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MULTIPLE COLORING OF CONE GRAPHS*

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Abstract. A k-fold coloring of a graph assigns to each vertex a set of k colors, and color sets assigned to adjacent vertices are disjoint. The kth chromatic number $\chi_k(G)$ of a graph G is the minimum total number of colors needed in a k-fold coloring of G. Given a graph $G = (V, E)$ and an integer $m \geq 0$, the m-cone of G, denoted by $\mu_m(G)$, has vertex set $(V \times \{0, 1, \ldots, m\}) \cup \{u\}$ in which u is adjacent to every vertex of $V \times \{m\}$, and $(x, i)(y, j)$ is an edge if $xy \in E$ and $i = j = 0$ or $xy \in E$ and $|i - j| = 1$. This paper studies the kth chromatic number of the cone graphs. An upper bound for $\chi_k(\mu_m(G))$ in terms of $\chi_k(G)$, k, and m are given. In particular, it is proved that for any graph G, if $m \geq 2k$, then $\chi_k(\mu_m(G)) \leq \chi_k(G) + 1$. We also find a surprising connection between the kth chromatic number of the cone graph of G and the circular chromatic number of G. It is proved that if $\chi_k(G)/k > \chi_c(G)$ and $\chi_k(G)$ is even, then for sufficiently large m, $\chi_k(\mu_m(G)) = \chi_k(G)$. In particular, if $\chi(G) > \chi_c(G)$ and $\chi(G)$ is even, then for sufficiently large m, $\chi(\mu_m(G)) = \chi(G)$.

Key words. multiple coloring, cone graphs, Mycielski graphs, fractional chromatic number, Kneser graphs

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1. Introduction.

Multiple coloring of graphs was introduced by Stahl in [14] and has been studied extensively in the literature. Suppose $n \geq k$ are positive integers. We denote by $[n]$ the set $\{0, 1, \ldots, n-1\}$ and denote by $\binom{[n]}{k}$ the family of all k-subsets of $[n]$. Given a graph G and positive integers k and n, a *k-fold n-coloring* of G is a mapping $f : V \to \binom{[n]}{k}$ such that for any edge xy of G, $f(x) \cap f(y) = \emptyset$. In other words, a k-fold coloring assigns to each vertex a set of k colors, and no color is assigned to two adjacent vertices. If all the colors are taken from a set of n colors, then it is a k-fold n-coloring. If $k = 1$, then each vertex is assigned one color, and adjacent vertices are assigned distinct colors. So a 1-fold n-coloring of G is simply an n-coloring of G. The *kth chromatic number* of G is defined as $\chi_k(G) = \min\{n : G$ has a k-fold n-coloring$\}$. As a 1-fold n-coloring is just an n-coloring, we have $\chi_1(G) = \chi(G)$. So $\chi_k(G)$ is a generalization of $\chi(G)$. It is well known [12] and easy to see that for any $k, k' \geq 1$, $\chi_{k+k'}(G) \leq \chi_k(G) + \chi_{k'}(G)$. This implies that $\chi_k(G) \leq k\chi(G)$. The *fractional chromatic number* of G is defined as

$$\chi_f(G) = \inf\left\{\frac{\chi_k(G)}{k} : k = 1, 2, \ldots,\right\}.$$

Thus $\chi_f(G) \leq \chi(G)$.

It is known that the gap $\chi(G) - \chi_f(G)$ can be arbitrarily large. However, as observed in [11], only a few types of graphs G are known to have arbitrarily large gaps between $\chi(G)$ and $\chi_f(G)$. One class of such graphs is the Kneser graphs. Given positive integers $n \geq k$, the *Kneser graph* $K(n, k)$ has vertex set $\binom{[n]}{k}$ in which $A \sim B$

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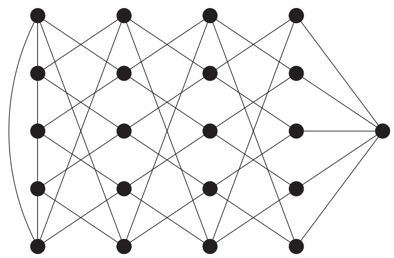

FIG. 1. *The graph* $\mu_3(C_5)$.

if and only if $A \cap B = \emptyset$. It is known [12] that for $n \geq 2k$, $\chi_f(K(n,k)) = n/k$. On the other hand, by the Lovász theorem [10], $\chi(K(n,k)) = n - 2k + 2$.

Another class of graphs G for which the gap $\chi(G) - \chi_f(G)$ is known to be arbitrarily large is obtained by cone constructions. Suppose $G = (V, E)$ is a graph and $m \geq 0$ is an integer. The *m-cone* $\mu_m(G)$ of G is the graph with vertex set $(V \times [m+1]) \cup \{u\}$ in which $(x,i)(y,j)$ is an edge if $xy \in E$ and $|i - j| = 1$ or $xy \in E$ and $i = j = 0$. Moreover, the vertex u is adjacent to (x, m) for all $x \in V$. If $m = 0$, then $\mu_m(G)$ is obtained from G by adding a universal vertex u. If $m = 1$, then the graph $\mu_m(G)$ is also called the *Mycielskian* of G and is denoted by $\mu(G)$. In the literature, the cone graph $\mu_m(G)$ for arbitrary integer $m \geq 0$ is also called the generalized Mycielskian of G. Figure 1 is the 3-cone of C_5.

The fractional chromatic number of $\mu_m(G)$ has been studied by Larsen, Propp and Ullman [6] and Tardif [15]. It is proved in [6] that for any graph G,

$$\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

This result is generalized in [15], where it is proved that for any graph G,

$$\chi_f(\mu_m(G)) = \chi_f(G) + \frac{1}{\sum_{i=0}^{m}(\chi_f(G) - 1)^i}.$$

We define $\mu_{m_1,m_2,\ldots,m_t}(G)$ recursively as $\mu_{m_1,m_2,\ldots,m_t}(G) = \mu_{m_1}(\mu_{m_2,m_3,\ldots,m_t}(G))$. Then by choosing the integers m_i to be sufficiently large, the formula above shows that $\chi_f(\mu_{m_1,m_2,\ldots,m_t}(K_2))$ is "close" to 2. On the other hand, it is known [2] that $\chi(\mu_{m_1,m_2,\ldots,m_t}(K_2)) = 2 + t$. So for $G = \mu_{m_1,m_2,\ldots,m_t}(K_2)$, the gap $\chi(G) - \chi_f(G)$ is close to t. Indeed, for any given graph H, it is known [13] that for $m_i \geq 1$ for $i \geq 1$, for $G_0 = H$ and $G_t = \mu_{m_t}(G_{t-1})$, $\lim_{t\to\infty}(\chi(G_t) - \chi_f(G_t)) = \infty$, although the exact value of the gap $\chi(G_t) - \chi_f(G_t)$ might be difficult to determine.

