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Optimal maintenance time for imperfect maintenance actions on repairable product ☆

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ABSTRACT

This paper develops a maintenance strategy for repairable products that combines imperfect maintenance actions at pre-scheduled times and minimal repair actions for failures. Under a power law process of failures, an expected total cost is developed that involves the sum of the total cost of imperfect preventive maintenances and the expected total cost of minimal repairs. Moreover, a searching procedure is provided to determine the optimal maintenance schedule within a finite time span of warranty. When the parameters of the power law process are unknown, the accuracy of the estimated maintenance schedule is evaluated based on data through an asymptotic upper bound for the difference of the true expected total cost and its estimate. The proposed method is applied to an example regarding the maintenance of power transformers and the performance of the proposed method is investigated through a numerical study. Numerical results show that the proposed maintenance strategy could save cost whether an imperfect maintenance action or the perfect maintenance action is implemented.

1. Introduction

Most products or systems are designed to be repaired rather than replaced after failure in the real world. Maintenance policies are fundamental under these conditions because a properly preventive maintenance (PM) strategy can save money and keep products running longer. A PM policy specifies the periodicity to maintain a product through the product whole lifetime. Pham and Wang (1996) mentioned that a maintenance action could be classified into perfect maintenance, minimal repair (MR) or imperfect maintenance. A perfect maintenance restores a product to be as good as new, an MR restores a product to have the same failure rate condition as it had just right before failure and an imperfect maintenance makes a product better than what it had before failure but not necessarily to be as good as new. Since the pioneer work of Barlow and Hunter (1960), the combination of a perfect PM and an MR has been of interest by many authors, for examples, Gerstack (1977), Block, Borges, and Savits (1990), Park, Jung, and Yum (2000), Lai, Leung, Tao, and Wang (2001) and Gilardoni and Colosimo (2007).

In an optimal maintenance policy setting, the nonhomogeneous Poisson process (NHPP) has played a key role in modeling the random occurrences of failures. Let $N(0,t)$ denote the number of failures in the interval $(0,t]$. A process $\{N(0,t): t \geqslant 0\}$, which has independent increments and $N(0,0) = 0$, is a Poisson process with intensity $\lambda(t)$, if the random variable $N(0,t)$ has a Poisson distribution and mean $M(t) = E(N(0,t)) = \int_0^t \lambda(u)du$ for $t \geqslant 0$. When the intensity function $\lambda(t)$ is not constant and depends on the time t, the Poisson process is called the NHPP. The most popular NHPP is the power law process (PLP) which has a Weibull intensity function,

$$\lambda(t) = \beta t^{\beta-1}/\theta^{\beta}, \tag{1.1}$$

where $\theta > 0$ is the scale parameter and $\beta > 0$ is the shape parameter.

The PLP had been successfully applied to model the occurrences of failures in a number of PM studies. Some good discussions regarding the applications of NHPP have been published by Crow (1974), Cox and Miller (1965), Ascher and Feingold (1984), Bain and Engelhardt (1991), Rigdon and Basu (2000) and Pulcini (2001). The model (1.1) is quite flexible in reliability studies because it includes the growth model when $0 < \beta < 1$, the decay model when $\beta > 1$ and the homogeneous Poisson process when $\beta = 1$.

Assuming infinite operation time for a repairable product, Gilardoni and Colosimo (2007) proposed an optimal perfect PM schedule which minimized the expected average total cost per unit time. Moreover, they provided a large sample estimation procedure for the determination of their PM schedule when the parameters of the PLP are unknown. However, the maintenance actions in practical situations could be imperfect and the operation time for a repairable product could be finite. This article relaxes the

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conditions of Gilardoni and Colosimo (2007) to develop a new PM plan in which repairable products undergo imperfect maintenance actions within a finite time span of warranty. The objective of the proposed PM policy is to minimize the expected total cost in a finite time span of warranty instead.

In Section 2, the proposed PM policy and a searching procedure to setup the optimal PM schedule are developed for repairable products when the time span of warranty is finite. In Section 3, an asymptotic upper bound for the difference of the true expected total cost and its estimate is provided to evaluate the accuracy of the estimated PM schedule based on data. The proposed method is illustrated via an example in Section 4. Moreover, the performance of the proposed PM policy is compared with the one proposed by Gilardoni and Colosimo (2007) in terms of the expected average total cost per unit time. In Section 5, a numerical study is conducted to evaluate the performance of the proposed PM policy for various combinations of parameters. Finally, concluding remarks are given in Section 6.

2. The proposed preventive maintenance model

Assume that a repairable product starts to operate at time zero and undergoes m times of imperfect maintenance actions within a finite time span of warranty, W. The imperfect maintenance action satisfies the following conditions:

1. PM check points are scheduled after every τ units of time such that $0 \leqslant m\tau \leqslant W$.
2. Each PM can return the product's age $x_i = x = r\tau$, where $i = 1,2,\ldots,m$ and $0 \leqslant r \leqslant 1$. If $r = 0$, then the maintenance action has no effect to the product; while $r = 1$, it represents a perfect maintenance action which instantly returns the product to a new condition.
3. The PM cost at the time t can be modeled as a linear function of the product's age t and the returned product's age x, $C_p(t,x) = a + c_1x + c_2t$, where a, c_1 and c_2 are nonnegative coefficients (see Yeh & Chen (2005)).
4. When a failure occurs between two PM check points, a MR is applied. The cost for a MR is denoted by c_{MR}.

