On reduced-order filter design for uncertain cascaded 2-1 sigma-delta modulators

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***Abstract*—In this paper, we present a new robust matching filter design method for uncertain 2-1 cascaded sigma-delta modulators. This method addresses a well known limitation of H-infinity loop shaping techniques that they yield filters of high order (equal to the sum of the plant order and the order of the weighting function), thus increasing the complexity of circuit implementation. In contrast, the new method yields filters whose order is equal to the plant order, independent of the weighting function. We compare the new method with other existing fixed-order designs, and establish its efficacy.**

***Keywords*—uncertainty, cascaded sigma-delta modulator, reduced-order filter, H-infinity loop shaping, linear matrix inequality.**

I. INTRODUCTION

Sigma-delta (ΣΔ) modulators [1] are important devices which have found widespread application in high-speed analog-to-digital (A/D) conversion for modern digital signal processing. Cascaded sigma-delta modulators are preferred over single loop modulators as they offer greater stability. In order to have a higher signal-to-noise ratio (SNR), cascaded sigma-delta modulators rely heavily on accurate matching of analog stages with a digital filter to prevent leakage of quantization noise. However, perfect matching is impossible in practice due to the limited accuracy of the implementation technologies as well as parameter variations. Techniques that do not explicitly account for this often yield designs that perform poorly [1].

Thus, in recent years, considerable effort has been devoted to the study of robust matching filters under the framework of "model matching" [2, 3, 4]. The basic idea is to recast the filter design problem as a problem of minimizing the worst-case value, over all possible uncertainties, of a measure of a certain model mismatch (we will present details in Section II). The specific mismatch measure that has been most often used is the H_∞ norm which measures the peak value of the mismatch over all frequencies. To achieve higher-order noise shaping, a weighting function was introduced in [2]; the uncertainties were of the "polytopic" type, and the problem became that of the minimization of the H_∞ norm of the weighted matching error over a linearized polytopic model. While the introduction of weighting functions is useful in shaping the noise transfer function (NTF) so as to increase the SNR, it also increases the order of the filter, which in turn leads to increased complexity of circuit implementation. To alleviate this problem, two fixed-order designs have been proposed [3, 4]. In [3], the central polynomial linear matrix inequality (LMI) method has been employed. In particular, the order of the resulting infinite impulse response (IIR) filter is independent of that of the introduced weighting function. Moreover, the filter order can be chosen to be any positive integer. In contrast to the mathematical approach in [2, 3], a design based more on engineering insight was presented in [4]. A low-frequency linearized model of a 2-1 cascaded modulator was derived and, again a fixed-order (but finite impulse response (FIR)) filter design was presented based on a formal optimization method.

In this paper, we revisit the weighted H_∞ norm minimization formulation in [2], and directly address the issue of high filter order. Our main contribution is a new reduced-order filter design procedure that yields filters whose order is equal to the plant order, independent of the weighting function. We show that this approach yields filters whose performance compares favorably with those presented in [3, 4]. The rest of this work is organized as follows. In Section II, an uncertain cascaded 2-1 ΣΔ modulator is briefly described and the problem formulation is presented. In Section III, the proposed reduced-order IIR filter design is provided. Section IV shows the simulation results with comparison to some of the existing fixed-order works. Section V is the conclusion.

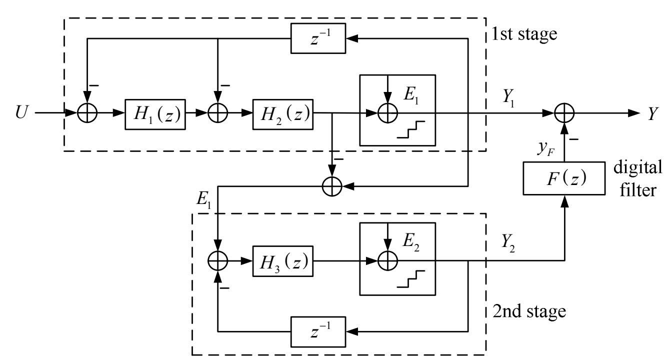

Figure 1. A cascaded 2-1 modulator with 1-bit quantizers

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II. CASCADED 2-1 SIGMA-DELTA MODULATOR