For both the Kneser graphs and the cone graphs, the determination of their chromatic numbers is nontrivial. The problem of determining the chromatic number of Kneser graphs remained an open problem for more than 20 years before it was settled by Lovász by an ingenious application of algebraic topology. The proof in [2] of the fact that $\chi(\mu_{m_1,m_2,\ldots,m_t}(K_2)) = 2 + t$ also uses the topological method.

As the kth chromatic number of graphs is a generalization of the chromatic number, it is natural to investigate the kth chromatic numbers of these graphs. The problem of determining the kth chromatic number of Kneser graphs, which is equivalent to the problem of determining existence of homomorphisms among Kneser graphs, has been studied in the literature [1, 4, 3, 14]. It is known that if $k = qk' - r$, where $q \geq 1$ and $0 \leq r < k'$ are integers, then $\chi_k(K(n', k')) \leq qn' - 2r$. It was conjectured in [14] that equality holds. The conjecture is verified for $r = 0$ or $q = 1$, but is open for other cases.

Multicoloring of Mysielskian of graphs are studied in [7, 8, 9]. In this paper, we are interested in the kth chromatic number of cone graphs. As $\mu_m(G)$ contains G as a subgraph, we have $\chi_k(\mu_m(G)) \geq \chi_k(G)$. On the other hand, given a k-fold coloring f of G, we obtain a k-fold coloring of $\mu_m(G)$ by letting $g(x,i) = f(x)$ for $x \in V(G)$ and for $i = 0, 1, \ldots, m$, and letting $g(u)$ be a set of k new colors. So for any graph G, for any $m \geq 0$, for any $k \geq 1$,

$$\chi_k(G) \leq \chi_k(\mu_m(G)) \leq \chi_k(G) + k.$$

Both the upper and lower bounds can be attained. Indeed, for any graph G, for any integer $k \geq 1$, $\chi_k(\mu_0(G)) = \chi_k(G) + k$, as $\mu_0(G) - u$ is a copy of G, and the k colors assigned to the vertex u cannot be assigned to any other vertex of $\mu_0(G)$. However, if m is large, then the upper bound can be improved. We shall prove an upper bound for $\chi_k(\mu_m(G))$ in terms of $\chi_k(G)$, k, and m. In particular, the upper bound implies that if $m \geq 2k$, then $\chi_k(\mu_m(G)) \leq \chi_k(G)+1$. This upper bound is the best in general because if $\chi_k(G) = k\chi_f(G)$, then it follows from the above-mentioned result of Tardif that for any m, $\chi_k(\mu_m(G)) \geq k\chi_f(\mu_m(G)) > k\chi_f(G) = \chi_k(G)$. However, for some graphs G and integers k, the upper bound can be further improved. Surprisingly, the kth chromatic number of $\mu_m(G)$ is related to the circular chromatic number of G. We prove that if $\chi_k(G)/k > \chi_c(G)$ and $\chi_k(G)$ is even, then for sufficiently large m, we have $\chi_k(\mu_m(G)) = \chi_k(G)$. In particular, if $\chi(G) > \chi_c(G)$ and $\chi(G)$ is even, then for sufficiently large m, $\chi(\mu_m(G)) = \chi(G)$. We also prove that if $\chi_c(G) = p/q$ and p is odd, then for any $k \geq q$, if $\chi_k(G)/k > \chi_c(G)$, then for sufficiently large m, we have $\chi_k(\mu_m(G)) = \chi_k(G)$.

2. An upper bound for $\chi_k(\mu_m(G))$. The m-cone of a graph G can be equivalently defined as follows: Given two graphs $G = (V,E)$ and $G' = (V',E')$, the *categorical product* $G \times G'$ of G and G' has vertex set $V \times V'$ in which $(x,x')(y,y')$ is an edge if $xy \in E$ and $x'y' \in E'$. We denote by P_m the path of length m with vertex set $\{0, 1, \ldots, m\}$, and we denote by P_m^* the graph obtained from P_m by adding a loop on vertex 0. Then $\mu_m(G)$ is the graph obtained from $G \times P_m^*$ by adding a vertex u which is adjacent to every vertex of $V \times \{m\}$. We call a k-fold n-coloring of $G \times P_m^*$ an *extendable k-fold n-coloring* if the restriction of f to $V \times \{m\}$ uses at most $n-k$ colors. The following lemma follows easily from the definitions.

LEMMA 1. *For any graph $G = (V,E)$ and integers k and m, $\chi_k(\mu_m(G)) \leq n$ if and only if there is an extendable k-fold n-coloring of $G \times P_m^*$.*

LEMMA 2. *For any graph G, for any integers $m \geq 0, k \geq 1$,*

$$\chi_k(\mu_{m+1}(G)) \leq \chi_k(\mu_m(G)).$$

Proof. Assume $\chi_k(\mu_m(G)) = n$. By Lemma 1, there is an extendable k-fold n-coloring f of $G \times P_m^*$. Let B be a k-subset of $\{0, 1, \ldots, n-1\}$ of colors not used by $V(G) \times \{m\}$. Then the mapping ϕ defined as $\phi(x,i) = f(x,i)$ for $i = 0, 1, \ldots, m$ and $\phi(x, m+1) = B$ is an extendable k-fold n-coloring of $G \times P_{m+1}^*$. □

The following theorem gives an upper bound for $\chi_k(\mu_m(G))$ in terms of $\chi_k(G), k$, and m.

THEOREM 3. *For any finite graph G, for any positive integers m, k, s, if $m \geq 2\lfloor k/s \rfloor$, then $\chi_k(\mu_m(G)) \leq \chi_k(G) + s$.*

Proof. Assume $\chi_k(G) = n$. Let ϕ be a k-fold n-coloring of G, using color set $[n] = \{0, 1, \ldots, n-1\}$. We shall construct an extendable k-fold $(n+s)$-coloring ψ of $G \times P_m^*$ using color set $\{c_1, c_2, \ldots, c_s, 0, 1, \ldots, n-1\}$.

