Optimal Warranty Length for a Rayleigh Distributed Product With Progressive Censoring

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***Abstract*—In an intensely competitive market, one way by which manufacturers attract consumers to their products is by providing warranties on the products. Consumers are willing to purchase a high-priced product only if they can be assured about the product's reliability. A longer warranty period usually indicates better reliability. However, offering an unlimited warranty is unrealistic because maintaining such a policy needs very high cost. In this article, we investigate a decision problem under the warranty which is a combination of free-replacement, and pro-rata policies. We use a Bayesian approach to determine the optimal warranty lengths. The Rayleigh distribution is employed to describe the product lifetime. An example with real data is presented for illustration.**

***Index Terms*—Bayesian analysis, dissatisfaction cost, economic benefit, posterior predictive distribution, warranty cost.**

ACRONYM

FRW	free-replacement warranty
PRW	pro-rata warranty

NOTATION

θ	scale parameter of the Rayleigh distribution
$f(\cdot)$	probability density function
$F(\cdot)$	cumulative distribution function
n	sample size
m	number of failures observed in a progressively censored sample of size n
r_i	number of surviving units withdrawn from the life test at the time of failure i
$L(\cdot)$	likelihood function
$\pi(\cdot)$	prior or posterior density function of parameter θ
a, b	hyperparameters of the prior distribution
p	probability $(0 < p < 1)$
t_p	$100p$-th percentile
$B(\cdot,\cdot)$	beta function
S	sales price
w_1, w_2	warranty lengths
$C_w(t)$	cost of reimbursing a unit with lifetime t
$u(t, w_1, w_2)$	utility function
$B(w_1, w_2)$	economic benefit function
$W(t, w_1, w_2)$	warranty cost function
$D(t, w_1, w_2)$	dissatisfaction cost function
M	potential number of units to be sold
A_1	manufacturer's profit
A_2	parameter to control the speed of increment in benefit
t_w	standard market warranty length under FRW
I	indicator function
q_1, q_2	proportion of the sales price

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I. INTRODUCTION

THE major goal for most manufacturers is to increase profits. Promoting the sales and reputations of the manufacturers is an effective way to achieve the goal. To stimulate purchase willingness, manufacturers must convince consumers of the product quality and reliability. A good warranty policy assists in leveraging the image of a high-quality product, and hence becomes a useful tool in an increasingly intense global competition.

A warranty is a formal commitment by a manufacturer to provide certain responsibilities for product quality after the sale of the product. Through warranties, customers are provided guarantees for failure free, acceptable service for a period of time following the purchase of a product. In general, buyers believe that a product with a long warranty period is higher quality, and more reliable than the one with a short warranty period. For manufacturers, a warranty program is an important tool in marketing products. It not only serves as a sales weapon to increase the sales volume, but also brings considerable profits. However, if the product quality is low, it can be expensive. Thus, a proper warranty plays an increasingly important role in commercial transactions.

Warranty length is often the most visible and marketed warranty element (Menezes & Currim [29]). If the manufacturer promises to give compensation to the buyer when failure occurs, the warranty length, and the reliability of the product play a key role on determining the cost of the product. Agrawal *et al.* [1] indicated that the warranty length, and product reliability definitely affect its profit. Therefore, the ideal would be to determine the warranty length by considering all the factors such as product reliability, and cost to fulfill the warranty. In the literature, the problem of determining warranty length has been investigated from several different directions. For example, in the warranty handbook edited by Blischke & Murthy [8], some statistical methods are used for the determination of warranty length. Blischke & Murthy [7] provided some analytical approaches dealing with warranty cost and optimization problems primarily from the manufacturer's point of view. Singpurwalla & Wilson [38] invoke the idea of decision theory to obtain the optimal warranty lengths of a two-dimensional warranty. Wu *et al.* [45] presented a decision model to determine the optimal price and warranty length to maximize profits based on the pre-determined life cycle. Some other related papers are, for example, Thomas [40], Gutiérrez-Pulido *et al.* [16], Huang *et al.* [17], Huang *et al.* [19], and Chien [9].