The linear model for a cascaded 2-1 ΣΔ modulator is presented in Fig. 1, where E_1 and E_2 are the quantization noises of the first stage and the second stage, respectively [1]. The first stage quantization noise E_1 is extracted and re-quantized at the second-stage. Accordingly, the NTF from E_1 to output Y is

$$T = NTF_1(z) - F(z) \times STF_2(z) \tag{1}$$

where

$$NTF_1(z) = \frac{1}{1 + z^{-1}H_2 + z^{-1}H_1H_2}, \quad STF_2 = \frac{H_3}{1 + z^{-1}H_3} \tag{2}$$

The noise effect from E_2 is less significant and hence can be neglected [2, 3, 4]. Ideally, the E_1 term can be completely eliminated from the modulator output Y by a matching filter

$$F(z) = \frac{NTF_1(z)}{STF_2(z)} \tag{3}$$

However, perfect cancellation of E_1 is not possible in practice owing to non-ideal analog components. Recall that two common sources of analog imperfections in a ΣΔ modulator are finite amplifier gain and mismatch in capacitor values. These factors can be modeled as parametric uncertainties in the gains and poles of the integrators. Therefore, a non-ideal integrator can be modeled as [2, 3, 4]

$$H_i(z) \cong \frac{1-\delta_a}{1-(1-\delta_b)z^{-1}}, \; i = 1,2,3 \tag{4}$$

where $\delta_a \in [0,1)$ and $\delta_b \in [0,1)$ are the parameter deviations from the nominal values. Here we assume the integrators H_2 and H_3 are ideal, and thus the uncertain $NTF_1(z)$ can be described by

$$NTF_1(z) = \frac{1-(2-\delta_b)z^{-1} + (1-\delta_b)z^{-2}}{1-(-\delta_b+\delta_a)z^{-1}} \tag{5}$$

and STF_2=1. Our work is to minimize the effect of quantization noise E_1 on the output Y in the signal band. This can be formulated as a weighted H_∞ norm minimization problem

$$\min_{F(z)} \left\| W(z)\left(NTF_1(z) - F(z)\right) \right\|_\infty \tag{6}$$

where $W(z)$ is a weighting function that is employed to shape the NTF for E_1.

For later use, we assume that the transfer functions $W(z)$, $F(z)$, and $NTF_1(z)$ have the following state-space realizations:

$$W(z) := C_W(zI - A_W)^{-1}B_W + D_W \tag{7}$$

$$F(z) := C_F(zI - A_F)^{-1}B_F + D_F \tag{8}$$

$$NTF_1(z) := C_1(zI - A_1)^{-1}B_1 + D_1 \tag{9}$$

where

$$A_1 = \begin{pmatrix} -\delta_b + \delta_a & 1 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} -2+\delta_a \\ 1-\delta_b \end{pmatrix}, C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, D_1 = 1 \tag{10}$$

In order to take the uncertainties δ_a and δ_b into account in problem (6), we assume the uncertain matrices (A_1, B_1, C_1, D_1) of $NTF_1(z)$ belong to the following uncertainty polytope:

$$\Omega = \Big\{ (A_1, B_1, C_1, D_1) \mid (A_1, B_1, C_1, D_1) = \sum_{i=1}^{m} \alpha_i \left(A_1^{(i)}, B_1^{(i)}, C_1^{(i)}, D_1^{(i)}\right), \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1 \Big\} \tag{11}$$

Here giving the values of $\alpha_i, i = 1, \cdots, m,$ with $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$ produces an element of Ω. Then, it is readily verified the state-space realization of the weighted matching error $W(z)\left(NTF_1(z) - F(z)\right)$ denoted by $H(z) := C_H(zI - A_H)^{-1}B_H + D_H$ is given by

$$H(z) := \sum_{i=1}^{m} \alpha_i \left(\begin{array}{c|c} A_H^{(i)} & B_H^{(i)} \\ \hline C_H^{(i)} & D_H^{(i)} \end{array} \right) = \sum_{i=1}^{m} \alpha_i \left(\begin{array}{cc|c} A_M^{(i)} & B_M^{(i)}C_W & B_M^{(i)}D_W \\ 0 & A_W & B_W \\ \hline C_M^{(i)} & D_M^{(i)}C_W & D_M^{(i)}D_W \end{array} \right) \tag{12}$$

where

$$A_M^{(i)} = \begin{bmatrix} A_1^{(i)} & 0 \\ 0 & A_F \end{bmatrix}, B_M^{(i)} = \begin{bmatrix} B_1^{(i)} \\ B_F \end{bmatrix}, \quad C_M^{(i)} = \begin{bmatrix} C_1^{(i)} & -C_F \end{bmatrix}, D_M^{(i)} = \begin{bmatrix} D_1^{(i)} - D_F \end{bmatrix} \tag{13}$$

