Unbiased MMSE vs. Biased MMSE Equalizers

Rainfield Y. Yen

Department of Electrical Engineering, Tamkang University, Tamsui, Taiwan 251, R.O.C.

Abstract

We systematically analyze the biased and unbiased minimum mean square error (MMSE) equalizers of finite as well as infinite length, with and without decision feedback sections. New closed-form expressions of optimum equalizer weights, the MMSE, and symbol error probabilities (SEP), solely in terms of channel response parameters and noise power, are derived for the above receivers. These new expressions have not appeared in the literature and should be included for completeness. We also prove analytically that the biased and unbiased MMSE equalizers have the same optimum weights and that an infinitely long unbiased MMSE equalizer approaches the optimum minimum error probability equalizer. Performance curves are presented and compared for all the receivers discussed. Moreover, for all the infinite length equalizers presented, alternative error probability expressions are provided to best suit computer simulations.

Key Words: Minimum Mean Square Error (MMSE), Channel Equalization, Unbiased Estimate, Symbol Error Probability, Decision Feedback Equalizers (DFEs)

1. Introduction

Although MMSE equalization has been well studied and has become an old subject in today's research, there are still certain areas we feel that need be perfected for more completeness such as those presented in this paper.

Conventional MMSE equalizers adopt a biased decision for signal estimation. That is, the desired signal term in the output estimate is multiplied by a proportionality factor that is the main cursor tap weight coefficient of the total system response [1−3]. As a result, the estimate error is biased. The error probability performance can actually be improved if we multiply the output estimate by the inverse of the above-mentioned proportionality factor, thus obtaining an unbiased MMSE equalizer. In the past, the unbiased MMSE equalizer has not been widely studied. To the author's knowledge, the earliest work of unbiased MMSE is due to Saltzberg [4] who derived a theoretical error bound for the unbiased linear MMSE equalizer. Although Saltzberg corrected the biased part in the equalizer output estimate, he did not use the term 'unbiased' just as no one today refers to the conventional MMSE receiver as biased. Then, until this last decade, Lee and Messerschmitt [5, Ch.10] started to mention about the unbiased MMSE/DFE equalizer. Concurrently, Cioffi, et al. [6] also treated the unbiased MMSE/DFE receiver. Both Lee and Cioffi et al. worked in the frequency domain to deal with the problem (Cioffi used the *D*-transform approach). They essentially addressed the output signal-to-noise ratio (SNR) and capacity issues. Other than the three mentioned above, no one else has ever since touched the problem of unbiased equalizers (at least to the author's knowledge).

There have been many versions of error probability expressions for the well-known biased linear MMSE equalizer dated back to 1960's [1,4,7−12] and many are only error bounds [1,4,10,12]. Perhaps the most recent exact error probability expression for the linear MSE equalizer is the brute force expression given in [1] in terms of all possible information symbol sequences, no-

*Corresponding author. E-mail: rainfieldy@yahoo.com

ise power, and system (channel and equalizer in cascade) parameters [1, (10.2-62)]. That expression proves to be cumbersome and too time consuming. Thus an upper bound is given in terms of system parameters [1, (10.2-64)]. However, the system parameters (q_k's in [1]) are still not explicitly provided. For decision feedback equalizer (DFE) performance, only Monte Carlo simulations are given in [1] for particular channels. As to unbiased MMSE receivers, error probability expressions are not available in the literature. In any case, an error probability expressed mainly in terms of the channel response should be more desirable. Furthermore, expressions for optimum equalizer weights and the MMSE mainly in terms of the channel response are also desirable and should be made available. But these expressions have not been systematically derived or presented in the literature.

In this work, we systematically analyze the biased MMSE and unbiased MMSE receivers of finite and infinite length, with and without DFE for baseband M-ary PAM transmission. We derive exact expressions of optimum MMSE equalizer weights, the MMSE, and the SEP for the various receivers addressed, altogether eight different MMSE cases. All these expressions will be found to be solely in terms of channel response parameters and noise power. We will also present performance comparisons between all the receivers. Furthermore, we prove analytically two important facts that have also not appeared in the literature. (a) The biased and unbiased MMSE equalizers have the same optimum weights. (b) An infinitely long unbiased MMSE equalizer approaches the optimum minimum error probability equalizer.

Section 2 reviews the biased linear MMSE receivers of finite and infinite length. Section 3 analyzes the unbiased linear MMSE receivers of finite and infinite length. The proofs of the two important facts mentioned above are provided in this section. Then, Section 4 discusses the biased and unbiased MMSE/DFE receivers of finite and infinite length. Section 5 presents simulated SEP results of various receivers given in previous sections with performance comparisons. Finally, Section 6 draws conclusions.

2. Biased MMSE Receivers

2.1 Finite Length Equalizers

We consider baseband M-ary PAM transmission. The source signal symbol x_k at any kth time instant randomly takes values with equal probabilities from the set $\{\alpha_m = (2m-1-M)d,\ m = 1, 2, \ldots, M\}$. The constant d is useful for power control. The signal power is thus $\sigma_x^2 = E(x_k^2) = (M^2-1)d^2/3$. The cascaded connection of the transmitter, the channel, a sampler, and a whitened matched filter [1] constitutes an equivalent discrete-time channel response h_k, $k = -L1, \ldots, -1, 0, 1, \ldots, L2$. The channel noise sequence n_k is additive Gaussian with zero mean and variance σ_n^2. The equalizer weights are w_k, $k = -N1, \ldots, -1, 0, 1, \ldots, N2$. The output estimate is given by

$$\begin{aligned}\hat{x}_k &= h_k * w_k * x_k + w_k * n_k = q_k * x_k + w_k * n_k \\ &= q_0 x_k + \sum_{\substack{j=-L1-N1 \\ j\neq 0}}^{L2+N2} q_j x_{k-j} + \sum_{j=-N1}^{N2} w_j n_{k-j}\end{aligned} \quad (1)$$

where * denotes convolution and $q_k = h_k * w_k$. Notice that the desired term in (1) is $q_0 x_k$ rather than x_k and this is why we call the estimate to be biased. It is more convenient to use vector formulation for analysis. Therefore, we define the following vectors:

Source data vector $\mathbf{x} = [x_{k+L1+N1} \ldots x_k \ldots x_{k-L2-N2}]^T$

Noiseless received signal vector $\mathbf{r} = [r_{k+N1} \ldots r_k \ldots r_{k-N2}]^T$

Noise vector $\mathbf{n} = [n_{-N1} \ldots n_0 \ldots n_{N2}]^T$

Equalizer weight vector $\mathbf{w} = [w_{-N1} \ldots w_0 \ldots w_{N2}]^T$

Channel response matrix of size $(N1+N2+1) \times (L1+L2+N1+N2+1)$

$$\mathbf{H} = \begin{bmatrix} h_{-L1} & \cdots & h_{L2} & 0 & \cdots & 0 \\ 0 & h_{-L1} & \cdots & h_{L2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & h_{-L1} & \cdots & h_{L2} \end{bmatrix}$$

Total system response vector

$$\mathbf{q} = \mathbf{H}^T \mathbf{w} = [q_{-L1-N1} \ \cdots q_0 \cdots q_{L2+N2}]^T \quad (2)$$

where T denotes transpose. With the above vector definitions, (1) can now be written as

$$\hat{x}_k = \mathbf{w}^T\mathbf{H}\mathbf{x} + \mathbf{w}^T\mathbf{n} = \mathbf{q}^T\mathbf{x} + \mathbf{w}^T\mathbf{n} \tag{3}$$

The receiver estimate error is

$$\begin{aligned} e_k &= x_k - \hat{x}_k = (1-q_0)x_k - \sum_{\substack{j=-L1-N1 \\ j\neq 0}}^{L2+N2} q_j x_{k-j} \\ &\quad - \sum_{j=-N1}^{N2} w_j n_{k-j} = x_k - \mathbf{q}^T\mathbf{x} - \mathbf{w}^T\mathbf{n} \end{aligned} \tag{4}$$

(1) Optimum Equalizer Weights for the Biased MMSE Receiver:

The mean square error (MSE) for the biased equalizer can be obtained from (4) as

$$J_B = E(e_k^2) = \sigma_x^2(1 - 2q_0 + \|\mathbf{q}\|^2 + N_0^{'}\|\mathbf{w}\|^2) \tag{5}$$

where $\|\mathbf{q}\|$ and $\|\mathbf{w}\|$ are respectively the norms of the vectors $\mathbf{q}$ and $\mathbf{w}$, and $N_0^{'} = \sigma_n^2/\sigma_x^2$. We note that

$$q_0 = \sum_{j=-N1}^{N2} w_j h_{-j} = \mathbf{w}^T\mathbf{h}_B = \mathbf{h}_B^T\mathbf{w} \tag{6}$$

$$\frac{\partial q_0}{\partial \mathbf{w}} = \mathbf{h}_B \tag{7}$$

where $\mathbf{h}_B = [0\ldots0\ h_{L2}\ldots h_0\ldots h_{-L1}\ 0\ldots0]^T$ is the backward vector of $\mathbf{h}$ with padded zeros on both sides to fit into the size of $N1+N2+1$ corresponding to the size of $\mathbf{w}$. Now, setting the gradient of J_B with respect to $\mathbf{w}$ to zero, i.e., $\nabla_{\mathrm{w}} J_B = 0$, we get

$$\begin{aligned} 2\frac{\partial q_0}{\partial \mathbf{w}} &= \frac{\partial \|\mathbf{q}\|^2}{\partial \mathbf{w}} + N_0^{'}\frac{\partial \|\mathbf{w}\|^2}{\partial \mathbf{w}} \\ &= \frac{\partial}{\partial \mathbf{w}}(\mathbf{w}^T\mathbf{H}\mathbf{H}^T\mathbf{w}) + N_0^{'}\frac{\partial \|\mathbf{w}\|^2}{\partial \mathbf{w}} \end{aligned}$$

$$2\mathbf{h}_B = 2\mathbf{H}\mathbf{H}^T\mathbf{w} + 2N_0^{'}\mathbf{w} = 2(\mathbf{H}\mathbf{H}^T + N_0^{'}\mathbf{I})\mathbf{w} \tag{8}$$

whence, the optimum equalizer weight vector is obtained as

$$\mathbf{w}_o = (\mathbf{H}\mathbf{H}^T + N_0^{'}\mathbf{I})^{-1}\mathbf{h}_B \tag{9}$$

and the corresponding optimum $\mathbf{q}$ and q_0 are

$$\mathbf{q}_o = \mathbf{H}^T\mathbf{w}_o = \mathbf{H}^T(\mathbf{H}\mathbf{H}^T + N_0^{'}\mathbf{I})^{-1}\mathbf{h}_B \tag{10}$$

$$q_{0,o} = \mathbf{h}_B^T\mathbf{w}_o = \mathbf{h}_B^T(\mathbf{H}\mathbf{H}^T + N_0^{'}\mathbf{I})^{-1}\mathbf{h}_B \tag{11}$$

(2) Biased MMSE:

Substituting the optimum $\mathbf{w}_o$ of (9) into $\mathbf{q}$ of (2) and q_0 of (6), then into (5), it is straightforward to show that the biased MMSE is given by

$$J_{B,\min} = [1 - \mathbf{h}_B^T(\mathbf{H}\mathbf{H}^T + N_0^{'}\mathbf{I})^{-1}\mathbf{h}_B]\sigma_x^2 = (1 - q_{0,o})\sigma_x^2 \tag{12}$$

It should be noted that (9) through (12) are all expressed in terms of the channel parameters $\mathbf{h}$ and $N_0^{'}$.