It should be noticed that if no PM action is implemented in the operating time interval $(0,W]$, then the expected total cost is given as

$$c_0 = c_{MR} \cdot E[N(0,W)]. \tag{2.1}$$

Otherwise, let the time interval $((0,W] = (0,\tau] \cup (\tau,2\tau] \cup \cdots \cup ((m-1)\tau, m\tau] \cup (m\tau, W]$ and y_i denote the i^{th} cumulative return time at the i^{th} PM action, where $i = 0,1,2,\ldots,m$, $y_0 = 0$ and $y_i = \sum_{k=1}^{i} x_i = ir\tau$ for $i = 1,2,\ldots,m$. Therefore, the expected MR cost in the i^{th} interval $((i-1)\tau, i\tau]$ is the c_{MR} multiple of the expected number of failures that occur within the interval. Moreover, the expected cost in the ith interval $((i-1)\tau, i\tau]$ is the sum of the expected MR cost in the interval and the PM cost happened at the end of the interval. Therefore, the expected cost in the ith interval can be mathematically represented as,

$$c_{MR} \cdot E[N((i-1)\tau - y_{i-1}, i\tau - y_{i-1})] + C_p(i\tau, x). \tag{2.2}$$

The expected total cost for the entire time interval, $(0,W]$ can be determined as follows:

$$C(\tau,m) = \sum_{i=1}^{m} \{c_{MR} \cdot E[N((i-1)\tau - y_{i-1}, i\tau - y_{i-1})] + C_p(i\tau, x)\} + c_{MR} \cdot E[N(m\tau - y_m, W - y_m)], \tag{2.3}$$

where $E[N(h_1,h_2)] = \int_{h_1}^{h_2} \lambda(u)du$, $h_1 < h_2$. When a perfect PM is applied and $c_1 = c_2 = 0$, $C(\tau,m)$ is reduced to Eq. (1) of Gilardoni & Colosimo (2007) with $T = W$. However, when $T = W$ is finite, the term $R = c_{MR} \cdot E[N(m\tau - y_m, W - y_m)]$ may not be negligible and the average expected total cost per unit of time $\frac{C(\tau,m)}{W}$ is, hence, different from the Eq. (2) of Gilardoni & Colosimo (2007) under a perfect PM with $c_1 = c_2 = 0$. The derivative of $C(\tau,m)$ with respect to τ is

$$\frac{dC(\tau,m)}{d\tau} = \sum_{i=1}^{m} \{c_{MR} \cdot [a_i\lambda(a_i\tau) - b_i\lambda(b_i\tau)] + (c_1r + c_2i)\} - c_{MR} \cdot [mr\lambda(W - mr\tau) + m(1-r)\lambda(m(1-r)\tau)], \tag{2.4}$$

where $a_i = i - ir + r$ and $b_i = (i-1)(1-r)$. Under the PLP of the Weibull intensity function, Eq. (2.4) can be rewritten as,

$$\frac{dC(\tau,m)}{d\tau} = \sum_{i=1}^{m} \left[\frac{c_{MR}\beta\tau^{\beta-1}}{\theta^\beta}\left(a_i^\beta - b_i^\beta\right) + c_1r + c_2i\right] - \frac{c_{MR}}{\theta^\beta}\left[\beta mr(W - mr\tau)^{\beta-1} + (m(1-r))^\beta \beta\tau^{\beta-1}\right] \tag{2.5}$$

and the second derivative of $C(\tau,m)$ with respect to τ is

$$\frac{d^2C(\tau,m)}{d\tau^2} = \left(\frac{c_{MR}\beta(\beta-1)}{\theta^\beta}\right)\left\{\tau^{\beta-2}\left[\sum_{i=1}^{m}\left(a_i^\beta - b_i^\beta\right) - (m(1-r))^\beta\right] + (mr)^2(W - mr\tau)^{\beta-2}\right\}. \tag{2.6}$$

It can be shown that $\sum_{i=1}^{m}\left(a_i^\beta - b_i^\beta\right) - (m(1-r))^\beta > 0$ for $\beta > 0$. Therefore, when $\beta > 1$, $\frac{d^2C(\tau,m)}{d\tau^2} > 0$ and $C(\tau,m)$ is a convex function of τ over $0 < \tau \leqslant W$ for a given m. Because the repairable product is assumed to decay in reliability, only the case of $\beta > 1$ is considered. Hence, the optimal τ_m^*, which minimizes the expected total cost in the time interval $(0,W]$ can be determined by solving $dC(\tau,m)/d\tau = 0$ over $0 < \tau < W$ or $\tau_m^* = W$. For each m, the optimal τ_m^* can be solved numerically. The optimal PM number m^* can be obtained by

$$m^* = \arg\min_{\{m=1,2,\ldots\}} \{C(\tau_m^*, m)\}.$$

In practical applications, it is common for the MR cost to greatly exceed the PM cost. In addition, the expected number of failures is increasing with respect to the length of operating time interval between two PMs when the PLP shape parameter $\beta > 1$ and each PM can improve the system state. Therefore, when the number of PMs starts to increase from zero, it is expected to decrease the expected total cost, intuitively. However, when the number of PMs increases to a certain level, the expected total cost would start to increase. Based on this principle, a search algorithm shown in Fig. 1 is proposed to find the optimal PM time schedule.

