailer's optimal trade credit period in years is:

$$N_2^* = \frac{1}{b}\ln\frac{(a-b)p}{a[c + \frac{(h+pI_e)T}{2} + pI_e(M-N-\frac{1}{a})]}, \tag{18}$$

and the retailer's optimal replenishment cycle time is

$$T_2^* = \sqrt{\frac{2A}{D(N)(h+pI_e)}}. \tag{19}$$

For the second-order derivatives of $TP_2(N, T)$ with respect to $N$and T, it is clear that if $(a-b)^2p < a^2c$ and $aM < 2$, then $TP_2(N, T)$ is a concave function at point (N_2^*, T_2^*). For detailed proof, please see Appendix B. Similarly, the right-hand side of (18) is a function of N. Hence, we do not have a closed-form solution to find the optimal credit period N_2^*. However, we can use the recursive algorithm above to obtain the optimal N_2^* and T_2^*. From the fact that $TP_2(N, T)$ is a concave function at point (N_2^*, T_2^*) if $(a-b)^2p < a^2c$ and $aM < 2$, we know the proposed algorithm will converge to an optimal solution.

Finally, let us discuss the last case in which $N \le M \le T+N$. Substituting $D(N) = Ke^{aN}$ into (5), and re-arranging terms, we obtain

$$\begin{aligned} TP_1(N,T) = {} & pKe^{(a-b)N} - cKe^{aN} - \frac{A}{T} - \frac{(h+cI_c)T}{2}Ke^{aN}. \\ & + cI_c(M-N)Ke^{aN} + \frac{pI_e - cI_c}{2T}(M-N)^2Ke^{aN}. \end{aligned} \tag{20}$$

Taking the first-order derivatives of $TP_1(N, T)$ with respect to N and T, and setting the results to be zero, we get

$$\begin{aligned} \frac{\partial TP_1(N,T)}{\partial N} = {} & Ke^{aN}\left\{(a-b)pe^{-bN} - ac - \frac{a(h+cI_c)T}{2}\right. \\ & \left. -cI_c[1-a(M-N)] - \frac{(pI_e-cI_c)}{2T}(M-N)[2-a(M-N)]\right\} = 0, \end{aligned} \tag{21}$$

and

$$\frac{\partial TP_1(N,T)}{\partial T} = \frac{A}{T^2} - \frac{1}{2}(h+cI_c)Ke^{aN} - \frac{(pI_e-cI_c)}{2T^2}(M-N)^2Ke^{aN} = 0, \tag{22}$$

respectively. Thus, if $N \le M \le T + N$, then the retailer's optimal trade credit period in years is:

$$N_1^* = \frac{1}{b}\ln\frac{(a-b)p}{a\{c + \frac{(h+cI_c)T}{2} + cI_c(\frac{1}{a} - M + N) + \frac{(pI_e-cI_c)(M-N)}{2T}(\frac{2}{a} - M + N)\}}, \tag{23}$$

and the retailer's optimal replenishment cycle time is

$$T_1^* = \sqrt{\frac{2A - (pI_e - cI_c)(M-N)^2 D(N)}{(h + cI_c)D(N)}}. \tag{24}$$

For the second-order derivatives of $TP_1(N, T)$ with respect to $N$and T, it is clear that if $(a-b)^2 p < a^2 c$, $a(M-N) < 1$, $[2-a(M-N)]^2 < 2$, and $2A-(pI_e-cI_c)(M-N)^2 D \geq 0$, then $TP_1(N, T)$ is a concave function at point (N_1^*, T_1^*). For detailed proof, please see Appendix C. Due to the complexity of the problem, we do not have a closed-form solution to find the optimal credit period N_1^* because the right-hand side of (23) is a function of N. However, we can use the recursive algorithm above to obtain the optimal N_1^* and T_1^*. Next, we try to explore the characteristics of the optimal solution.