To design a cost-effective warranty, a manufacturer must have some information about product reliability. To gain this information, life-testing experiments are performed before products are put on the market. Censoring is very common in life tests. It usually arises in a life test whenever the experimenter does not observe the lifetimes of all test units. The most common censoring schemes are type I censoring, and type II censoring. These two censoring schemes have been studied rather extensively by a number of authors including Mann *et al.* [27], Meeker & Escobar [28], and Lawless [24].

One important characteristic of these two censoring schemes is that they do not allow for units to be removed from the test at the points other than the final termination point. However, this allowance may be desirable when a compromise between reduced time of experimentation and the observation of at least some extreme lifetimes is sought. This reason leads us into the area of progressive censoring. In this topic, much work of statistical inference has been done by several authors including, for example, Tse & Yuen [41], Guilbaud [14], Ali Mousa & Jaheen [2], Wu [44], Ng *et al.* [31], Gouno *et al.* [13], Soliman [39], Ng [30], Lin *et al.* [26], Huang & Wu [18], Balakrishnan *et al.* [5], and Kim & Han [22]. A recent account on progressive censoring can be found in the book by Balakrishnan & Aggarwala [4], or in the review article by Balakrishnan [3].

The purpose of this article is to provide an approach for the determination of optimal warranty length. We use the concept of utility function to measure the monetary utility under a specified warranty policy. The information of product reliability is obtained by conducting a progressively type-II censored life test. These utility function and information are used for the purpose of determining the warranty length under the Bayesian framework. The rest of this paper is organized as follows. Section II uses the Bayesian approach to inference the parameter of lifetime distribution, and obtain some related results. Section III provides a brief discussion of the warranty policies. Section IV gives the formulation of a utility function, and obtains the optimal warranty lengths that maximize the expected utility function. Section V applies the proposed approach to a numerical example. Some conclusions are in Section VI.

II. Lifetime Model and Parameter Estimation

The Rayleigh distribution is a special case of the Weibull distribution, and has wide applications in areas such as communication engineering (Dyer & Whisenand [10], [11]), and life testing of electrovacuum devices (Polovko [34]). The probability density function is given by

$$f(x|\theta) = \frac{x}{\theta^2}\exp\left\{-\frac{x^2}{2\theta^2}\right\}, \quad x > 0. \tag{1}$$

An important characteristic of the Rayleigh distribution is that its failure rate is an increasing linear function of time. When the failure times are distributed according to the Rayleigh law, an intense aging of the equipment takes place. Then, as time increases, the reliability function decreases at a much higher rate than in the case of an exponential distribution (see Polovko [34]).

Suppose n s-independent units are placed on a test with the corresponding lifetimes being identically distributed with a Rayleigh distribution. Let $X_1, \ldots, X_m$ be a progressively type II censored sample from a Rayleigh distribution with censoring scheme $(r_1, \ldots, r_m)$. The likelihood function is given by

$$L(\theta) \propto \frac{1}{\theta^{2m}}\exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^{m}(r_i+1)x_i^2\right\}. \tag{2}$$

In the Bayesian approach, the parameter θ is a random variable with a specified density distribution. As indicate by Berger [6], conjugate priors have the intuitively appealing feature of allowing one to begin with a certain functional form for the prior, and end up with a posterior of the same functional form, but with parameters updated by the sample data. Here we consider the conjugate prior distribution of the form

$$\pi(\theta) = \frac{a^b}{\Gamma(b)2^{b-1}}\theta^{-2b-1}\exp\left\{-\frac{a}{2\theta^2}\right\}, \quad \theta > 0, \tag{3}$$

where $a > 0$, and $b > 0$. This density is known as the square-root inverted-gamma distribution. Fernández [12] indicated that this prior has advantages over many other distributions because of its analytical tractability, and easy interpretability. Most often, the hyperparameters a and b are obtained from history. Waller *et al.* [42] presented a method by which engineering experiences, judgments, and beliefs can be used to assign values to the hyperparameters of a prior distribution. This method requires an engineer to provide two distinct percentiles which are used to determine values for the hyperparameters. Gutiérrez-Pulido *et al.* [15] provided a similar procedure based on putting the first two moments of the time to failure in terms of the first two moments implied by the prior distribution to obtain the values of hyperparameters. This method is based on characteristics easy to communicate by the user in terms of time to failure.