Subsets of $\{0, 1, \ldots, n-1\}$ will be denoted by capital letters $A, B, \ldots$. The elements of a k-element set A will be denoted as $A = \{a_1, a_2, \ldots, a_k\}$ in such a way that $a_1 < a_2 < \cdots < a_k$, and elements of a k-element set B will be denoted as $B = \{b_1, b_2, \ldots, b_k\}$ with $b_1 < b_2 \cdots < b_k$, etc. The complement $[n] \setminus A$ of A is denoted by $\overline{A} = \{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{n-k}\}$, where $\overline{a}_1 < \overline{a}_2 < \cdots < \overline{a}_{n-k}$.

Assume $k = qs + r$, where $0 \le r \le s - 1$. By Lemma 2, it suffices to consider the case where $m = 2\lfloor k/s \rfloor = 2q$. We construct an extendable k-fold coloring of $G \times P^*_{2q}$ as follows: Suppose $x \in V$ and $\phi(x) = A = \{a_1, a_2, \ldots, a_k\}$. Then $\psi(x, 0) = \phi(x)$. For $1 \le i \le q$, let

$$\psi(x, 2i-1) = \{c_1, c_2, \ldots, c_s, 0, 1, \ldots, (i-1)s - 1\} \cup \{a_{is+1}, a_{is+2}, \ldots, a_k\}$$

and

$$\psi(x, 2i) = \{a_{t+1}, a_{t+2}, \ldots, a_k\} \cup \{\overline{a}_{is-t+1}, \overline{a}_{is-t+2}, \ldots, \overline{a}_{is}\},$$

where $t = |A \cap \{0, 1, \ldots, is - 1\}|$.

Now we shall prove that ψ is indeed an extendable k-fold $(n+s)$-coloring of $G \times P^*_m$. It follows easily from the definition that for any $x \in V$ and for any $0 \le i \le m$, $\psi(x, i)$ is a k-subset of $[n] \cup \{c_1, c_2, \ldots, c_s\}$. Assume $(x, j) \sim_{G \times P^*_m} (y, j')$. Then either $j = j' = 0$ and $x \sim_G y$, or $j' = j + 1$ and $x \sim_G y$. We need to show that $\psi(x, j) \cap \psi(y, j') = \emptyset$. Assume $\phi(x) = A$ and $\phi(y) = B$. As ϕ is a k-fold coloring of G, $A \cap B = \emptyset$. If $j = j' = 0$, then $\psi(x, j) = A$ and $\psi(y, j') = B$, and we are done. Assume $j' = j + 1$.

If $j = 2i$ is even, then

$$\psi(x, 2i) = \{a_{t+1}, a_{t+2}, \ldots, a_k\} \cup \{\overline{a}_{is-t+1}, \overline{a}_{is-t+2}, \ldots, \overline{a}_{is}\},$$

where $|A \cap \{0, 1, \ldots, is - 1\}| = t$ and

$$\psi(y, 2i+1) = \{c_1, c_2, \ldots c_s, 0, 1, \ldots, is - 1\} \cup \{b_{(i+1)s+1}, b_{(i+1)s+2}, \ldots, b_k\}.$$

To prove that $\psi(x, 2i) \cap \psi(y, 2i+1) = \emptyset$, it suffices to show that

$$\begin{aligned}
\{a_{t+1}, a_{t+2}, \ldots, a_k\} \cap \{b_{(i+1)s+1}, b_{(i+1)s+2}, \ldots, b_k\} &= \emptyset, \\
\{a_{t+1}, a_{t+2}, \ldots, a_k\} \cap \{c_1, c_2, \ldots c_s, 0, 1, \ldots, is - 1\} &= \emptyset, \\
\{\overline{a}_{is-t+1}, \overline{a}_{is-t+2}, \ldots, \overline{a}_{is}\} \cap \{b_{(i+1)s+1}, b_{(i+1)s+2}, \ldots, b_k\} &= \emptyset, \\
\{\overline{a}_{is-t+1}, \overline{a}_{is-t+2}, \ldots, \overline{a}_{is}\} \cap \{c_1, c_2, \ldots, c_s, 0, 1, \ldots, is - 1\} &= \emptyset.
\end{aligned}$$

The first equality holds because $A \cap B = \emptyset$. The second equality holds because $|A \cap \{0, 1, \ldots, is - 1\}| = t$, which implies that $a_{t+1} \ge is$. The third equality holds because $B \subseteq \overline{A}$, which implies that $\overline{a}_{is} \le b_{is} < b_{(i+1)s+1}$. The fourth equality holds because $\overline{A} \cap \{0, 1, \ldots, is - 1\}| = is - t$, which implies that $\overline{a}_{is-t+1} \ge is$.

Assume $j = 2i - 1$ is odd. Then

$$\psi(x, 2i-1) = \{c_1, c_2, \ldots, c_s, 0, 1, \ldots, (i-1)s - 1\} \cup \{a_{is+1}, a_{is+2}, \ldots, a_k\}$$

and

$$\psi(y, 2i) = \{b_{t+1}, b_{t+2}, \ldots, b_k\} \cup \{\overline{b}_{is-t+1}, \overline{b}_{is-t+2}, \ldots, \overline{b}_{is}\},$$

where $t = |B \cap \{0, 1, \ldots, is - 1\}|$.

Again, since $|B \cap \{0, 1, \ldots, is - 1\}| = t$, we have $\overline{B} \cap \{0, 1, \ldots, is - 1\}| = is - t$. Therefore $b_{t+1} \ge is$ and $\overline{b}_{is-t+1} \ge is$. Since $A \subseteq \overline{B}$, we have $\overline{b}_{is} \le a_{is} < a_{is+1}$. Similarly as above, this implies that $\psi(x, 2i-1) \cap \psi(y, 2i) = \emptyset$.

Moreover, the colors in the set $\{c_1, c_2, \ldots, c_s\} \cup \{0, 1, \ldots, qs-1\}$ are not used in $V \times \{2q\}$. So ψ is indeed an extendable k-fold $(n+s)$-coloring of $G \times P_m^*$, and hence $\chi_k(\mu_m(G)) \le n+s$. ∎

Corollary 4. *For any integer $k \ge 1$ and for any graph $G = (V, E)$, if $m \ge 2k$, then $\chi_k(\mu_m(G)) \le \chi_k(G) + 1$.*

For the Mycielskian $\mu(G)$ of a graph G, the upper bound given in Theorem 3 is not better than the trivial bound $\chi_k(\mu(G)) \le \chi_k(G) + k$. The following theorem shows that if $\chi_k(G) \le 3k-2$, then we do have a better bound.

