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TRAVELING PLANE WAVE SOLUTIONS OF DELAYED LATTICE DIFFERENTIAL SYSTEMS IN COMPETITIVE LOTKA-VOLTERRA TYPE

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Abstract. In this work we consider the existence of traveling plane wave solutions of systems of delayed lattice differential equations in competitive Lotka-Volterra type. Employing iterative method coupled with the explicit construction of upper and lower solutions in the theory of weak quasi-monotone dynamical systems, we obtain a speed, c^*, and show the existence of traveling plane wave solutions connecting two different equilibria when the wave speeds are large than c^*.

1. **Introduction.** The purpose of this work is to investigate the existence of traveling plane wave solutions of systems of N delayed 2-dimensional lattice differential equations (2D-LDEs) in competitive Lotka-Volterra type. The nth 2D-LDE in the systems is of the form

$$\frac{d}{dt}u_{n;i,j}(t) = L_n[u_{n;i,j}](t) + u_{n;i,j}(t) f_n\big(\boldsymbol{u}_{i,j}(t), (\boldsymbol{u}_{i,j})_t^{\widehat{n}}\big), \tag{1}$$

for $(i,j) \in \mathbb{Z}^2$ and $1 \le n \le N$, where $\boldsymbol{u}_{i,j}(t) := (u_{1;i,j}(t), \cdots, u_{N;i,j}(t))$,

$$\begin{aligned}&(\boldsymbol{u}_{i,j})_t^{\widehat{n}}(\tau_1, \cdots, \tau_{n-1}, \tau_{n+1}, \cdots, \tau_N)\\ :=&(u_{1;i,j}(t-\tau_1), \cdots, u_{n-1;i,j}(t-\tau_{n-1}), u_{n+1;i,j}(t-\tau_{n+1}), \cdots, u_{N;i,j}(t-\tau_N)),\end{aligned}$$

and

$$\begin{aligned}L_n[u_{n;i,j}](t) = d_{n,1}u_{n;i+1,j}(t) &+ d_{n,2}u_{n;i,j+1}(t)\\ &+ d_{n,3}u_{n;i-1,j}(t) + d_{n,4}u_{n;i,j-1}(t) - d_{n,0}u_{n;i,j}(t).\end{aligned}$$

Here τ_i and $d_{i,j}$ are positive real constants which represent the time delays and coupling coefficients respectively. Let $\tau := \max\{\tau_1, \cdots, \tau_{n-1}, \tau_{n+1}, \cdots, \tau_N\}$. All f_n are C^1 functions from $\mathbb{R}^N \times C^1([-\tau, 0], \mathbb{R})^{N-1}$ to $\mathbb{R}$ where $C^1([-\tau, 0], \mathbb{R})^{N-1}$ is

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the Banach space of continuous differentiable $N-1$ dimensional functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$ with supremum norm. For any C^1 function $\boldsymbol{u} : \mathbb{R}^1 \to \mathbb{R}^N$, $(\boldsymbol{u})_t^{\widehat{n}} \in C^1([-\tau, 0], \mathbb{R})^{N-1}$ means that $(\boldsymbol{u})_t^{\widehat{n}}(\boldsymbol{s}) = \big(u_1(t+s_1), \cdots, u_{n-1}(t+s_{n-1}), u_{n+1}(t+s_{n+1}), \cdots, u_N(t+s_N)\big)$ for $s_n \in [-\tau, 0]$ and $1 \le n \le N$. Here the natation "$\widehat{n}$" means that in the n-layer there is no delay effect from the same layer.

Systems (1) are infinite dimensional, consisting of infinitely many ordinary differential equations, indexed by points in a three-dimensional lattice which consist of N layers of two-dimensional plane lattice. In the position (i, j) of nth-layer, the state $u_{n;i,j}$ is linear coupling with nearest neighbor states, $u_{n;i+1,j}$, $u_{n;i,j+1}$, $u_{n;i-1,j}$, and $u_{n;i,j-1}$. Interactions between different layers are governed by the nonlinear function

$$F_n\big((\boldsymbol{u}_{i,j})_t\big) = u_{n;i,j}(t) f_n\big(\boldsymbol{u}_{i,j}(t), (\boldsymbol{u}_{i,j})_t^{\widehat{n}}\big) \text{ for } 1 \le n \le N. \tag{2}$$

Such systems arise from the study of dynamics of multi-layer neural networks [28], material science [4], chemical reaction theory [12], image processing and pattern recognition [9, 10, 33, 34], and population dynamics of multiple species in biology [31]. We also refer to the papers [7, 25] for the detailed account of the theory and applications of lattice differential equations.

On the other hand, it is often that when one discretizes some partial differential equations one ends up with a lattice differential equation to solve. For example, if $d_{n,i} = 1$ for $i = 1, \cdots, 4$, and $d_{n,0} = 4$ then the operator L_n represents the discrete two-dimensional Laplacian operator. Thus equations (1) can be viewed as the spatial discretization of the following partial differential equations defined in the plane

$$\frac{\partial u_n(\boldsymbol{x}, t)}{\partial t} = d_n \Delta u_n(\boldsymbol{x}, t) + u_n(\boldsymbol{x}, t) f_n\big(\boldsymbol{u}(\boldsymbol{x}, t), \boldsymbol{u}_t(\boldsymbol{x})^{\widehat{n}}\big),$$

with $\boldsymbol{x} \in \mathbb{R}^2$ and $1 \le n \le N$. Specifically, if

$$f_n\big(\boldsymbol{u}(\boldsymbol{x}, t), \boldsymbol{u}_t(\boldsymbol{x})^{\widehat{n}}\big) = (r_n - p_n u_n(\boldsymbol{x}, t) - \sum_{m=1, m\neq n}^{N} s_{n,m} u_m(\boldsymbol{x}, t - \tau_m))$$

for some positive constants r_n, p_n and $s_{n,m}$, then systems (1) can be viewed as the spatial discretization of the diffusive competitive Lotka-Volterra systems of N-species equations with delay effects in the plane. The systems model the interaction among various competing species, has been studied extensively, and various sufficient conditions for the coexistence and extinction of the competing species are obtained, cf. [11, 22, 29, 36].

Our aim is to study the existence of traveling plane wave solutions of (1). A traveling plane wave solution of systems (1) is a solution of the form

$$u_{n;i,j}(t) = \phi_n(t - ic\cos\theta - jc\sin\theta), \quad n = 1, \cdots, N, \tag{3}$$

where $1/c > 0$ is the wave speed; $\theta \in [0, \pi/2]$ is the direction of waves propagation; ϕ_n are continuously differentiable functions. According to (3), the profile equations of systems (1) can be written as

$$\begin{aligned} \phi_n'(t) &= \mathcal{L}_n[\phi_n](t) + F_n(\Phi_t), \\ F_n(\Phi_t) &= \phi_n(t) f_n(\Phi(t), \Phi_t^{\widehat{n}}) \end{aligned} \tag{4}$$

for $n = 1, \cdots, N$, with $c_1 := c\cos\theta$, $c_2 := c\sin\theta$, $\Phi(t) = (\phi_1(t), \cdots, \phi_N(t))$, $\Phi_t^{\widehat{n}} \in C^1([-\tau, 0], \mathbb{R})^{N-1}$, and

$$\begin{aligned}\mathcal{L}_n[\phi_n](t) := d_{n,1}\phi_n(t-c_1) &+ d_{n,2}\phi_n(t-c_2)\\ &+ d_{n,3}\phi_n(t+c_1) + d_{n,4}\phi_n(t+c_2) - d_{n,0}\phi_n(t).\end{aligned}$$

Typically, traveling wavefront solutions arise from the competition between two equilibria. To find a traveling plane wave solution of (1) connecting two equilibria is equivalent to find a heteroclinic trajectory of (4) with asymptotically boundary conditions. It is obvious that (4) has a trivial solution $\mathbf{0} := (0, \cdots, 0)$. Some sufficient conditions for the uniqueness of positive equilibrium and global asymptotic stability for Lotka-Volterra competition-diffusion systems with discrete time delays were given in [30]. Generalizing the ideas of [30], we will give sufficient conditions to guarantee the existence of a positive equilibrium $\Phi^\star$ of (4) in Section 2. Then we look for the existence of heteroclinic orbits of (4) that satisfies the following asymptotically boundary conditions:

$$\lim_{t\to-\infty} \Phi(t) = \mathbf{0} \quad \text{and} \quad \lim_{t\to\infty} \Phi(t) = \Phi^\star. \tag{5}$$

Traveling wave solutions for a *single* lattice differential equation without or with delay have drawn considerable attention in the past decades, see, e.g., [2, 3, 5, 6, 8, 13, 17, 18, 21, 26, 24, 35, 37, 38] and many references cited therein. Particularly, Wu and Zou [35] developed a *monotone iterative* scheme and used a non-standard ordering, quasi-monotone or exponentially quasi-monotone, in the profile set to prove the existence of traveling wave solutions LDEs with asymptotical boundary conditions of by an upper-lower solution method. This technique was generalized to a delayed LDEs on higher dimensional lattices [38]. From another point of view, to show the existence of traveling wave solutions with asymptotically boundary conditions is equivalent to find a heteroclinic orbit connecting two equilibria of the corresponding profile equation which is a mixed type functional differential equation. Hence, by the same approach, Hsu et al. [17, 18] generalized the results of [35] to a general scalar functional differential equation in delay, advance or mixed type with some suitable conditions.