5 Characteristics of the optimal solution

We first establish the characteristics of the optimal replenishment time, and then the optimal credit period. For the case of $N \leq M$, to ensure the condition of $T_1^* + N \geq M$, we substitute T_1^* in (24) into inequality $T + N \leq M$, and obtain that

$$\text{if and only if } \Delta \equiv 2A - (h + pI_e)D(N)(M-N)^2 \geq 0, \text{ then } T_1^* + N \geq M. \tag{25}$$

Likewise, we substitute T_2^* in (19) into inequality $M \geq T + N$, and know that

$$\text{if and only if } \Delta \equiv 2A - (h + pI_e)D(N)(M-N)^2 \leq 0, \text{ then } T_2^* + N \geq M. \tag{26}$$

From the above arguments, we obtain the following results.

Theorem 1 *When $N \leq M$,*

(A) *if $\Delta \geq 0$, then $T^* = T_1^* \geq M - N$, and $Q^* = Q_1^* = DT_1^*$.*
(B) *if $\Delta \leq 0$, then $T^* = T_2^* \leq M - N$, and $Q^* = Q_2^* = DT_2^*$.*
(C) *if $\Delta = 0$, then $T^* = T_1^* = T_2^* = M - N$, and $Q^* = D(M - N)$.*

Proof It immediately follows from (25) and (26). □

Note that Theorem 1 is a general form of the corresponding theoretical result in Chung (1998), in which it requires $I_c \geq I_e$, $p = c$, and $N = 0$. In addition, Theorem 1 is also an extension of Theorem 1 in Teng and Goyal (2007), in which it assumes D is not affected by N. A simple economical interpretation of Theorem 1 is as follows. It is clear from the traditional EOQ model that the optimal order quantity is obtained when the ordering cost is equal to the holding cost. Whenever the retailer orders items from the supplier, it receives the benefit of $D(N)(M-N)^2 pI_e/2$ from the supplier's up-stream trade credit of M minus his down-stream trade credit of $N$to its customers. As a result, the true ordering cost is reduced to $A - [D(N)(M-N)^2 pI_e]/2$. On the other hand, we know that the holding cost (excluding interest charges) for order $D(N)(M-N)$ units is $D(N)(M-N)^2 h/2$. Therefore if the true ordering cost, $A - [D(N)(M-N)^2 pI_e]/2$, is higher than the holding cost for order $D(N)(M-N)$ units, $D(N)(M-N)^2 h/2$, then the optimal lot size $Q^* = T^* D(N)$ must be higher than $D(N)(M-N)$ units. Hence, if $\Delta \equiv 2A - (h + pI_e)D(N)(M-N)^2 \geq 0$, then $T^* > M - N$, and vice versa.

In the classical EOQ model, both the retailer and the customer are assumed to pay for the products as soon as they receive them. Hence, it is a special case of Sub-case 1.2 with $M = N = 0$. Therefore, the classical optimal EOQ is

$$Q_4^* = \sqrt{2AD(N)/(h + cI_c)}. \tag{27}$$

As a result, we can easily obtain the following theoretical result.

Theorem 2 *When $N \le M$ and $2A - (pI_e - cI_c)(M - N)^2 D(N) \ge 0$,*

(A) if $pI_e < cI_c$, then both $Q_1^*$and Q_2^* are larger than Q_4^*.
(B) if $pI_e > cI_c$, then both $Q_1^*$and Q_2^* are smaller than Q_4^*.
(C) if $pI_e = cI_c$, then $Q_1^* = Q_2^* = Q_4^*$.

Proof The reader can easily prove it from (19), (24), and (27). □

Note that Theorem 2 is a generalization of Theorem 2 in Teng (2002), in which it assumes $N = 0$. It is clear from (24) that if $2A - (pI_e - cI_c)(M - N)^2 D(N) < 0$, then T_1^* as well as Q_1^* does not exist. A simple economical interpretation of Theorem 2 is as follows. It makes economic sense for a well-established retailer (i.e., $pI_e > cI_c$) to order less quantity and take the benefits of the permissible delay more frequently.

Next, let us discuss the case in which $N \ge M$. When $N \ge M$, we know from (14) that T_3^* is

$$T_3^* = \sqrt{2A/[D(N)(h + cI_c)]}. \tag{28}$$

Therefore, the optimal order quantity Q_3^* is

$$Q_3^* = T_3^* D(N) = \sqrt{2AD(N)/(h + cI_c)} = Q_4^*. \tag{29}$$

As a result, if $N \ge M$, then the retailer's optimal order quantity is exactly the same as the classical economic order quantity.