and $A_F \in R^{nf}$, $A_W \in R^{nw}$, $A_1^{(i)} \in R^{np}$, $B_1^{(i)} \in R^{np\times 1}$, $C_1^{(i)} \in R^{1\times np}$, $D_1 \in R^{1\times 1}$. This recasts the design problem (6) to that of finding a filter of form (8) via the solution of the following optimization problem:

$$\min_{A_F, B_F, C_F, D_F} \gamma \tag{14}$$

subject to $\| C_H(zI - A_H)^{-1}B_H + D_H \|_\infty < \gamma$. Referring to (6), it is known that the conventional H-infinity loop-shaping technique derives a full-order filter where $nf = np + nw$. In the following section, we present a reduced-order filter design method and the filter order nf is reduced to be np.

III. MAIN RESULT

In this section, we present the main result. The following lemma is useful in the development.

Lemma 1 [2]. For all (A_1, B_1, C_1, D_1) belonging to Ω, the condition $\| C_H(zI - A_H)^{-1}B_H + D_H \|_\infty < \gamma$ holds if there exist a matrix G and matrices $P^{(i)} = P^{(i)T}$ ($i = 1, \ldots, m$) satisfying

$$\begin{bmatrix} G + G^T - P^{(i)} & 0 & GA_H^{(i)} & GB_H^{(i)} \\ 0 & I & C_H^{(i)} & D_H^{(i)} \\ A_H^{(i)T}G^T & C_H^{(i)T} & P^{(i)} & 0 \\ B_H^{(i)T}G^T & D_H^{(i)T} & 0 & \gamma^2 I \end{bmatrix} > 0, \; i = 1, \ldots, m. \tag{15}$$

We now present Theorem 1 that states that there exists a robust matching filter with order equal to that of the first stage of the cascaded modulator if certain matrix inequality constraints (16) are satisfied.

Theorem 1: Assume $nw = np$. For all m vertices

$$\left(A_1^{(i)}, B_1^{(i)}, C_1^{(i)}, D_1^{(i)}\right) \quad (\, i = 1,\ldots,\, m\,),$$

there exists a suboptimal filter (8) of order $nf = np$ to problem (14) if optimization problem (16) is feasible for i =1,…, m. In the case, the filter is given by

$$F(z) := Q_C\left(zI - M^{-T}Q_A\right)^{-1} M^{-T}Q_B + Q_D \tag{17}$$

Proof: We will invoke Lemma 1 to derive the solvability condition for the filter with order $nf = np$. To proceed, partition the matrices G and P as follows:

$$P^{(i)} = \begin{bmatrix} P_{11}^{(i)} & P_{12}^{(i)} & P_{13}^{(i)} \\ P_{12}^{(i)T} & P_{22}^{(i)} & P_{23}^{(i)} \\ P_{13}^{(i)T} & P_{23}^{(i)T} & P_{33}^{(i)} \end{bmatrix}. \; G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \tag{18}$$

where all the submatrices $P_{11}^{(i)}$, $P_{22}^{(i)}$, G_{11} , G_{22} have the dimension $np \times np$. Without loss of generality, G_{22} is assumed to be nonsingular. Define $\lambda = \alpha \times I_{nf}$, where α is a scalar parameter. Under the constraint $G_{32} = \lambda G_{12}$, apply congruence transformation $J = diag(T, I, T, I)$ to (15) (i.e., multiplying (15) on the left by J and on the right by J^T) where $T = diag\left(I, G_{12}G_{22}^{-1}, I\right)$ and define

$$P_{g12}^{(i)} = P_{12}^{(i)}G_{22}^{-T}G_{12}^T,\;\; P_{g22}^{(i)} = G_{12}G_{22}^{-1}P_{22}^{(i)}G_{22}^{-T}G_{12}^T,\;\; P_{g23}^{(i)} = G_{12}G_{22}^{-1}P_{23}^{(i)} \tag{19}$$