Theorem 5. *Suppose $k \ge 2$ and G is a graph with $\chi_k(G) = n \le 3k-2$. Then $\chi_k(\mu(G)) \le n + k - 1$.*

Proof. Let ϕ be a k-fold n-coloring of G, using colors $\{0, 1, \ldots, n-1\}$. We shall construct a k-fold $(n+k-1)$-coloring f of $\mu(G)$. Suppose $x \in V(G)$ and $\phi(x) = \{a_1, a_2, \ldots, a_k\} \in \binom{[n]}{k}$, where $a_1 < a_2 < \cdots < a_k$. Let

$$f(x,0) = \begin{cases} \phi(x) & \text{if } a_1 = 0, \\ \{a_2, a_3, \ldots, a_k, n + a_1 - 1\} & \text{if } 1 \le a_1 \le n-2k, \\ \{a_2, a_3, \ldots, a_k, n+k-2\} & \text{if } a_1 \ge n-2k+1, \end{cases}$$

$$f(x,1) = \begin{cases} \phi(x) & \text{if } 0 \notin \phi(x), \\ (\phi(x) \setminus \{0\}) \cup \{\overline{a}_1\} & \text{otherwise,} \end{cases}$$

$$f(u) = \{0, n, n+1, \ldots, n+k-2\}.$$

The argument similar to the proof of Theorem 3 shows that f is a k-fold $(n+k-1)$-coloring of $\mu(G)$. We omit the details. ∎

It was conjectured in [8] that if $n \ge 3k-1$, then there is a graph G with $\chi_k(G) = n$ and $\chi_k(\mu(G)) = n+k$. The conjectured is confirmed when $k = 2, 3$, or when n is a multiple of k or $n \ge 3k^2/\ln k$.

3. Graphs with $\chi_k(\mu_m(G)) = \chi_k(G)$. The upper bound $\chi_k(\mu_m(G)) \le \chi_k(G) + 1$ for sufficiently large m is tight in general. As observed in the introduction, if $\chi_k(G) = k\chi_f(G)$, then for any integer m, $\chi_k(\mu_m(G)) \ge k\chi_f(\mu_m(G)) > k\chi_f(G) = \chi_k(G)$, and hence $\chi_k(\mu_m(G)) \ge \chi_k(G) + 1$. It is natural to ask if $\chi_k(G) > k\chi_f(G)$, then is it true that for sufficiently large m, $\chi_k(\mu_m(G)) = \chi_k(G)$? The answer is no in general. However, for some graphs G, we do have a positive answer. In this section, we present a sufficient condition under which a graph G has $\chi_k(\mu_m(G)) = \chi_k(G)$ for sufficiently large m.

Suppose $p \ge 2q$ are positive integers. The graph $K_{p/q}$ has vertex set $\{0, 1, \ldots, p-1\}$, in which $i \sim j$ if $q \le |i-j| \le p-q$. The graphs $K_{p/q}$ are called *circular complete graphs* and are used to define the circular chromatic number of graphs. A homomorphism f from a graph G to $K_{p/q}$ is also called a (p,q)-*coloring* of G. The *circular chromatic number* of a graph G is defined as

$$\chi_c(G) = \inf\{p/q : G \text{ has a } (p,q)\text{-coloring}\}.$$

It is known [16] that for any graph G, $\chi_f(G) \le \chi_c(G) \le \chi(G)$ and $\chi_c(G) > \chi(G) - 1$. For circular complete graphs $K_{p/q}$, we have $\chi_f(K_{p/q}) = \chi_c(K_{p/q}) = p/q$ [16].

We view the set $\{0, 1, \ldots, p-1\}$ as a circle, and we denote it by $S(p)$. For $a, b \in S(p)$, $[a,b]_p = \{[a+i]_p : 0 \le i \le [b-a]_p\}$ denotes the interval of the circle $S(p)$ from a to b along the increasing direction. For example, $[2,5]_8 = \{2,3,4,5\}$ and $[5,2]_8 = \{5,6,7,0,1,2\}$. The *length* of an interval A is denoted by $\ell(A)$, which is defined to be the number of elements in A. It is easy to verify that the interval

$[a,b]_p$ has length $\ell([a,b]_p) = [b-a]_p + 1$. We say two intervals $[a,b]_p$ and $[c,d]_p$ are *consecutive* if $[b+1]_p = [c]_p$.

LEMMA 6. *For any $p/q \geq 2$ and for any integer $k \geq 1$,*

$$\chi_k(K_{p/q}) = \min\{n : nq \geq pk\}.$$

Proof. Suppose $\chi_k(K_{p/q}) = n$. Since $n/k = \chi_k(K_{p/q})/k \geq \chi_f(K_{p/q}) = p/q$, we have $nq \geq pk$. It remains to prove that if $nq \geq pk$, then $\chi_k(K_{p/q}) \leq n$.

To prove that $\chi_k(K_{p/q}) \leq n$, it suffices to construct a family of independent sets $I_0, I_1, \ldots, I_{n-1}$ of $K_{p/q}$ such that each vertex of G is contained in k of these independent sets. Indeed, if such a family of independent sets are found, then we assign color j to all the vertices in I_j. As each vertex is contained in k of the I_j's, each vertex receives k colors. Hence we obtain a k-fold n-coloring of $K_{p/q}$. Let $t^* = nq - pk \geq 0$, and let

$$I_0 = [0, q - t^* - 1]_p, \ I_j = [jq - t^*, (j+1)q - t^* - 1]_p, \text{ for } j = 1, 2, \ldots, n-1.$$

Each I_j is an interval of length at most q. Hence each I_j is an independent set. Moreover, I_j and I_{j+1} are consecutive intervals for $j = 0, 1, \ldots, n-1$ (where summation in the indices are modulo n). As

$$\sum_{j=0}^{n-1} \ell(I_j) = kp,$$

so the union of the intervals $I_0, I_1, \ldots, I_{n-1}$ winds around the circle $S(p)$ exactly k times. Thus each vertex of $K_{p/q}$ is contained in exactly k of the independent sets I_j. So $\chi_k(K_{p/q}) \leq n$. ∎

In the following, we consider k-fold coloring of the cone graph $\mu_{m'}(K_{p/q})$ when m' is large.

In the remaining part of this section, we assume n, k, p, q are positive integers, $n = 2s$ is even, and $n/k > p/q$. We shall construct an extendable k-fold n-coloring of $K_{p/q} \times P^*_m$, where m' is sufficiently large. The proof is complicated. However, the idea of the coloring is simple. For each $0 \leq i \leq m'$, let $V_i = V(K_{p/q}) \times \{i\} = S(p) \times \{i\}$. The vertices in $S(p) \times \{i\}$ are viewed to form a circle. We color $V_0, V_1, \ldots, V_{m'}$ in order.