(3) Biased SEP:

We consider a given signal vector $\mathbf{x} = [x_{k+L1+N1}\ldots x_k\ldots x_{k-L2-N2}]^T$. Then, e_k is a Gaussian random variable with mean $\mu_B = (1-q_0)x_k - \sum_{\substack{j=-L1-N1 \\ j\neq 0}}^{L2+N2} q_j x_{k-j}$ and variance $\sigma_B^2 = \sigma_n^2\|\mathbf{w}\|^2$. The signal vector $\mathbf{x}$ has $D = M^{L1+L2+N1+N2+1}$ possible outcomes, so does the mean μ_B. It is therefore proper to use an additional subscript i for distinctive outcomes. Thus, we shall now use $\mathbf{x}_i = [x_{k+L1+N1,i}\ldots x_{k,i}\ldots x_{k-L2-N2,i}]^T$ and $\mu_{B,i}$ for the ith outcome, $i = 1, 2, \ldots, D$. Assume $p(\mathbf{x}_i) = 1/D$ for all i. We need to consider 3 situations.

Situation 1: If the component $x_{k,i}$ in a given vector $\mathbf{x}_i$ is $x_{k,i} = \alpha_m = (2m-1-M)d$, $m \neq 1, M$, the SEP is

$$p_i(|e_k| > d) = Q(\frac{d+\mu_{B,i}}{\sigma_B}) + Q(\frac{d-\mu_{B,i}}{\sigma_B}) \tag{13}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-x^2/2}dx$. There are $D(1-2/M)$ such errors out of D possibilities, each with probability $1/D$. Thus, the total SEP for Situation 1 is

$$\begin{aligned} &\frac{1}{D}\sum_{\substack{i=1 \\ x_{k,i}\neq\alpha_1,\alpha_M}}^{D(1-2/M)}\left[Q(\frac{d+\mu_{B,i}}{\sigma_B}) + Q(\frac{d-\mu_{B,i}}{\sigma_B})\right] \\ &= \frac{2}{D}\sum_{\substack{i=1 \\ x_{k,i}\neq\alpha_1,\alpha_M}}^{D(1-2/M)}\left[Q(\frac{d+\mu_{B,i}}{\sigma_B})\right] \\ &= \frac{2}{D}\sum_{\substack{i=1 \\ x_{k,i}\neq\alpha_1,\alpha_M}}^{D(1-2/M)}\left[Q(\frac{d-\mu_{B,i}}{\sigma_B})\right] \end{aligned} \tag{14}$$

The latter two equalities in (14) arise from the symmetry property of the M-PAM constellation.

Situation 2: There are D/M possibilities that $x_{k,i} = \alpha_1 = -(M-1)d$, each with probability $1/D$. The total SEP in this situation is

$$\frac{1}{D}\sum_{\substack{i=1+D(1-2/M)\\ x_{k,i}=\alpha_1}}^{D(1-1/M)} p_i(e_k < -d) = \frac{1}{D}\sum_{\substack{i=1+D(1-2/M)\\ x_{k,i}=\alpha_1}}^{D(1-1/M)} \left[Q(\frac{d+\mu_{B,i}}{\sigma_B})\right] \tag{15}$$

Situation 3: There are D/M possibilities that $x_{k,i} = \alpha_M = (M-1)d$, each with probability $1/D$. The total SEP in this situation is

$$\frac{1}{D}\sum_{\substack{i=1+D(1-1/M)\\ x_{k,i}=\alpha_M}}^{D} p_i(e_k > d) = \frac{1}{D}\sum_{\substack{i=1+D(1-1/M)\\ x_{k,i}=\alpha_M}}^{D} \left[Q(\frac{d-\mu_{B,i}}{\sigma_B})\right] \tag{16}$$

It is easy to see that (15) = (16) due to constellation symmetry. Thus, adding (14), (15), and (16) together, we obtain the total SEP as

$$P_M = \frac{2}{D}\sum_{\substack{i=1\\ x_{k,i}\neq\alpha_M}}^{D(1-1/M)} \left[Q(\frac{d+\mu_{B,i}}{\sigma_B})\right] = \frac{2}{D}\sum_{\substack{i=1\\ x_{k,i}\neq\alpha_1}}^{D(1-1/M)} \left[Q(\frac{d-\mu_{B,i}}{\sigma_B})\right] \tag{17}$$

Now, using vector notations, we write

$$\begin{aligned}\mu_{B,i} &= (1-q_0)x_{k,i} - \sum_{\substack{j=-L1-N1\\ j\neq 0}}^{L2+N2} q_j x_{k-j,i} \\ &= x_{k,i} - \mathbf{q}^T\mathbf{x}_i = x_{k,i} - \mathbf{w}^T\mathbf{H}\mathbf{x}_i\end{aligned} \tag{18}$$