3. Statistical methods

In practical applications, parameters in the NHPP may not be known in advance. It is necessary to estimate τ based on data. Assuming that a product could be operated for infinite time, Gilardoni & Colosimo (2007) discussed a procedure for the large sample maximum likelihood estimation of their optimal maintenance schedule based on the failure times observed from one or more identical products under their proposed perfect PM policy. The basic ways for collecting data from a repairable product could be the failure truncated or the time-truncated sampling. The failure truncated sampling means that the data collection is ceased after a specified number, k, of failures. The time truncated sampling means that the data collection is ceased at a predetermined time T.

Let $0 < t_1 < t_2 < \cdots < t_k < T$ denote the times to failures observed until a predetermined time T for a NHPP with intensity function $\lambda(t) = \lambda(t|\boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is a vector of the unknown parameters.

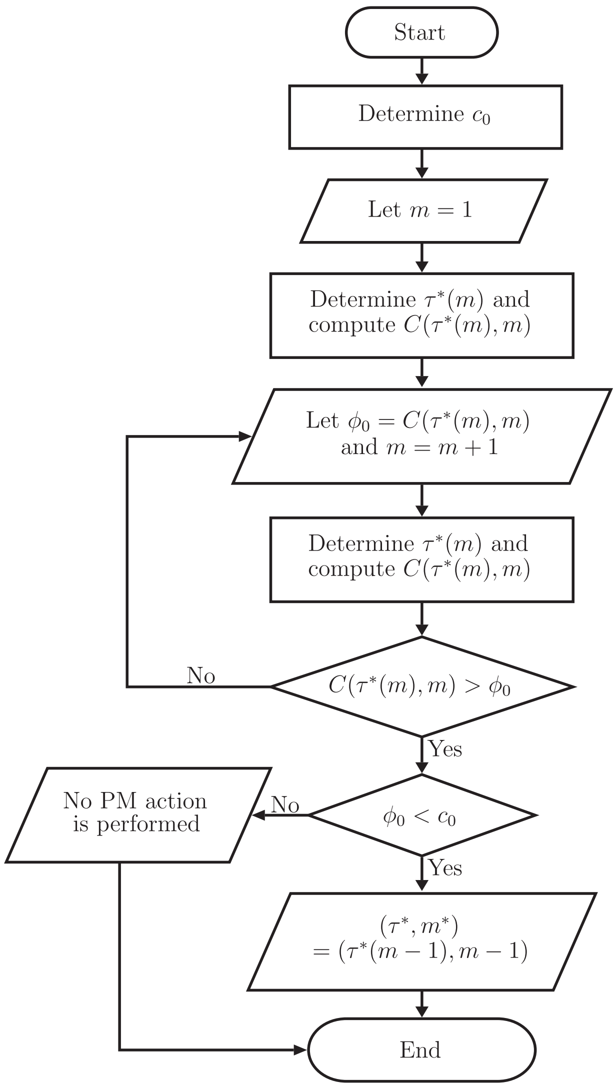

Fig. 1. Flow chart of the searching process.

Following Rigdon & Basu (2000), the likelihood function for $\boldsymbol{\mu}$ is given by

$$L(\boldsymbol{\mu}) = \exp\left(-\int_0^T \lambda(u)du\right)\prod_{j=1}^{k}\lambda(t_j). \tag{3.1}$$

For the failure truncated sampling, t_k will be used to replace T in Eq. (3.1), where t_k is the k^{th} failure time. If n independent products are observed, involving n_1 time-truncated products at time T_i for $i = 1,\ldots,n_1$, respectively and $n - n_1$ failure truncated products at the k_i^{th} failure for $i = n_1 + 1,\ldots,n$, respectively, then the likelihood function can be presented as

$$L(\boldsymbol{\mu}) = \exp\left(-\sum_{i=1}^{n_1}\int_0^{T_i}\lambda(u)du - \sum_{i=n_1+1}^{n}\int_0^{t_{ik_i}}\lambda(u)du\right)\prod_{i,j}\lambda(t_{ij}), \tag{3.2}$$

where t_{ij} is the jth failure time for the ith product. Therefore, the maximum likelihood estimator (MLE) of $\boldsymbol{\mu}$ can be determined by

$$\hat{\boldsymbol{\mu}} = \arg\max_{\boldsymbol{\mu}\in\Omega}\{\ln L(\boldsymbol{\mu})\},$$

where Ω is the parameter space. Using the PLP with the $\lambda(t)$ of Eq. (1.1) and letting $\boldsymbol{\mu} = (\theta, \beta)$ in Eq. (3.2), the likelihood function can be rewritten as

$$L(\theta,\beta) = \exp\left\{-\sum_{i=1}^{n_1}\int_0^{T_i}\frac{\beta u^{\beta-1}}{\theta^\beta}du - \sum_{i=n_1+1}^{n}\int_0^{t_{ik_i}}\frac{\beta u^{\beta-1}}{\theta^\beta}du\right\}\prod_{i,j}\frac{\beta t_{ij}^{\beta-1}}{\theta^\beta}$$
$$= \exp\left\{-\sum_{i=1}^{n_1}\left(\frac{T_i}{\theta}\right)^\beta - \sum_{i=n_1+1}^{n}\left(\frac{t_{ik_i}}{\theta}\right)^\beta\right\}\prod_{i,j}\frac{\beta t_{ij}^{\beta-1}}{\theta^\beta}.$$

