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Quantum entanglement, unitary braid representation and Temperley-Lieb algebra

C.-L. Ho[1,2], A. I. Solomon[3,4] and C.-H. Oh[1]

[1] *Department of Physics, and Center for Quantum Technologies, National University of Singapore 117543, Singapore*
[2] *Department of Physics, Tamkang University - Tamsui 251, Taiwan, ROC* [(a)]
[3] *Department of Physics and Astronomy, The Open University - Walton Hall, Milton Keynes MK7 6AA, UK*
[4] *LPTMC, Université Pierre et Marie Curie - 75252 Paris Cedex 05, France*

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Abstract – Important developments in fault-tolerant quantum computation using the braiding of anyons have placed the theory of braid groups at the very foundation of topological quantum computing. Furthermore, the realization by Kauffman and Lomonaco that a specific braiding operator from the solution of the Yang-Baxter equation, namely the Bell matrix, is universal implies that in principle all quantum gates can be constructed from braiding operators together with single qubit gates. In this paper we present a new class of braiding operators from the Temperley-Lieb algebra that generalizes the Bell matrix to multi-qubit systems, thus unifying the Hadamard and Bell matrices within the same framework. Unlike previous braiding operators, these new operators generate *directly*, from separable basis states, important entangled states such as the generalized Greenberger-Horne-Zeilinger states, cluster-like states, and other states with varying degrees of entanglement.

Introduction. – Recent developments in fault-tolerant quantum computation using the braiding of anyons [1], have stimulated interest in applying the theory of braid groups to the fields of quantum information and quantum computation. In this respect, an interesting result is the realization that a specific braiding operator is a universal gate for quantum computing in the presence of local unitary transformations [2]. This operator involves a unitary matrix R that generates the four maximally entangled Bell states from the standard basis of separable states. This has led to further investigation on the possibility of generating other entangled states by appropriate braiding operators [3–5]. In [4], unitary braiding operators were used to realize entanglement swapping and generate the Greenberger-Horne-Zeilinger (GHZ) state [6], as well as the linear cluster states [7]. Further generalizations of the braiding operators to bipartite quantum systems with states of arbitrary dimension, *i.e.*, qudits, were obtained by the approach of Yang-Baxterization [8,9].

The GHZ state was not directly generated by the braiding operator in [4]. The resulting state was transformed, by use of a local unitary transformation, to the GHZ state. We argue here that this state does not, in fact, possess the same entanglement properties as the GHZ state. In this note we show how the Bell states, the generalized GHZ states and some cluster-like states may be generated *directly* from a braiding operator. We adopt a different approach, based on the Temperley-Lieb algebra (TLA) [10], to obtain a class of unitary representations of the braid group, and with it the required braiding operator. We first obtain an explicit representation of the TLA, and then find the braid group representation via the Jones representation [11].

Braid group and quantum entanglement. – The m-stranded braid group B_m is generated by a set of elements $\{b_1, b_2, \dots, b_{m-1}\}$ with defining relations:

$$\begin{aligned} &b_i b_j = b_j b_i, \qquad |i-j|>1;\\ &b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \qquad 1 \leqslant i < m. \end{aligned} \tag{1}$$

Quantum computing requires that quantum gates be represented by unitary operators. Thus, for applications of

[(a)] Permanent address.

the braid group in quantum computation, one requires its unitary representations. For an m-qubit system the usual $2^m \times 2^m$ unitary representation of B_m employed in the literature is

$$b_i = I \otimes \ldots \otimes I \otimes \; R \otimes I \otimes \ldots \otimes I \qquad (i = 1 \ldots m-1), \tag{2}$$

where I is the 2×2 unit matrix and R is a 4×4 unitary matrix that acts on both the i-th and $(i+1)$-th qubits; that is, occupying the $(i, i+1)$ position. The first of the two braid group relations in (1) is automatically satisfied by the form (2). To fulfill the second relation, R must satisfy

$$(R \otimes I)\,(I \otimes R)\,(R \otimes I) = (I \otimes R)\,(R \otimes I)\,(I \otimes R). \tag{3}$$