we obtain (20), i.e.,

$$\begin{bmatrix} G_{11} + G_{11}^T - P_{11}^{(i)} & \Xi_{12} & \Xi_{13} & 0 & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & \Xi_{26} & \Xi_{27} & \Xi_{28} \\ * & * & \Xi_{33} & 0 & \Xi_{35} & \Xi_{36} & \Xi_{37} & \Xi_{38} \\ * & * & * & I & \Xi_{45} & \Xi_{46} & \Xi_{47} & \Xi_{48} \\ * & * & * & * & P_{11}^{(i)} & P_{g12}^{(i)} & P_{13}^{(i)} & 0 \\ * & * & * & * & * & P_{g22}^{(i)} & P_{g23}^{(i)} & 0 \\ * & * & * & * & * & * & P_{33}^{(i)} & 0 \\ * & * & * & * & * & * & * & \gamma^2 I \end{bmatrix} > 0 \tag{20}$$

where

$$\Xi_{12} = G_{12}G_{22}^{-T}G_{12}^T + G_{21}^T G_{22}^{-T}G_{12}^T - P_{g12}^{(i)}$$
$$\Xi_{13} = G_{13} + G_{31}^T - P_{13}^{(i)}$$
$$\Xi_{15} = G_{11}A_1^{(i)}$$
$$\Xi_{16} = G_{12}A_F G_{22}^{-T}G_{12}^T$$
$$\Xi_{17} = G_{11}B_1^{(i)}C_W + G_{12}B_F C_W + G_{13}A_W$$
$$\Xi_{18} = G_{11}B_1^{(i)}D_W + G_{12}B_F D_W + G_{13}B_W$$
$$\Xi_{22} = G_{12}G_{22}^{-T}G_{12}^T + G_{12}G_{22}^{-1}G_{12}^T - P_{g22}^{(i)}$$
$$\Xi_{23} = G_{12}G_{22}^{-1}G_{23} + \lambda G_{12}G_{22}^{-1}G_{12}^T - P_{g23}^{(i)}$$
$$\Xi_{25} = G_{12}G_{22}^{-1}G_{21}A_1^{(i)}$$
$$\Xi_{26} = G_{12}A_F G_{22}^{-T}G_{12}^T$$
$$\Xi_{27} = G_{12}G_{22}^{-1}G_{21}B_1^{(i)}C_W + G_{12}B_F C_W + G_{12}G_{22}^{-1}G_{23}A_W$$
$$\Xi_{28} = G_{12}G_{22}^{-1}G_{21}B_1^{(i)}D_W + G_{12}B_F D_W + G_{12}G_{22}^{-1}G_{23}B_W$$
$$\Xi_{33} = G_{33} + G_{33}^T - P_{33}^{(i)}$$
$$\Xi_{35} = G_{31}A_1^{(i)}$$
$$\Xi_{36} = \lambda G_{12}A_F G_{22}^{-T}G_{12}^T$$
$$\Xi_{37} = G_{31}B_1^{(i)}C_W + \lambda G_{12}B_F C_W + G_{33}A_W$$
$$\Xi_{38} = G_{31}B_1^{(i)}D_W + \lambda G_{12}B_F D_W + G_{33}B_W$$
$$\Xi_{45} = C_1^{(i)}$$
$$\Xi_{46} = -C_F G_{22}^{-T}G_{12}^T$$
$$\Xi_{47} = D_1^{(i)}C_W - D_F C_W$$
$$\Xi_{48} = D_1^{(i)}D_W - D_F D_W$$

Now, define new variables as follows:

$$\begin{aligned} &M = G_{12}G_{22}^{-1}G_{12}^T, N = G_{12}G_{22}^{-1}G_{23}, S = G_{12}G_{22}^{-1}G_{21} \\ &Q_A = G_{12}A_F G_{22}^{-T}G_{12}^T, Q_B = G_{12}B_F, Q_C = C_F G_{22}^{-T}G_{12}^T, Q_D = D_F \end{aligned} \tag{21}$$

We obtain the matrix constraints in problem (16). Furthermore, if (16) is feasible, it implies the positive definiteness of the (2,2) block of (16), i.e.,

$$M^T + M > 0 \tag{22}$$

It follows that

$$G_{12}G_{22}^{-1}G_{12}^T + G_{12}G_{22}^{-T}G_{12}^T > 0 \tag{23}$$

$$\min_{\lambda, P_{11}^{(i)} = (P_{11}^{(i)})^T, P_{g12}^{(i)}, P_{g22}^{(i)} = (P_{g22}^{(i)})^T, P_{13}^{(i)}, P_{g23}^{(i)}, P_{33}^{(i)} = (P_{33}^{(i)})^T, G_{11}, G_{13}, G_{31}, G_{33}, M, N, S, Q_A, Q_B, Q_C, Q_D \;\; (i=1,\ldots,m)} \gamma \tag{16}$$