We thus obtain the biased SEP as

$$P_{M,\mathrm{B}}(\mathbf{w}) = \frac{2}{D}\sum_{\substack{i=1\\ x_{k,i}\neq\alpha_1}}^{D(1-1/M)} Q\left(\frac{d - x_{k,i} + \mathbf{w}^T\mathbf{H}\mathbf{x}_i}{\sigma_n \|\mathbf{w}\|}\right) \tag{19a}$$

or

$$P_{M,\mathrm{B}}(\mathbf{w}) = \frac{2}{D}\sum_{\substack{i=1\\ x_{k,i}\neq\alpha_M}}^{D(1-1/M)} Q\left(\frac{d + x_{k,i} - \mathbf{w}^T\mathbf{H}\mathbf{x}_i}{\sigma_n \|\mathbf{w}\|}\right) \tag{19b}$$

Equation (19a) or (19b) are valid for any weight vector $\mathbf{w}$. If we use the optimum $\mathbf{w}_o$ of (9) for $\mathbf{w}$, we get the exact SEP for the biased MMSE equalizer of finite length. Although (19a) and (19b) are still pretty much brute force expressions, but the number of terms has been drastically reduced as compared to (10.2-62) in [1]. Moreover, with the replacement of $\mathbf{w}_o$, only channel parameters in addition to the noise power are involved in the SEP expression rather than the unspecified system parameters involved in (10.2-62) of [1].

2.2 Infinite Length Equalizers

For a biased MMSE equalizer of infinite length, the optimum equalizer weight vector and the MMSE respectively take the same forms of (9) and (12) except that $\mathbf{H}$ and $\mathbf{q}_o$ are now of infinite size (by filling in more zeros) and $\mathbf{I}$ is an infinite size identity matrix. As to the SEP, the infinite length equalizer will yield an expression slightly different as will be seen below.

If the equalizer is of infinite length, we have $N1$, $N2 \to \infty$ in (4) and the signal vector $\mathbf{x}$ is also infinitely long. In (4), we shall hold x_k fixed and treat all other components $x_j, j \neq k$, in $\mathbf{x}$ as random variables. Then, the e_k in (4) is composed of infinite number of independent random variables. By the central limit theorem, e_k approaches (or approximates) a Gaussian random variable with mean $\mu_B = (1 - q_0)x_k$ and variance $\sigma_B^2 = [\sigma_x^2(\|\mathbf{q}\|^2 - q_0^2) + \sigma_n^2\|\mathbf{w}\|^2]$. The x_k has M possible values. Therefore, by the similar fashion in arriving at (19a) and (19b), we can obtain the SEP for the biased equalizer of infinite length as

$$P_{M,B}(\mathbf{w}) = \frac{2}{M}\sum_{x_k=\alpha_2}^{\alpha_M} Q\left(\frac{d-(1-q_0)x_k}{\sigma_B}\right) \tag{20a}$$

$$= \frac{2}{M}\sum_{x_k=\alpha_1}^{\alpha_{M-1}} Q\left(\frac{d+(1-q_0)x_k}{\sigma_B}\right) \tag{20b}$$

Replacing the $\mathbf{w}$, $\mathbf{q}$, and q_0 within σ_B by the infinite size optimum $\mathbf{w}_o$ of (9) and the optimum $\mathbf{q}_o$ and $q_{0,o}$ given by (10) and (11), we get the SEP in terms of channel information for the biased MMSE equalizer of infinite length. However, (20a) or (20b) is not practical for computer simulations since the quantity σ_B contains the infinitely long vectors $\mathbf{q}$ and $\mathbf{w}$. To overcome this difficulty, we can carry one step further by noting that, under the optimum MMSE condition, we can apply (5) and (12) to get

$$\sigma_B^2 = J_{B,\min} - (1-q_{0,o})^2\sigma_x^2 = q_{0,o}(1-q_{0,o})\sigma_x^2 \qquad (21)$$

Substituting (21) into (20) and replacing $d = \sigma_x\sqrt{\frac{3}{M^2-1}}$, we get the alternative SEP expressions for the biased MMSE equalizer of infinite length as

$$P_{M,B}(\mathbf{w}_o) = \frac{2}{M}\sum_{x_k=\alpha_2}^{\alpha_M} Q\left(\sqrt{\frac{3}{q_{0,o}(1-q_{0,o})(M^2-1)}} - \frac{x_k}{q_{0,o}\sigma_x^2}\right) \qquad (22a)$$

$$= \frac{2}{M}\sum_{x_k=\alpha_1}^{\alpha_{M-1}} Q\left(\sqrt{\frac{3}{q_{0,o}(1-q_{0,o})(M^2-1)}} - \frac{x_k}{q_{0,o}\sigma_x^2}\right) \qquad (22b)$$

These expressions become useful for computer simulations as will be explained below.

It is well known that the frequency domain solution for an infinite length MMSE equalizer (biased) is given by [1]

$$W(\theta) = \frac{H^*(\theta)}{|H(\theta)|^2 + N_0'} \qquad (23)$$

where * denotes complex conjugation, θ is the discrete-time frequency in radians, and $W(\theta)$ and $H(\theta)$ are respectively the discrete-time Fourier transform (DTFT) of w_k and h_k. Further, we know that

$$q_{0,o} = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta)W(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|H(\theta)|^2}{|H(\theta)|^2 + N_0'}d\theta \qquad (24)$$

where $Q(\theta)$is the DTFT of q_k. The integral term in (24) is in terms of the channel frequency response $H(\theta)$ and can certainly be evaluated by computer simulation. We note here that, by experience, if using a sufficiently long equalizer, using the optimum $q_{0,o}$ in (21) and then in (20) or simply using the optimum $q_{0,o} = \mathbf{h}_B^T(\mathbf{HH}^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B$ of (11) in (22) will give quite satisfactory results, almost undetected by eye but actually less accurate than the results obtained by substituting (24) into (22). Moreover, computing $q_{0,o}$ by (11) in the time domain will take much longer time than computing $q_{0,o}$ by (24) in the frequency domain. This is because computing the inverse of an extremely large matrix will take much time. For our system examples, computing $q_{0,o}$ using (10) for the extremely long equalizer system takes about 40–80 times longer time than using (24).