Recently, researchers have started to investigate *systems* of LDEs [1, 12, 20, 27, 31]. Huang et al. [20] considered the following systems of two delayed LDEs:

$$\begin{aligned}\frac{du_n}{dt} &= \sum_{j=1}^{m} a_j[g(u_{n+j}(t)) - 2g(u_n(t)) + g(u_{n-j}(t))] + f_1(u_n(t), v_n(t-\tau)),\\ \frac{dv_n}{dt} &= \sum_{j=1}^{m} b_j[g(v_{n+j}(t)) - 2g(v_n(t)) + g(v_{n-j}(t))] + f_2(u_n(t-\tau), v_n(t)).\end{aligned} \tag{6}$$

Let the nonlinear reaction terms of (6) satisfy the quasi monotonicity condition [35].

(QM) There exist two positive constants β_1 and β_2 such that

$$\begin{aligned}f_1(\psi_1,\psi_2) - f_1(\phi_1,\phi_2) + \beta_1[\psi_1(0) - \phi_1(0)] \geq 2A[g(\psi_1(0)) - g(\phi_1(0))],\\ f_2(\psi_1,\psi_2) - f_2(\phi_1,\phi_2) + \beta_2[\psi_2(0) - \phi_2(0)] \geq 2B[g(\psi_2(0)) - g(\phi_2(0))]\end{aligned}$$

for (ψ_1,ψ_2), $(\phi_1,\phi_2) \in C([-\tau,0],\mathbb{R})^2$ with $0 \preceq (\phi_1(s),\phi_2(s)) \preceq (\psi_1(s),\psi_2(s)) \preceq (k_1,k_2)$, for $s \in [-\tau, 0]$ and some positive constant k_1, k_2, where $A = \sum_{j=1}^{m} a_j$ and

$B = \sum_{j=1}^m b_j$. Then the existence of traveling wave solutions of (6) can be established by the results of Wu and Zou [35]. Here the notation "$\prec$ ($\preceq$)" denote the standard order in high dimension, that is, $\Phi = (\phi_1, \cdots, \phi_N)$ and $\Psi = (\psi_1, \cdots, \psi_N) \in C([-\tau, 0], \mathbb{R})^N$, denote $\Phi \preceq (\prec) \Psi$ if $\phi_n(s) \leq (<) \psi_n(s)$ for $n = 1, \cdots, N$ with $s \in [-\tau, 0]$.

However, the nonlinear reaction terms of some important examples from practical problems may not satisfy the condition (QM). Hence the methods used in [35] cannot be applied. Here is a example which is the discrete diffusive predator-prey model with delay for ecological systems:

$$\begin{aligned} \frac{du_n}{dt} &= d_1[u_{n+1} - 2u_n + u_{n-1}] + u_n[1 - u_n] - a u_n v_n(t - \tau_1), \\ \frac{dv_n}{dt} &= d_2[v_{n+1} - 2v_n + v_{n-1}] - v_n + b u_n(t - \tau_2) v_n, \end{aligned} \tag{7}$$

Hence a modified (QM) condition for the reaction terms called partial quasi monotonicity (PQM) was introduced simultaneously in delayed LDEs [20] and in delayed reaction diffusion systems [19].

(PQM) There exist two positive constants β_1 and β_2 such that

$$\begin{aligned} &f_1(\psi_1, \psi_2) - f_1(\phi_1, \psi_2) + \beta_1[\psi_1(0) - \phi_1(0)] \geq 2A[g(\psi_1(0)) - g(\phi_1(0))], \\ &f_1(\psi_1, \psi_2) - f_1(\psi_1, \phi_2) \leq 0, \\ &f_2(\psi_1, \psi_2) - f_2(\phi_1, \phi_2) + \beta_2[\psi_2(0) - \phi_2(0)] \geq 2B[g(\psi_2(0)) - g(\phi_2(0))] \end{aligned}$$

for (ψ_1, ψ_2), $(\phi_1, \phi_2) \in C([-\tau, 0], \mathbb{R})^2$ with $\mathbf{0} \preceq (\phi_1(s), \phi_2(s)) \preceq (\psi_1(s), \psi_2(s)) \preceq (k_1, k_2)$, for $s \in [-\tau, 0]$ and some positive constant k_1, k_2, where $A = \sum_{j=1}^m a_j$ and $B = \sum_{j=1}^m b_j$. We remark that the functions f_2 satisfies the same monotone condition (QM). A new *cross-iteration* scheme was given to show the existence of traveling wave solutions of (6) if a pair of upper-lower solutions can be constructed and the condition (PQM) are satisfied for reaction terms. In particular, this results can be applied to (7) to show the existence of traveling wave solutions.

The same story happened to the following two species delayed competition systems.

$$\begin{aligned} \frac{\partial}{\partial t} u_1(x,t) &= d_1 \frac{\partial^2}{\partial x^2} u_1(x,t) + r_1 u_1(x,t)[1 - a_1 u_1(x,t) - b_1 u_2(x, t - \tau_1)], \\ \frac{\partial}{\partial t} u_2(x,t) &= d_2 \frac{\partial^2}{\partial x^2} u_2(x,t) + r_2 u_2(x,t)[1 - b_2 u_1(x, t - \tau_2) - a_2 u_2(x,t)], \end{aligned} \tag{8}$$

where d_i and τ_i are positive constants. It is easy to check that the reaction terms of (8) satisfies neither the condition (QM) nor the condition (PQM). Thus Li et al. [23] provided a condition on the reaction terms called the weak quasi monotone condition (WQM) stated in the following:

(WQM) There exist $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\begin{aligned} &f_1(\psi_1(0), \psi_2(-\tau_1)) - f_1(\phi_1(0), \psi_2(-\tau_1)) + \beta_1[\psi_1(0) - \phi_1(0)] \geq 0, \\ &f_1(\psi_1(0), \psi_2(-\tau_1)) - f_1(\psi_1(0), \phi_2(-\tau_1)) \leq 0, \\ &f_2(\psi_1(-\tau_2), \psi_2(0)) - f_2(\psi_1(-\tau_2), \phi_2(0)) + \beta_2[\psi_2(0) - \phi_2(0)] \geq 0, \\ &f_2(\psi_1(-\tau_1), \psi_2(0)) - f_2(\phi_1(-\tau_2), \psi_2(0)) \leq 0 \end{aligned}$$

for (ϕ_1, ϕ_2), $(\psi_1, \psi_2) \in C([-\tau, 0], \mathbb{R})^2$ with $\mathbf{0} \preceq (\phi_1(s), \phi_2(s)) \preceq (\psi_1(s), \psi_2(s)) \preceq (M_1, M_2)$, and $s \in [-\max\{\tau_1, \tau_2\}, 0]$. Based on the assumption (WQM), they reduced the existence of traveling wave solutions of (8) to the existence of an admissible pair of upper and lower solutions (cf. Definition 3.1). By assuming the existence of an admissible pair of upper and lower solutions of (8), they applied the cross-iterative method to establish the existence of traveling wave solutions.

Motivated by the previous works of [23, 20, 19, 35], we will provide a condition which we still denote it by the same name (WQM) on the nonlinear reaction terms of N-*systems* of LDEs in Section 2. In the strategy of iterative scheme, the construction of lower and upper solutions is nontrivial for any specific model. Following the ideas of [23, 24], we can *explicitly* construct upper and lower solutions of the corresponding systems of the wave profiles of (1) for some classes of nonlinear reaction functions F satisfying the condition (WQM). Applying the technique of the cross-iterative method and Schauder's fixed point theorem, we show the existence of traveling plane wave solutions of (1). The results can be applied to many models, e.g. the Lotka-Volterra competition systems with distributive time delays.