subject to

$$\begin{bmatrix} G_{11} + G_{11}^T - P_{11}^{(i)} & M^T + S^T - P_{g12}^{(i)} & G_{13} + G_{31}^T - P_{13}^{(i)} & 0 & G_{11}A_1^{(i)} & Q_A & G_{11}B_1^{(i)}C_W + Q_B C_W + G_{13}A_W & G_{11}B_1^{(i)}D_W + Q_B D_W + G_{13}B_W \\ * & M^T + M - P_{g22}^{(i)} & N + M\lambda^T - P_{g23}^{(i)} & 0 & SA_1^{(i)} & Q_A & SB_1^{(i)}C_W + Q_B C_W + NA_W & SB_1^{(i)}D_W + Q_B D_W + NB_W \\ * & * & G_{33} + G_{33}^T - P_{33}^{(i)} & 0 & G_{31}A_1^{(i)} & \lambda Q_A & G_{31}B_1^{(i)}C_W + \lambda Q_B C_W + G_{33}A_W & G_{31}B_1^{(i)}D_W + \lambda Q_B D_W + G_{33}B_W \\ * & * & * & I & C_1^{(i)} & -Q_C & D_1^{(i)}C_W - Q_D C_W & D_1^{(i)}D_W - Q_D D_W \\ * & * & * & * & P_{11}^{(i)} & P_{g12}^{(i)} & P_{13}^{(i)} & 0 \\ * & * & * & * & * & P_{g22}^{(i)} & P_{g23}^{(i)} & 0 \\ * & * & * & * & * & * & P_{33}^{(i)} & 0 \\ * & * & * & * & * & * & * & \gamma^2 I \end{bmatrix} > 0$$

Hence both M and G_{12} are invertible. It follows from (8) and (21) that the digital filter is given by

$$\begin{aligned} F(z) &= C_F\left(zI - A_F\right)^{-1} B_F + D_F \\ &= Q_C G_{12}^{-T} G_{22}^{T}\left(zI - G_{12}^{-1} Q_A G_{12}^{-T} G_{22}^{T}\right)^{-1} G_{12}^{-1} Q_B + Q_D \\ &= Q_C\left(zM^T - Q_A\right)^{-1} Q_B + Q_D \\ &= Q_C\left(zI - M^{-T} Q_A\right)^{-1} M^{-T} Q_B + Q_D \end{aligned} \tag{24}$$

With the defined change of variables, we obtain the synthesis condition given in (16) and the filter recovery procedure shown in (24). In order to confirm the correctness of the results, we shall further verify that the matrices $P^{(i)}$ and G in (18) can be recovered from any solution of problem (16). Specifically, we need to show that the matrices $P_{12}^{(i)}$, $P_{22}^{(i)}$, $P_{23}^{(i)}$, G_{12}, G_{21}, G_{22}, G_{23}, G_{32} can be recovered since the matrices $P_{11}^{(i)}$, $P_{13}^{(i)}$, $P_{33}^{(i)}$, G_{11}, G_{13}, G_{33} were obtained as part of the solution. For this purpose, we recall that

$$M = G_{12}G_{22}^{-1}G_{12}^{T},\ N = G_{12}G_{22}^{-1}G_{23},\ S = G_{12}G_{22}^{-1}G_{21} \tag{25}$$

where M, N, S can be determined when (16) is feasible. With the solution and let $G_{12}G_{22}^{-1}$ be a given nonsingular matrix X, we obtain G_{12}, G_{23}, and G_{21} via the following formulas:

$$G_{12} = M^T X^{-T},\ G_{23} = X^{-1}N,\ G_{21} = X^{-1}S\ .$$

Then, it is easily found that $G_{22} = X^{-1}G_{12}$. With a prior determined parameter λ, we immediately obtain $G_{32} = \lambda G_{12}$. Next, we can obtain $P_{12}^{(i)}$, $P_{22}^{(i)}$, $P_{23}^{(i)}$ by reversing (19), i.e.,

$$P_{12}^{(i)} = P_{g12}^{(i)} X^{-T}\ ,\quad P_{22}^{(i)} = X^{-1} P_{g22}^{(i)} X^{-T}$$

and

$$P_{23}^{(i)} = X^{-1} P_{g23}^{(i)}\ .$$

This completes the proof. ■

Remark 1. The proposed design has the following advantages. First, Theorem 1 provides a new solvability condition for deriving robust matching filters for uncertain cascaded modulators. In particular, the order of the proposed filters is independent of that of the weighting functions. This overcomes the well-known limitation of the state-space H-infinity loop shaping method where the resulting filters are of the same order as the plant plus weighting functions. Second, when λ in (16) is specified, problem (16) reduces to a linear objective minimization problem over LMI constraints, which can be efficiently solved by existing software [8].