This relation is sometimes called the (algebraic) Yang-Baxter equation. One of the simplest solutions of (3) that produces entanglement of states is the matrix

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

When acting on the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, R generates the four maximally entangled Bell states $(|00\rangle \pm |11\rangle)/\sqrt{2}$ and $(|01\rangle \pm |10\rangle)/\sqrt{2}$. Here we adopt the convention $|0\rangle = (1,0)^t$ and $|1\rangle = (0,1)^t$, where t denotes the transpose. Following [5] and [8], we shall call R the Bell matrix[1]. In the presence of local unitary transformations, R is a universal gate [2].

The representation (2) can also be used to generate maximally entangled n-qubit states which are equivalent, up to local unitary transformation, to the GHZ states [4]. To see this, let us take the $n = 3$ qubit case, and consider the action of $b_1 b_2$ on the separable state $|000\rangle$:

$$|\psi\rangle = b_1 b_2 |000\rangle = \frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle + |110\rangle\right). \tag{5}$$

$|\psi\rangle$ is related to the GHZ state $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ by a local unitary transformation as

$$|\psi\rangle = H \otimes H \otimes H |GHZ\rangle, \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6}$$

where H is the Hadamard matrix (or gate).

That the state $|\psi\rangle$ is said to be equivalent to the GHZ state is based on the fact that local unitary transformations do not alter the degree of entanglement[2]. Nevertheless, it is evident that they have very different entanglement properties. For instance, after making a measurement on any one of the three qubits, the other two qubits of the GHZ state become separable, whereas those of $|\psi\rangle$ are still in one of the maximally entangled Bell states! It would be more desirable if one could generate the GHZ states *directly* from the braiding operators without recourse to any local unitary transformation.

A common feature of the Bell states and the GHZ states is that they have the form of the superposition of a separable product state $|a_1 a_2 \ldots a_k \cdots a_n\rangle \equiv |a_1\rangle|a_2\rangle \cdots |a_n\rangle$ with its conjugate state $|\bar{a}_1 \bar{a}_2 \ldots \bar{a}_k \cdots \bar{a}_n\rangle$, which has all a_k's changed from 0 to 1, and 1 to 0, *i.e.*, $\bar{a}_k = 1, 0$ if $a_k = 0, 1$, respectively. Thus, the state $|00\rangle$ is conjugate to $|11\rangle$, $|001\rangle$ is conjugate to $|110\rangle$, etc. As pointed out after eq. (4), the Bell matrix essentially superimposes each two-qubit basis state on its conjugate, as does the Hadamard matrix in the one-qubit case.

We wish to generalize the Hadamard and Bell matrices to higher dimensions (*i.e.*, to n-qubits), so that they generate generalized GHZ states from separable states directly. We want these matrices to be representatives of certain braiding operators of the braid group. Hence the main task is to find an appropriate unitary representation of the braid group, and to determine the correct combination of the braid generators that gives the required matrix. We find that a very simple way to achieve this task is by means of the Jones representation of the braid group, which we describe below.

Unitary Jones representation of B_3. – In his construction of certain polynomial invariants, the Jones polynomials, for knots and links, Jones [11] provided a new representation of the braid group based on what is essentially the TLA. The TLA, more specifically denoted by $TL_m(d)$, is defined, for an integer m and a complex number d, to be the algebra generated by the unit element I and the elements $h_1, h_2, \ldots, h_{m-1}$ satisfying the relations

$$\begin{aligned} h_i h_j &= h_j h_i, \qquad |i-j| > 1; \\ h_i h_{i\pm 1} h_i &= h_i, \qquad 1 \leqslant i < m, \\ h_i^2 &= d h_i. \end{aligned} \tag{7}$$

Given a TLA, the Jones representation of the braid group is defined by (see *e.g.*, [12])

$$b_i = A h_i + A^{-1} I, \qquad b_i^{-1} = A^{-1} h_i + A I, \tag{8}$$

where A is a complex number given by $d = -A^2 - A^{-2}$. It is easily checked that the b_i's so defined do satisfy the braid group relation (1).