The remainder of this paper is organized as follows. In Section 2, some necessary notations and definitions are introduced. Then the existence of strictly positive equilibrium is obtained under suitable assumptions. We also examine the weak quasi monotone properties of the systems (4). In Section 3, with the aid of real roots of the corresponding characteristic function of (4) at the trivial solution, we construct the upper and lower solutions of (4). Based on the results of Sections 2 and 3, we show the existence of traveling plane wave solutions of (4) and (5) in Section 4 by using the cross-iterative method and Schauder's fixed point theorem. In the last section, we apply our main results to the Lotka-Volterra competition systems with distributive time delays, and obtain the existence of traveling plane wave solutions.

2. **Preliminary.**

In this section, we have three main purposes. The first one is to show the existence of a positive equilibrium of (4) under some sufficient conditions on reaction terms. The second one is to state a general (WQM) condition on N-systems of LDEs. Then some sufficient conditions are imposed on reaction functions to guarantee that the condition (WQM) is satisfied. Finally, we investigate the characteristic equations about the trivial solution $\mathbf{0}$. The characteristic roots will help us to construct a pair of upper-lower solutions of (4).

From now on in this paper, we just consider the existence of a heteroclinic orbit of (4) with asymptotical boundary condition (5) instead of finding the traveling wavefront solutions of (1). And the nonlinear reaction functions F_n are always of the form (2).

2.1. **The existence of a positive equilibrium.**

We use the notation $[\Psi|\phi_n] = [\psi_1, \cdots, \psi_N|\phi_n]$ to denote a vector or a vector function Ψ which the nth component is replaced by ϕ_n, that is,

$$[\Psi|\phi_n] := (\psi_1, \cdots, \psi_{n-1}, \phi_n, \psi_{n+1}, \cdots, \psi_N).$$

And the notation $\Psi^{\widehat{n}} = (\psi_1, \cdots, \psi_N)^{\widehat{n}}$ is denoted a vector or a vector function which is removed the nth component, that is,

$$\Psi^{\widehat{n}} = (\psi_1, \cdots, \psi_{n-1}, \psi_{n+1}, \psi_N)$$

for $1 \leq n \leq N$.

To find the equilibrium $\Phi^\star$ of (4) satisfying $\Phi^\star \succ \mathbf{0}$, we have to solve the following systems of nonlinear algebraic equations,

$$(\sum_{i=1}^{4} d_{n,i} - d_{n,0})\phi_n^* + \phi_n^* f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}}) = 0 \quad \text{for} \quad n = 1, \cdots, N.$$

For mathematical simplicity, we assume that $\sum_{i=1}^{4} d_{n,i} = d_{n,0}$. Then the above equation can be simplified as

$$f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}}) = 0 \quad \text{for} \quad n = 1, \cdots, N.$$

Lemma 2.1. *Let $K = (k_1, \cdots, k_N)$ and $L = (\ell_1, \cdots, \ell_N)$ be two constant vectors such that $0 \prec K \preceq L$. If $g_n \in C^1(\mathbb{R}^{2N-1}, \mathbb{R})$,*

$$\partial g_n / \partial x_m \le 0 \quad \text{and} \quad g_n([K|\ell_n], K^{\widehat{n}}) \le 0 \le g_n([L|k_n], L^{\widehat{n}}),$$

for $1 \le n \le N$ and $1 \le m \le 2N-1$, then there exist two constant vectors $\Phi = (\phi_1, \cdots, \phi_N)$ and $\Psi = (\psi_1, \cdots, \psi_N)$ in $\mathbb{R}^N$ such that $K \preceq \Phi \preceq \Psi \preceq L$ and

$$g_n([\Phi|\psi_n], \Phi^{\widehat{n}}) = 0 = g_n([\Psi|\phi_n], \Psi^{\widehat{n}})$$

for all $1 \le n \le N$.

Proof. Let $\Lambda := \{X = (x_1, \cdots, x_N) \in \mathbb{R}^N : K \preceq X \preceq L\}$ and M_n be positive constants such that

$$M_n > \max\left\{ -\frac{\partial}{\partial x_n}(x_n g_n(X, X^{\widehat{n}})) : X \in \Lambda \right\}, \tag{9}$$

for all $n = 1, \cdots, N$. For any $X, Y \in \Lambda$, define

$$\begin{aligned} G_n(X) &:= x_n g_n(X, X^{\widehat{n}}), \\ Z_n(\alpha; X, Y) &:= \big((1-\alpha)y_1, \cdots, (1-\alpha)y_{n-1}, \alpha y_n, (1-\alpha)y_{n+1}, \cdots, (1-\alpha)y_N\big) \\ &\quad + \big(\alpha x_1, \cdots, \alpha x_{n-1}, (1-\alpha)x_n, \alpha x_{n+1}, \cdots, \alpha x_N\big) \\ &= [(1-\alpha)Y|\alpha y_n] + [\alpha X|(1-\alpha)x_n], \end{aligned}$$

for $n = 1, \cdots, N$ and $\alpha \in [0,1]$. Obviously, $Z_n(\alpha; X, Y) \in \Lambda$ if X and Y in Λ. If $X \succeq Y$, by (9) and the Mean Value Theorem, we have

$$\begin{aligned} & y_n g_n([X|y_n], X^{\widehat{n}}) - x_n g_n([Y|x_n], Y^{\widehat{n}}) \\ =& G_n([X|y_n]) - G_n([Y|x_n])) \\ =& G_n(Z_n(1; X, Y)) - G_n(Z_n(0; X, Y)) \\ =& \frac{d}{d\alpha}\Big|_{\alpha = \widetilde{\alpha}} G_n([(1-\alpha)Y|\alpha y_n] + [\alpha X|(1-\alpha)x_n]) \\ \le& M_n(x_n - y_n) \end{aligned}$$

for some $\widetilde{\alpha} \in [0,1]$. Hence,

$$M_n x_n + x_n g_n([Y|x_n], Y^{\widehat{n}}) \ge M_n y_n + y_n g_n([X|y_n], X^{\widehat{n}}). \tag{10}$$

Let $\Phi^{(0)} = K$, $\Psi^{(0)} = L$, and consider the following iterations:

$$\begin{aligned} M_n \phi_n^{(m)} &= M_n \phi_n^{(m-1)} + \phi_n^{(m-1)} g_n\big([\Psi^{(m-1)}|\phi_n^{(m-1)}], (\Psi^{(m-1)})^{\widehat{n}}\big), \\ M_n \psi_n^{(m)} &= M_n \psi_n^{(m-1)} + \psi_n^{(m-1)} g_n\big([\Phi^{(m-1)}|\psi_n^{(m-1)}], (\Phi^{(m-1)})^{\widehat{n}}\big). \end{aligned}$$

We claim that

$$K \preceq \Phi^{(m-1)} \preceq \Phi^{(m)} \preceq \Psi^{(m)} \preceq \Psi^{(m-1)} \preceq L,$$

for all $m \geq 1$. Once the claim is true, then the assertion of this lemma follows by letting $m \to \infty$.

Now we prove the above claim by using the induction method. From (9) and (10), it is obviously that

$$K \preceq \Phi^{(0)} \preceq \Phi^{(1)} \preceq \Psi^{(1)} \preceq \Psi^{(0)} \preceq L.$$

Suppose that

$$K \preceq \Phi^{(m-1)} \preceq \Phi^{(m)} \preceq \Psi^{(m)} \preceq \Psi^{(m-1)} \preceq L$$

for some $m \geq 1$. Applying (10) to the inequalities:

$$[\Psi^{(m-1)}|\phi_n^{(m)}] \succeq [\Psi^{(m)}|\phi_n^{(m-1)}], \ \Psi^{(m)} \succeq \Phi^{(m)} \text{ and } [\Phi^{(m)}|\psi_n^{(m-1)}] \succeq [\Phi^{(m-1)}|\psi_n^{(m)}],$$

the claim holds obviously. Hence the proof is complete. □

2.2. **Weak quasi monotonicity.** Now the conditions (WQM) conditions are stated for the nonlinear reaction functions $F = (F_1, \cdots, F_N)$ for general N as follows.

(WQM): there exist N positive constants $\beta_1, \cdots, \beta_N$ such that
(i) $F_n([\Psi|\phi_n]) - F_n\big(\Phi\big) \leq 0$,
(ii) $F_n([\Phi|\psi_n]) - F_n(\Phi) + \beta_n\big(\psi_n(0) - \phi_n(0)\big) \geq 0$
for $1 \leq n \leq N$, $\Psi, \Phi \in C^1([-\tau, 0], \mathbb{R})^N$ with $\mathbf{0} \preceq \Phi(s) \preceq \Psi(s) \preceq M$ for $s \in [-\tau, 0]$ and a positive constant vector M.