Each V_i will be colored as follows: Find a family of intervals I^i_j of the circle $S(p)$ so that interval I^i_j is followed by interval I^i_{j+1} and the union $\cup_{j=0}^{n-1} I^0_j$ covers $S(p)$ exactly k times. We assign color j to all the vertices of $I^i_j \times \{i\}$. Because $\cup_{j=0}^{n-1} I^0_j$ covers $S(p)$ exactly k times, each vertex of V_i receives k colors. To make sure that no color is assigned to two adjacent vertices, the intervals V^i_j need to satisfy the following conditions:

(A) Each I^0_j is an independent set of $K_{p/q}$.

(B) If $a \in I^i_j$ and $b \in I^{i+1}_j$, then a and b are not adjacent in $K_{p/q}$, i.e., $|a-b|_p \leq q-1$.

To make the k-fold coloring of $K_{p/q} \times P^*_{m'}$ extendable, we need to have k colors, say, $j_1, j_2, \ldots, j_k$, not used by any vertex of $V_{m'}$, i.e., $I^{m'}_{j_s} = \emptyset$ for $s = 1, 2, \ldots, k$.

The intervals $I^0_0, I^0_1, \ldots, I^0_{n-1}$ are chosen in the same way as in the proof of Lemma 6.

Note that each interval I^0_j has length q, except that I^0_0 has length $q - t^*$, where $t^* = nq - pk \geq 1$. This allows us to choose I^1_0 to be $[0 - t^*, q-1]_p$ so that condition

(B) above is still satisfied. The interval I_0^1 has length $q + t^*$. Now we can choose I_1^1 and I_{n-1}^1 to be of length $q - t^*$ and choose $I_j^1 = I_j^0$ for $j \neq 0, 1, n-1$, so that conditions (A) and (B) are satisfied.

Then we color $S(p) \times \{2\}$. The intervals I_j^2 will be chosen so that $I_0^2 = I_0^0$, I_1^2 and I_{n-1}^2 will be of length $q + t^*$, I_2^2 and I_{n-2}^2 will be of length $q - t^*$, and I_j^2 for other j will be of length q.

We say an interval of length q is a *normal interval*; other intervals are either *short intervals* or *long intervals*. So V_0 has only one short interval, and all other intervals are normal, V_1 has two short intervals and one long interval, V_2 has two long intervals and two short intervals, and so on. As i increases, the part occupied by normal intervals will become smaller and smaller, and the part occupied by short and long intervals expands. As i increases, two short intervals of V_i move toward the "center." Eventually, there will be two shorter intervals that meet each other and will create an even shorter interval. As i continue to increase, we will have shorter and shorter intervals, and eventually one interval will disappear; i.e., one color will not be used by vertices of V_i for some i. We continue this process. As i increases, there will be more colors not used by vertices of V_i. When m' is large enough, the coloring of $K_{p/q} \times P_{m'}^*$ will be become extendable; i.e., there are k colors not used by vertices of $V_{m'}$.

To write a detailed proof that realizes the simple idea above, we need to keep track of all the intervals. The complication arises from such bookkeeping.

Let $t^* = nq - kp \geq 1$, $s^* = \lceil q/t^* \rceil$, and $m = (s^* - 1)s$. Let $\delta_{i,j}$ be integers defined as follows: For $i \geq 0$, we write i in the form $i = zs + i'$, where z, i' are integers with $0 \leq i' \leq s - 1$. We partition pairs (i, j) into two sets: $A = \{(i, j) : z \text{ is even and } |j|_n > i' \text{ or } z \text{ is odd and } |j|_n < s - i'\}$ and $B = \{(i, j) : z \text{ is even and } |j|_n \leq i' \text{ or } z \text{ is odd and } |j|_n \geq s - i'\}$. Then

$$\delta_{i,j} = \begin{cases} (-1)^{i+j+1} z & \text{if } (i,j) \in A, \\ (-1)^{i+j+1}(z+1) & \text{if } (i,j) \in B. \end{cases} \tag{1}$$

LEMMA 7. *The integers $\delta_{i,j}$ defined above satisfy the following equations:*

$$\begin{aligned} &\delta_{0,0} = -1, \\ &\delta_{0,j} = 0 \text{ for } j = 1, 2, \ldots, n-1, \\ &\delta_{i,j} = \delta_{i-1,[j-1]_n} + \delta_{i-1,j} + \delta_{i-1,[j+1]_n} \text{ if } i \geq 1 \text{ and } i + j \text{ is even}, \\ &\delta_{i,j} = -\delta_{i-1,j} \text{ if } i \geq 1 \text{ and } i + j \text{ is odd}. \end{aligned}$$

Proof. The proof is a straightforward but tedious case-by-case check. If $i = 0$, then $z = 0$ and $i' = 0$. It follows from the definition that $\delta_{0,0} = -1$ and $\delta_{0,j} = 0$ for $j = 1, 2, \ldots, n-1$.

Assume $i = zs + i' \geq 1$. We need to divide the argument into the following cases: (1) z is even and $1 \leq i' \leq s-1$. (2) z is even and $i' = 0$. (3) z is odd and $1 \leq i' \leq s-1$. (4) z is odd and $i' = 0$. All the checking are similar. We give only the details for case 1.

Assume z is even and $1 \leq i' \leq s - 1$. If $i + j$ is odd, then since z is even, we conclude that $i' + j$ is odd. Hence $[j]_n \neq i'$. If $|j|_n < i'$, then $\delta_{i,j} = z + 1 = -\delta_{i-1,j}$. If $|j|_n > i'$, then $\delta_{i,j} = z = -\delta_{i-1,j}$.