In general the Jones representation is not unitary. However, it is obvious from (8) that if $A = e^{i\theta}$ $(\theta \in [0, 2\pi))$ and all the h_i's are Hermitian $(h_i^\dagger = h_i)$, then indeed the Jones representation is unitary[3].

[1]Not to be confused with the Bell matrix of combinatorial mathematics (after E.T. Bell).

[2]A more precise statement is that for bipartite states entanglement is preserved under LOCC (local operations and classical communication).

[3]This representation is not faithful in that more than one group element can be represented by the same matrix. It is easily checked using the TLA and the binomial theorem that if m is the least integer such that $A^m = 1$, then $b_i^m = \left(\frac{(-1)^m - 1}{d}\right) h_i + I$. Hence $b_i^m = I$ for

Based on this fact, in what follows we shall provide a class of unitary representation of the 3-stranded braid group B_3, and show that a subclass of it gives nonlocal unitary transformations that generate conjugate-state entanglements from separable basis states.

For $A = e^{i\theta}$, $d = -2\cos 2\theta$ is real. A simple unitary representation of B_3 is given by the Jones representation with TLA elements $h_i = dE_i$ $(i = 1, 2)$, where

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} a^2 & e^{-i\phi}ab \\ e^{i\phi}ab & b^2 \end{pmatrix}, \qquad a^2 + b^2 = 1. \tag{9}$$

Here ϕ is a phase angle. The E_i's satisfy

$$\begin{aligned} E_i^2 &= E_i, \\ E_1E_2E_1 &= a^2E_1, \\ E_2E_1E_2 &= a^2E_2. \end{aligned} \tag{10}$$

With $a^2 = d^{-2}$, h_i's as constructed from E_i's satisfy the TLA. Now as d and a are real, in order that h_i's be Hermitian, we must have $b^2 = 1 - 1/d^2 \geqslant 0$. This implies $d^2 \geqslant 1$, and hence $\theta(\mathrm{mod}\, 2\pi)$ is restricted to be in the range $|\theta| \leqslant \pi/6$ or $|\theta - \pi| \leqslant \pi/6$. We shall assume θ to be in these domains below. The special case of this representation with $\phi = 0$ was employed previously in exploring the relation between quantum computing and the Jones polynomials [12] (see also [13]).

A very simple way to generalize the above representation of TLA to higher dimensions is as follows. Let

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}. \tag{11}$$

Define

$$E_1^{(n,k)} \equiv \otimes_{j=1}^{k-1} I \otimes e_1 \otimes_{j=k+1}^{n} I, \tag{12}$$

$$\begin{aligned} E_2^{(n,k)} \equiv\ & \otimes_{j=1}^{k-1} I \otimes e_2 \otimes_{j=k+1}^{n} I \\ & + ab \otimes_{j=1}^{k-1} s_j \otimes e_3 \otimes_{j=k+1}^{n} s_j, \end{aligned} \tag{13}$$

where $\otimes_{j=1}^{m} s_j = s_1 \otimes s_2 \otimes \cdots \otimes s_m$. Here s_j is any Hermitian operator satisfying $s_j^2 = 1$. For example, s_j can be I, any one of the Pauli matrices $\sigma_m (m = 1, 2, 3)$, or the Hadamard matrix H. The integer n is the number of 2×2 matrices in the tensor products, and k indicates the position of e_1, e_2 and e_3. The $E_i^{(n,k)}$'s are $2^n \times 2^n$ matrices, and they reduce to (9) in the case $n = k = 1$. One can easily check that $E_i^{(n,k)}$'s satisfy (10). Hence, the operators $h_i^{(n,k)} = dE_i^{(n,k)}$ form a $2^n \times 2^n$ matrix realization of $TL_3(d)$(see footnote [4]). A unitary braid group representation is then obtained from the h_i's by the Jones representation.