Now we explore the condition (WQM) on the reaction functions of the form

$$\begin{aligned} F_n(\Phi) =&\phi_n(0) f_n(\Phi(0), \Phi^{\widehat{n}}) \\ =&\phi_n(0) f_n(\phi_1(0), \cdots, \phi_N(0), \\ &\phi_1(-\tau_1), \cdots, \phi_{n-1}(-\tau_{n-1}), \phi_{n+1}(-\tau_{n+1}), \cdots, \phi_N(-\tau_N)) \end{aligned} \tag{11}$$

for $\Phi = (\phi_1, \cdots, \phi_N) \in C^1([-\tau, 0], \mathbb{R})^N$.

Lemma 2.2. *Assume that Φ and $\Psi \in C^1([-\tau, 0], \mathbb{R})^N$ such that $\mathbf{0} \preceq \Phi(s) \preceq \Psi(s) \preceq M$ for some constant vector $M = (M_1, \cdots, M_N) \succ \mathbf{0}$. If the functions f_n of (11) satisfies $\partial f_n / \partial x_m \leq 0$ for $1 \leq m \leq 2N - 1$, then reaction function F_n satisfies the condition (WQM).*

Proof. (i). It is sufficiently to check that

$$f_n\big([\Psi|\phi_n](0), \Psi^{\widehat{n}}\big) - f_n\big(\Phi(0), \Phi^{\widehat{n}}\big) \leq 0$$

since ϕ_n is nonnegative. According to the assumptions of f_n, it is obviously that

$$f_n([\Psi|\phi_n](0), \Psi^{\widehat{n}}) - f_n(\Phi(0), \Phi^{\widehat{n}}) = Df_n(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \cdot ([\Psi - \Phi|0_n](t), \Psi^{\widehat{n}} - \Phi^{\widehat{n}}) \leq 0,$$

for some $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Hence the results follows.

(ii). Based on the assumptions, we have

$$\begin{aligned} &\psi_n(0) f_n([\Phi|\psi_n](0), \Phi^{\widehat{n}}) - \phi_n(0) f_n(\Phi(0), \Phi^{\widehat{n}}) \\ =&\psi_n(0) f_n([\Phi|\psi_n](0), \Phi^{\widehat{n}}) - \psi_n(0) f_n(\Phi(0), \Phi^{\widehat{n}}) \\ &\qquad + \psi_n(0) f_n(\Phi(0), \Phi^{\widehat{n}}) - \phi_n(0) f_n(\Phi(0), \Phi^{\widehat{n}}) \\ =&\psi_n(0) Df(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \cdot ([\mathbf{0}|\psi_n - \phi_n](0), \mathbf{0}) + \big(\psi_n(0) - \phi_n(0)\big) f_n(\Phi(0), \Phi^{\widehat{n}}) \\ \geq&\big(\psi_n(0) - \phi_n(0)\big)\big(-M_n \max_{0 \preceq \boldsymbol{\xi}_1 \preceq M, \ 0 \preceq \boldsymbol{\xi}_2 \preceq M^{\widehat{n}}} \|Df(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\| + f_n(M, M^{\widehat{n}})\big), \end{aligned}$$

for some $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Thus, the assertion of this part follows by taking

$$\beta_n > M_n \max_{\mathbf{0}\preceq\boldsymbol{\eta}_1\preceq M,\mathbf{0}\preceq\boldsymbol{\eta}_2\preceq M^{\widehat{n}}} \|Df(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\| - f_n(M, M^{\widehat{n}}), \quad i = 1,\cdots,N.$$

The proof is complete. □

According to Lemma 2.2, we can define operators $H = (H_1,\cdots,H_N)$, $G = (G_1,\cdots,G_N) : C^1([-\tau,0],\mathbb{R})^N \to C^1(\mathbb{R},\mathbb{R}^N)$ by

$$H_n(\Phi)(t) := \mathcal{L}_n[\phi_n](t) + \phi_n(t) f_n(\Phi(t), \Phi_t^{\widehat{n}}) + \beta_n\phi_n(t),$$
$$G_n(\Phi)(t) := e^{-\beta_n t}\int_{-\infty}^{t} e^{\beta_n s} H_n(\Phi)(s)ds, \quad t\in\mathbb{R},\ i = 1,\cdots,N.$$

Then the profile equation (4) can be represented as

$$\phi_n'(t) + \beta_n\phi_n(t) - H_n(\Phi)(t) = 0, \quad n = 1,\cdots,N,$$

and a fixed point of G is equivalent to the solutions of (4). By Lemma 2.2, G_n and H_n have the following properties.

Lemma 2.3. *Assume Φ and Ψ satisfy the assumptions of Lemma 2.2. Then*

(1) $H_n([\Psi|\phi_n]) \le H_n(\Phi) \le H_n([\Phi|\psi_n])$, for $n = 1,\cdots,N$.
(2) $G_n([\Psi|\phi_n]) \le G_n(\Phi) \le G_n([\Phi|\psi_n])$, for $n = 1,\cdots,N$.

Proof. The results follow obviously from Lemma 2.2. We omit the details. □

By Lemma 2.3, it motivates us to use the iterative scheme to obtain the existence of traveling plane wave solutions. But some properties of characteristic equations and characteristic roots should be investigated.

2.3. **Characteristic functions and characteristic roots.** First, we give the definition of the characteristic functions of (4). The characteristic function arises from the linearized equation of (4) at the equilibrium $\mathbf{0}$, and its roots play crucial roles in studying the behavior of solutions of (4) near $\mathbf{0}$.

Definition 2.4. Let $c > 0$ and $\theta \in [0,\pi/2]$. The characteristic function of (4) at $\mathbf{0}$ is defined by

$$\Delta(\lambda, c) = \prod_{n=1}^{N} \Delta_n(\lambda, c),$$

where $\Delta_n(\lambda, c)$, $n = 1,\cdots,N$, are of the form

$$\Delta_n(\lambda, c) = -\lambda + d_{n,1}e^{-\lambda c_1} + d_{n,2}e^{-\lambda c_2} + d_{n,3}e^{\lambda c_1} + d_{n,4}e^{\lambda c_2} - d_{n,0} + f_n(\mathbf{0},\mathbf{0}),$$

where $c_1 = c\cos\theta$ and $c_2 = c\sin\theta$.

Now, we explore some properties of the functions $\Delta_n(\lambda, c)$.

Lemma 2.5. *Assume that for all n $d_{n,3} \ge d_{n,1} > 0$, $d_{n,4} \ge d_{n,2} > 0$, $\sum_{i=1}^4 d_{n,i} = d_{n,0}$, and $f_n(\mathbf{0},\mathbf{0}) > 0$. There exists a $c^* > 0$ which depends on $\{d_{n,i}\}_{i=1}^4$ such that if $0 < c < c^*$ we can find two real characteristic roots $\lambda_{n,1}(c)$ and $\lambda_{n,2}(c)$ such that $0 < \lambda_{n,1}(c) < \lambda_{n,2}(c)$ and*

$$\Delta_n(\lambda, c) = \begin{cases} = 0, & \text{if } \lambda = \lambda_{n,1}, \lambda_{n,2}, \\ > 0, & \text{if } 0 < \lambda < \lambda_{n,1}(c), \\ < 0, & \text{if } \lambda_{n,1}(c) < \lambda < \lambda_{n,2}(c), \\ > 0, & \text{if } \lambda > \lambda_{n,2}(c). \end{cases} \tag{12}$$

Proof. We first define

$$\bar{c} = \min\{\tilde{c},\ 1/(d_{n,3} - d_{n,1} + d_{n,4} - d_{n,2})\}$$

where

$$\tilde{c} := \inf\{c > 0 : \Delta_n(\lambda, c) > 0, \quad \text{for all } \lambda > 0\}.$$

(If $d_{n,1} = d_{n,3}$ and $d_{n,2} = d_{n,4}$, then we take $1/(d_{n,3} - d_{n,1} + d_{n,4} - d_{n,2}) = \infty$.) It is clear that $\tilde{c} > 0$. Hence $\bar{c} > 0$. For $0 < c < \bar{c}$ and $\lambda > 0$, we have

$$\frac{\partial \Delta_n(\lambda, c)}{\partial c} = \lambda \cos\theta(d_{n,3}e^{\lambda c_1} - d_{n,1}e^{-\lambda c_1}) + \lambda \sin\theta(d_{n,4}e^{\lambda c_2} - d_{n,2}e^{-\lambda c_2}) > 0,$$
$$\frac{\partial \Delta_n(\lambda, c)}{\partial \lambda} = -1 + c_1(d_{n,3}e^{\lambda c_1} - d_{n,1}e^{-\lambda c_1}) + c_2(d_{n,4}e^{\lambda c_2} - d_{n,2}e^{-\lambda c_2}),$$
$$\frac{\partial^2 \Delta_n(\lambda, c)}{\partial \lambda^2} = c_1^2(d_{n,3}e^{\lambda c_1} + d_{n,1}e^{-\lambda c_1}) + c_2^2(d_{n,4}e^{\lambda c_2} + d_{n,2}e^{-\lambda c_2}) > 0.$$

Then we yield

$$\frac{\partial \Delta_n(0, c)}{\partial \lambda} = -1 + c_1(d_{n,3} - d_{n,1}) + c_2(d_{n,4} - d_{n,2}) < 0,$$
$$\Delta_n(0, c) = d_{n,1} + d_{n,2} + d_{n,3} + d_{n,4} - d_{n,0} + f_n(0, 0) > 0.$$

Thus there exists a $\lambda^*(c) > 0$ such that $\Delta_n(\lambda, c)$ attains its global minimum at $\lambda = \lambda^*(c)$. Moreover, λ^* satisfy the equation

$$1 = c_1(d_{n,3}e^{\lambda^* c_1} - d_{n,1}e^{-\lambda^* c_1}) + c_2(d_{n,4}e^{\lambda^* c_2} - d_{n,2}e^{-\lambda^* c_2}), \tag{13}$$

and hence we have $\dfrac{d\lambda^*}{dc} < 0$ by implicit differentiation of (13).