From (2) and (3), Wu *et al.* [45] obtained the posterior distribution of the parameter θ as

$$\pi(\theta|\mathbf{x}) = \frac{[a+\sum_{i=1}^{m}(r_i+1)x_i^2]^{b+m}}{2^{b+m-1}\Gamma(b+m)}\theta^{-2(b+m)-1} \times \exp\left\{-\frac{1}{2\theta^2}\left[a+\sum_{i=1}^{m}(r_i+1)x_i^2\right]\right\} \quad (4)$$

for $\theta > 0$, and zero elsewhere. By forming the product of (1) and (4), and integrating out θ, the posterior predictive distribution is given by

$$f(t|\mathbf{x}) = \int_0^\infty f(t|\theta)\pi(\theta|\mathbf{x})d\theta = \frac{2t(b+m)}{a+\sum_{i=1}^{m}(r_i+1)x_i^2} \times \left[1+\frac{t^2}{a+\sum_{i=1}^{m}(r_i+1)x_i^2}\right]^{-(b+m+1)}, \quad t > 0. \quad (5)$$

Furthermore, the expectation, and $100\,p$-th percentile of the posterior predictive distribution are, respectively,

$$E(t|\mathbf{x}) = \frac{1}{2}B\left(b+m-\frac{1}{2},\frac{1}{2}\right) \times\sqrt{a+\sum_{i=1}^{m}(r_i+1)x_i^2}, \quad b+m>\frac{1}{2},$$

and

$$t_p|\mathbf{x} = \sqrt{\left[\left(\frac{1}{1-p}\right)^{\frac{1}{b+m}}-1\right]\left[a+\sum_{i=1}^{m}(r_1+1)x_i^2\right]}.$$

III. Warranty Policy

A warranty is a manufacturer's assurance to a buyer that a product is or will be as represented. Several manufacturers offer a warranty on their product as a sales enhancement. The warranty can be considered to be a contractual agreement between the manufacturer and buyer entered into upon the sale of the product. Thus, the problem of determining a warranty on a product that is optimal, according to some criteria, becomes of interest.

An important feature of a warranty policy is the form of compensation to a buyer as a product fails during the offered warranty period. The two commonly used policies are the free-replacement warranty (FRW), and pro-rata warranty (PRW). Under the FRW policy, a product is replaced by an identical one free of charge if it fails during the warranty period. If the manufacturer agrees to refund the full purchase price, such policy is also called rebate FRW. In both cases, the buyer receives full compensation for all failures occurring during the warranty period. Under the PRW policy, the manufacturer gives the buyer a prorated compensation if the product fails during the warranty period. The simplest such compensation to the buyer is a linear function of the remaining time in the warranty period.

A combination of FRW and PRW is called the combined FRW/PRW policy. This policy is a combination of FRW and PRW in which the FRW is used during the period $[0, w_1)$, and the PRW is used during the period $[w_1, w_2)$, where $w_1 \le w_2$ are positive values. Note that the FRW policy and PRW policy are both the special cases of the combined FRW/PRW policy. If $w_1 = w_2$, the combined policy reduces to the FRW policy. When $w_1 = 0$, the combined policy becomes the PRW policy.

Let S be the sales price of a product. According to the combined FRW/PRW policy, the cost of reimbursing an item with lifetime t is

$$C_w(t) = \begin{cases} S, & \text{if } 0 \le t < w_1 \\ S\left(\frac{w_2-t}{w_2-w_1}\right), & \text{if } w_1 \le t < w_2 \\ 0, & \text{if } t \ge w_2 \end{cases} \quad (6)$$

This is the manufacturer loss associated with setting up a warranty.