Our new unitary braid representation generalizes the 2×2 matrices of (9) to $2^n \times 2^n$ matrices of (13) within the TLA $TL_3(d)$. Other routes of generalization are possible. For instance, in [14] the 2×2 representation of $TL_3(d)$ were generalized to higher dimensional matrices for $TL_m(d)$ with $m > 3$, where the dimension of representation varies with the number of strands m according to the Fibonacci numbers, or with the number of independent bit-strings of certain path model proposed in [15].

Generalized GMZ states. – From now on we will be mainly concerned with the unitary braiding transformation representing the action of the braid b_1b_2. This braiding operator can be evaluated to be

$$\begin{aligned} & b_1^{(n,k)} b_2^{(n,k)} = \\ & \otimes_{j=1}^{k-1} I \otimes \begin{pmatrix} da^2 & 0 \\ 0 & db^2 + A^{-2} \end{pmatrix} \otimes_{j=k+1}^{n} I \\ & + \otimes_{j=1}^{k-1} s_j \otimes \begin{pmatrix} 0 & -e^{-i\phi}A^4dab \\ e^{i\phi}dab & 0 \end{pmatrix} \otimes_{j=k+1}^{n} s_j. \end{aligned} \tag{14}$$

Its action on the separable n-qubit states $|a_1a_2 \ldots a_{k-1}0a_{k+1} \ldots a_n\rangle$ and $|a_1a_2 \ldots a_{k-1}1a_{k+1} \ldots a_n\rangle$ $(a_j = 0, 1,\ j = 1, 2, \ldots, k-1, k+1, \ldots n)$ is given by

$$\begin{aligned} & b_1^{(n,k)} b_2^{(n,k)} |a_1a_2 \ldots a_{k-1}0a_{k+1} \cdots a_n\rangle = \\ & (da^2)|a_1a_2 \ldots a_{k-1}0a_{k+1} \cdots a_n\rangle \\ & + (e^{i\phi}dab)|\tilde{a}_1\tilde{a}_2 \ldots \tilde{a}_{k-1}1\tilde{a}_{k+1} \cdots \tilde{a}_n\rangle, \end{aligned} \tag{15}$$

and

$$\begin{aligned} & b_1^{(n,k)} b_2^{(n,k)} |a_1a_2 \ldots a_{k-1}1a_{k+1} \cdots a_n\rangle = \\ & (db^2 + A^{-2})|a_1a_2 \ldots a_{k-1}1a_{k+1} \cdots a_n\rangle \\ & + (-e^{-i\phi}A^4dab)|\tilde{a}_1\tilde{a}_2 \ldots \tilde{a}_{k-1}0\tilde{a}_{k+1} \cdots \tilde{a}_n\rangle, \end{aligned} \tag{16}$$

where $|\tilde{a}_j\rangle \equiv s_j|a_j\rangle$ $(j = 1, \ldots, k-1, k+1, \ldots, n)$. Thus, under the action of $b_1^{(n,k)} b_2^{(n,k)}$, the separable n-qubit state $|a_1a_2 \ldots a_k \ldots a_n\rangle$ is superimposed on the state $|\tilde{a}_1\tilde{a}_2 \ldots \tilde{a}_k \cdots \tilde{a}_n\rangle$ in either the form (15) or (16), depending on whether the k-th qubit $|a_k\rangle$ is $|0\rangle$ or $|1\rangle$. The states in (15) and (16) are normalized, as $(da^2)^2 + |e^{i\phi}dab|^2 = 1$, and $|db^2 + A^{-2}|^2 + |-e^{-i\phi}A^4dab|^2 = 1$, which can be easily checked. Depending on the choice of the set of s_j's, the resulting state (15) or (16) will have varying degrees of entanglement. In particular, if all $s_j = I$, then the resulting state is separable, and $b_1^{(n,k)} b_2^{(n,k)}$ is simply a local unitary transformation.

m even and $b_i^{2m} = I$ for m odd. And so b_i^k and b_i^l have the same matrix representation if k and l differ by a multiple of m (m even) or $2m$ (m odd). Similarly, the commonly used representation (2) with R given by (4) is also not a faithful representation, since $R^8 = I$ implies $b_i^8 = I$. However, one can obtain a faithful representation $\hat{b}_i$ by defining $\hat{b}_i \equiv e^{\theta} b_i$, where θ/π is irrational but otherwise arbitrary.