It follows that

$$\frac{\partial \ln L(\theta,\beta)}{\partial\theta} = \beta\left\{\sum_{i=1}^{n_1}\frac{T_i^\beta}{\theta^{\beta+1}} + \sum_{i=n_1+1}^{n}\frac{t_{ik_i}^\beta}{\theta^{\beta+1}} - \sum_{i,j}\frac{1}{\theta}\right\}, \tag{3.3}$$

$$\frac{\partial \ln L(\theta,\beta)}{\partial\beta} = -\sum_{i=1}^{n_1}\left(\frac{T_i}{\theta}\right)^\beta\ln\left(\frac{T_i}{\theta}\right) - \sum_{i=n_1+1}^{n}\left(\frac{t_{ik_i}}{\theta}\right)^\beta\ln\left(\frac{t_{ik_i}}{\theta}\right) + \sum_{i,j}\left(\frac{1}{\beta} + \ln t_{ij} - \ln\theta\right). \tag{3.4}$$

The MLEs, $\hat\theta$ and $\hat\beta$, can be determined numerically by equating Eqs. (3.3) and (3.4) to zero and solving θ and β simultaneously.

Let $\hat{\boldsymbol{\mu}} = (\hat\theta, \hat\beta)$, it can been shown that $\hat{\boldsymbol{\mu}} \to MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ as the number of failures grows to infinity, where $MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and variance–covariance matrix $\boldsymbol{\Sigma}$. The variance-covariance matrix $\boldsymbol{\Sigma}$ can be estimated by $\hat{\boldsymbol{\Sigma}}$ which is obtained by evaluating the negative inverse of the Hessian matrix of $\ln L(\boldsymbol{\mu})$ at $\hat{\boldsymbol{\mu}}$. According to Eq. (2.5), τ_m^* can be given by a function of $\boldsymbol{\mu}$, say $\tau_m^* = g(\boldsymbol{\mu})$. So the MLE of τ_m^* will be $\hat\tau_m^* = g(\hat{\boldsymbol{\mu}})$ by the invariance property. Moreover, it follows that

$$\hat\tau_m^* \to N\left(\tau_m^*, \sigma_{\hat\tau_m^*}^2\right),$$

as the number of failures is large, where $\sigma_{\hat\tau_m^*}^2 = [\nabla g(\hat{\boldsymbol{\mu}})]^{\mathrm{T}}\boldsymbol{\Sigma}[\nabla g(\hat{\boldsymbol{\mu}})]$ and ∇g is the gradient of g. The approximate $100(1-\alpha)\%$ confidence interval for τ_m^* can be obtained by $\hat\tau_m^* \pm z_{\alpha/2}\hat\sigma_{\hat\tau_m^*}$, where $z_{\alpha/2}$ is the quantile of standard normal distribution with $\alpha/2$ upper-tail probability and $\hat\sigma_{\hat\tau_m^*}^2$ is $[\nabla g(\hat{\boldsymbol{\mu}})]^{\mathrm{T}}\hat{\boldsymbol{\Sigma}}[\nabla g(\hat{\boldsymbol{\mu}})]$.

Assuming that $\lambda(t)$ is of Eq. (1.1), the gradient, $\nabla g(\theta,\beta) = \left(\partial\tau_m^*/\partial\theta, \partial\tau_m^*/\partial\beta\right)^{\mathrm{T}}$, of $\tau_m^* = g(\theta,\beta)$ can be obtained as follows: Let $F(\theta,\beta,\tau_m^*) = dC(\tau,m)/d\tau = 0$,

$$F_1(\theta,\beta,\tau_m^*) = \frac{\partial}{\partial\theta}F(\theta,\beta,\tau_m^*),$$
$$F_2(\theta,\beta,\tau_m^*) = \frac{\partial}{\partial\beta}F(\theta,\beta,\tau_m^*),$$
$$F_3(\theta,\beta,\tau_m^*) = \frac{\partial}{\partial\tau_m^*}F(\theta,\beta,\tau_m^*).$$

From Eq. (2.5), we get

$$\frac{\partial F(\theta,\beta,\tau_m^*)}{\partial\theta} = -\frac{c_{MR}\beta^2}{\theta^{\beta+1}}\left[\sum_{i=1}^{m}(\tau_m^*)^{\beta-1}\kappa_i - mr\phi_1^{\beta-1} - \frac{\phi_2^\beta}{\tau_m^*}\right], \tag{3.5}$$

$$\frac{\partial F(\theta,\beta,\tau_m^*)}{\partial\beta} = \frac{c_{MR}}{\theta^\beta}\left\{(\tau_m^*)^{\beta-1}\left\{\sum_{i=1}^{m}\left[\kappa_i + \beta\left(a_i^\beta\ln\left(\frac{\tau_m^* a_i}{\theta}\right)\right)\right] - \sum_{i=2}^{m}\beta b_i^\beta\ln\left(\frac{\tau_m^* b_i}{\theta}\right)\right\} - mr\phi_1^{\beta-1}\left[1 + \beta\ln\left(\frac{\phi_1}{\theta}\right)\right] - \frac{\phi_2^\beta}{\tau_m^*}\left[1 + \beta\ln\left(\frac{\phi_2}{\theta}\right)\right]\right\}, \tag{3.6}$$