Now, we establish the characteristics of the optimal credit period N^*. It is clear from the optimal solution N^* in (13), (18), and (23) that we have the following results.

Theorem 3 *A higher value of p causes a higher value of N^*, while a higher value of b and c causes a lower value of N^*.*

Proof It immediately follows from (13), (18), and (23). □

A simple economical interpretation of Theorem 3 is obvious. The higher the selling price, the higher the sales revenue increases by the credit period. Meanwhile if the default risk b and the purchase cost c are very huge, then it makes no sense to offer a long credit period.

6 Numerical examples

In order to illustrate the previous results, let us apply the theoretical results to solve the following examples.

Example 1 A product sells at a store for \$2.40. The retailer buys the product from a supplier at \$1.00 a piece. The supplier offers a permissible delay if the payment is made within 60 days (i.e., $M = 2/12 = 1/6$). This credit term in finance management is usually denoted as "net 60" (e.g., see Brigham 1995). However, if the payment is not paid in full by the end of 60 days, then 6% interest (i.e., $I_c = 0.06$) is charged on the outstanding amount. We assume that $D(N) = 3600e^{2N}$ (i.e., $a = 2$) units, $F(N) = 1 - e^{-N}$ (i.e., $b = 1$), h=\$0.50/unit/year, A = \$15.00 per order, and I_e = 5% if the store deposits its revenue into a mutual fund account.

Table 1 The optimal solution for the case of $N \leq M$

i	0	1	2	3	4	5
N_i	0.0800	0.1080	0.1127	0.1135	0.1137	0.1137
$D_i(N)$	4225	4468	4510	4517	4519	4519
T_i	0.1090	0.1078	0.1075	0.1075	0.1075	0.1075

Table 2 The optimal solution for the case of $N \geq M$

i	0	1	2	3
N_i	0.2000	0.2041	0.2040	0.2040
$D_i(N)$	5371	5415	5414	5414
T_i	0.0999	0.0995	0.0995	0.0995

The case of $N \leq M$ is more interesting than the other case of $N \geq M$, which has no sub-cases. Let us start with the case of $N \leq M$. We start here with $N_0 = 0.0800$ simply because it is a simple number about $M/2 = 0.0833$. Since

$$\Delta \equiv 2A - (h + pI_e)D(M - N)^2 = 30 - (0.5 + 0.12)(4225)(0.0867)^2 > 0,$$

we know from Theorem 1 that the optimal replenishment interval is $T^* = T_1^*$. By using the proposed algorithm, (23) and (24), we can easily obtain the following Table 1. Hence, the retailer's optimal solution is $N^* = 0.1137$, and $T^* = 0.1075$.

Example 2 Next, let us run the numerical example for the case of $N \geq M$. We assume that all parameters here are the same as in Example 1 except p = \$2.60. Since $N \geq M$, we may start with $N_0 = 0.2000$, which is larger than $M = 0.1667$. By using the proposed algorithm, (13) and (14), we get Table 2. Hence, the retailer's optimal solution in Example 2 is $N^* = 0.2040$, and $T^* = 0.0995$.

7 Conclusions

In this paper, we have developed an EOQ model to reflect the following facts: (1) both the supplier and the retailer often offer trade credits to their buyers in order to increase sales, (2) the credit period has a positive impact on demand, and (3) the longer the credit period, the higher the default risk as well as the cost. Then we have derived the necessary and sufficient conditions to obtain the optimal solution. Although we have not obtained a closed-form solution to the optimal credit period for the retailer, we have proposed an algorithm to obtain it. In addition, we have characterized the influence of the parameters to the optimal solution. Finally, we have provided some numerical examples to illustrate the proposed model and its optimal solution.

For further research, this paper can be extended in several ways. For instance, we may add the constant deterioration rate for the items. Also, we could generalize the model to allow for shortages. Finally, we could consider the effect of inflation rates on the economic order quantity.

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Appendix A

If $(a-b)^2p < a^2c$, then $TP_3(N, T)$ is a concave function at point (N_3^*, T_3^*).