If $i + j$ is even, then $i' + j$ is even. Hence $[j]_n \neq i' - 1$. If $|j|_n > i'$, then $\delta_{i,j} = -z$. Since $|j|_n, |j-1|_n, |j+1|_n > i' - 1$, we have $\delta_{i-1,j-1} = \delta_{i-1,j+1} = -z$ and $\delta_{i-1,j} = z$. Hence $\delta_{i,j} = \delta_{i-1,j-1} + \delta_{i-1,j} + \delta_{i-1,j+1}$. If $|j|_n \leq i' - 2$, then similarly we have

$$\delta_{i,j} = -(z+1) = -(z+1) + (z+1) - (z+1) = \delta_{i-1,j-1} + \delta_{i-1,j} + \delta_{i-1,j+1}.$$

If $|j|_n = i'$, then $\delta_{i,j} = -(z+1) = \delta_{i-1,j-1}$, $\delta_{i-1,j} = z$, and $\delta_{i-1,j+1} = -z$. So again we have $\delta_{i,j} = \delta_{i-1,j-1} + \delta_{i-1,j} + \delta_{i-1,j+1}$. ∎

The following is the matrix $[\delta_{i,j}]_{t\times n}$ for $n = 2s = 6$ and $t = 9$ (note that our indices starts with 0, so the first entry is $\delta_{0,0}$):

$$\begin{pmatrix} \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 1 & \mathbf{-2} & 1 & -1 \\ -1 & 1 & -2 & 2 & -2 & 1 \\ 1 & -2 & 2 & -2 & 2 & -2 \\ \mathbf{-3} & 2 & -2 & 2 & -2 & 2 \\ 3 & -3 & 2 & -2 & 2 & -3 \\ -3 & 3 & -3 & 2 & -3 & 3 \\ 3 & -3 & 3 & \mathbf{-4} & 3 & -3 \end{pmatrix}.$$

Note that the absolute values of the numbers $\delta_{i,j}$ becomes larger and larger when i increases. The boldfaced numbers are the "first" places where the numbers $\delta_{i,j}$ have absolute values $1, 2, 3, 4$, respectively.

COROLLARY 8. *For any i, j, $\delta_{i,j} + \delta_{i,j+1} \in \{0, -1\}$ and for any i, $\sum_{j=0}^{n-1} \delta_{i,j} = -1$.*

Let I_j^i ($i = 0, 1, \ldots, m-1, j = 0, 1, \ldots, n-1$) be intervals of $S(p)$ defined as follows:

$$I_0^0 = [0, q - t^* - 1]_p,$$
$$I_j^0 = [jq - t^*, (j+1)q - t^* - 1]_p \text{ for } j = 1, 2, \ldots, n-1.$$

Assume $I_j^{i-1} = [a, b]_p$ has been constructed. Then let

$$I_j^i = \begin{cases} [a - \delta_{i-1,j-1}t^*, b + \delta_{i-1,j+1}t^*]_p & \text{if } i+j \text{ is even,} \\ [a + \delta_{i-1,j}t^*, b - \delta_{i-1,j}t^*]_p & \text{if } i+j \text{ is odd.} \end{cases}$$

Here the calculation in the second indices are modulo n.

LEMMA 9. *For $i = 0, 1, \ldots, m-1, j = 0, 1, \ldots, n-1$, we have*

$$\ell(I_j^i) = q + \delta_{i,j}t^*.$$

Proof. We prove this lemma by induction on i. The lemma is obviously true for $i = 0$. Assume $i \geq 1$ and the lemma holds for $i-1$. It follows from the definition that

$$\ell(I_j^i) = \begin{cases} \ell(I_j^{i-1}) + \delta_{i-1,j-1}t^* + \delta_{i-1,j+1}t^* & \text{if } i+j \text{ is even,} \\ \ell(I_j^{i-1}) - 2\delta_{i-1,j}t^* & \text{if } i+j \text{ is odd.} \end{cases}$$

If $i+j$ is even, then $\ell(I_j^i) = q + \delta_{i-1,j}t^* + \delta_{i-1,j-1}t^* + \delta_{i-1,j+1}t^* = q + \delta_{i,j}t^*$. If $i+j$ is odd, then $\ell(I_j^i) = q + \delta_{i-1,j}t^* - 2\delta_{i-1,j}t^* = q + \delta_{i,j}t^*$. ∎

Observe that it follows from Lemma 7 that for $i \leq m-1 < (s^*-1)s$, $\delta_{i,j}t^* < q$, and hence $\ell(I_j^i) > 0$.

LEMMA 10. *For $i = 0, 1, \ldots, m-1$, for $j = 0, 1, \ldots, n-1$, the intervals I_j^i, I_{j+1}^i are consecutive intervals, and the union $\cup_{j=0}^{n-1} I_j^i$ winds around $S(p)$ exactly k times.*

Proof. For $i = 0$, this is trivial (and is shown in the proof of Lemma 6). Assume the lemma holds for $i-1$. Assume $I_j^{i-1} = [a,b]_p$ and $I_{j+1}^{i-1} = [b+1,c]_p$. If $i+j$ is even, then

$$I_j^i = [a - \delta_{i-1,j-1}t^*, b + \delta_{i-1,j+1}t^*]_p$$

and

$$I_{j+1}^i = [b + 1 + \delta_{i-1,j+1}t^*, c - \delta_{i-1,j+1}t^*]_p.$$

If $i+j$ is odd, then

$$I_j^i = [a + \delta_{i-1,j}t^*, b - \delta_{i-1,j}t^*]_p$$

and

$$I_{j+1}^i = [b + 1 - \delta_{i-1,j}t^*, c + \delta_{i-1,j+2}t^*]_p.$$

In any case, they are consecutive intervals. Since

$$\sum_{j=0}^{n-1} \ell(I_j^i) = \sum_{j=0}^{n-1}(q + \delta_{i,j}t^*),$$

by Lemma 8, we have

$$\sum_{j=0}^{n-1} \ell(I_j^i) = kp.$$

Thus the union $\cup_{j=0}^{n-1} I_j^i$ winds around $S(p)$ exactly k times. □

LEMMA 11. *Assume $i \in \{0,1,\dots,m-2\}$ and $j \in \{0,1,\dots,n-1\}$. If $x \in I_j^{i-1}$ and $y \in I_j^i$, then x and y are not adjacent in $K_{p/q}$.*

Proof. First we consider the case where $i+j$ is even. Assume $I_j^{i-1} = [a,b]$. Then $I_j^i = [a - \delta_{i-1,j-1}t^*, b + \delta_{i-1,j+1}t^*]_p$. For any $x \in I_j^{i-1}$ and $y \in I_j^i$, we have

$$[y - x]_p \le \max\{[b + \delta_{i-1,j+1}t^* - a]_p, [b - a + \delta_{i-1,j-1}t^*]_p\}.$$

As $[a,b]_p$ has length $q + \delta_{i-1,j}t^*$, we have

$$[b - a]_p = q + \delta_{i-1,j}t^* - 1.$$

Thus

$$[y - x]_p \le \max\{[q + \delta_{i-1,j}t^* + \delta_{i-1,j+1}t^*]_p - 1, [q + \delta_{i-1,j}t^* + \delta_{i-1,j-1}t^*]_p - 1\}.$$

By Corollary 8, we have $[y-x]_p \le q-1$. Similarly $[x-y]_p \le q-1$. So $|x-y|_p \le q-1$.