3. Unbiased MMSE Receivers

Figure 1 presents the configuration of an unbiased equalizer system. The major difference of an unbiased system from a biased system is the multiplication by 1/ q'_0 at the equalizer output. Note also that, in Figure 1, $\hat{x}_k$, w_k, and q_0 have been replaced by $\hat{x}'_k$, w'_k, and q'_0 to distinguish unbiased quantities from the biased counterparts.

3.1 Finite Length Equalizers

For an unbiased equalizer, (3) must be modified as

$$\hat{x}'_k = \frac{1}{q'_0}(\mathbf{w}'^T\mathbf{Hx} + \mathbf{w}'^T\mathbf{n}) = \frac{1}{q'_0}(\mathbf{q}'^T\mathbf{x} + \mathbf{w}'^T\mathbf{n}) = x_k + \frac{1}{q'_0}\left(\sum_{\substack{j=-L1-N1\\ j\neq 0}}^{L2+N2} q'_j x_{k-j} + \sum_{j=-N1}^{N2} w'_j n_{k-j}\right) \qquad (25)$$

As in Figure 1, we have used a prime superscript on the relevant quantities to distinguish them from the biased counterparts. Note now that the desired term in (25) is x_k and hence the estimate is unbiased. The unbiased estimate error and MSE are then given by

$$e'_k = x_k - \hat{x}'_k = -\frac{1}{q'_0}\left(\sum_{\substack{j=-L1-N1\\ j\neq 0}}^{L2+N2} q'_j x_{k-j} + \sum_{j=-N1}^{N2} w'_j n_{k-j}\right) \qquad (26)$$

$$J_U = E(e'^2_k) = \frac{\sigma_x^2}{q'^2_0}(\|\mathbf{q}'\|^2 - q'^2_0 + N_0'\|\mathbf{w}'\|^2) \qquad (27)$$

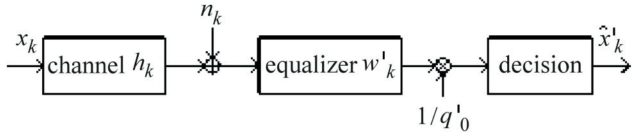

Figure 1. The unbiased equalizer model.

(1) Optimum Weighs for the Unbiased MMSE Equalizer:

To obtain optimum weight vector of the unbiased MMSE equalizer, we note that to minimize J_U with respect to $\mathbf{w}'$ is the same as to minimize the following quantity with respect to $\mathbf{w}'$

$$\frac{1}{q'^2_0}(\|\mathbf{q}'\|^2 + N_0' \|\mathbf{w}'\|^2) \tag{28}$$

Setting the gradient of (28) with respect to $\mathbf{w}'$ to equal zero, we get

$$q'_0(\mathbf{H}\mathbf{H}^T + N_0'\mathbf{I})\mathbf{w}' = \mathbf{w}'^T(\mathbf{H}\mathbf{H}^T + N_0'\mathbf{I})\mathbf{w}'\frac{\partial q'_0}{\partial \mathbf{w}'} \tag{29}$$

or

$$\mathbf{w}'^T\mathbf{h}_B(\mathbf{H}\mathbf{H}^T + N_0'\mathbf{I})\mathbf{w}' = \mathbf{w}'^T(\mathbf{H}\mathbf{H}^T + N_0'\mathbf{I})\mathbf{w}'\mathbf{h}_B \tag{30}$$

A solution to (29) can be found by inspection as

$$\mathbf{w}'_o = C(\mathbf{H}\mathbf{H}^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B = C\mathbf{w}_o \tag{31}$$

where C is an arbitrary constant. The reader can verify (31) by substituting (31) into (30). Thus, the unbiased MMSE receiver has infinitely many solutions as C can have infinitely many values. Simply choose $C = 1$. Then, the optimum weights of the unbiased MMSE equalizer is exactly the same as those of the biased MMSE equalizer as given by (a).

(2) Unbiased MMSE:

Choosing, $C = 1$, $\mathbf{w}'_o = \mathbf{w}_o$, $q'_{0,o} = q_{0,o}$, and $\mathbf{q}'_o = \mathbf{q}_o = \mathbf{H}^T\mathbf{w}_o$, we can easily prove that the unbiased MMSE is

$$\begin{aligned} J_{U,\min} &= E(e'^2_k) = \frac{\sigma_x^2}{q_{0,o}^2}(\|\mathbf{q}_o\|^2 - q_{0,o}^2 + N_0'\|\mathbf{w}_o\|^2) \\ &= \frac{J_{B,\min} - (1-q_{0,o})^2\sigma_x^2}{q_{0,o}^2} = \frac{(1-q_{0,o})\sigma_x^2}{q_{0,o}} \end{aligned} \tag{32}$$

where, in terms of channel response information, $q_{0,o} = \mathbf{h}_B^T(\mathbf{H}\mathbf{H}^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B$ as given by (11). Since $q_{0,o} < 1$ [see (24)] and $q_{0,o}$ is the same for both biased and unbiased MMSE receiver as indicated above, comparing (32) with (12), we find that $J_{B,\min}/J_{U,\min} = q_{0,o} < 1$. Hence, the unbiased MMSE is greater than the biased MMSE.

