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Retailer's Optimal Ordering Policy for Deteriorating Items with Ramp-Type Demand under Stock-Dependent Consumption Rate

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Abstract

In this paper, we investigated an inventory model for deteriorating items with a ramp-type demand under stock-dependent consumption rate. The model allows for shortage and complete backlogging of unfilled demand. The purpose of this article is to develop an optimal replenishment policy which maximizes the total profit per unit of time for the retailer. The necessary and sufficient conditions of the existence and uniqueness of the optimal solution are shown. Further, we establish a useful theorem to determine the optimal order quantity and replenishment time. Finally, several numerical examples are provided to illustrate the theoretical results, and sensitivity analysis of major parameters involved in the model is also examined.

Keywords: Inventory Control, Deteriorating Items, Ramp-Type Demand, Stock-Dependent Consumption Rate.

1. Introduction

Maintenance of inventories of deteriorating items in a modern business environment is challenging for a decision maker. In general, deterioration can be defined as the damage, spoilage, dryness, vaporization, etc., that result in decrease of usefulness of the original one. The effect of deterioration of goods can not be disregarded in inventory system because there are sufficient amounts of deterioration of goods taking place during

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the normal storage period. Ghare and Schrader [8] first developed an EOQ model which took into account the effect of deteriorating items in store. In this model, it is assumed that deterioration of stock occurs at a constant rate. Later, Cover and Philip [2] extended the model by considering a variable deterioration rate of two-parameter Weibull distribution. Philip [20] then proposed the inventory model with a three-parameter Weibull distribution and no shortage. Shah [25] extended Philip's [20] model by allowing shortages. Following, there is a vast inventory literature on deteriorating items, the outline which can be found in review articles by Goyal [12], Sarma [24], Raafat et al. [21], Pakkala and Achary [17], Wee [23] and others.

However, all the above models are limited to the constant demand. The assumption of constant demand is not always applicable to real situations. In real life, the demand rate of product is different phases of product life cycle in the market. For example, the demand for inventory increases over time in the growth phase and decreases in the decline phase. Donaldson [5] initially developed an inventory model with a linear trend in demand. After that, many researches such as Silver [26], Dave and Pal [4], Giri et al. [9], Pal and Mandal [18] have been devoted to incorporating a time-varying demand rate into their models for deteriorating under a variety of circumstances. Especially, Mandal and Pal [14] investigated an order-level inventory model for deteriorating items, where the demand rate is a ramp-type function of time. This type of demand pattern is generally seen in the case of any new brand of consumer goods coming to the market. The demand rate for such items increases with time up to a certain time and then ultimately stabilizes and becomes constant. Up to now several papers such as Wu et al. [28], Wu and Ouyang [29], Wu [30], Giri et al. [10], Deng[6], have considered this kind inventory model.

Besides, it is usually observed in the supermarket that display of the consumer goods in large quantities attracts more customers and generates higher demand. In the last several years, many researchers have given considerable attention to the situation where the demand rate is dependent on the level of the on-hand inventory. Gupta and Vrat [13] were first to develop models for stock-dependent consumption rate. Later, Baker and Urban [1] also established an economic order quantity model for a power-form inventory-level-dependent demand pattern. Mandal and Phaujdar [15] then developed an economic production quantity model for deteriorating items with constant production rate and linearly stock-dependent demand. Other researches related to this area such as Pal et al. [19], Padmanabhan and Vrat [16], Giri et al. [11], Ray and Chaudhuri [22], Datta et al. [3], Ray et al. [23], Dye and Ouyang [7] and so on.

Reviewing previous inventory literatures with varying demand rate, most of them considered that the demand rate for product is dependent on different phases of product life cycle or the level of the on-hand inventory. But none of these review works considered the inventory level's influence to the ramp-type demand rate. In real life, the real selling rate of the item not only depend on theoretical demand rate (i.e., various demand rate for different phases of product life cycle), but also the level of the on-hand inventory. For precision, this paper provides an inventory model for deteriorating items with a ramp-type demand rate under stock-dependent selling rate. In the model, shortages are allowed and completely backlogged. The necessary and sufficient conditions of the existence and uniqueness of the optimal solution are shown and a useful theorem to determine the optimal order quantity and replenishment time is developed. Furthermore, we provide several numerical examples to illustrate the theoretical results and obtain some observations from sensitivity analysis with respect to major parameters.

The remainder of this paper is organized as follows. In the next section, we describe the assumptions and notation, which are used throughout this article. In Section 3, we establish the complete mathematical model to maximize the total profit per unit of time under two cases. In Section 4, we develop some useful results to characterize the optimal solutions and provide a simple theorem to find the optimal replenishment cycle time and order quantity. Several numerical examples are provided in Section 5 to illustrate the results. Finally, conclusions and suggestions for future research are given in the last Section.