The case $i+j$ is odd can be proved similarly (and is easier). We omit the details. □

COROLLARY 12. *Suppose $n/k > p/q$, $n = 2s$ is even. Let $t^* = nq - kp \ge 1$, and let $s^* = \lceil q/t^* \rceil$ and $m = (s^*-1)s$. Let I_j^i be the intervals of $S(p)$ defined as above. Then the mapping $\phi : V(K_{p/q} \times P_{m-1}^*) \to \binom{[n]}{k}$ defined as $j \in \phi(x,i)$ if and only if $x \in I_j^i$ is a k-fold coloring of $K_{p/q} \times P_{m-1}^*$.*

Proof. It follows from Lemma 10 that $\phi(x,i)$ contains exactly k colors, and it follows from Lemma 11 that each color class is an independent set. □

In the following, we shall extend the coloring ϕ to an extendable k-fold n-coloring of $K_{p/q} \times P^*_{m+k-1}$ and hence prove that $\chi_k(\mu_{m+k-1}(K_{p/q})) \leq n$.

LEMMA 13. *For the intervals I^i_j defined as above, there is an index j such that $I^{m-1}_{j-1}, I^{m-1}_j, I^{m-1}_{j+1}$ have lengths $q-(s^*-1)t^*, q+(s^*-2)t^*, q-(s^*-1)t^*$, respectively. (The calculation in the indices are modulo n.)*

Proof. By Lemma 9, $\ell(I^i_j) = q + \delta_{i,j}t^*$. Thus Lemma 13 is equivalent to the statement that for some j, $\delta_{m-1,j-1} = \delta_{m-1,j+1} = -(s^*-1)$ and $\delta_{m-1,j} = s^*-2$. This follows from Lemma 7. Indeed, if s^* is odd, then it follows from Lemma 7 that $\delta_{m-1,n-1} = \delta_{m-1,1} = -(s^*-1)$ and $\delta_{m-1,0} = s^*-2$. If s^* is even, then $\delta_{m-1,s-1} = \delta_{m-1,s+1} = -(s^*-1)$ and $\delta_{m-1,s} = s^*-2$. ∎

LEMMA 14. *Suppose $n/k > p/q \geq 2$, $n = 2s$ is even. Let t^*, s^*, and m be integers defined as above. Then for $m' \geq m+k-1$,*

$$\chi_k(\mu_{m'}(K_{p/q})) \leq n.$$

Proof. For $i = 0, 1, \ldots, m+k-1$, let $V_i = V(K_{p/q}) \times \{i\}$. The mapping ϕ defined in Corollary 12 colors the vertices of $V_0, V_1, \ldots, V_{m-1}$. Each of the sets V_i for $i = 0, 1, \ldots, m-1$ uses all the colors. We shall extend ϕ to $V_m, V_{m+1}, \ldots, V_{m+k-1}$. For convenience, for $i \geq m+1$ and $x \in V(K_{p/q})$, we allow $\phi(x,i)$ to contain more than k colors. Of course, color sets assigned to adjacent vertices must be disjoint.

By Lemma 13, there is an index j^* such that $\ell(I^{m-1}_{j^*-1}) = \ell(I^{m-1}_{j^*+1}) = q-(s^*-1)t^*$ and $\ell(I^{m-1}_{j^*}) = q+(s^*-2)t^*$.

Assume

$$\begin{aligned} I^{m-1}_{j^*-1} &= [a, a+q-(s^*-1)t^*-1]_p, \\ I^{m-1}_{j^*} &= [a+q-(s^*-1)t^*, a+2q-t^*-1]_p, \\ I^{m-1}_{j^*+1} &= [a+2q-t^*, a+3q-s^*t^*-1]_p. \end{aligned}$$

If $I^{m-2}_{j^*-1} = [b, b']_p$ and $I^{m-2}_{j^*+1} = [c, c']_p$, then let

$$I^m_j = \begin{cases} [b, a+q-1]_p & \text{if } j = j^*-1, \\ \emptyset & \text{if } j = j^*, \\ [a+q, c']_p & \text{if } j = j^*+1, \\ I^{m-2}_j & \text{otherwise.} \end{cases}$$

A straightforward calculation shows for any $x \in I^m_{j^*-1} \setminus I^{m-2}_{j^*-1}$ and any $y \in I^{m-1}_{j^*-1}$, $|x-y|_p \geq q$. Similarly, for any $x \in I^m_{j^*+1} \setminus I^{m-2}_{j^*+1}$ and any $y \in I^{m-1}_{j^*+1}$, $|x-y|_p \geq q$. Together with Lemma 11, this implies that for any j, if $x \in I^{m-1}_j$ and $y \in I^m_j$, then x and y are not adjacent in $K_{p/q}$.

Now by letting $j \in \phi(x,m)$ if $x \in I^m_j$, we extend the k-fold coloring ϕ to the subset V_m. Note that the color j^* is not assigned to any vertex of V_m.

Let $I^{m+1}_{j^*} = S(p)$, $I^{m+1}_{j^*-1} = I^{m+1}_{j^*+1} = \emptyset$, and let $I^{m+1}_j = I^{m-1}_j$ for $j \neq j^*-1, j^*, j^*+1$. By Corollary 8, the sum of the lengths of the intervals $I^{m-1}_{j^*-1}, I^{m-1}_{j^*}$ and $I^{m-1}_{j^*+1}$ is at most $2q$. Since $p \geq 2q$, $I^{m-1}_{j^*-1}, I^{m-1}_{j^*}$ and $I^{m-1}_{j^*+1}$ are pairwise disjoint. This implies that each element $x \in V(K_{p/q})$ is contained in at least k of the sets I^{m+1}_j. Hence, by letting $j \in \phi(x, m+1)$ if $x \in I^j_{m-1}$, ϕ is extended to a k-fold n-coloring of $K_{p/q} \times P^*_{m+1}$. (Some vertices may have received more than k colors.) Observe that colors j^*-1, j^*+1 are not assigned to any vertex of V_{m+1}.

We repeat the process. The vertices in V_{m+2j} are colored in the same way as V_m, except that colors $j^*-2j, j^*-2j+2, \ldots, j^*, j^*+2, \ldots, j^*+2j$ are not assigned to any vertices of V_{m+2j}, and colors $j^*-2j+1, j^*-2j+3, \ldots, j^*+2j-1$ are assigned to all vertices of V_{m+2j}. The vertices in V_{m+2j+1} are colored in the same way as V_{m-1}, except that colors $j^*-2j-1, j^*-2j+1, \ldots, j^*+2j+1$ are not assigned to any vertices of V_{m+2j+1}, and colors $j^*-2j, j^*-2j+2, \ldots, j^*+2j$ are assigned to all vertices of V_{m+2j}.