$$\frac{\partial F(\theta,\beta,\tau_m^*)}{\partial\tau_m^*} = \frac{c_{MR}\beta(\beta-1)}{\theta^\beta}\left[\sum_{i=1}^{m}(\tau_m^*)^{\beta-2}\kappa_i + (mr)^2\phi_1^{\beta-2} - \frac{\phi_2^\beta}{(\tau_m^*)^2}\right], \tag{3.7}$$

where $\phi_1 = W - mr\tau_m^*$, $\phi_2 = m(1-r)\tau_m^*$ and $\kappa_i = a_i^\beta - b_i^\beta$. Therefore, it can be shown that

$$\frac{\partial}{\partial\theta}g(\theta,\beta) = -\frac{F_1(\theta,\beta,\tau_m^*)}{F_3(\theta,\beta,\tau_m^*)}, \tag{3.8}$$

$$\frac{\partial}{\partial\beta}g(\theta,\beta) = -\frac{F_2(\theta,\beta,\tau_m^*)}{F_3(\theta,\beta,\tau_m^*)}. \tag{3.9}$$

Even though the confidence interval for τ_m^* may be short, the total cost function $C(\tau,m)$ could be very peaked around its minimum so that a small deviation from the optimal PM schedule could have a large impact on the expected total cost. Hence, the error bound for $C(\hat{\tau}_m^*,m) - C(\tau_m^*,m)$ could provide useful information to evaluate the accuracy of $\hat{\tau}_m^*$. Since $C'(\tau_m^*,m) = 0$, by taking a second-order approximation, we have

$$\begin{aligned} C(\hat{\tau}_m^*,m) - C(\tau_m^*,m) &\approx C'(\tau_m^*,m)(\hat{\tau}_m^* - \tau_m^*) \\ &\quad + \frac{C''(\tau_m^*,m)}{2}(\hat{\tau}_m^* - \tau_m^*)^2 \\ &= \sigma^2_{\hat{\tau}_m^*}\frac{C''(\tau_m^*,m)}{2}\left(\frac{\hat{\tau}_m^* - \tau_m^*}{\sigma_{\hat{\tau}_m^*}}\right)^2, \end{aligned} \tag{3.10}$$

where $C''(\tau_m^*,m) = d^2C(\tau_m^*,m)/d(\tau_m^*)^2 > 0$.

Since $\hat{\tau}_m^*$ is asymptotically normally distributed with mean τ_m^* and variance $\sigma^2_{\hat{\tau}_m^*}$, $(\hat{\tau}_m^* - \tau_m^*)/\sigma_{\hat{\tau}_m^*}$ has a standard normal distribution approximately. Thus, the factor $\left(\frac{\hat{\tau}_m^* - \tau_m^*}{\sigma_{\hat{\tau}_m^*}}\right)^2$ in Eq. (3.10) has a χ^2 distribution with one degree of freedom asymptotically. The upper bound of $C(\hat{\tau}_m^*,m) - C(\tau_m^*,m)$ can be approximated by

$$C(\hat{\tau}_m^*,m) - C(\tau_m^*,m) \leqslant \frac{\sigma^2_{\hat{\tau}_m^*}}{2}C''(\tau_m^*,m)\chi^2_{1-\alpha}(1) \tag{3.11}$$

with a confidence level $(1-\alpha)$, where $\chi^2_{1-\alpha}(1)$ denotes the $(1-\alpha)$ quantile of the χ^2 distribution with one degree of freedom and

$$\begin{aligned} C''(\tau_m^*,m) = \frac{c_{MR}\beta(\beta-1)}{\theta^\beta}\Bigg\{&\sum_{i=1}^{m}(\tau_m^*)^{\beta-2}\left(a_i^\beta - b_i^\beta\right) \\ &+ (mr)^2(W - mr\tau_m^*)^{\beta-2} - (m(1-r))^\beta(\tau_m^*)^{\beta-2}\Bigg\}. \end{aligned}$$

Because the $C(\tau_m^*,m)$ depends on the length of warranty time, W, the average difference of $C(\hat{\tau}_m^*,m)$ and $C(\tau_m^*,m)$ per unit time is used, instead. Hence, inequality (3.11) can be rewritten as $(C(\hat{\tau}_m^*,m) - C(\tau_m^*,m))/W \leqslant \frac{\sigma^2_{\hat{\tau}_m^*}}{2W}C''(\tau_m^*,m)\chi^2_{1-\alpha}(1)$. The $100(1-\alpha)\%$ upper bound for $(C(\hat{\tau}_m^*,m) - C(\tau_m^*,m))/W$, under the PLP, becomes

$$\begin{aligned} UB = \left(\frac{\chi^2_{1-\alpha}(1)}{2W}\right)\left(\frac{\sigma^2_{\hat{\tau}_m^*}c_{MR}\beta(\beta-1)}{\theta^\beta}\right)\Bigg\{&\sum_{i=1}^{m}(\tau_m^*)^{\beta-2}\left(a_i^\beta - b_i^\beta\right) \\ &+ (mr)^2(W - mr\tau_m^*)^{\beta-2} - (m(1-r))^\beta(\tau_m^*)^{\beta-2}\Bigg\}. \end{aligned} \tag{3.12}$$

4. An example

A data set regarding the power transformers which was reported by Gilardoni & Colosimo (2007) is used for illustration. The power transformers are basic components of an electrical power-transmission system. They are complex and most of their repairs involve the replacement of only a small fraction of their constituent parts. Hence, it is reasonable to assume that the system's reliability after a transformer repair is essentially the same as it was right before failure.