2. Assumptions and Notation

The following assumptions and notation are used throughout this paper:

(1) Replenishment occurs instantaneously at an infinite rate and the lead time is assumed to be zero.

(2) The distribution of time for deterioration of the items following exponential distribution with a constant deteriorating rate θ. There is no replacement or repair of deteriorated units during the period under consideration.

(3) Shortage are allowed and completely backlogged.

(4) T is the fixed length of each ordering cycle.

(5) $R(t)$ is the theoretical demand rate at time t, and it is assumed to be a ramp-type function of time:

$$R(t) = D_0[t - (t-\mu)H(t-\mu)], \quad D_0 > 0,$$

where $H(t-\mu)$ is the well known Heaviside's function as following:

$$H(t-\mu) = \begin{cases} 1, & t \geq \mu, \\ 0, & t < \mu. \end{cases}$$

(6) $D(t)$ is the real selling rate at time t, and it is influenced by the theoretical demand rate and the on-hand inventory according to relation

$$D(t) = \begin{cases} R(t) + \alpha I(t), & I(t) > 0, \\ R(t), & I(t) \leq 0, \end{cases}$$

where α is positive constant and $I(t)$ is the on-hand inventory level at time t.

(7) t_1 is the length of time in which the inventory level falls to zero. Q is the order quantity per cycle.

(8) A, s, c, h, c_s denote the ordering cost per order, the selling price per unit, the purchasing cost per unit, inventory holding cost per unit per unit of time and the shortage cost per unit per unit of time for backlogged items, respectively. All of the cost parameters are positive constant.

(9) $Z(t_1)$ is the total profit per unit of time of inventory system.

3. Model Formulation

In this article, an inventory for deteriorating items with a ramp-type demand and stock-dependent selling rate is considered. The objective of the inventory problem here is to determine the optimal order quantity in order to keep the total profit per unit of time as high as possible. The inventory system evolves as follows: the replenishment at the beginning of each cycle brings the inventory level up to $I_{\max}$. During the time interval $[0, t_1]$, the inventory level gradually depletes due to demand and deterioration of items. Thereafter shortages occur and keep to the end of the current order cycle, which are completely backlogged. The whole process is repeated.

As described above, the inventory level, $I(t)$, $0 \leq t \leq T$ satisfies the following differential equations:

$$\frac{dI(t)}{dt} + \theta I(t) = -D(t), \quad 0 \leq t \leq t_1 \tag{1}$$

with the boundary condition $I(0) = I_{\max}$ and

$$\frac{dI(t)}{dt} = -D(t), \quad t_1 \le t \le T \tag{2}$$

with the boundary condition $I(t_1) = 0$.

The solutions of these differential equations depend on the real selling rate. There are two cases considering in this paper: (a) $t_1 \ge \mu$ and (b) $t_1 \le \mu$. The fluctuation of the inventory level for the two cases is depicted in Figures 1, respectively.

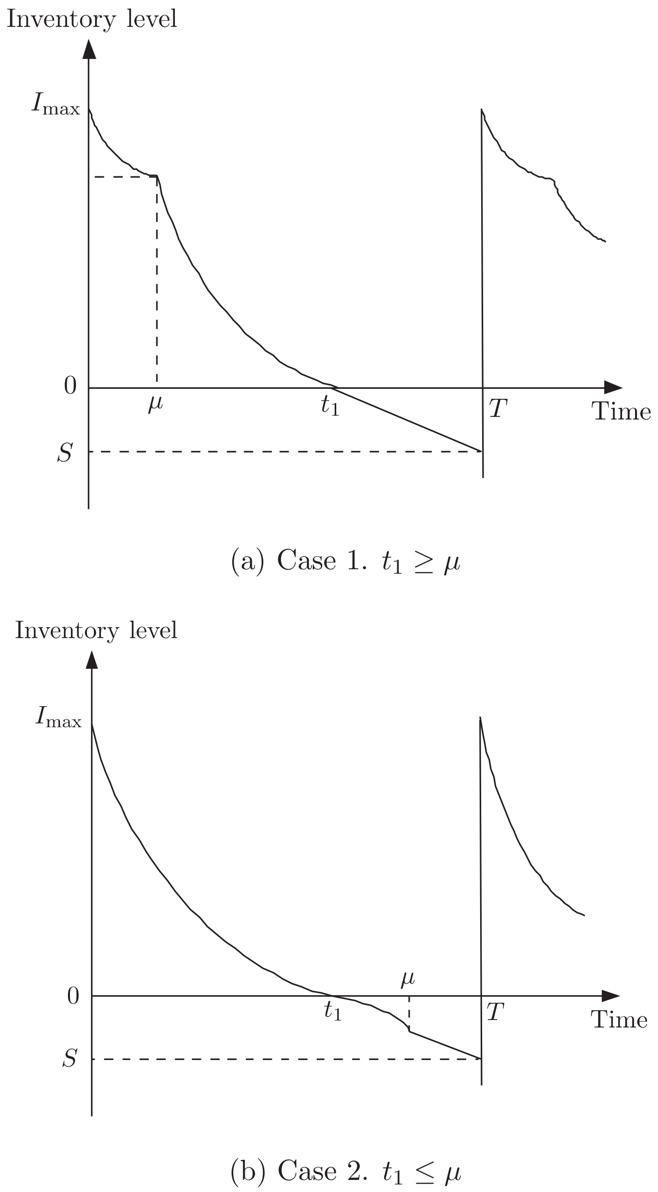

Figure 1. Graphical presentation of inventory system for two cases.