Now we study the behavior of the curve $\lambda^*(c)$. From (13) it is easy to see that $\lambda^*(c) \to \infty$ as $c \to 0^+$, and $\lambda^*(c) \to 0$ as $c \to \infty$. These imply that

$$\lim_{c \to 0^+} \Delta_n(\lambda^*(c), c) = -\infty \quad \text{and} \quad \lim_{c \to \infty} \Delta_n(\lambda^*(c), c) > 0.$$

Furthermore, $\Delta_n(\lambda^*(c), c)$ is monotone increasing with respect to c. Since if $c_1 > c_2$, then

$$\Delta_n(\lambda^*(c_1), c_1) > \Delta_n(\lambda^*(c_1), c_2) \geq \Delta_n(\lambda^*(c_2), c_2).$$

Note that $\Delta_n(\lambda^*(c_2), c_2)$ is a global minimum for such fixed c_2. Hence we can find a particular $c^* \in (0, \tilde{c})$ such that the statement of lemma is true. The proof is complete. ☐

Summarize the above results, we make the following assumptions on reaction terms and coupling coefficients of (4).

(A_1) There exist $K = (k_1, \cdots, k_N)$ and $L = (\ell_1, \cdots, \ell_N)$ in $\mathbb{R}^N$ such that $\mathbf{0} \prec K \preceq L$ and $f_n([K|\ell_n], K^{\widehat{n}}) \leq 0 \leq f_n([L|k_n], L^{\widehat{n}})$ for all n.

(A_2) The functions f_n are C^1 functions from $\mathbb{R}^N \times C^1([-\tau, 0], \mathbb{R})^{N-1}$ to $\mathbb{R}$ and $\partial f_n/\partial x_m \leq 0$ for $1 \leq m \leq 2N - 1$.

(A_3) Assume that for all n $d_{n,3} \geq d_{n,1} > 0$, $d_{n,4} \geq d_{n,2} > 0$, $\sum_{i=1}^4 d_{n,i} = d_{n,0}$, and $f_n(\mathbf{0}, \mathbf{0}) > 0$.

Remark 1. (i) Let us reexamine (4) with the assumptions (A_1)∼(A_3). The assumption (A_1) implies that the systems (4) have a positive equilibrium. The conditions (WQM) is satisfied for the reaction terms if (A_2) is hold. The assumption (A_3) help us to understand the linear behavior of the heteroclinic solution near the trivial solution $\mathbf{0}$. This is crucial to construct a pair of upper-lower solutions in next section.

(ii) The reaction terms of many classical and typical models have the form F_n in (11) and satisfy the assumption (A_2). Here we list some continuous and discrete delayed reaction diffusion equations which we know:

- The logistic scalar equation.

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2}{\partial x^2}u(x,t) + ru(x,t)[1-au(x,t)].$$

- A diffusive delay equation which can be used to model the growth of the population of Daphnia. [32, 16]

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial^2 x} + ru(x,t)\big(\frac{1-au(x,t)}{1+bu(x,t)}\big)$$

- The Belousov-Zhabotinskii reaction model with delay.

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= D\frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)[1-u(x,t)-rv(x,t-\tau_1)],\\ \frac{\partial v(x,t)}{\partial t} &= D\frac{\partial^2}{\partial x^2}v(x,t) - bu(x,t-\tau_2)v(x,t).\end{aligned}$$

- The Lotka-Volterra competition-diffusion systems of N-species equations in the plane.

$$\begin{aligned}\frac{\partial u_n(x,t)}{\partial t} =&d_n\Delta u_n(x,t)\\ &+ u_n(x,t)\big(r_n - p_n u_n(x,t) - \sum_{m=1,m\neq n}^{N} s_{n,m}u_m(x,t-\tau_m)\big),\end{aligned}$$

where $x\in\mathbb{R}^2$, and r_n, p_n and $s_{n,m}$ are nonnegative constants.

3. **Construction of upper and lower solutions.** This section is devoted to the construction of upper and lower solutions of (4). First, we give the definition.

Definition 3.1. Assume $\Phi = \big(\phi_1,\cdots,\phi_N\big)$ and $\Psi = \big(\psi_1,\cdots,\psi_N\big)$ belong to $C(\mathbb{R},\mathbb{R})^N$ such that $\mathbf{0}\preceq \Phi,\Psi\preceq M = (M_1,\cdots,M_N)\succeq 0$. Then Ψ and Φ are called an upper solution and a lower solution of (4) respectively, if they are differentiable almost everywhere and satisfy

(1) $\phi_n'(t)\le \mathcal{L}_n[\phi_n](t) + \phi_n(t)f_n([\Psi|\phi_n](t),\Psi_t^{\widehat{n}})$, $\quad n=1,\cdots,N,\quad$ a.e.;
(2) $\psi_n'(t)\ge \mathcal{L}_n[\psi_n](t) + \psi_n(t)f_n([\Phi|\psi_n](t),\Phi_t^{\widehat{n}})$, $\quad n=1,\cdots,N,\quad$ a.e..

Now we construct a pair of upper-lower solutions of (4). First, let η be the number satisfying

$$1<\eta<\min\Big\{\frac{\lambda_{n,2}}{\lambda_{n,1}},\frac{\lambda_{n,1}+\lambda_{m,1}}{\lambda_{n,1}}\Big|\ m,n=1,\cdots,N\Big\}\le 2. \tag{14}$$

For $\delta>1$, we define functions $h_n(t)$ by

$$h_n(t) := e^{\lambda_{n,1}t} - \delta e^{\eta\lambda_{n,1}t},\ \ n=1,\cdots,N. \tag{15}$$

Then it is easy to see that

$$\lim_{t\to-\infty}h_n(t)=0,\ \lim_{t\to\infty}h_n(t)=-\infty \text{ and } h_n'(t)=\lambda_{n,1}(e^{\lambda_{n,1}t}-\delta\eta e^{\eta\lambda_{n,1}t}).$$

Thus there exists a unique $t_n^\star(\delta)<0$ such that

$$h_n^\star := h_n(t_n^\star) = \max_{t\in\mathbb{R}} h_n(t) > 0 \ \text{ and } \ \lim_{\delta\to\infty} t_n^\star(\delta) = -\infty.$$

If δ is large enough then $h_n(0) < 0$ and there exists $\sigma_n > 1$ such that the length of the intervals $I_n := \{t|\ h_n(t) \geq h_n^\star/\sigma_n,\ t \in \mathbb{R}\}$ is equal to $\max\{c^*, \tau\}$ for $n = 1, \cdots, N$. Denote $\sigma := \max\{\sigma_n|\ n = 1, \cdots, N\}$ and $t_n(\delta)$ by

$$t_n(\delta) := \max\{t|\ h_n(t) = h_n^\star/\sigma\}, \quad n = 1, \cdots, N,$$

then $\lim_{\delta\to\infty} t_n(\delta) = -\infty$ and $h_n^\star < \phi_n^\star$ for all n when δ is large enough. Furthermore, for any $\gamma > 0$, let $\varepsilon_n > 0$, $n = 1, \cdots, N$ be such that

$$h_n(t_n) = \phi_n^\star - \varepsilon_n e^{-\gamma t_n},\ (\text{or } \varepsilon_n = (\phi_n^\star - h_n^\star/\sigma)e^{\gamma t_n}), \tag{16}$$

then is easy to see that $h_n(t) > h_n(t_n)$ for $t_n - c^* < t < t_n$.