Remark 2. While the weighting functions play an important role in arriving at a design with good SNR, it is difficult to determine their order a priori, as this will be application-dependent. Theorem 1 was derived under the assumption that $nf = nw = np$ (where np is the order of the first stage of the modulator) and the imposed constraint $G_{32} = \lambda G_{12} = \left(\alpha \times I_{nf}\right) G_{12}$, $\alpha \in R$. In the exceptional case that a candidate weighting function W_0 has order n_0 less than np, a new weighting function can be formed by multiplying W_0 with the delay elements $z^{-(np-n_0)}$ to fulfill the requirement $nw = np$ without altering its magnitude response; see e.g., Section IV. On the contrary, if W_0 has order greater than np, then the proposed method may be extended by replacing λ with a $n_0 \times np$ matrix. For simplicity, λ can be chosen to be $\lambda = \left[\alpha \times I_{np}\quad 0_{np\times(n_0-np)}\right]^T$.

IV. SIMULATION

Nonlinear simulations are carried out and validated with MATLAB/SIMULINK [8],[9] for a cascaded 2-1 ΣΔ modulator with 1-bit quantizer. Specifically, the modulator of this experiment is aimed at applying to an audio system. The experimental parameters are set up as follows. The signal bandwidth (BW) is 25KHz. A 8KHz sinusoidal wave is used to perform a standard test. The oversampling ratio (OSR) is chosen to be 64. The sampling frequency f_s is 3.2MHz and the number of time points used for FFT is 16384. The following paragraphs consist of two parts. In Part A, we consider three weighting functions, each of which has order less than or equal to or greater than the plant order ($np = 2$). We will numerically verify that the resulting (reduced-order) filters obtained by applying Theorem 1 and Remark 2 have order the same as that of the plant, independent of that of the introduced weighting functions. In Part B, we compare the best filter obtained in Part A with some of the existing fixed-order filter designs.

A. Filter Design with Weights of Different Order

Our work is to minimize the effect of the leaky quantization noise E_1 on the output Y in the signal band. To achieve the goal, it's important to design the digital filter such that the magnitudes of NTF T is relatively small in the frequency range [0,25K] Hz, i.e. we want the NTF T be a high-pass one. As far as the noise effect beyond the signal band is concerned, it can be reduced by a subsequent decimation filter [1]. In discrete-time domain, the cut-off frequency of the desired NTF T can be computed by the following formula [7, pp. 541]:

$$2\pi \frac{BW}{f_s} = 2\pi \frac{25\times 10^3}{3.2\times 10^6} = 0.0490625 \text{ (rad/s)}. \tag{26}$$

Accordingly, three low-pass weighting functions (27), (28), and (29) are considered.

$$W_1(z) = \frac{z^{-1}}{1-0.9z^{-1}}\ , \tag{27}$$

$$W_2(z) = \frac{2z^{-2}}{1-1.88z^{-1}+0.8832z^{-2}}\ , \tag{28}$$

$$W_3(z) = \frac{0.85z^{-3}}{1-2.62z^{-1}+2.2767z^{-2}-0.6556z^{-3}} \tag{29}$$

To apply Theorem 1, an augmented $W_1(z)$ is given by

$$W_{1a}(z) = \frac{z^{-1}}{1-0.9z^{-1}} \times z^{-1} \tag{30}$$

TABLE I. GIVEN DATA & OUTCOME

Case	Parameter	Weighting function Eq.	Weighting function Order	Filter	Performance index	SNR (dB)
A	$\lambda_1 = \begin{bmatrix} -0.006 & 0 \\ 0 & -0.006 \end{bmatrix}$	(30)	1+1	$F_{w1}(z) = \frac{1.0033 - 1.9831z^{-1} + 0.9798z^{-2}}{1 + 0.0062z^{-1} + 0.0003z^{-2}}$	$\gamma_1 = 0.0412$	83.71
B	$\lambda_2 = \begin{bmatrix} -0.006 & 0 \\ 0 & -0.006 \end{bmatrix}$	(28)	2	$F_{w2}(z) = \frac{0.8973 - 1.7721z^{-1} + 0.8748z^{-2}}{1 - 0.0616z^{-1} - 0.0019z^{-2}}$	$\gamma_2 = 2.4396$	88.10
C	$\lambda_3 = \begin{bmatrix} -0.35 & 0 \\ 0 & -0.35 \\ 0 & 0 \end{bmatrix}$	(29)	3	$F_{w3}(z) = \frac{0.5029 - 0.9947z^{-1} + 0.4918z^{-2}}{1 - 0.4525z^{-1} - 0.0110z^{-2}}$	$\gamma_3 = 3.1639$	92.46