Case 1. $t_1 \geq \mu$

In this case, the real selling rate $D(t)$ is

$$D(t) = \begin{cases} D_0 t + \alpha I(t), & 0 \leq t \leq \mu, \\ D_0 \mu + \alpha I(t), & \mu \leq t \leq t_1, \\ D_0 \mu, & t_1 \leq t \leq T. \end{cases}$$

Hence, Equation (1) leads to the following two equations:

$$\frac{dI(t)}{dt} + \theta I(t) = -[D_0 t + \alpha I(t)], \quad 0 \leq t \leq \mu, \tag{3}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -[D_0 \mu + \alpha I(t)], \quad \mu \leq t \leq t_1, \tag{4}$$

with the boundary conditions $I(0) = I_{\max}$ and $I(t_1) = 0$. Solving Equations (3) and (4) for the inventory over time, $I(t)$, yields:

$$I(t) = I_{\max} e^{-(\theta+\alpha)t} - \frac{D_0}{(\theta+\alpha)^2}\left[e^{-(\theta+\alpha)t} + (\theta+\alpha)t - 1\right], \quad 0 \leq t \leq \mu, \tag{5}$$

$$I(t) = \frac{D_0 \mu}{\theta+\alpha}\left[e^{(\theta+\alpha)(t_1-t)} - 1\right], \quad \mu \leq t \leq t_1. \tag{6}$$

Considering continuity of $I(t)$ at $t = \mu$, it follows from Equations (5) and (6) that

$$I_{\max} e^{-(\theta+\alpha)\mu} - \frac{D_0}{(\theta+\alpha)^2}\left[e^{-(\theta+\alpha)\mu} + (\theta+\alpha)\mu - 1\right] = \frac{D_0 \mu}{\theta+\alpha}\left[e^{(\theta+\alpha)(t_1-\mu)} - 1\right],$$

which implies that the maximum inventory level for each cycle is

$$I_{\max} = \frac{D_0 \mu}{\theta+\alpha} e^{(\theta+\alpha)t_1} - \frac{D_0}{(\theta+\alpha)^2}\left[e^{(\theta+\alpha)\mu} - 1\right]. \tag{7}$$

Putting Equation (7) into (5), it gets

$$I(t) = \frac{D_0 \mu}{\theta+\alpha} e^{(\theta+\alpha)(t_1-t)} - \frac{D_0}{(\theta+\alpha)^2}\left[e^{(\theta+\alpha)(\mu-t)} + (\theta+\alpha)t - 1\right], \quad 0 \leq t \leq \mu. \tag{8}$$

Following, Equation (2) becomes

$$\frac{dI(t)}{dt} = -D_0 \mu, \quad t_1 \leq t \leq T \tag{9}$$

with the boundary condition $I(t_1) = 0$. Solving Equation (9) for the inventory over time, $I(t)$, yields:

$$I(t) = D_0 \mu (t_1 - t), \quad t_1 \leq t \leq T. \tag{10}$$

Putting $t = T$ in Equation (10), we obtain the maximum amount of demand backlogged per cycle as follows:

$$S \equiv -I(T) = D_0 \mu (T - t_1). \tag{11}$$

From Equations (7) and (11), we can obtain the order quantity, Q, as

$$Q = I_{\max} + S = \frac{D_0\mu}{\theta+\alpha}e^{(\theta+\alpha)t_1} - \frac{D_0}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)\mu} - 1\Big] + D_0\mu(T - t_1). \tag{12}$$

Next, the total profit per cycle consists of the following five elements:

(a) The ordering cost per cycle is A.