(3) Unbiased SEP:

Starting with (26) and following the same procedure as for the biased receiver case, we can easily derive the SEP for the unbiased receiver. First, we find the mean and variance of the unbiased random error variable e'_k for an ith outcome of signal vector $\mathbf{x}_i$ as

$$\mu_{U,i} = -\frac{1}{q_0}\sum_{\substack{j=-L1-N1 \\ j\neq 0}}^{L2+N2} q_j x_{k-j,i} \tag{33}$$

$$\sigma_U^2 = \frac{\sigma_n^2}{q_0^2}\|\mathbf{w}\|^2 \tag{34}$$

Then, the unbiased SEP is calculated as

$$P_{M,U}(\mathbf{w}) = \frac{2}{D}\sum_{\substack{i=1 \\ x_k\neq\alpha_1}}^{D(1-1/M)} Q\left(\frac{q_0 d - q_0 x_{k,i} + \mathbf{w}^T\mathbf{H}\mathbf{x}_i}{\sigma_n\|\mathbf{w}\|}\right) \tag{35a}$$

$$= \frac{2}{D}\sum_{\substack{i=1 \\ x_k\neq\alpha_M}}^{D(1-1/M)} Q\left(\frac{q_0 d + q_0 x_{k,i} - \mathbf{w}^T\mathbf{H}\mathbf{x}_i}{\sigma_n\|\mathbf{w}\|}\right) \tag{35b}$$

Replacing the $\mathbf{w}$ in the above equations by the optimum $\mathbf{w}_o$, we get the exact SEP in term of channel response parameters for the unbiased MMSE equalizer of finite length.

3.2 Infinite Length Equalizers

For an unbiased MMSE equalizer of infinite length, the optimum equalizer weight vector and the MMSE take the same forms of (31) and (32) respectively. As to SEP, the infinite length unbiased equalizer will yield an expression much neater.

From (26) and (27), we see that, for any fixed x_k with all other components x_j, $j \neq k$ treated as random variables, the e'_k approximates a Gaussian random variable with zero mean and variance $\sigma_U^2 = \frac{1}{q_0^2}[\sigma_x^2(\|\mathbf{q}\|^2 - q_0^2) + \sigma_n^2\|\mathbf{w}\|^2] = J_U$. Thus, for all M possible values of x_k, we have the identical e'_k. Following the same fashion in deriving (35), we obtain the SEP for the infinite length unbiased equalizer as

$$P_{M,U}(\mathbf{w}) = \frac{2(M-1)}{M} Q\left(\frac{d}{\sigma_U}\right) = \frac{2(M-1)}{M} Q\left(\frac{q_0 d}{\sqrt{\sigma_x^2(\|\mathbf{q}\|^2 - q_0^2) + \sigma_n^2 \|\mathbf{w}\|^2}}\right) \quad (36)$$

Notice that, instead of a sum of Q functions, we now have only one Q function term. This offers another advantage. We can now minimize the SEP by simply maximizing the argument within the Q function. Moreover, this argument is simply $\sqrt{\frac{3\sigma_x^2}{(M^2-1)J_U}}$, i.e., the argument of the Q function is inversely proportional to the square-root of the unbiased MSE. We conclude that the minimum error probability is

$$P_{M,U,\min} = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{3\sigma_x^2}{(M^2-1)J_{U,\min}}}\right) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{3q_{0,o}}{(M^2-1)(1-q_{0,o})}}\right) \quad (37)$$

where $q_{0,o} = \mathbf{h}_B^T(\mathbf{H}\mathbf{H}^T + N_0^{'}\mathbf{I})^{-1}\mathbf{h}_B$. We thus found that an unbiased MMSE receiver of infinite length approaches the minimum error probability receiver. Again, for computer simulations, $q_{0,o}$ is better expressed in frequency-domain as given by (24) to avoid dealing with infinite size vectors.

4. Biased and Unbiased MMSE/DFE Receivers

A considerable gain in SEP performance can be achieved relative to the linear equalizer by use of DFE [1]. Essentially, a DFE can help reduce spectral null effect to mitigate noise enhancement [1]. We shall now perform similar analysis on biased and unbiased MMSE/DFE receivers as done for the linear MMSE receivers in the previous sections.

4.1 Biased MMSE/DFE Receivers

Let the decision feedback section have tap weights b_k, $k = 1, 2, \ldots, K$, and $K = L2 + N2$ [13]. Then the estimate output $\hat{x}_k$ is given by

$$\begin{aligned}\hat{x}_k &= w_k * h_k * x_k - b_k * \hat{x}_k + w_k * n_k \\ &= q_k * x_k - b_k * \hat{x}_k + w_k * n_k \\ &= q_0 x_k + \sum_{\substack{j=-L1-N1 \\ j\neq 0}}^{L2+N2} q_j x_{k-j} - \sum_{j=1}^{K} b_j x_{k-j} + \sum_{j=-N1}^{N2} w_j n_{k-j}\end{aligned} \quad (38)$$

We further define a feedback weight vector of size $(L1+L2+N1+N2+1) \times 1$ as

$$\mathbf{b} = [0 \cdots 0 \; b_1 \cdots b_{L2+N2}]^T \quad (39)$$

Assuming prior decisions are correct, (38) can now be written as

$$\hat{x}_k = \mathbf{w}^T\mathbf{H}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathbf{w}^T\mathbf{n} = \mathbf{q}^T\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathbf{w}^T\mathbf{n} \quad (40)$$