Table 1 shows the repair and failure records, in which 30 transformers and 21 failure times were recorded between January 1999 and July 2001 for a group of 300-kV and 345-kV transformers belonging to the electrical power company. Censoring due to the implementation of a perfect PM is indicated by an asterisk(*) following the corresponding time in Table 1. Ten units censored at the 21,888th hour without failures are not included in the table. For more information about the data, please refer to Gilardoni & Colosimo (2007).

This company was working under a restrictive policy concerning nonscheduled maintenance. This policy makes the cost of an MR performed after a failure to be 15 times of the cost from a scheduled perfect PM. Under the PLP, the MLEs of the intensity function parameters can be found as $\hat{\beta} = 1.988$ and $\hat{\theta} = 24844$. Since the maintenance policy proposed by Gilardoni & Colosimo (2007) was under the perfect PM of a given fixed cost, for the purpose of comparison, the coefficients $a = 1$, $c_1 = 0$, and $c_2 = 0$ for the PM cost are set accordingly. Therefore, the MR cost $c_{MR} = 15$ is set to be $c_{MR}/C_p(t,x) = 15$. A PM plan with a perfect maintenance action, which instantly returns the product to a like-new condition, results in $r = 1$.

Using the proposed methodologies in Sections 2 and 3, the statistically estimated optimal PM schedules proposed here, $\hat{\tau}$'s, along with the corresponding approximated 95% confidence upper bounds, UB's, for various finite warranties or finite lifetime periods are obtained, respectively. And the results are presented in Table 2. It can be seen that, as the warranty time or lifetime W increases, the estimated optimal schedule $\hat{\tau}$ decreases to 6400 which was the perfect PM schedule suggested by Gilardoni & Colosimo (2007) for $W = \infty$. It should be mentioned that the PM schedule proposed by Gilardoni & Colosimo (2007) is not the optimal maintenance schedule in terms of the expected total cost within a finite time span of warranty. Table 2 also indicates that the

Table 1
Power transformers data.

Unit	Failures and PMs times (h) Censoring times are enclosed between parentheses			
1	8839	17,057	(21,887)	
2	9280	16,442	(21,887)	
3	10,445	(13,533)[a]	(21,435)	
4	(8414)[a]	(21,745)		
5	17,156	(21,887)		
6	16,305	(21,887)		
7	16,802	(21,887)		
8	(4881)[a]	(21,506)		
9	7396	7541	(19,590)[a]	(21,711)
10	15,821	19,746	(19,877)[a]	(21,804)
11	15,813	(21,886)		
12	15,524	(21,886)		
13	(21,440)[a]	(21,809)		
14	11,664	17,031	(21,857)	
15	(7544)[a]	(13,583)[a]	15,751	(20,281)
16	18,840	(21,879)		
17	(2288)[a]	(4787)[a]	(21887)	
18	10,668	(16,838)		
19	15,550	(21,887)		
20	(1616)[a]	15,657	(21,620)	

[a] Censoring due to a perfect PM.

Table 2
Optimal PM plans for various lengths of operation time under warranty.

W	$\hat{\tau}$	UB	Ratio
10^4	5000.000	1.135×10^{-4}	1.070088
10^5	6250.000	3.645×10^{-5}	1.012358
10^6	6410.256	4.079×10^{-6}	1.000204
10^7	6402.049	4.121×10^{-7}	1.000399
10^8	6401.229	4.626×10^{-8}	1.000000
∞[a]	6400.000	4.600×10^{-5}	1.000000

[a] denotes the optimal PM plan of Gilardoni and Colosimo (2007).

Table 3
Profermance of the proposed PM plans.

			(c_1, c_2)						
			(0.0001, 0.00015)		(0.00015, 0.0001)		(0.0000, 0.0000)		
r	β	W	τ^*	R_1	τ^*	R_1	τ^*	τ^*_{GC} [a]	R_2
0.7	1.3	15,000	6162.318	1.359	6205.678	1.361			
		20,000	8286.557	1.535	8339.767	1.536			
		30,000	12564.03	1.849	12635.00	1.851			
	1.6	15,000	6823.906	1.502	6846.983	1.504			
		20,000	6102.934	1.724	6146.200	1.732			
		30,000	9257.780	2.200	6984.714	2.213			
	1.9	15,000	7182.722	1.664	7199.367	1.666			
		20,000	6393.063	2.018	6423.013	2.028			
		30,000	7235.468	2.724	7273.115	2.744			
0.9	1.3	15,000	6637.276	1.421	6647.940	1.422			
		20,000	8906.163	1.607	8919.225	1.608			
		30,000	8993.476	1.982	9051.666	1.987			
	1.6	15,000	7067.254	1.637	7073.010	1.638			
		20,000	6303.259	1.974	6326.360	1.981			
		30,000	7132.096	2.632	7166.830	2.650			
	1.9	15,000	4807.118	1.888	4823.712	1.898			
		20,000	6462.479	2.440	6479.552	2.452			
		30,000	5821.436	3.568	5848.976	3.614			
1	1.3	15,000	7034.907	1.469	7034.907	1.469	7500.000	6681.809	1.052
		20,000	6288.372	1.678	6317.929	1.682	6666.667	6681.809	1.000
		30,000	7115.096	2.122	7164.542	2.133	6000.000	6681.809	1.009
	1.6	15,000	4811.640	1.758	4826.222	1.764	5000.000	5321.399	1.003
		20,000	6455.567	2.167	4847.171	2.177	5000.000	5321.399	1.004
		30,000	5820.559	3.049	5849.288	3.085	5000.000	5321.399	1.007
	1.9	15,000	4840.620	2.080	4852.897	2.089	5000.000	5298.297	1.004
		20,000	4852.688	2.746	4871.147	2.773	5000.000	5298.297	1.005
		30,000	5857.393	4.310	5879.974	4.373	5000.000	5298.297	1.008