(b) The inventory holding cost per cycle is given by

$$\begin{aligned}
&h\Big[\int_0^{\mu} I(t)dt + \int_{\mu}^{t_1} I(t)dt\Big] \\
&= h\Big\{\frac{D_0 e^{(\theta+\alpha)\mu}}{(\theta+\alpha)^3}\Big[e^{-(\theta+\alpha)\mu}+(\theta+\alpha)\mu-1\Big] + \frac{D_0\mu[e^{(\theta+\alpha)\mu}-1]}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)(t_1-\mu)}-1\Big] \\
&\quad -\frac{D_0\mu^2}{2(\theta+\alpha)} + \frac{D_0\mu}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)(t_1-\mu)} - (\theta+\alpha)(t_1-\mu) - 1\Big]\Big\}.
\end{aligned} \tag{13}$$

(c) The purchase cost per cycle is given by

$$cQ = c\Big\{\frac{D_0\mu}{\theta+\alpha}e^{(\theta+\alpha)t_1} - \frac{D_0}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)\mu} - 1\Big] + D_0\mu(T - t_1)\Big\}. \tag{14}$$

(d) The shortage cost per cycle is given by

$$c_s\int_{t_1}^{T} -I(t)dt = \frac{c_s D_0\mu(T-t_1)^2}{2}. \tag{15}$$

(e) The sale revenue per cycle is given by

$$\begin{aligned}
s\int_0^T D(t)dt &= s\Big\{\Big[\int_0^{\mu} D_0 t dt + \int_{\mu}^{T} D_0\mu dt\Big] + \alpha\Big[\int_0^{\mu} I(t)dt + \int_{\mu}^{t_1} I(t)dt\Big]\Big\} \\
&= sD_0\mu\Big\{T - \frac{\mu}{2} + \alpha\Big\{\frac{e^{(\theta+\alpha)\mu}}{(\theta+\alpha)^3\mu}\Big[e^{-(\theta+\alpha)\mu} + (\theta+\alpha)\mu - 1\Big] \\
&\quad + \frac{[e^{(\theta+\alpha)\mu} - 1]}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)(t_1-\mu)} - 1\Big] - \frac{\mu}{2(\theta+\alpha)} \\
&\quad + \frac{1}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)(t_1-\mu)} - (\theta+\alpha)(t_1-\mu) - 1\Big]\Big\}\Big\}.
\end{aligned} \tag{16}$$

Therefore, the total profit per unit of time when $t_1 \geq \mu$ which denoted by $Z_1(t_1)$, is given by

$$\begin{aligned}
Z_1(t_1) &= \{\text{sale revenue} - \text{ordering cost} - \text{holding cost} - \text{purchase cost} - \text{shortage cost}\}/T \\
&= \frac{D_0\mu}{T}\Big\{(s-c)T - \frac{s\mu}{2} + (\alpha s - h)\Big\{\frac{(\theta+\alpha)\mu e^{(\theta+\alpha)\mu} - e^{(\theta+\alpha)\mu} + 1}{(\theta+\alpha)^3\mu}
\end{aligned}$$

$$+\frac{[e^{(\theta+\alpha)\mu}-1]}{(\theta+\alpha)^2}\Big[e^{(\theta+\alpha)(t_1-\mu)}-1\Big]-\frac{\mu}{2(\theta+\alpha)}+\frac{e^{(\theta+\alpha)(t_1-\mu)}-(\theta+\alpha)(t_1-\mu)-1}{(\theta+\alpha)^2}\Bigg\}$$
$$-c\left[\frac{1-e^{(\theta+\alpha)\mu}}{(\theta+\alpha)^2\mu}+\frac{e^{(\theta+\alpha)t_1}}{\theta+\alpha}-t_1\right]-\frac{c_s(T-t_1)^2}{2}-\frac{A}{D_0\mu}\Bigg\}. \tag{17}$$

Case 2. $t_1 \leq \mu$

In this case, the real selling rate $D(t)$ is

$$D(t)=\begin{cases} D_0t+\alpha I(t), & 0\leq t\leq t_1,\\ D_0t, & t_1\leq t\leq \mu,\\ D_0\mu, & \mu\leq t\leq T.\end{cases}$$

Hence, Equation (1) becomes

$$\frac{dI(t)}{dt}+\theta I(t)=-[D_0t+\alpha I(t)],\quad 0\leq t\leq t_1, \tag{18}$$

with the boundary condition $I(0)=I_{\max}$. Solving Equation (18) the inventory over time, $I(t)$, yields:

$$I(t)=I_{\max}e^{-(\theta+\alpha)t}-\frac{D_0}{(\theta+\alpha)^2}\Big[e^{-(\theta+\alpha)t}+(\theta+\alpha)t-1\Big],\quad 0\leq t\leq t_1. \tag{19}$$

Considering the boundary condition $I(t_1)=0$, we obtain the maximum inventory level for each cycle is

$$I_{\max}=\frac{D_0}{(\theta+\alpha)^2}\Big[(\theta+\alpha)t_1e^{(\theta+\alpha)t_1}-e^{(\theta+\alpha)t_1}+1\Big]. \tag{20}$$

Putting Equation (20) into Equation (19), it gets

$$I(t)=\frac{D_0}{(\theta+\alpha)^2}\Big[(\theta+\alpha)t_1e^{(\theta+\alpha)(t_1-t)}-e^{(\theta+\alpha)(t_1-t)}+1\Big]-\frac{D_0t}{(\theta+\alpha)},\quad 0\leq t\leq t_1. \tag{21}$$