Next, we assume that there exist $\widehat{\varepsilon}_n > 0$, $n = 1, \cdots, N$ satisfying the following assumption:

$$(\mathrm{A}_4)\ \begin{cases} \varepsilon_n \dfrac{\partial f_n}{\partial x_n}(X,Y) < \displaystyle\sum_{m=1, m\neq n}^{N} \widehat{\varepsilon}_m \big(\dfrac{\partial f_n}{\partial x_m}(X,Y) + \dfrac{\partial f_n}{\partial y_m}(X,Y)\big), \\ \widehat{\varepsilon}_n \dfrac{\partial f_n}{\partial x_n}(X,Y) < \displaystyle\sum_{m=1, m\neq n}^{N} \varepsilon_m \big(\dfrac{\partial f_n}{\partial x_m}(X,Y) + \dfrac{\partial f_n}{\partial y_m}(X,Y)\big), \end{cases}$$

for $X, Y \in \mathbb{R}^N \times \mathbb{R}^{N-1}$. We further define the numbers $\widehat{t}_n$, $n = 1, \cdots, N$ by

$$\phi_n^\star + \widehat{\varepsilon}_n e^{-\gamma \widehat{t}_n} = e^{\lambda_{n,1}\widehat{t}_n}, \quad n = 1, \cdots, N. \tag{17}$$

Then it is obvious that $\min\{\widehat{t}_n|\ n = 1, \cdots, N\} > \tau + \max\{t_n|\ n = 1, \cdots, N\}$ if δ is large enough, since $\widehat{t}_n$ is bounded below by (17).

Finally, we define the functions ϕ_n^-, ϕ_n^+, $n = 1, \cdots, N$ by

$$\phi_n^-(t) := \begin{cases} e^{\lambda_{n,1}t} - \delta e^{\eta\lambda_{n,1}t}, & t \leq t_n, \\ \phi_n^\star - \varepsilon_n e^{-\gamma t}, & t > t_n, \end{cases} \qquad \phi_n^+(t) := \begin{cases} e^{\lambda_{n,1}t}, & t \leq \widehat{t}_n, \\ \phi_n^\star + \widehat{\varepsilon}_n e^{-\gamma t}, & t > \widehat{t}_n, \end{cases}$$

and $\Phi^- := (\phi_1^-, \cdots, \phi_N^-)$ and $\Phi^+ := (\phi_1^+, \cdots, \phi_N^+)$. Then

$$\lim_{t\to-\infty} \Phi^-(t) = \lim_{t\to-\infty} \Phi^+(t) = \mathbf{0} \ \text{ and } \ \lim_{t\to\infty} \Phi^-(t) = \lim_{t\to\infty} \Phi^+(t) = \Phi^\star, \tag{18}$$

and $\Phi^-(t) < \Phi^+(t)$ for $t \leq t_n$ or $t \geq \widehat{t}_n$, see Figure 1. On the interval $[t_n, \widehat{t}_n]$, $\phi_n^-(t)$ and $\phi_n^+(t)$ are concave downwards and concave upwards respectively. Consequently, if $(\phi_n^-(t_n))' \leq (\phi_n^+(t_n))'$ then $\phi_n^-(t) \leq \phi_n^+(t)$ for all $t \geq t_n$ by taking γ satisfying $0 < \gamma \leq \min\limits_{1\leq n\leq N}\{\lambda_{n,1}e^{\lambda_{n,1}t_n}/\varepsilon_n\}$.

Lemma 3.2. *Assume that δ is large enough, $0 < \gamma \leq \min_{1\leq n\leq N}\{\lambda_{n,1}e^{\lambda_{n,1}t_n}/\varepsilon_n\}$ small enough, and there exists positive numbers $\{\widehat{\varepsilon}_n\}_{n=1}^N$ satisfying (A_4). Then Φ^- and Φ^+ are lower and upper solutions of (4) respectively.*

Proof. We only have to show that Φ^- and Φ^+ satisfy the differential inequalities of Definition 3.1 for $t \in \mathbb{R} \setminus \{t_n, \widehat{t}_n|\ n = 1, \cdots, N\}$. To simplify the computations, we introduce the notation

$$(\bar{d}_{n,0}, \bar{d}_{n,1}, \bar{d}_{n,2}, \bar{d}_{n,3}, \bar{d}_{n,4}) := (-d_{n,0}, d_{n,1}, d_{n,2}, d_{n,3}, d_{n,4}),$$
$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) := (0, -c_1, -c_2, c_1, c_2).$$

Then the function $\Delta_n(\lambda, c)$ in the characteristic equations can be rewritten as

$$\Delta_n(\lambda_{n,1}, c) = \ -\lambda_{n,1} + \textstyle\sum_i \bar{d}_{n,i} e^{\lambda_{n,1}\alpha_i} + f_n(0,0) = 0. \tag{19}$$

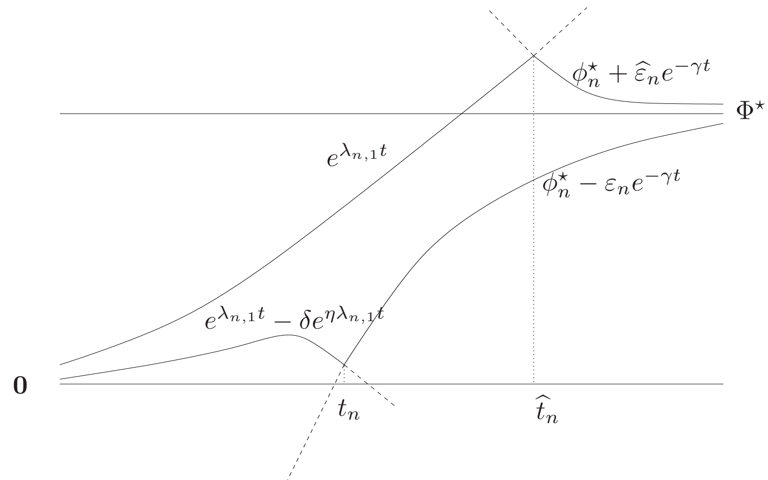

FIGURE 1. Graphs of a pair of upper-lower solutions Φ^+ and Φ^-.

Note that

$$\Delta_n(\eta\lambda_{n,1}, c) = \quad -\eta\lambda_{n,1} + \textstyle\sum_i \bar{d}_{n,i} e^{\eta\lambda_{n,1}\alpha_i} + f_n(0,0) < 0. \tag{20}$$

Now we start the proof of the first differential inequality of Definition 3.1.

If $t \le t_n$, then $(\phi_n^-)'(t) = \lambda_{n,1}e^{\lambda_{n,1}t} - \delta\eta\lambda_{n,1}e^{\eta\lambda_{n,1}t}$. According to equations (19), (20) and the Mean Value Theorem, we have

$$\begin{aligned}
&\mathcal{L}_n[\phi_n^-](t) + \phi_n^-(t) f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) \\
\ge& \sum_i \bar{d}_{n,i}(e^{\lambda_{n,1}(t+\alpha_i)} - \delta e^{\eta\lambda_{n,1}(t+\alpha_i)}) + \phi_n^-(t) f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) \\
=&(\phi_n^-)'(t) - \delta e^{\eta\lambda_{n,1}t}\Delta(\eta\lambda_{n,1}, c) + \phi_n^-(t)\big(f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) - f_n(0,0)\big) \\
=&(\phi_n^-)'(t) - \delta e^{\eta\lambda_{n,1}t}\Delta(\eta\lambda_{n,1}, c) + \phi_n^-(t) Df_n(\Psi_1, \Psi_2) \cdot \big([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}\big),
\end{aligned} \tag{21}$$

for some Ψ_1 and Ψ_2. By (14), we have $-\delta e^{\eta\lambda_{n,1}t}\Delta(\eta\lambda_{n,1}, c) = O(e^{\eta\lambda_{n,1}t})$ and

$$\begin{aligned}
&\phi_n^-(t) Df_n(\Psi_1, \Psi_2) \cdot ([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) \\
=&\phi_n^-(t)\Big(\frac{\partial f_n}{\partial x_n}(\Psi_1, \Psi_2)\phi_n^-(t) + \sum_{m \ne n}\big(\frac{\partial f_n}{\partial x_m}(\Psi_1, \Psi_2)e^{\lambda_{m,1}t} + \frac{\partial f_n}{\partial y_m}(\Psi_1, \Psi_2)e^{\lambda_{m,1}(t-\tau_m)}\big)\Big) \\
=&O(e^{2\lambda_{n,1}t}) + O(e^{(\lambda_{n,1}+\lambda_{m,1})t}),
\end{aligned}$$

as $t \to -\infty$. Since $\Delta(\eta\lambda_{n,1}, c) < 0$, the summation in equation (21) is positive if t_n is small enough.