Proof Taking the second-order derivatives of $TP_3(N, T)$ with respect to $N$and T, we get

$$\frac{\partial^2 TP_3(N,T)}{\partial N^2} = D\left\{(a-b)^2pKe^{-bN} - a^2c - \frac{a^2hT}{2} - \frac{a^2cI_c}{2}[2(N-M)+T] - 2acI_c\right\} < 0, \tag{A1}$$

$$\frac{\partial^2 TP_3(N,T)}{\partial T^2} = -\frac{2A}{T^3} < 0, \tag{A2}$$

$$\frac{\partial^2 TP_3(N,T)}{\partial T\partial N} = -\frac{a}{2}(h+cI_c)Ke^{aN}, \tag{A3}$$

and

$$\begin{aligned}&\left[\frac{\partial^2 TP_3(N,T)}{\partial N^2}\right]\left[\frac{\partial^2 TP_3(N,T)}{\partial T^2}\right] - \left[\frac{\partial^2 TP_3(N,T)}{\partial T\partial N}\right]^2 \quad \text{at } (N_3^*, T_3^*)\\ &= -D\{(a-b)^2pe^{-bN} - a^2c - a^2cI_c(N-M) - 2acI_c\}[2A/T^3]\\ &\quad + a^2A(h+cI_c)D/T^2 - [a(h+cI_c)D]^2/4\\ &= -D\{(a-b)^2pe^{-bN} - a^2c - a^2cI_c(N-M) - 2acI_c\}[2A/T^3]\\ &\quad + [a(h+cI_c)D]^2/4 > 0.\end{aligned} \tag{A4}$$

Consequently, if$(a-b)^2p < a^2c$, then the Hessian matrix associated with $TP_3(N, T)$ is negative definite at (N_3^*, T_3^*), which implies that $TP_3(N, T)$ is a concave function at point (N_3^*, T_3^*).

Appendix B

If $(a-b)^2p < a^2c$ and $aM < 2$, then $TP_2(N, T)$ is a concave function at point (N_2^*, T_2^*).

Proof

$$\frac{\partial^2 TP_2(N,T)}{\partial N^2} = Ke^{aN}\left\{(a-b)^2pe^{-bN} - a^2c - \frac{a^2(h+pI_e)T}{2} + pI_e[a^2(M-N) - 2a]\right\} < 0, \tag{B1}$$

$$\frac{\partial^2 TP_2(N,T)}{\partial T^2} = -\frac{2A}{T^3} < 0, \tag{B2}$$

$$\frac{\partial^2 TP_2(N,T)}{\partial T\partial N} = -\frac{a}{2}(h+pI_e)Ke^{aN}, \tag{B3}$$

and

$$\left[\frac{\partial^2 TP_2(N,T)}{\partial N^2}\right]\left[\frac{\partial^2 TP_2(N,T)}{\partial T^2}\right]-\left[\frac{\partial^2 TP_2(N,T)}{\partial T\partial N}\right]^2 \quad \text{at } (N_2^*, T_2^*)$$
$$= -D\{(a-b)^2pe^{-bN}-a^2c+pI_e[a^2(M-N)-2a\}[2A/T^3]$$
$$+a^2A(h+pI_e)D/T^2-[a(h+pI_e)D]^2/4$$
$$= -D\{(a-b)^2pe^{-bN}-a^2c+pI_e[a^2(M-N)-2a\}[2A/T^3]$$
$$+[a(h+cI_c)D]^2/4>0. \tag{B4}$$

As a result, if$(a-b)^2p < a^2c$ and $aM < 2$, then the Hessian matrix associated with $TP_2(N,T)$ is negative definite at (N_2^*, T_2^*), which implies that $TP_2(N,T)$ is a concave function at point (N_2^*, T_2^*). □

Appendix C

If $(a-b)^2p < a^2c, a(M-N) < 2, [2-a(M-N)]^2 < 2$, and $2A-(pI_e-cI_c)(M-N)^2 D \geq 0$, then$TP_1(N,T)$ is a concave function at point (N_1^*, T_1^*).