Following, Equation (2) leads to the following two equations:

$$\frac{dI(t)}{dt}=-D_0t,\quad t_1\leq t\leq \mu, \tag{22}$$
$$\frac{dI(t)}{dt}=-D_0\mu,\quad \mu\leq t\leq T, \tag{23}$$

with the boundary conditions $I(t_1)=0$ and $-I(T)=S$. Solving Equations (22) and (23) for the inventory over time, $I(t)$, yields:

$$I(t)=\frac{D_0}{2}(t_1^2-t^2),\quad t_1\leq t\leq \mu, \tag{24}$$

$$I(t) = D_0\mu(T-t) - S, \quad \mu \le t \le T. \tag{25}$$

Considering continuity of $I(t)$ at $t = \mu$, it follows from Equations (24) and (25) that

$$\frac{D_0}{2}(t_1^2 - \mu^2) = D_0\mu(T-\mu) - S \tag{26}$$

which implies that the maximum amount of demand backlogged per cycle is

$$S = D_0\Big(\mu T - \frac{\mu^2}{2} - \frac{t_1^2}{2}\Big). \tag{27}$$

Substituting Equation (27) into (25), it gets

$$I(t) = D_0\Big(\frac{\mu^2}{2} + \frac{t_1^2}{2} - \mu t\Big), \quad \mu \le t \le T. \tag{28}$$

From Equations (20) and (27), we can obtain the order quantity, Q, as

$$Q = I_{\max} + S = \frac{D_0}{(\theta+\alpha)^2}\Big[(\theta+\alpha)t_1 e^{(\theta+\alpha)t_1} - e^{(\theta+\alpha)t_1} + 1\Big] + D_0\Big(\mu T - \frac{\mu^2}{2} - \frac{t_1^2}{2}\Big). \tag{29}$$

Similar to Case 1, the total profit per unit of time when $t_1 \le \mu$ which denoted by $Z_2(t_1)$, is given by

$$\begin{aligned} Z_2(t_1) = \frac{D_0\mu}{T}\Bigg\{&(s-c)\Big(T-\frac{\mu}{2}\Big) + (\alpha s - h)\Bigg[\frac{(\theta+\alpha)t_1 e^{(\theta+\alpha)t_1} - e^{(\theta+\alpha)t_1} + 1}{(\theta+\alpha)^3\mu} - \frac{t_1^2}{2(\theta+\alpha)\mu}\Bigg] \\ &- c\Bigg\{\frac{(\theta+\alpha)t_1 e^{(\theta+\alpha)t_1} - e^{(\theta+\alpha)t_1} + 1}{(\theta+\alpha)^2\mu} - \frac{t_1^2}{2\mu}\Bigg\} \\ &- c_s\Big(\frac{\mu^2}{6} + \frac{t_1^3}{3\mu} - \frac{\mu T}{2} - \frac{T t_1^2}{2\mu} + \frac{T^2}{2}\Big) - \frac{A}{D_0\mu}\Bigg\}. \end{aligned} \tag{30}$$

Consequently, the total profit function per unit of time of the system over the time interval $[0, T]$ is given by

$$Z(t_1) = \begin{cases} Z_1(t_1), & \text{if } t_1 \ge \mu, \\ Z_2(t_1), & \text{if } t_1 \le \mu. \end{cases}$$

It is obvious that $Z_1(\mu) = Z_2(\mu)$. Therefore, $Z(T)$ is continuous at μ and well-defined.

4. Theoretical Results

In this section, we will derive the results which ensure the necessary and sufficient conditions of the existence and uniqueness of the optimal solution to maximize the total profit per unit of time. For Case 1, the necessary condition for the function $Z_1(t)$ in Equation (17) to be maximum is $dZ_1(t_1)/dt_1 = 0$, which gives

$$\frac{[\alpha(s-c)-(h+\theta c)]}{\theta+\alpha}\Big[e^{(\theta+\alpha)t_1}-1\Big]+c_s(T-t_1)=0. \tag{31}$$

It is not easy to find the closed-form solution of t_1 from Equation (31). But we can show that the value of t_1 which satisfies Equation (31) not only exists but also is unique under some conditions. Further, for notational convenience, we let $\Delta \equiv F(\mu) = \dfrac{\alpha(s-c)-(h+\theta c)}{\theta+\alpha}[e^{(\theta+\alpha)\mu}-1]+c_s(T-\mu)$, then we have following lemma.

Lemma 1. *When $t_1 \geq \mu$,*

(a) *for $\alpha(s-c)-(h+\theta c) < 0$,*

  i) *if $\Delta \geq 0$, then the solution of $t_1 \in [\mu, T]$ (say t_{11}) in Equation* (31) *not only exists but also is unique.*

  ii) *If $\Delta < 0$, then the solution of $t_1 \in [\mu, T]$ in Equation* (31) *does not exist.*

(b) *For $\alpha(s-c)-(h+\theta c) \geq 0$, the solution of $t_1 \in [\mu, T]$ in Equation* (31) *does not exist.*

Proof. See the Appendix A.