[4]See [9] for an $n^2 \times n^2$ matrix realization of the TLA. The braiding operator (called the Yang-Baxter matrix in these works) was obtained there through a Yang-Baxterization process. This latter process was also employed in [8], but not related to TLA, to generalize the Bell matrix to $(2n)^2 \times (2n)^2$ braid matrices.

We now consider a subclass of the representation obtained by setting $\phi=0$ in (13) (*i.e.*, $e_3=\sigma_1$), $s_j=I$ for $j<k$, and $s_j=\sigma_1$ for $j>k$. In this case, $|\tilde{a}_j\rangle=|a_j\rangle$ for $j<k$ and $|\tilde{a}_j\rangle=\sigma_1|a_j\rangle=|\bar{a}_j\rangle$ for $j>k$. Hence, under the action of $B(n,k)\equiv b_1^{(n,k)}b_2^{(n,k)}$ (with the above-mentioned choice of the s_j's in $b_i^{(n,k)}$ understood), the separable n-qubit state $|a_1a_2\dots a_{k-1}a_ka_{k+1}\dots a_n\rangle$ is superimposed on the state $|a_1a_2\dots a_{k-1}\bar{a}_k\bar{a}_{k+1}\cdots\bar{a}_n\rangle$ in either the form (15) or (16) (with the appropriate change in the $\tilde{a}_j$), depending on whether the k-th qubit $|a_k\rangle$ is $|0\rangle$ or $|1\rangle$. The resulting states are separable in the first $(k-1)$ qubits, but entangled in the other $(n-k+1)$ qubits. In particular, for $k=1$, the operator $B(n,1)$ entangles the state $|a_1a_2\dots a_k\dots a_n\rangle$ with its conjugate state $|\bar{a}_1\bar{a}_2\dots\bar{a}_k\cdots\bar{a}_n\rangle$, thus giving the generalized GHZ states. We see that these states can indeed be obtained from separable basis states by the braiding operator.

We now give a few examples of the braiding operator $B(n,1)$ for $k=1$ and $n=1,2,3$. From now on we choose $\theta=\pi/8$. This gives $d=-\sqrt{2}$, and $a,b=\pm1/\sqrt{2}$. Without loss of generality, we take $a=b=1/\sqrt{2}$. The four matrix elements in (14) are $da^2=dab=-1/\sqrt{2}$ and $db^2+A^{-2}=A^4dab=-i/\sqrt{2}$. Explicitly, $B(n,1)$ has the form

$$\begin{aligned}B(n,1) \;=\; &\begin{pmatrix}-\frac{1}{\sqrt{2}} & 0\\ 0 & -\frac{i}{\sqrt{2}}\end{pmatrix}\otimes_{j=2}^{n} I\\ &+\begin{pmatrix}0 & \frac{i}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & 0\end{pmatrix}\otimes_{j=2}^{n}\sigma_1.\end{aligned}\qquad(17)$$

For $n=1$, $B(1,1)=-\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$ is, up to global phases, equivalent to the Hadamard gate. For $n=2$:

$$B(2,1)=-\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & -i\\ 0 & 1 & -i & 0\\ 0 & 1 & i & 0\\ 1 & 0 & 0 & i\end{pmatrix}.\qquad(18)$$

This is equivalent to the Bell matrix up to global phases, and it gives all four Bell states from the separable standard basis. For example, when acting on the states $|00\rangle$ and $|10\rangle$, it gives $-(|00\rangle+|11\rangle)/\sqrt{2}$ and $-i(|10\rangle-|01\rangle)/\sqrt{2}$, respectively.