If $t > t_n$ then

$$\begin{aligned}
&\mathcal{L}_n[\phi_n^-](t) + \phi_n^-(t) f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) - (\phi_n^-)'(t) \\
\ge\ & \sum_i \bar{d}_{n,i}(\phi_n^\star - \varepsilon_n e^{-\gamma(t+\alpha_i)}) + \phi_n^-(t) f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) - \gamma\varepsilon_n e^{-\gamma t} \\
=\ & -\varepsilon_n e^{-\gamma t} I(\gamma) + \phi_n^-(t)\big(f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) - f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}})\big)
\end{aligned}$$

where $I(\gamma) := \sum_i d_{n,i}e^{-\gamma\alpha_i} + f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}}) + \gamma$. Note that $I(\gamma) \to 0$ as $\gamma \to 0^+$. For $t_n < t \le \widehat{t}_n$, we have

$$\begin{aligned}
& f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) - f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}}) \\
= \;& Df_n(\Psi_1, \Psi_2) \cdot ([\Phi^+|\phi_n^-](t) - \Phi^\star, (\Phi_t^+)^{\widehat{n}} - (\Phi^\star)^{\widehat{n}}) \\
\ge \;& -\varepsilon_n e^{-\gamma \widehat{t}_n} \frac{\partial f_n}{\partial x_n}(\Psi_1, \Psi_2) + \sum_{m\neq n} \frac{\partial f_n}{\partial x_m}(\Psi_1, \Psi_2)\widehat{\varepsilon}_m e^{\gamma \widehat{t}_n} + \sum_{m\neq n} \frac{\partial f_n}{\partial y_m}(\Psi_1, \Psi_2)\widehat{\varepsilon}_m e^{\gamma \widehat{t}_n} \\
= \;& e^{-\gamma \widehat{t}_n}\big(- \varepsilon_n \frac{\partial f_n}{\partial x_n}(\Psi_1, \Psi_2) + \sum_{m\neq n} \frac{\partial f_n}{\partial x_m}(\Psi_1, \Psi_2)\widehat{\varepsilon}_m + \sum_{m\neq n} \frac{\partial f_n}{\partial y_m}(\Psi_1, \Psi_2)\widehat{\varepsilon}_m\big) \\
> \;& 0.
\end{aligned}$$

Similarly, for $t \in (\widehat{t}_n, \infty)$ we have

$$\begin{aligned}
& f_n([\Phi^+|\phi_n^-](t), (\Phi_t^+)^{\widehat{n}}) - f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}}) \\
= \;& Df_n(\Psi_1, \Psi_2) \cdot ([\Phi^+|\phi_n^-](t) - \Phi^\star, (\Phi_t^+)^{\widehat{n}} - (\Phi^\star)^{\widehat{n}}) \\
= \;& e^{-\gamma t}\big(- \varepsilon_n \frac{\partial f_n}{\partial x_n}(\Psi_1, \Psi_2) + \sum_{m\neq n} \frac{\partial f_n}{\partial x_m}(\Psi_1, \Psi_2)\widehat{\varepsilon}_m + \sum_{m\neq n} \frac{\partial f_n}{\partial y_m}(\Psi_1, \Psi_2)\widehat{\varepsilon}_m e^{\gamma\tau_m)}\big) \\
> \;& 0.
\end{aligned}$$

Combining the above discussions, if γ is small enough then we obtain the first differential inequality of Definition 3.1.

Next, we prove of the second differential inequality of Definition 3.1. If $t \le \widehat{t}_n$, then $(\phi_n^+)'(t) = \lambda_{n,1}e^{\lambda_{n,1}t}$ and

$$\begin{aligned}
& \mathcal{L}_n[\phi_n^+](t) + \phi_n^+(t) f_n([\Phi^-|\phi_n^+](t), (\Phi_t^-)^{\widehat{n}}) \\
\le \;& \sum_i \bar{d}_{n,i} e^{\lambda_{n,1}(t+\alpha_i)} + e^{\lambda_{n,1}t} f_n([\Phi^-|\phi_n^+](t), (\Phi_t^-)^{\widehat{n}}) \\
= \;& (\phi_n^+)'(t) + e^{\lambda_{n,1}t}\big(f_n([\Phi^-|\phi_n^+](t), (\Phi_t^-)^{\widehat{n}}) - f_n(0, 0^{\widehat{n}})\big) \le (\phi_n^+)'(t)
\end{aligned}$$

for $n = 1, \cdots, N$. On the other hand, if $t > \widehat{t}_n$, then $(\phi_n^+)'(t) = -\gamma\widehat{\varepsilon}_n e^{-\gamma t}$ and

$$\begin{aligned}
& \mathcal{L}_n[\phi_n^+](t) + \phi_n^+(t) f_n([\Phi^-|\phi_n^+](t), (\Phi_t^-)^{\widehat{n}}) - (\phi_n^+)'(t) \\
\le \;& \sum_i \bar{d}_{n,i}(\phi_n^\star + \widehat{\varepsilon}_n e^{-\gamma(t+\alpha_i)}) + \phi_n^+(t) f_n([\Phi^-|\phi_n^+](t), (\Phi_t^-)^{\widehat{n}}) + \gamma\widehat{\varepsilon}_n e^{-\gamma t} \\
= \;& \widehat{\varepsilon}_n e^{-\gamma t} I(\gamma) + \phi_n^+(t)\big(f_n([\Phi^-|\phi_n^+](t), (\Phi_t^-)^{\widehat{n}}) - f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}})\big).
\end{aligned}$$

Similar to previous estimation, $I(\gamma) \to 0$ as $\gamma \to 0^+$ and

$$\begin{aligned}
& f_n([\Phi^-|\phi_n^+](t), (\Phi_t^-)^{\widehat{n}}) - f_n(\Phi^\star, (\Phi^\star)^{\widehat{n}}) \\
= \;& Df_n(\Psi_1, \Psi_2) \cdot ([\Phi^-|\phi_n^+](t) - \Phi^\star, (\Phi_t^-)^{\widehat{n}} - (\Phi^\star)^{\widehat{n}}) \\
= \;& e^{-\gamma t}\big(\widehat{\varepsilon}_n \frac{\partial f_n}{\partial x_n}(\Psi_1, \Psi_2) - \sum_{m\neq n} \frac{\partial f_n}{\partial x_m}(\Psi_1, \Psi_2)\varepsilon_m - \sum_{m\neq n} \frac{\partial f_n}{\partial y_m}(\Psi_1, \Psi_2)\varepsilon_m e^{\gamma\tau_m)}\big) \\
< \;& 0.
\end{aligned}$$

Therefore, the second differential inequality holds when γ is small enough. The proof is complete. ∎

4. **Existence of traveling plane waves.** After constructing the upper and lower solutions of (4) in previous section, now we start to show the existence of traveling plane wave solutions by using the iterative method and Schauder's fixed point theorem.

Denote $C_M(\mathbb{R},\mathbb{R}^N) := \{(u_1,\cdots,u_N)|u_n \in C(\mathbb{R},\mathbb{R}),\, 0 \le u_n \le M,\ n = 1,\cdots,N\}$ where $M = \max_{t\in\mathbb{R},1\le n\le N} \phi_n^+$, and subspace Γ of $C_M(\mathbb{R},\mathbb{R}^N)$ by

$$\Gamma := \{\Phi \in C_M(\mathbb{R},\mathbb{R}^N) \mid \Phi^- \preceq \Phi \preceq \Phi^+\}. \tag{22}$$

Then Γ is closed, convex and bounded under the supremum norm. To apply the Schauder's fixed point theorem for the existence of traveling plane wave solutions, we need the following properties of the operator G on the space Γ.

Lemma 4.1.

(i) G is a continuous operator from $C_M(\mathbb{R},\mathbb{R}^N)$ to $C(\mathbb{R},\mathbb{R}^N)$.

(ii) G is an invariant and compact operator on Γ.