Remark 1. We see that $\alpha(s-c)-(h+\theta c)$, where $\alpha(s-c)$ is the benefit received from an additional unit of inventory and $(h+\theta c)$ is the cost due to an additional unit of inventory. Thus, when the condition $\alpha(s-c)-(h+\theta c) < 0$ is satisfied, it implies building up inventory is unprofitable; and when $\alpha(s-c)-(h+\theta c) \geq 0$, it is profitable to build up inventory.

According to Lemma 1, we can obtain the following result.

Lemma 2. *When $t_1 \geq \mu$,*

(a) *for $\alpha(s-c)-(h+\theta c) < 0$,*

  i) *if $\Delta \geq 0$, then the total profit per unit of time $Z_1(t_1)$ has the global maximum value at the point $t_1 = t_{11}$, where $t_{11} \in [\mu, T]$ and satisfies Equation* (31).

  ii) *If $\Delta < 0$, then the total profit per unit of time $Z_1(t_1)$ has a maximum value at the lower boundary point $t_1 = \mu$.*

(b) *For $\alpha(s-c)-(h+\theta c) \geq 0$, the total profit per unit of time $Z_(t_1)$ has a maximum value at the upper boundary point $t_1 = T$.*

Proof. See the Appendix B.

When considering Case 2, we can easily obtain the first-order necessary condition for the total profit per unit of time in Equation (30) to be maximum is $dZ_2(t_1)/dt_1 = 0$, which leads to

$$\frac{[\alpha(s-c)-(h+\theta c)]}{\theta+\alpha}\left[e^{(\theta+\alpha)t_1}-1\right]+c_s(T-t_1)=0. \tag{32}$$

It is also not easy to find the closed-form solution of t_1 from Equation (32). But we can show that the value of t_1 which satisfies Equation (32) not only exists but also is unique under some conditions. Note that Equations (31) and (32) will be the same. Therefore, by the similar argument and proof as in Case 1, we can obtain the following results. Here, the proof is omitted.

Lemma 3. *When $t_1 \le \mu$,*

(a) *for $\alpha(s-c)-(h+\theta c)<0$,*

i) *if $\Delta \le 0$, then the solution of $t_1 \in (0,\mu]$ (say t_{12}) in Equation (32) not only exists but also is unique.*

ii) *If $\Delta < 0$, then the solution of $t_1 \in (0,\mu]$ in Equation (32) does not exist.*

(b) *For $\alpha(s-c)-(h+\theta c)\ge 0$, the solution of $t_1 \in (0,\mu]$ in Equation (32) does not exist.*

Lemma 4. *When $t_1 \le \mu$,*

(a) *for $\alpha(s-c)-(h+\theta c)<0$,*

i) *if $\Delta \le 0$, then the total profit per unit of time $Z_2(t_1)$ has the global maximum value at the point $t_1 = t_{12}$, where $t_{12} \in (0,\mu]$ and satisfies Equation (32).*

ii) *If $\Delta > 0$, then the total profit per unit of time $Z_2(t_1)$ has a maximum value at the upper boundary point $t_1 = \mu$.*

(b) *For $\alpha(s-c)-(h+\theta c)\ge 0$, the total profit per unit of time $Z_2(t_1)$ has a maximum value at the upper boundary point $t_1 = \mu$.*

Combining the mentioned-above situations and summarizing the results in Lemmas 2 and 4, we can obtain the following theorem to determine the optimal length of time in which the inventory level falls to zero t_1 (denoted by t_1^*) and total profit per unit of time $Z(t_1^*)$ (denoted by Z^*).

Theorem 1.

(a) *For $\alpha(s-c)-(h+\theta c)<0$,*

i) *if $\Delta > 0$, then $t_1^* = t_{11}$, and $Z^* = Z_1(t_{11})$.*

ii) *If* $\Delta < 0$, *then* $t_1^* = t_{12}$, *and* $Z^* = Z_1(T_{12})$.

iii) *If* $\Delta = 0$, *then* $t_1^* = \mu$, *and* $Z^* = Z_1(\mu) = Z_2(\mu)$.

(b) *For* $\alpha(s-c) - (h+\theta c) \geq 0$, $t_1^* = T$ *and* $Z^* = Z_1(T)$.

Proof. It immediately follows from Lemmas 2 and 4, and the fact that $Z_1(\mu) = Z_2(\mu)$.

Note that Equations (31) and (32) will be the same, i.e., $t_{11} = t_{12}$. After obtaining the value of t_1^*, the optimal order quantity Q (denoted by Q^*) can be determined by using Equation (12) or (29).