Proof

$$\frac{\partial^2 TP_1(N,T)}{\partial N^2} = Ke^{aN}\left\{(a-b)^2pe^{-bN}-a^2c-\frac{a^2(h+cI_c)T}{2}-acI_c[2-a(M-N)]\right.$$
$$\left.-\frac{(pI_e-cI_c)}{2T}(M-N)\{2-[2-a(M-N)]^2\}\right\}<0, \tag{C1}$$

$$\frac{\partial^2 TP_1(N,T)}{\partial T^2} = -\frac{2A-(pI_e-cI_c)(M-N)^2D}{T^3} = -\frac{(h+cI_c)D}{T}<0, \tag{C2}$$

$$\frac{\partial^2 TP_1(N,T)}{\partial T\partial N} = -\left\{\frac{a}{2}(h+cI_c)+\frac{(pI_e-cI_c)}{2T^2}(M-N)[a(M-N)-2]\right\}D, \tag{C3}$$

and

$$\left[\frac{\partial^2 TP_1(N,T)}{\partial N^2}\right]\left[\frac{\partial^2 TP_1(N,T)}{\partial T^2}\right]-\left[\frac{\partial^2 TP_1(N,T)}{\partial T\partial N}\right]^2 \quad \text{at } (N_1^*, T_1^*)$$
$$=\left[U-\frac{a^2(h+cI_c)TD}{2}\right]-\left[\frac{(h+cI_c)D}{T}\right]-\left[\frac{a}{2}(h+cI_c)+V\right]^2D^2$$
$$>\left[\frac{a^2(h+cI_c)TD}{2}\right]\left[\frac{(h+cI_c)D}{T}\right]-\left[\frac{a}{2}(h+cI_c)\right]^2D^2=\frac{[a(h+cI_c)]^2}{4}D^2>0,$$

where $U = D\{(a-b)^2pe^{-bN}-a^2c-acI_c[2-a(M-N)]-\frac{(pI_e-cI_c)}{2T}(M-N)\{2-[2-a(M-N)]^2\}\}$ is a negative number, and $V=\frac{(pI_e-cI_c)}{2T^2}(M-N)[a(M-N)-2]$ is also a negative number. Therefore, if $(a-b)^2p < a^2c, a(M-N) < 2, [2-a(M-N)]^2 < 2$, and $2A-(pI_e-cI_c)(M-N)^2D \geq 0$, then the Hessian matrix associated with $TP_1(N,T)$ is negative definite at (N_1^*, T_1^*), which implies $TP_1(N,T)$ is a concave function at point (N_1^*, T_1^*). □

References

Abad, P.L., Jaggi, C.K.: A joint approach for setting unit price and the length of the credit period for a seller when end demand is price sensitive. Int. J. Prod. Econ. **83**, 115–122 (2003)

Aggarwal, S.P., Jaggi, C.K.: Ordering policies of deteriorating items under permissible delay in payments. J. Oper. Res. Soc. **46**, 658–662 (1995)

Brigham, E.F.: Fundamentals of Financial Management. The Dryden Press, Florida (1995)

Chang, C.T., Ouyang, L.Y., Teng, J.T.: An EOQ model for deteriorating items under supplier credits linked to ordering quantity. Appl. Math. Model. **27**, 983–996 (2003)

Chang, C.T., Teng, J.T.: Retailer's optimal ordering policy under supplier credits. Math. Method. Oper. Res. **60**, 471–483 (2004)

Chung, K.J.: A theorem on the determination of economic order quantity under conditions of permissible delay in payments. Comput. Oper. Res. **25**, 49–52 (1998)

Chung, K.J., Huang, Y.F.: The optimal cycle time for EPQ inventory model under permissible delay in payments. Int. J. Prod. Econ. **84**, 307–318 (2003)

Chung, K.J., Liao, J.J.: Lot-sizing decisions under trade credit depending on the order quantity. Comput. Oper. Res. **31**, 909–928 (2004)

Dave, U.: Letters and viewpoints on" Economic order quantity under conditions of permissible delay in payments". J. Oper. Res. Soc. **46**, 1069–1070 (1985)

Goyal, S.K.: Economic order quantity under conditions of permissible delay in payments. J. Oper. Res. Soc. **36**, 335–338 (1985)