The receiver estimate error is given by

$$\begin{aligned}e_k &= x_k - \hat{x}_k = (1-q_0)x_k - \sum_{\substack{j=-L1-N1 \\ j\neq 0}}^{L2+N2} q_j x_{k-j} \\ &+ \sum_{j=1}^{K} b_j x_{k-j} - \sum_{j=-N1}^{N2} w_j n_{k-j} = x_k - \mathbf{q}^T\mathbf{x} + \mathbf{b}^T\mathbf{x} - \mathbf{w}^T\mathbf{n}\end{aligned} \quad (41)$$

The MSE can be obtained from (41) as

$$J_B = E(e_k^2) = \sigma_x^2(1 - 2q_0 + \|\mathbf{q} - \mathbf{b}\|^2 + N_0^{'}\|\mathbf{w}\|^2) \quad (42)$$

We want to find the optimum weight vectors $\mathbf{b}_o$ and $\mathbf{w}_o$. We will first define two vectors of size $(L1+N1+N2+L2+1) \times 1$ as

$$\mathbf{q}_+ = [0 \cdots 0 \; q_1 \cdots q_{L2+N2}]^T \quad (43a)$$

$$\mathbf{q}_- = [q_{-L1-N1} \cdots q_0 \; 0 \cdots 0]^T \quad (43b)$$

Then, define $\mathbf{H}_L$ as the left $(N1+N2+1) \times (L1+N1+1)$ sub-matrix of $\mathbf{H}$ and $\mathbf{H}_R$ as the right $(N1+N2+1) \times (L2+N2)$ sub-matrix of $\mathbf{H}$. Further note that [see definition of $\mathbf{b}$ in (39)]

$$\mathbf{q}^T\mathbf{b} = \mathbf{q}_+^T\mathbf{b} \tag{44}$$

Now, by setting the gradients $\nabla_b J_B = 0$, $\nabla_w J_B = 0$, we get

$$\mathbf{b}_o = \mathbf{q}_+ \tag{45}$$

$$\begin{aligned}(\mathbf{HH}^T + N_0'\mathbf{I})\mathbf{w}_o &= \mathbf{h}_B + \mathbf{Hb}_o = \mathbf{h}_B + \mathbf{Hq}_+ \\ &= \mathbf{h}_B + \mathbf{Hq}_+ = \mathbf{h}_B + \mathbf{H}_R\mathbf{H}_R^T\mathbf{w}_o\end{aligned} \tag{46}$$

so

$$\begin{aligned}\mathbf{w}_o &= (\mathbf{HH}^T - \mathbf{H}_R\mathbf{H}_R^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B \\ &= (\mathbf{H}_L\mathbf{H}_L^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B = (\mathbf{H}_L\mathbf{H}_L^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B\end{aligned} \tag{47}$$

and

$$\mathbf{q}_o = \mathbf{H}^T\mathbf{w}_o = \mathbf{H}^T(\mathbf{H}_L\mathbf{H}_L^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B \tag{48a}$$

$$q_{0,o} = \mathbf{h}_B^T\mathbf{w}_o = \mathbf{h}_B^T(\mathbf{HH}^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B \tag{48b}$$

Equations (45) and (47), (48a), (48b) can all be expressed in terms of channel response parameters and are applicable to biased DFE of both finite and infinite length.

Now, substituting (45) and (47), (48a), (48b) into (42), we get

$$\begin{aligned}J_{B,\min} &= (1 - 2q_{0,o} + \|\mathbf{q}_-\|^2 + N_o'\|\mathbf{w}_o\|^2)\sigma_x^2 \\ &= (1 - 2\mathbf{w}_o^T\mathbf{h}_B + \mathbf{w}_o^T\mathbf{H}_L\mathbf{H}_L^T\mathbf{w}_o + N_0'\mathbf{w}_o^T\mathbf{w}_o)\sigma_x^2 \\ &= [1 - 2\mathbf{w}_o^T\mathbf{h}_B + \mathbf{w}_o^T(\mathbf{H}_L\mathbf{H}_L^T + N_0'\mathbf{I})\mathbf{w}_o]\sigma_x^2 \\ &= [1 - \mathbf{w}_o^T\mathbf{h}_B]\sigma_x^2 = [1 - \mathbf{h}_B^T(\mathbf{H}_L\mathbf{H}_L^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B]\sigma_x^2 \\ &= (1 - q_{0,o})\sigma_x^2\end{aligned} \tag{49}$$

This result is similar to that of the biased linear equalizer given by (12) and is applicable to the biased DFE of both finite and infinite length except that now $q_{0,o} = \mathbf{h}_B^T(\mathbf{H}_L\mathbf{H}_L^T + N_0'\mathbf{I})^{-1}\mathbf{h}_B$ as given by (48b). For the infinite length case, the vectors in (48b) are of infinite size. In (49), we have used the facts

$$\mathbf{q} - \mathbf{b} = \mathbf{q}_- \tag{50}$$

$$\mathbf{q}_- = \mathbf{H}_L^T\mathbf{w} \tag{51}$$

Finally, by the same fashion as done for the biased linear equalizer, we can find the SEP for the finite length biased DFE as

$$P_{M,B}(\mathbf{w},\mathbf{b}) = \frac{2}{D}\sum_{\substack{i=1\\ x_k \neq \alpha_1}}^{D(1-1/M)} Q\left(\frac{d - x_{k,i} + \mathbf{w}^T\mathbf{Hx}_i - \mathbf{b}^T\mathbf{x}_i}{\sigma_n\|\mathbf{w}\|}\right) \tag{52a}$$

$$= \frac{2}{D}\sum_{\substack{i=1\\ x_k \neq \alpha_M}}^{D(1-1/M)} Q\left(\frac{d + x_{k,i} - \mathbf{w}^T\mathbf{Hx}_i + \mathbf{b}^T\mathbf{x}_i}{\sigma_n\|\mathbf{w}\|}\right) \tag{52b}$$