Remark 2. When $\alpha = 0$, it means the real selling rate of the item is independent of on the level of the on-hand inventory, and the model can be reduced the inventory model with ramp-type demand where the inventory starts without shortages in Mandal and Pal [14].

5. Numerical Examples

In order to illustrate the proposed model, we consider the following examples:

Example 1. Given $D_0 = 400$, $A = 50$, $s = 20$, $c = 15$, $h = 3$, $c_s = 5$, $\alpha = 0.1$, $\theta = 0.05$ and $T = 1$ in appropriate units. We obtain that $\alpha(s-c)-(h+\theta c) = -3.25 < 0$. Consequently, if $\mu = 0.4$, then $\Delta = 1.660 > 0$. We know from Theorem 1 that the optimal replenishment cycle time $t_1^* = t_{11} = 0.5933$, the optimal order quantity $Q^* = 132.36$ and the maximum total profit per unit of time $Z^* = 443.62$. If $\mu = 0.6$, then $\Delta = -0.404 < 0$. Hence $t_1^* = t_{12} = 0.5953$, $Q^* = 172.36$ and $Z^* = 597.19$. If $\mu = 0.5953$, then $\Delta = 0$. Hence $t_1^* = \mu = 0.5953$, $Q^* = 171.60$ and $Z^* = 594.15$.

Example 2. Let us consider another inventory system with the following data: $D_0 = 400$, $A = 50$, $s = 30$, $c = 15$, $h = 5$, $c_s = 5$, $\alpha = 0.4$, $\theta = 0.05$; $\mu = 0.5$ and $T = 1$ in appropriate units. We obtain that $\alpha(s-c) - (h+\theta c) = 0.25 > 0$. From Theorem 1, the optimal replenishment cycle time $t_1^* = T = 1$, the optimal order quantity $Q^* = 198.612$ and the maximum total profit per unit of time $Z^* = 2227.01$. The computational results for different situations in Examples 1 and 2 are shown in Table 1.

As shown in Table 1, when the condition $\alpha(s-c)-(h+\theta c) < 0$ is satisfied, it can be found that when μ increases and other parameters remain unchanged, the optimal order quantity per cycle Q^* and the maximum total profit per unit of time Z^* will increase.

Table 1. The optimal solutions in Examples 1 and 2.

Case	μ	Situation	t_1^*	Q^*	Z^*
Example 1 $\alpha(s-c)-(h+\theta c)<0$	0.4	$\Delta > 0$	$t_1^* = t_{11} = 0.5953$	132.36	443.62
	0.5953	$\Delta = 0$	$t_1^* = \mu = 0.5953$	171.60	594.15
	0.6	$\Delta < 0$	$t_1^* = t_{12} = 0.5953$	172.36	597.19
Example 2 $\alpha(s-c)-(h+\theta c)>0$	0.5	–	$t_1^* = T = 1$	198.61	2227.01

Example 3. From Equation (31) or (32), we can see that the optimal length of order cycle t_1^* is influenced by the parameters s, c, h, c_s, α, θ and T. Therefore we discuss the influences of changes in these parameters on the optimal solution of the Example 1. For convenience, we only consider the case for fixed $\mu = 0.6$. The sensitivity analysis is performed by changing each of the parameters by -20%, -10%, $+10\%$ and $+20\%$, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 2.

On the basis of the results in Table 2, the following observations can be made.

(a) The optimal length of time in which the inventory level falls to zero t_1^*, the optimal order quantity per cycle Q^* and the maximum total profit per unit of time Z^* increase with increases in the values of parameters s, α and T. Moreover, t_1^* and Q^* are higher sensitive to change in the parameter T, and Z^* is higher sensitive to change in the parameter s.

(b) All the optimal values of t_1^*, Q^* and Z^* decrease with increases in the values of parameters c and h. Furthermore, Z^* is higher sensitive to change in the parameter c.

(c) With the augment of the parameter c_s, t_1^* and Q^* increase but Z^* decreases.

(d) With increase in the value of parameter θ, t_1^* and Z^* decrease while Q^* increase.

6. Conclusions

In this article, a deterministic inventory model has been proposed for deteriorating items with ramp-type demand rate under stock-dependent selling rate. Shortages are allowed and completely backlogged is considered in the model. In Lemmas 1-4, the necessary and sufficient conditions of the existence and uniqueness of the optimal solution for two cases are shown. Theorem 1 identifies the best circumstance between the two cases based on the maximum total profit per unit of time. Further, several numerical

Table 2. Effect of changes in various parameters of the Example 1 ($\mu = 0.6$).