Goyal, S.K., Teng, J.T., Chang, C.T.: Optimal ordering policies when the supplier provides a progressive interest-payable scheme. Eur. J. Oper. Res. **179**, 404–413 (2007)

Huang, Y.F.: Optimal retailer's ordering policies in the EOQ model under trade credit financing. J. Oper. Res. Soc. **54**, 1011–1015 (2003)

Huang, Y.F.: retailer's replenishment policy for the EPQ model under the supplier's trade credit policy. Prod. Plan. Control **15**, 27–33 (2004)

Huang, Y.F.: Economic order quantity under conditionally permissible delay in payments. J. Oper. Res. Soc. **176**, 911–924 (2007)

Huang, Y.F., Hsu, K.H.: An EOQ model under retailer partial trade credit policy in supply chain. Int. J. Prod. Eco. **112**, 655–664 (2008)

Hwang, H., Shinn, S.W.: Retailer's pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payments. Comput. Oper. Res. **24**, 539–547 (1997)

Jaggi, C.K., Goyal, S.K., Goel, S.K.: Retailer's optimal replenishment decisions with credit-linked demand under permissible delay in payments. Eur. J. Oper. Res. **190**, 130–135 (2008)

Jamal, A.M.M., Sarker, B.R., Wang, S.: An ordering policy for deteriorating items with allowable shortages and permissible delay in payment. J. Oper. Res. Soc. **48**, 826–833 (1997)

Jaruphongsa, W., Lee, C.Y.: Dynamic lot-sizing problem with demand time windows and container-based transportation cost. Optim. Lett. **2**(1), 39–51 (2008)

Liao, H.C., Tsai, C.H., Su, C.T.: An inventory model with deteriorating items under inflation when a delay in payment is permissible. Int. J. Prod. Econ. **63**, 207–214 (2000)

Migdalas, A., Baourakis, G., Pardalos, P.M.: Supply Chain and Finance. World Scientific, Singapore (2004)

Ouyang, L.Y., Chang, C.T., Teng, J.T.: An EOQ model for deteriorating items under trade credits. J. Oper. Res. Soc. **56**, 719–726 (2005)

Ouyang, L.Y., Teng, J.T., Chen, L.H.: Optimal ordering policy for deteriorating items with partial backlogging under permissible delay in payments. J. Glob. Optim. **34**, 245–271 (2006)

Pardalos, P.M., Tsitsiringos, V.: Financial Engineering, Supply Chain and E-commerce. Kluwer, Boston (2002)

Shah, N.H.: Probabilistic time-scheduling model for an exponentially decaying inventory when delay in payment is permissible. Int. J. Prod. Econ. **32**, 77–82 (1993)

Sharma, S.: A composite model in the context of a production-inventory system. Optim. Lett. **3**(2), 239–251 (2009)

Shinn, S.W., Hwang, H.: Optimal pricing and ordering policies for retailers under order-size dependent delay in payments. Comput. Oper. Res. **30**, 35–50 (2003)

Teng, J.T.: On the economic order quantity under conditions of permissible delay in payments. J. Oper. Res. Soc. **53**, 915–918 (2002)

Teng, J.T., Chang, C.T.: Optimal manufacturer's replenishment policies in the EPQ model under two levels of trade credit policy. Eur. J. Oper. Res. **195**, 358–363 (2009)

Teng, J.T., Chang, C.T., Chern, M.S., Chan, Y.L.: Retailer's optimal ordering policies in the EOQ models with trade credit financing. Int. J. Syst. Sci. **38**, 269–278 (2007)

Teng, J.T., Chang, C.T., Goyal, S.K.: Optimal pricing and ordering policy under permissible delay in payments. Int. J. Prod. Econ. **97**, 121–129 (2005)

Teng, J.T., Goyal, S.K.: Optimal ordering policies for a retailer in a supply chain with up-stream and down-stream trade credits. J. Oper. Res. Soc. **58**, 1252–1255 (2007)

Teng, J.T., Ouyang, L.Y., Chen, L.H.: Optimal manufacturer's pricing and lot-sizing policies under trade credit financing. Int. Trans. Oper. Res. **13**, 515–528 (2006)