Note, however, the difference between the appearance of this matrix in our approach, and the Bell matrix R in (4). There the Bell matrix R is the solution of the algebraic Yang-Baxter equation (3), and is the basic building block of the braid generators b_i in (2). In our approach the matrix (18) is obtained from the product of the matrices representing the braid generators b_1 and b_2, *i.e.*, it represents the braid b_1b_2. In a sense, we have factorized R.

It was mentioned in the introduction that the main impetus to using braid group representations in quantum computing is that the Bell matrix is a universal gate [2]. Since $B(2,1)$ is equivalent to R in generating the Bell states, it should also be a universal gate. To prove that, it suffices to show, following [2], that the universal CNOT gate can be generated from $B(2,1)$ and local unitary transformations. This is indeed the case, as we have $\mathrm{CNOT}=(\alpha\otimes\beta)B(2,1)(\gamma\otimes\delta)$, where

$$\begin{aligned}\alpha&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}, &\beta&=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ i & -1\end{pmatrix},\\ \gamma&=\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & i\\ 1 & i\end{pmatrix}, &\delta&=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.\end{aligned}\qquad(19)$$

Generalized GHZ states for larger n can be obtained accordingly.

Cluster-like states. – As mentioned in the introduction, the GHZ state is rather fragile in its entanglement, as it becomes separable after one of its qubits is measured. Multi-qubit systems which possess more robust entanglement can in fact be generated using $B(n,k)$. As an example, we consider the result of applying a braiding operator $B(n,k)$ on a generalized GHZ state $|\phi\rangle$ generated from $|00\dots00\rangle$ with the braiding operator $B^{-1}(n,1)=B^{\dagger}(n,1)$. We have $|\Phi\rangle=B^{-1}(n,1)|00\dots00\rangle=(|00\dots00\rangle+i|11\dots11\rangle)/\sqrt{2}$. Upon applying $B(n,k)$ to $|\phi\rangle$, we get

$$\begin{aligned}B(n,k)|\Phi\rangle \;=\; \frac{1}{2}\,(&|00\cdots00\rangle_{k-1}|00\cdots00\rangle_{n-k+1}\\ &+|00\cdots00\rangle_{k-1}|11\cdots11\rangle_{n-k+1}\\ &+|11\cdots11\rangle_{k-1}|00\cdots00\rangle_{n-k+1}\\ &-|11\cdots11\rangle_{k-1}|11\cdots11\rangle_{n-k+1}).\end{aligned}\qquad(20)$$

Here $|00\dots00\rangle_{k-1}\equiv|0\rangle_1|0\rangle_2\dots|0\rangle_{k-1}$, $|00\dots00\rangle_{n-k+1}\equiv|0\rangle_k|0\rangle_{k+1}\dots|0\rangle_n$, etc. This state is an entangled state for $n\geqslant2$ and $k>1$. Unlike the GHZ states, when it loses one of its qubits, the remaining state is still partially entangled when $n>2$. For $n=4$ and $k=3$, the state (20) is just the 4-qubit linear cluster state given in [7].

By acting with $B(n,k)B^{-1}(n,1)$ on any one of the 2^n separable basis state $|a_1a_2\dots a_n\rangle$, one can in fact generate 2^n orthogonal cluster-like states similar to those of (20).

Summary. – In summary, we have obtained a new class of unitary representation of the three-stranded braid group by the Jones representation. The construction is based on a new matrix realization of the Temperley-Lieb algebra. A subclass of the representation provides a braiding operator that can superimpose states on their conjugate states, thus giving the generalized GHZ states. This braiding operator becomes the Hadamard matrix and the Bell matrix in the one-qubit and two-qubit case, respectively. Certain cluster-like states with robust entanglement can also be generated from separable basis states with two such braiding operators.

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