as alluded to in Remark 2. As shown in Fig. 2, the Bode plots for these weights (27), (28), (29), and (30) are low-pass and have cut-off frequency around 0.0491 (rad/s).

Afterward, we suppose that the uncertainties in the gain and pole of the integrator $H_1(z)$ are within the ranges $0 \le \delta_a \le 0.01$, and $0 \le \delta_b \le 0.01$ [3, 4]. By mapping the uncertain parameters δ_a and δ_b to (A_1, B_1, C_1, D_1) of (10), these uncertain matrices can be described by a four-vertex polytope, i.e. m =4. By using Theorem 1 with m=4, and the weighting functions (28)-(30), three filters of the same order as that of the plant were derived. Table 1 shows the results and the searched parameters λ_i ($i = 1, 2, 3$) which were computed by the function *fminsearch* of MATLAB. As shown in Fig. 3, the design with $W_3(z)$ yields filter $F_{w3}(z)$ which performs better than that with $W_1(z)$ and $W_2(z)$ in terms of lower magnitudes for the NTF T at low-frequencies, especially for frequency interval [0.01,0.0491] (rad/s). This implies the lowest noise power in the signal band by using filter $F_{w3}(z)$. Specifically, using filter $F_{w3}(z)$ results in the best SNR value, 92.46 dB, when a –20 dBFS input signal is given.

From the above simulation results, we conclude that the proposed method can be used to design filters whose order is equal to the plant order, even if the order of the employed weighting functions is different.

B. Comparison with Existing Methods

We compare the performance of the proposed filter $F_{w3}(z)$ with the following filters:

Method A [5]: $F_t(z) = \frac{1 - 1.9952z^{-1} + 0.9941z^{-2}}{1 + 0.00028z^{-1} + 2.0912\times10^{-5} z^{-2}}$;

Method B [1]: $F_n(z) = 1 - 2z^{-1} + z^{-2}$;

Method C [3]: $F_c(z) = \frac{0.9963 - 1.967z^{-1} + 0.9709z^{-2}}{1 + 0.01815z^{-1} + 0.01194z^{-2}}$;

Method D [4]: $F_o(z) = 0.97465 - 1.9392z^{-1} + 0.9646z^{-2}$.

It should be noted that method A derived a full-order H-infinity filter $F_t(z)$ for problem (14) without using weighting functions.

The Bode plots of the NTF T matched by all filters are provided in Fig. 4. As expected, $F_{w3}(z)$ which employed a weighting function in the design produces lower magnitude than $F_t(z)$ does at low-frequencies. Similarly, $F_{w3}(z)$ outperforms the other filters, in terms of the lower magnitudes for the NTF T, in a large portion of the frequency interval [0,0.0491] (rad/s). With a –20 dBFS input signal, the resulting SNR values for the modulators matched with filters $F_t(z)$, $F_n(z)$, $F_c(z)$, $F_o(z)$, and $F_{w3}(z)$ are 63.25, 66.28, 80.04, 71.38, and 92.46 dB, respectively. In Fig. 5, it can be seen that the proposed filter $F_{w3}(z)$ achieves the best SNR performance for different input levels.

V. CONCLUSION

In this work, we have studied the synthesis problem of robust matching filters for uncertain 2-1 cascaded sigma-delta modulators. A new design method which involves minimizing the worst case H_∞ norm of a certain weighted matching error over linearized polytopic model has been presented. In particular, the method overcomes a limitation of the well known H-infinity loop shaping techniques in terms of filter order, i.e., the filter derived by the proposed method has order *independent* of the weighting function. This reduces the complexity of circuit implementation. The simulation results demonstrate the effectiveness of the proposed method. Finally, the proposed method is general which is applicable to the other cases.