Proof. (i) First, we show that G maps $C_M(\mathbb{R},\mathbb{R}^N)$ into $C(\mathbb{R},\mathbb{R}^N)$. By the definitions of G_n, for any $t \in \mathbb{R}$ and $h > 0$, we have

$$\begin{aligned}
&|G_n(\Phi)(t+h) - G_n(\Phi)(t)| \\
= \;& |e^{-\beta_n(t+h)}\int_{-\infty}^{t+h} e^{\beta_n s}H_n(\Phi)(s)ds - e^{-\beta_n t}\int_{-\infty}^{t} e^{\beta_n s}H_n(\Phi)(s)ds| \\
\le \;& (1-e^{-\beta_n h})\int_{-\infty}^{t} e^{\beta_n(s-t)}|H_n(\Phi)(s)|ds + \int_{t}^{t+h} e^{\beta_n(s-t-h)}|H_n(\Phi)(s)|ds.
\end{aligned}$$

Therefore,

$$\lim_{h\to 0^+} |G_n(\Phi)(t+h) - G_n(\Phi)(t)| = 0, \quad \text{uniformly for } t \in \mathbb{R}.$$

The above result holds similarly for $h < 0$. Hence $G(\Phi) \in C(\mathbb{R},\mathbb{R}^N)$. Moreover, $\{G(\Phi)|\Phi \in C_M(\mathbb{R},\mathbb{R}^N)\}$ is uniformly equicontinuous.Next, by the assumption (A_2) of f_n, if $\Phi,\, \Psi \in C_M(\mathbb{R},\mathbb{R}^N)$ then there exists a constant $L_f(M) > 0$ such that

$$|\phi_n f_n(\Phi,\Psi_t^{\widehat{n}}) - \psi_n f_n(\Psi,\Psi_t^{\widehat{n}})| \le L_f \sum_{n=1}^{N} \|\phi_n - \psi_n\|.$$

Hence

$$\begin{aligned}
|G_n(\Phi)(t) - G_n(\Psi)(t)| \;&\le\; \int_{-\infty}^{t} e^{\beta_n(s-t)}\Big(\big|\phi_n f_n(\Phi,\Phi_t^{\widehat{n}}) - \psi_n f_n(\Psi,\Psi_t^{\widehat{n}})\big| \\
&\qquad\qquad + \beta_n|\phi_n - \psi_n| + |\mathcal{L}_n[\phi_n] - \mathcal{L}_n[\psi_n]|\Big)(s)ds \\
&\le\; (L_f + \beta_f + 2\sum_{n=0}^{4} d_{n,i})(\sum_{n=1}^{N}\|\phi_n - \psi_n\|)\int_{-\infty}^{t} e^{\beta_n(s-t)}ds \\
&\le\; (L_f + \beta_f + 2\sum_{n=0}^{4} d_{n,i})(\sum_{n=1}^{N}\|\phi_n - \psi_n\|)/\beta_n
\end{aligned} \tag{23}$$

for all $n = 1,\cdots,N$, where $\beta_f := \max_{1\le n\le N}\beta_n$. Since the right hand side of above inequality is independent on t, this implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\Phi - \Psi\| < \delta$ then

$$\|G_n(\Phi) - G_n(\Psi)\| < \varepsilon.$$

Therefore the assertion of part (1) follows.(ii) By Lemma 2.3 and the properties of upper and lower solutions, if $t \in \mathbb{R} \setminus \{t_n, \widehat{t}_n\}_{n=1}^N$ then

$$\phi_n^-(t) \le G_n([\Phi^+|\phi_n^-])(t) \le G_n(\Phi^-)(t) \le G_n([\Phi^-|\phi_n^+])(t) \le \phi_n^+(t). \quad (24)$$

Following the same arguments, we have $\Phi^- \le G(\Phi) \le \Phi^+$ for any $\Phi \in \Gamma$. Hence, G is invariant on Γ.

The proof of compactness property for the iterations can be found in [23], so we omit it. The proof is complete. □

By Lemma 4.1, we obtain the following main results.

Main Theorem. *Assume (A_1)-(A_3) and the same assumptions of Lemma 3.2 Then for any $0 < c < c^*$, there exists traveling plane wave solutions of (4) and (5).*

5. **Applications.** In this section we will apply the main theorem to show the existence of traveling plane wave solutions of the Lotka-Volterra N-species competition systems on two-dimensional lattices with discrete diffusion and distributive time delays. The dynamics of this competition model is governed by the following systems:

$$\begin{aligned}\frac{du_{n;i,j}}{dt} =& L_n[u_{n;i,j}] \\ &+ u_{n;i,j}\Big(r_n - p_n u_{n;i,j} - \sum_{m=1,m\neq n}^{N} s_{n,m} \int_{-\tau_n}^{0} k_n(s) u_{m;i,j}(t+s)ds\Big),\end{aligned} \quad (25)$$

for $(i,j) \in \mathbb{Z}^2$, $1 \le n \le N$, where $L_n[u_{n;i,j}](t)$ are defined as before in Section 1, r_n are the natural growth rates; p_n account for self-regulation of each spicy; $s_{n,m}$ are the competing rates; $k_n(s) \in C([-\tau_n, 0], (0,\infty))$ are delay kernels and normalized such that

$$\int_{-\tau_n}^{0} k_n(s)ds = 1, \quad \text{for } n = 1, \cdots, N.$$

The systems (25) model the interaction among various competing species, has been studied extensively, and various sufficient conditions for the coexistence and extinction of the competing species are obtained, cf. [11, 22, 29, 36]. The coefficients $r_n, p_n, s_{n,m}$ play a fundamental row in its asymptotic behavior. In particular, if $N = 2$ and $d_{n,i} = 0$ for all n and i, that is no diffusion term, Gopalsamy studied the stability of the equilibrium of systems (25). For the ecological significance of (25), one can refer to [14, 15] and the references cited therein.

To find a strictly positive stationary solution $\Phi^\star = (\phi_1^\star, \cdots, \phi_N^\star)$ of (25), we should solve the following systems of linear algebraic equations.

$$0 \;=\; d_n + r_n - p_n\phi_n^\star - \sum_{m=1,m\neq n}^{N} s_{n,m}\phi_m^\star, \quad (26)$$

where $d_n := \sum_{i=1}^4 d_{n,i} - d_{n,0}$. Denote the matrices S, P and the vector $\boldsymbol{r}$, $\boldsymbol{d}$ by

$$\begin{aligned}&P = (p_{n,m}) \;\text{ with }\; p_{n,n} = p_n \;\text{ and }\; p_{n,m} = 0, \;\text{ if }\; n \neq m;\\ &S = (s_{n,m}) \;\text{ with }\; s_{n,n} = 0,\; n, m = 1, \cdots, N;\\ &\boldsymbol{r} = (r_1, \cdots, r_N)^T \quad \text{and} \quad \boldsymbol{d} = (d_1, \cdots, d_N)^T.\end{aligned}$$

In matrix form, we have to solve a linear systems

$$(P + S)\Phi^\star \;=\; \boldsymbol{d} + \boldsymbol{r}. \quad (27)$$

It is obvious that the solution $\Phi^\star$ exists and is unique if and only if $P+S$ is nonsingular. More additional conditions on the parameters should be imposed to ensure the strictly positivity of $\Phi^\star$. Hence we assume the following conditions hold for systems (25).

(LV_1) assume $d_{n,3} \geq d_{n,1} > 0$, $d_{n,4} \geq d_{n,2} > 0$ and $d_n + r_n > 0$ for all n;

(LV_2) assume the matrix $P+S$ is nonsingular, and there exists two constant vectors $L \succeq K \succ 0$ such that

$$PK + SL \preceq \boldsymbol{d} + \boldsymbol{r} \preceq PL + SK.$$

Next, let we check assumptions (A_1)$\sim$(A_4). According the conditions (LV_1) and (LV_2), conditions (A_1)$\sim$(A_3) hold obviously. The following we illustrate some specific examples which satisfy (A_4).

Example. Assume $N \geq 2$, $r_n = r$, $d_n = d$, $r + d > 0$, $p_n = 1$ for $n = 1, \cdots, N$, $s_{n,m} = \alpha > 0$ for all $n \neq m$ and $s_{n,n} = 0$ for all n. Let α be small enough, e.g. $(N-1)\alpha < 1$, then there exists a unique positive equilibrium $\Phi^\star = (\phi_1^\star, \cdots, \phi_N^\star)$ with

$$\phi_n^\star = \frac{r_n + d_n}{1 + (N-1)\alpha}, \quad n = 1, \cdots, N.$$

The condition (A_4) is equivalent to

$$\alpha \sum_{m \neq n} \varepsilon_m < \widehat{\varepsilon}_n < \frac{\varepsilon_n}{(N-1)\alpha}, \quad n = 1, \cdots, N.$$

Let us fix any numbers $\varepsilon_1, \cdots, \varepsilon_N$. If α is small enough then it is easy to see that there exist positive numbers $\widehat{\varepsilon}_1, \cdots, \widehat{\varepsilon}_N$ satisfying the above conditions.

Therefore, we have the following results.

Theorem 5.1. *Assume the systems (25) satisfies the assumptions (LV_1)$\sim$(LV_2) and (A_4). Then there exists $c^* > 0$ such that for any $0 < c < c^*$, (25) has a traveling plane wave solution satisfying the asymptotically boundary conditions (5).*

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