[a] τ^*_{GC} denotes the optimal value of τ proposed by Gilardoni and Colosimo (2007).

approximated upper bound for the average difference of the expected optimal total cost and its estimate per unit time is almost negligible for all cases. Hence, the proposed statistic estimate of the optimal PM schedule works properly, accordingly.

Let $C_{(0,W]}(\tau)$ be the expected total cost which is the Equation (1) of Gilardoni & Colosimo (2007), $R = c_{MR}E(N(0, W - m\tau))$ and $H(\tau) = \lim_{W\to\infty} C_{(0,W]}(\tau)/W$. The average expected total cost per unit time can be represented as

$$C^*_{(0,W]}(\tau) = H(\tau) + \frac{R}{W} = \frac{c_{MR}}{\tau}\left[\frac{c_{PM}}{c_{MR}} + \int_0^{\tau} \lambda(t)dt\right] + \frac{c_{MR}}{W}\int_0^{W-m\tau} \lambda(t)dt.$$

When W is very large and approaches infinity, the last term of $C^*_{(0,W]}(\tau)$ could be neglected. Using the first two terms of $C^*_{(0,W]}(\tau)$, a perfect PM schedule had been obtained as 6400 by Gilardoni & Colosimo (2007). Apparently, 6400 is not the minimizer of $C^*_{(0,W]}(\tau)$ when W is finite. Let $\tau = 6400$, the average expected total cost per unit time from the PM policy of Gilardoni & Colosimo (2007) within the time interval $(0, W]$ is labeled by $C^*_{(0,W]}(6400)$. The ratio of $C^*_{(0,W]}(6400)$ over the average expected total cost per unit time using the proposed schedule $\hat{\tau}^*$ is calculated. Table 2 shows the ratios for various lengths of warranty times. It can be seen that all ratios are slightly larger than 1. This implies that the proposed PM policy has a lower average expected total cost than the one proposed by Gilardoni & Colosimo (2007) per unit time under a perfect PM and $c_1 = c_2 = 0$. For example, when $W = 10{,}000$, the ratio is 1.070088 which means the average expected total cost per unit time by using the PM policy of Gilardoni & Colosimo (2007) is around 7% over the average expected total cost of the proposed PM policy for $W = 10{,}000$ when the conditions for the proposed PM policy is reduced to the conditions given by Gilardoni & Colosimo (2007). Accordingly, the difference of these two expected total costs over the whole lifetime interval $(0, 10{,}000]$ could be very huge, if the maintenance cost and the repair cost are very high.

When the perfect PM action is applied and the coefficients $c_1 > 0$ and $c_2 = 0$, the Eq. (2) of Gilardoni & Colosimo (2007) should be added an extra constant term c_1. Hence, the perfect PM schedule of Gilardoni & Colosimo (2007) would not be changed. Hence, the comparison will not be changed.

When the perfect PM action is applied and the coefficients $c_1 = 0$ and $c_2 > 0$, the Eq. (2) of Gilardoni & Colosimo (2007) should have an extra non-constant term added, which is a function of c_2 and the number of PM actions within $(0, W]$. Therefore, the derivation for setting the perfect PM schedule by using Eq. (2) of Gilardoni & Colosimo (2007) could not be accomplished via letting the derivative equal to 0. In this situation, the perfect PM schedule must be studied through a different algorithm. At the present time, this perfect PM schedule is not available for comparison.

5. The numerical study

In this section, the performances for the proposed PM policy are investigated numerically under different maintenance parameter settings. Assume that the lifetimes of repairable products have a Weibull distribution with parameters β and θ. The NHPP of the PLP is used to model the number of failures during the whole lifetime period. The median lifetime for the products can be represented as

$$t_M = \theta(\log(2))^{1/\beta}. \tag{5.1}$$

Assume that this product is manufactured with a nominal median lifetime $t_m = 20{,}000$. We are interested in studying the performances of the proposed PM policy for this product under different warranty times, $W = 15{,}000$, 20,000, 30,000, and different intensities of failure models. In the study of the PLP, most estimates of β are in the interval of (1,2). Therefore, we take $\beta = 1.3$, 1.5, 1.9 for the numerical study. The value of θ is able to be determined based on the value of β and Eq. (5.1). Let the MR cost be 20 times of the PM cost, the maintenance cost increase linearly with coefficients $a = 20$, $c_1 = 0$, 0.00010 or 0.00015, and $c_2 = 0$, 0.00010 or 0.00015. The maintenance action can be an imperfect one with $r = 0.7$ or 0.9 or the perfect one with $r = 1$. All numerical results are given in Table 3.