IV. Determination of the Optimal Warranty Lengths

In the combined FRW/PRW policy, two warranty lengths of a product must be specified: w_1, and w_2. Before deciding the values of w_1 and w_2, one needs to consider the function of a warranty policy that measures the monetary utility when the product fails at time t. The utility function $u(t, w_1, w_2)$ defined in this paper consists of three parts. They are the economic benefit function $B(w_1, w_2)$, the warranty cost function $W(t, w_1, w_2)$, and the dissatisfaction cost function $D(t, w_1, w_2)$. The proposed utility function is then given by

$$u(t, w_1, w_2) = B(w_1, w_2) - W(t, w_1, w_2) - D(t, w_1, w_2). \quad (7)$$

The following three subsections provide detailed discussions of these three parts.

A. Economic Benefit Function

A warranty encourages sales of product because it ensures the buyer of a certain level of compensation should the product be faulty. With a warranty, the manufacturer may expect an increase in market share for the product, and hence gain the monetary benefit. Although it is difficult to quantify the benefit as a particular function, it should be unrealistic to suppose that the benefit is an increasing function of the warranty length. Because there are two-stage warranty lengths in a combined FRW/PRW policy, we assume that the benefit is increasing in the average of two-stage warranty lengths. We further assume that the benefit function should be bounded as the warranty lengths tend to infinity. We propose the function

$$B(w_1, w_2) = A_1 M\left(1 - e^{-A_2\left(\frac{w_1+w_2}{2}\right)}\right).$$

The parameter A_2 can be derived from the ratio of two special cases in the combined FRW/PRW policy. Consider the ratio

$$\frac{B(0,t_w)}{B(t_w,t_w)} = \frac{1-e^{-\frac{A_2 t_w}{2}}}{1-e^{-A_2 t_w}} \tag{8}$$

which means that the percentage of benefit remains when the manufacturer changes the warranty policy from FRW to PRW. Let the function $h(A_2)$ denote the ratio of (8). It is easy to show that h is a strictly monotone increasing function with $h(0^+) = 1/2$, and $h(\infty) = 1$. Hence, for a given standard market warranty t_w under a FRW policy, and a percentage between $(1/2)$ and 1, one can solve for A_2 uniquely.

B. Warranty Cost Function

The warranty cost function is equal to the cost of reimbursing an item $C_w(t)$ times the expected number of items that fail under the warranty period. The $C_w(t)$ was defined in (6). For the expected number of failures, we consider a multinomial distribution with probabilities of failure equal to $F(w_1|\mathbf{x})$, $F(w_2|\mathbf{x}) - F(w_1|\mathbf{x})$, and $1 - F(w_2|\mathbf{x})$, where $F(\cdot|\mathbf{x})$ is the posterior predictive cumulative distribution function. Thus, the warranty cost function can be written as

$$\begin{aligned} W(t,w_1,w_2) &= MF(w_1|\mathbf{x})SI_{[0,w_1)}(t) \\ &\quad + M\left[F(w_2|\mathbf{x}) - F(w_1|\mathbf{x})\right] S\left(\frac{w_2-t}{w_2-w_1}\right) I_{[w_1,w_2)}(t). \end{aligned}$$

C. Dissatisfaction Cost Function

The dissatisfaction cost is the manufacturer's indirect cost when the product fails during the warranty period, or fails during the time just after warranty. Patankar & Mitra [33] indicated that there may be some consumer dissatisfaction when the product fails under warranty. For example, in 1995, 17.8 million vehicles were recalled in the USA (Inman & Gonsalvez [20]). Such inconvenience caused to the car owners certainly caused loss of future sales. Kelly [21] also mentioned that the customer does not expect the products to last forever, but has certain expectation of the product. If the failure occurs somewhat early after the expiration of warranty, then the consumer may be dissatisfied.