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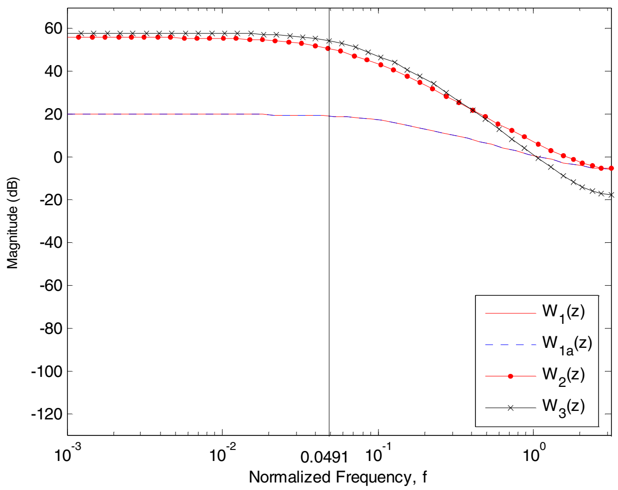

Figure 2. Bode plots of weighting functions.
plots of weighting functions. plots of weighting functions.

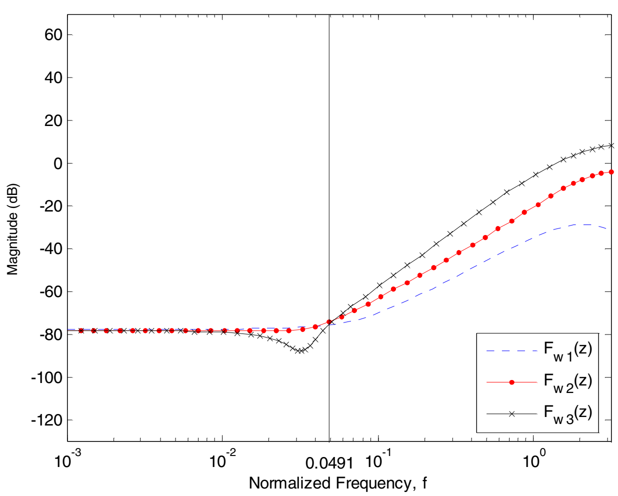

Figure 3. Bode plots of the uncertain NTF T matched by filters; parameter deviations δ_a=δ_b=0.01 in $H_i(z)$ (i=1,2,3).

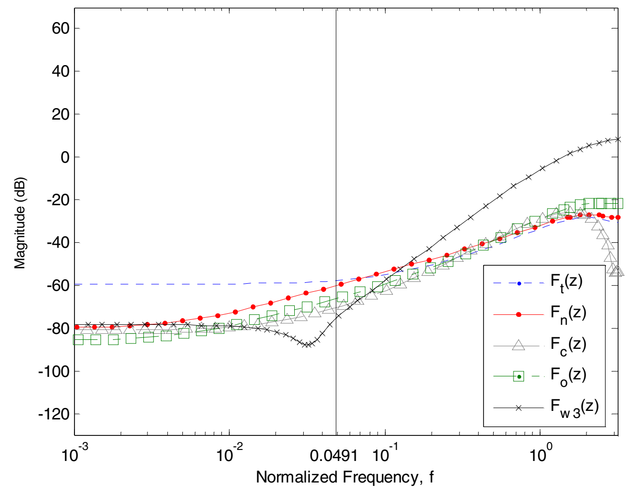

Figure 4. Bode plots of the uncertain NTF T matched by all filters; parameter deviations δ_a=δ_b=0.01 in $H_i(z)$ (i=1,2,3).

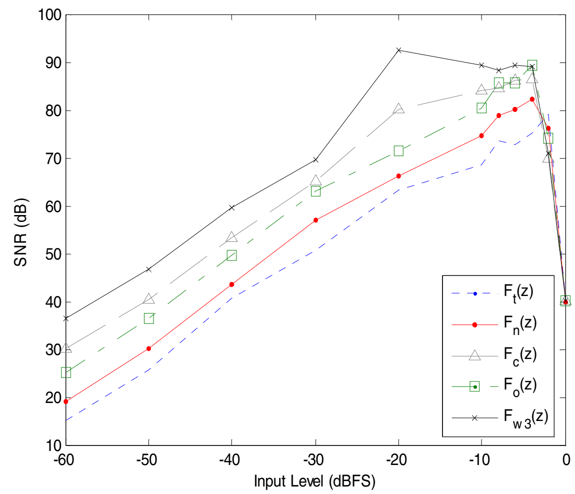

Figure 5. SNR performance vs. input amplitude; parameter deviations. ; parameter deviations δ_a=δ_b=0.01 in $H_i(z)$ (i=1,2,3).