Parameter	% change	% change in		
		t_1^*	Q^*	Z^*
s	−20	−4.66	−0.35	−114.34
	−10	−2.39	−0.18	−57.20
	+10	2.51	0.20	57.28
	+20	5.15	0.42	114.63
c	−20	5.83	0.48	86.79
	−10	2.83	0.22	43.34
	+10	−2.68	−0.20	−43.25
	+20	−5.21	−0.39	−86.41
h	−20	7.94	0.66	3.28
	−10	3.81	0.30	1.55
	+10	−3.54	−0.27	−1.38
	+20	−6.82	−0.51	−2.62
c_s	−20	−9.02	−0.67	3.73
	−10	−4.22	−0.32	1.75
	+10	3.75	0.30	−1.55
	+20	7.11	0.59	−2.94
α	−20	−0.97	−0.41	−0.41
	−10	−0.49	−0.21	−0.21
	+10	0.49	0.22	0.21
	+20	1.00	0.44	0.42
θ	−20	1.99	−0.03	0.79
	−10	0.99	−0.01	0.39
	+10	−0.97	0.01	−0.38
	+20	−1.92	0.02	−0.75
T	−20	−19.71	−29.22	−6.98
	−10	−9.84	−14.65	−2.64
	+10	9.80	14.76	1.42
	+20	19.57	29.67	1.98

examples are presented to illustrate the theoretical results, and some observations are obtained from sensitivity analysis with respect to major parameters.

In the future study, it is hoped to extend incorporate the proposed model into several situations, such as varying deterioration rate and a finite rate of replenishment. Furthermore, we could generalize the model to allow for shortages and partial backlogged, quantity discounts, inflation rates, and others.

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Appendix A: Proof of Lemma 1.

Proof of part (a). To prove it, we first let

$$F(x) = \frac{[\alpha(s-c)-(h+\theta c)]}{\theta+\alpha}\Big[e^{(\theta+\alpha)x}-1\Big]+c_s(T-x) \tag{A1}$$

for $x \in [\mu, T]$. Taking the first derivative of $F(x)$ in Equation (A1) with respect to $x \in (\mu, T)$, we obtain

$$\frac{dF(x)}{dx} = [\alpha(s-c)-(h+\theta c)]e^{(\theta+\alpha)x}-c_s. \tag{A2}$$

When $\alpha(s-c)-(h+\theta c) < 0$, we have $\frac{dF(x)}{dx} < 0$, and it gets $F(x)$ is a strictly decreasing function of x in the interval $[\mu, T]$ and $F(T) < 0$. (i) If $\Delta \geq 0$ which implies $F(\mu) \geq 0$, then by applying the Intermediate Value Theorem, there exists an unique $t_{11} \in [\mu, T]$ such that $F(t_{11}) = 0$. (ii) If $\Delta < 0$ which implies $F(\mu) < 0$, then there does not exist a value $t_1 \in [\mu, T]$ such that $F(t_1) = 0$.

Proof of part (b). When $\alpha(s-c)-(h+\theta c) \geq 0$, from (A1) we know that $F(x) > 0$ for all $x \in [\mu, T]$ which implies there does not exist a value $t_1 \in [\mu, T]$ such that $F(t_1) = 0$. This completes the proof.

Appendix B: Proof of Lemma 2.

Proof of part (a). For $\alpha(s-c)-(h+\theta c) < 0$, (i) if $\Delta \geq 0$, from Lemma 1(a-(i)), it can be seen that $t_{11} \in [\mu, T]$ is the unique solution of Equation (31). Now, taking the second derivative of $Z_1(t_1)$ with respect to t_1, and then finding the value of the function at point t_{11}, we obtain

$$\left.\frac{d^2 Z_1(t_1)}{dt_1^2}\right|_{t_1=t_{11}} = \frac{D_0\mu}{T}\Big\{[\alpha(s-c)-(h+\theta c)]e^{(\theta+\alpha)t_{11}}-c_s\Big\} < 0. \tag{B1}$$

Thus, t_{11} is the global maximum point of $Z_1(t_1)$. (ii) If $\Delta < 0$, then we have

$$\frac{dZ_1(t_1)}{dt_1} = \frac{D_0\mu}{T}\left\{\frac{[\alpha(s-c)-(h+\theta c)]}{\theta+\alpha}\Big[e^{(\theta+\alpha)t_1}-1\Big]+c_s(T-t_1)\right\} = \frac{D_0\mu}{T}F(t_1). \tag{B2}$$

By the proof process in Lemma 1(a-(ii)), we see that $F(t_1) < 0$, for $t_1 \in [\mu, T]$. It gets $\frac{dZ_1(t_1)}{dt_1} < 0$, for $t_1 \in (\mu, T)$. Consequently, $Z_1(t_1)$ has a maximum value at the lower boundary point $t_1 = \mu$.

Proof of part (b). For $\alpha(s-c)-(h+\theta c)\geq 0$, we have

$$\frac{dZ_1(t_1)}{dt_1}=\frac{D_0\mu}{T}F(t_1)>0, \quad \text{for} \quad t_1\in(\mu,T). \tag{B3}$$

Hence, $Z_1(t_1)$ has a maximum value at upper boundary point $t_1=T$. This completes the proof.

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