Under the combined FRW/PRW policy, and assuming that the consumer's expected life of the product is L, we propose a dissatisfaction cost function which consists of three components. The first component is that an item fails in the time period $[0, w_1)$. Because the FRW is used in this period, we propose that the dissatisfaction cost is a proportion q_1 $(0 < q_1 < 1)$ of the sales price, multiplied by the expected number of failures. That is,

$$D_1(t,w_1) = MF(w_1|\mathbf{x})Sq_1 I_{[0,w_1)}(t).$$

The second component is that a failure occurs in the time period $[w_1, w_2)$. If an item fails in this period, we assume that the dissatisfaction cost of an item is a linearly decreasing function of time with maximum Sq_1, and minimum Sq_2, where $0 < q_2 < q_1 < 1$. So the dissatisfaction cost function is

$$\begin{aligned} D_2(t,w_1,w_2) &= M\left[F(w_2|\mathbf{x}) - F(w_1|\mathbf{x})\right] \\ &\quad \times \left[Sq_1 - (Sq_1 - Sq_2)\left(\frac{t-w_1}{w_2-w_1}\right)\right] I_{[w_1,w_2)}(t). \end{aligned}$$

The last component is that the product fails after the expiration of warranty, but the customer may still be unsatisfied with the product unless its lifetime exceeds a specific value L, where $L > w_2$. We propose that the dissatisfaction cost decreases linearly with lifetime, reaching zero when lifetime is L. That is,

$$D_3(t,w_2) = M\left[F(L|\mathbf{x}) - F(w_2|\mathbf{x})\right] Sq_2\left(\frac{L-t}{L-w_2}\right) I_{[w_2,L]}(t).$$

Note that the value of L can be considered to be the mean, median, or percentile of the posterior predictive distribution. It may also be chosen based on historical information or market surveys.

Finally, the dissatisfaction cost function is given by

$$D(t,w_1,w_2) = D_1(t,w_1) + D_2(t,w_1,w_2) + D_3(t,w_2).$$

D. Expected Utility Function, and Optimal Warranty

The optimal warranty (w_1^*, w_2^*) is that which maximizes the expected value of the utility function, with expectation over the posterior predictive distribution. That is,

$$\begin{aligned} &E\left(u(T,w_1,w_2)\right) \\ &= \int_0^\infty u(t,w_1,w_2) f(t|\mathbf{x})dt \propto A_1\left(1 - e^{-A_2\left(\frac{w_1+w_2}{2}\right)}\right) \\ &\quad - F(w_1|\mathbf{x})S(1+q_1)\int_0^{w_1} f(t|\mathbf{x})dt \\ &\quad - \left[F(w_2|\mathbf{x}) - F(w_1|\mathbf{x})\right] \\ &\quad \times S\int_{w_1}^{w_2}\left[\left(\frac{w_2-t}{w_2-w_1}\right) + q_1 - (q_1-q_2)\left(\frac{t-w_1}{w_2-w_1}\right)\right] f(t|\mathbf{x})dt \\ &\quad - \left[F(L|\mathbf{x}) - F(w_2|\mathbf{x})\right] Sq_2\int_{w_2}^{L}\left(\frac{L-t}{L-w_2}\right) f(t|\mathbf{x})dt. \end{aligned} \tag{9}$$

Thus, the optimal warranty (w_1^*, w_2^*) is given by the solution to the optimization problem

$$(w_1^*, w_2^*) = \arg\left(\max_{w_1 \le w_2 \in R^+} E\left(u(T,w_1,w_2)\right)\right),$$

where R^+ denotes the set of positive real numbers.