When an imperfect maintenance action is implemented, the PM policy proposed by Gilardoni & Colosimo (2007) is improper to be used. Therefore, the comparison between the proposed maintenance schedule and the one by Gilardoni & Colosimo (2007) could not be executed under the imperfect maintenance policy. Since our proposed schedule is based on the global optimization criterion which minimizes the expected total cost within the whole finite operation time interval, no other PM policy would be used to compare with the current proposed PM policy under the cost function (2.3). However, the impact to the expected total cost from the input parameters for the proposal PM policy could be evaluated numerically. Here, the ratio between the expected total cost without a maintenance action over the expected total cost with the proposed PM policy under each different parameter setting is calculated and denoted by R_1 in Table 3. When the maintenance action is perfect and the PM cost is constant which indicates the case of $r = 1$ and $c_1 = c_2 = 0$, the ratio of the expected total cost from the PM schedule proposed by Gilardoni & Colosimo (2007) over the expected total cost from the proposed PM schedule under each different setting of β and W could be evaluated and denoted by R_2 in Table 3.

Table 3 shows that all R_1s are significantly greater than 1. It indicates that the expected total cost could be reduced significantly whether the perfect PM or an imperfect PM is applied during the finite warranty time period or the finite operation time interval. For example, if $r = 0.7$, $\beta = 1.3$, $W = 15000$, the proposed policy suggests only a PM action is conducted at time $\tau^* = 6162.318$ when $(c_1, c_2) = (0.00010, 0.00015)$. The proportion of cost reduction relative to the cost without a maintenance action can be found as 0.359/1.359 = 26.4%. Moreover, the expected total cost of the proposed PM policy is close to but slightly lower than the expected total cost of the PM policy proposed by Gilardoni & Colosimo (2007). Table 2 also indicates that the proposed PM policy performs better than the PM policy proposed by Gilardoni & Colosimo (2007). However, these two policies are getting close when W increases. This behavior can be a validation of the proposed policy.

In this numerical study, the maintenance cost, $C_p(t,x)$, increases linearly with respect to the product operation time, t. That is, the maintenance cost increases when the product operation time, t, increases. This assumption is more realistic than keeping the maintenance cost unchanged regardless of the usage time of the product. The proposed PM model is flexible, and users can adjust the parameters of the maintenance policy according to a practical situation.

6. Conclusions

Given a finite operation time interval for repairable products, an imperfect PM model has been established. A searching procedure is provided to find the optimal PM schedule which minimizes the expected total cost over the operation time interval. It should be mentioned again that the proposed optimal PM schedule is to minimize the expected total cost within a given finite operation time period. An estimator of the proposed PM schedule is also developed. The application to the power transformers data shows that the estimated optimal maintenance schedule performs well with a small upper bound for the difference of the true expected total cost and its estimate. Meanwhile the PM schedule provided by Gilardoni & Colosimo (2007) originally for infinite operating time interval could improperly estimate the perfect PM schedule when the product operation time is finite.

With the advent of modern powerful and accessible computers available, the computation times needed for the calculation of the proposed PM schedule and for the calculation of the PM schedule proposed by Gilardoni & Colosimo (2007) are negligible when the total cost saving is considered. Moreover, the proposed process to find the PM schedule can be applied to any imperfect PM action. Considering a different failure process and different cost components for the total cost function to develop a PM plan would be a fruitful area of future research.

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References

Ascher, H., & Feingold, H. (1984). *Repairable systems reliability modeling, inference, misconception, and their causes*. New York: Marcel Dekker.

Bain, L. J., & Engelhardt, M. (1991). *Statistical analysis of reliability and life-testing models. Theory and methods*. New York: Marcel Dekker.

Barlow, R. E., & Hunter, L. C. (1960). Optimum preventive maintenance policies. *Operations Research, 8*, 90–100.

Block, H. W., Borges, W. S., & Savits, T. H. (1990). A general age replacement with minimal repair. *Naval Research Logistics, 35*, 365–372.

Cox, D. R., & Miller, H. D. (1965). *The theory of stochastic processes*. London: Methuen.

Crow, L. R. (1974). Reliability analysis or complex systems. In F. Proschan & J. Serfling (Eds.), *Reliability and bimetry: Statistical analysis of lifelength* (pp. 379–410). Philadelphia, PA: SIAM.

Gerstack, I. B. (1977). *Models of preventive maintenance*. Amsterdam: North Holland.

Gilardoni, G. L., & Colosimo, E. A. (2007). Optimal maintenance time for repairable systems. *Journal of Quality Technology, 39*(1), 48–53.

Lai, K. K., Leung, K. N. F., Tao, B., & Wang, S. Y. (2001). A sequential method for preventive maintenance and replacement of a repairable single-unit system. *Journal of the Operational Research Society, 52*, 1276–1283.

Park, D. H., Jung, G. M., & Yum, J. K. (2000). Cost minimization for periodic maintenance policy of a system subject to slow degradation. *Reliability Engineering & System Safety, 6*, 105–112.

Pham, H., & Wang, H. (1996). Imperfect maintenance. *European Journal of Operational Research, 94*, 425–438.

Pulcini, G. (2001). A bounded intensity process for the reliability of repairable equipment. *Journal of Quality Technology, 33*, 480–492.

Rigdon, S. E., & Basu, A. P. (2000). *Statistical methods for the reliability of repairable systems*. New York: Wiley.

Yeh, R. H., & Chen, M.-Y. (2005). Optimal preventive-maintenance warranty policies for repairable products with age-dependent maintenance costs. *International Journal of Reliability, Quality and Safety Engineering, 12*(2), 111–125.