In particular, k colors are not assigned to any vertex of V_{m+k-1}. Thus we obtain an extendable k-fold n-coloring of $K_{p/q} \times P^*_{m+k-1}$, and $\chi_k(\mu_{m+k-1}(K_{p/q})) \leq n$. By Lemma 2, for any $m' \geq m+k-1$, $\chi_k(\mu_{m'}(K_{p/q})) \leq n$. ∎

THEOREM 15. *If G is a graph with $\chi_c(G) = p/q$ and $\chi_k(G) = n = 2s > k\chi_c(G)$ is even, then $\chi_k(\mu_m(G)) = \chi_k(G)$ for $m \geq (\lceil q/(nq-kp) \rceil - 1)s + k - 1$.*

Proof. Suppose $\chi_k(G) = n = 2s$ is even and $\chi_c(G) = p/q < n/k$. Then G admits a homomorphism to $K_{p/q}$, and hence for any positive integer m, $\chi_k(\mu_m(G)) \leq \chi_k(\mu_m(K_{p/q}))$. By Lemma 14, if $m \geq (\lceil q/(nq-kp) \rceil - 1)s+k-1$, then $\chi_k(\mu_m(K_{p/q})) \leq n$. Therefore $\chi_k(\mu_m(G)) \leq n$. As $\chi_k(\mu_m(G)) \geq \chi_k(G) = n$, we conclude that $\chi_k(\mu_m(G)) = n = \chi_k(G)$. ∎

The $k = 1$ case of Theorem 15 is of special interest.

COROLLARY 16. *If $\chi(G) > \chi_c(G)$ and $\chi(G)$ is even, then for sufficiently large m,*

$$\chi(\mu_m(G)) = \chi(G).$$

In [13], the notion that "a graph has chromatic number n for a topological reason" is defined. We shall not define this notion; however, we remark that it is shown in [2, 13] that if G has chromatic number n for a "topological" reason, then $\chi(\mu_m(G)) = \chi(G) + 1$ for any m. Thus the following result of [13] is a corollary of Corollary 16.

COROLLARY 17. *If G has chromatic number n for a topological reason and n is even, then $\chi(G) = \chi_c(G)$.*

It is proved in [5] that for any positive even number n and any nonnegative integer m, $\chi_c(\mu_m(K_n)) = n + \frac{1}{t}$, where $t = \lfloor 2m/n \rfloor + 1$. Let $G = \mu_m(K_n)$, where $2m \geq n$. Then $\chi(G) > \chi_c(G)$. It is known that G has chromatic number $n+1$ for a topological reason [13]. Therefore for any integer m', $\chi(\mu_{m'}(G)) = \chi(G) + 1$. This implies that the condition in Theorem 15 that $\chi_k(G)$ be even is necessary (at least when $k = 1$). However, in some cases, even if $\chi_k(G)$ is odd, the conclusion of Theorem 15 is still true.

THEOREM 18. *Assume G is a graph with $\chi_k(G) = n$. Suppose there are positive integers k_1, k_2, n_1, n_2 such that $k = k_1 + k_2$, $n = n_1 + n_2$, n_1 is even, $n_1/k_1 > \chi_c(G)$, and $n_2/k_2 \geq \chi_c(G)$. Then for sufficiently large m' we have $\chi_k(\mu_{m'}(G)) = n$.*

Proof. Assume $\chi_c(G) = p/q$. Then $G \to K_{p/q}$, and hence $\chi_k(G) \leq \chi_k(K_{p/q})$. Thus it suffices to prove that $\chi_k(\mu_{m'}(K_{p/q}) \leq n$. Assume $n_1 = 2s$, $t^* = qn_1 - k_1 p$, $s^* = \lceil q/t^* \rceil$, and $m = (s^*-1)s$. Using the proof of Lemma 14, one can prove that there is a k_1-fold n_1-coloring ϕ of $K_{p/q} \times P^*_{m+n_1/2}$, using color set C, such that the set $V_{m+n_1/2}$ uses only half of the colors. Let D be the set of colors not used by vertices in $V_{m+n_1/2}$. As $n_1/2 > k_1$, we can choose a proper subset B of D of k_1-colors and assign it to all the vertices of $V_{m+n_1/2+1}$. Let $c \in C \setminus B$ be a color not used by any vertices in $V_{m+n_1/2} \cup V_{m+n_1/2+1}$.

Let ψ be a k_2-fold n_2-coloring of $K_{p/q}$, using color set C'. By Theorem 3, ψ can be extended to an extendable k_2-fold (n_2+1)-coloring of $K_{p/q} \times P_{2k_2}$, using color set $C' \cup \{c\}$. Now we define an extendable k-fold n-coloring f of $K_{p/q} \times P^*_{m+n_1/2+1+2k_2}$

as follows: For $x \in V(K_{p/q})$ for $i \leq m + n_1/2 + 1$, let $f(x,i) = \phi(x,i) \cup \psi(x,0)$. For $1 \leq i \leq 2k_2$, let $f(x, m + n_1/2 + 1 + i) = \phi(x, m + n_1/2 + (1 + (-1)^i)/2) \cup \psi(x,i)$. It is easy to verify that f is an extendable k-fold n-coloring of $K_{p/q} \times P^*_{m+n_1/2+1+2k_2}$. □

As an example of a graph where Theorem 15 does not apply but Theorem 18 applies, we consider the graph $G = K_{13/4}$. We have $\chi_5(G) = 17$. As $\frac{17}{5} = \frac{4+13}{1+4}$, by Theorem 18, we conclude that sufficiently large m, $\chi_5(\mu_m(G)) = 17$. This conclusion cannot be derived from Theorem 15.

COROLLARY 19. *Suppose $\chi_c(G) = p/q$ and p is odd. Then for any $k \geq q$, if $\chi_k(G) > k\chi_c(G)$, then for sufficiently large m, $\chi_k(\mu_m(G)) = \chi_k(G)$.*

Proof. Assume $\chi_k(G) = n$. If n is even, then the conclusion follows from Theorem 15. If n is odd, then let $n_1 = n - p, n_2 = p, k_1 = k - q$, and $k_2 = q$. The conclusion follows from Theorem 18. □

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