In general, this constrained optimization problem is difficult to solve analytically, but it is relatively easy to implement maximization methods using standard optimization software. Extensive accounts of optimization (maximization or minimization) methods and software are given in Press *et al.* [35], Lange [23], and Nocedal & Wright [32]. One useful method to minimize a function subject to linear inequality constraints is called an adaptive barrier algorithm. This method discusses an adaptive version of the logarithmic barrier method motivated by the EM

TABLE I
PROGRESSIVELY TYPE II CENSORED SAMPLE

i	1	2	3	4	5	6	7
x_i	0.1788	0.2892	0.3300	0.4212	0.4560	0.4848	0.5184
r_i	0	0	3	0	0	2	0
i	8	9	10	11	12	13	
x_i	0.5196	0.6780	0.6864	0.8412	0.9312	1.2792	
r_i	0	2	0	2	1	0	

algorithm from computational statistics. Many software packages for numerical computation possess good optimization procedures. In fact, function (9) can be programmed with R (R Development Core Team [36]), and then it is optimized with the routine *constrOptim* of the same package.

V. Numerical Example

We apply the proposed method to a real data set from tests on the endurance of deep groove ball bearings as originally discussed by Leiblein & Zelen [25]. The data indicate the number of revolutions to failure for each of $n = 23$ ball bearings in the life test. Raqab & Madi [37] indicated that a one parameter Rayleigh distribution is acceptable for these data. Wu *et al.* [45] generated a progressively type II censored sample from this data set. The progressively censored sample size is $m = 13$. The observations (in hundreds of millions), and removed numbers are reported in Table I.

Raqab & Madi [37] assumed that the prior distribution of θ is a square-root inverted-gamma distribution with hyperparameters $a = b = 2$. Suppose that the sales price of this product is $S = \$150$, and its production cost is \$50. Thus, the profit of one product is $A_1 = \$100$. The manufacturer gives a standard warranty of $t_w = 0.285$ (which is the 10-th percentile of the posterior predictive distribution) under the FRW policy. The manufacturer is interested in a combined FRW/PRW policy, and wishes to set the percentage of benefit remaining to be 0.75 (*i.e.*, (8) equals to 0.75). In this case, the unique solution to (8) is $A_2 = 7.7093$. The manufacturer further assumes that the proportions of loss from consumer dissatisfaction are $(q_1, q_2) = (0.1, 0.05)$. The consumer is satisfied with the product if its lifetime is at least $L = 0.7383$ (which is the median of the posterior predictive distribution).

Under the combined FRW/PRW policy, the optimal warranty lengths is $w_1^* = 0.3297$, and $w_2^* = 0.4785$. The maximum value of the expected utility function is $\$91.2039M$, where M is defined in Section IV-A These optimal values of warranty lengths and expected utility function are found by using a program written in R with the routine *constrOptim*. The program is available from the authors.

In addition, we can also obtain the optimal warranty lengths under FRW, and PRW policies by setting $w_1 = w_2$, and $w_1 = 0$ in (9), respectively. The optimal warranty length under the FRW policy is $w_1^* = 0.3759$, and the maximum value of the expected utility function is $\$89.4733\ M$. The optimal warranty length under the PRW policy is $w_2^* = 0.5703$, and the optimal value of the expected utility function is $\$81.2677\ M$. We can find that the combined FRW/PRW policy can create the largest value of expected utility among the three policies under consideration. Therefore, the manufacturer may offer a warranty with the combined FRW/PRW policy on this product. After the product is sold, the FRW is used during the period [0,0.3297), and the PRW is used during the period [0.3297,0.4785).

VI. Conclusion

With today's high technology, manufacturers face increasingly intense global competition. To remain profitable, they are challenged to design, develop, and produce highly reliable products. To attract consumers to their products, the manufacturers usually provide warranties on product lifetimes. Therefore, determining the appropriate warranty length becomes an important decision problem for manufacturers. We provide an approach to determine the optimal warranty lengths under the combined FRW/PRW policy. This approach is based on the Bayesian method, and the formulation of a utility function. It is not necessary that data structure be progressively censored. It can be any other complete or censored data set. This approach is intuitive, and can be useful to manufacturers.

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