For the infinite length biased DFE, the SEP is

$$P_{M,B}(\mathbf{w},\mathbf{b}) = \frac{2}{M}\sum_{x_k=\alpha_2}^{\alpha_M} Q\left(\frac{d - (1 - q_0)x_k}{\sigma_B}\right) \tag{53a}$$

$$= \frac{2}{M}\sum_{x_k=\alpha_1}^{\alpha_{M-1}} Q\left(\frac{d + (1 - q_0)x_k}{\sigma_B}\right) \tag{53b}$$

where $\sigma_B^2 = [\sigma_x^2(\|\mathbf{q} - \mathbf{b}\|^2 - q_0^2) + \sigma_n^2\|\mathbf{w}\|^2]$. Note that although (53a) and (53b) look like (20a) and (20b), but the σ_B is different. Now, using the optimum $q_{0,o}$, $\mathbf{q}_o$, $\mathbf{b}_o$, and $\mathbf{w}_o$ obtained above in (52) and (53), we get the corresponding SEP's in terms of channel information for the biased MMSE/DFE receivers. Again, as for the biased linear MMSE equalizer of infinite length, alternative expressions for (53) can be also obtained where only $q_{0,o}$ (not $\mathbf{b}_o$ and $\mathbf{w}_o$) needs be computed. It has been shown by Salz [13] that, for MMSE/DFE receivers (biased), the MMSE is given by

$$J_{B,\min} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left[\frac{N_0'}{|H(\theta)|^2 + N_0'}\right]d\theta\right\} \tag{54}$$

Hence, by (49), we can obtain $q_{0,o}$ in frequency domain expression as

$$q_{0,o} = 1 - \frac{J_{B,\min}}{\sigma_x^2} = 1 - \frac{1}{\sigma_x^2}\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left[\frac{N_0'}{|H(\theta)|^2 + N_0'}\right]d\theta\right\} \tag{55}$$

Then, by noting that $\sigma_B^2 = J_{B,\min} - (1 - q_{0,o})^2\sigma_x^2 = q_{0,o}(1 - q_{0,o})\sigma_x^2$, we can modify (53a) and (53b) into the forms just like (22a) and (22b) but with $q_{0,o}$ in terms of channel frequency response given by (55) for computer simulations.

4.2 Unbiased MMSE/DFE Receivers

Using the same procedure as done before, we can readily obtain the optimum feedback and feedforward weight vectors for the unbiased MMSE/DFE receivers of both finite and infinite length to be the same as given by (45) and (47). The MMSE is

$$J_{U,\min} = \frac{(1-q_{0,o})\sigma_x^2}{q_{0,o}} \tag{56}$$

where $q_{0,o} = \mathbf{h}_B^T(\mathbf{H}_L\mathbf{H}_L^T + N_0^{'}\mathbf{I})^{-1}\mathbf{h}_B$. Equation (56) also holds for finite as well as infinite length MMSE/DFE receivers.

The SEP for the finite length unbiased DFE can be found as

$$P_{M,U}(\mathbf{w},\mathbf{b}) = \frac{2}{D}\sum_{\substack{i=1\\ x_k \neq \alpha_1}}^{D(1-1/M)} Q\left(\frac{q_0 d - q_0 x_{k,i} + \mathbf{w}^T\mathbf{H}\mathbf{x}_i - \mathbf{b}^T\mathbf{x}_i}{\sigma_n \|\mathbf{w}\|}\right) \tag{57a}$$

$$= \frac{2}{D}\sum_{\substack{i=1\\ x_k \neq \alpha_M}}^{D(1-1/M)} Q\left(\frac{q_0 d + q_0 x_{k,i} - \mathbf{w}^T\mathbf{H}\mathbf{x}_i + \mathbf{b}^T\mathbf{x}_i}{\sigma_n \|\mathbf{w}\|}\right) \tag{57b}$$

If optimum $\mathbf{b}_o$ and $\mathbf{w}_o$ given by (45) and (47) are used for (57), we will get the MMSE SEP.

For the infinite length unbiased DFE, the SEP is

$$P_{M,U}(\mathbf{w},\mathbf{b}) = \frac{2(M-1)}{M} Q\left(\frac{d}{J_U}\right) \tag{58}$$

The *minimum* error probability for the infinite length unbiased DFE receiver (also MMSE) is

$$\begin{aligned} P_{M,U,\min} &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{3\sigma_x^2}{(M^2-1)J_{U,\min}}}\right) \\ &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{3q_{0,o}}{(M^2-1)(1-q_{0,o})}}\right) \end{aligned} \tag{59}$$

which bears the exact resemblance to (37) for the infinite length unbiased linear MMSE receiver except that $q_{0,o}$ is now different. Here, $q_{0,o} = \mathbf{h}_B^T(\mathbf{H}_L\mathbf{H}_L^T + N_0^{'}\mathbf{I})^{-1}\mathbf{h}_B$. Since the unbiased $q_{0,o}$ and biased $q_{0,o}$ are the same, we can also express the $q_{0,o}$ in (59) by the frequency-domain expression of (55) for computer simulations.

5. Simulation Results

We now compare the performances of the various equalizers discussed in previous sections by computer simulations. We will present two channel models for 4-PAM transmission, one with length 2 and one with length 4. Although length 2 is short, we purposely select a high sub-cursor so that it will give rise to high intersymbol interferences. This will be sufficient for the demonstration purpose. To make it more convincing, the channel of longer length of 4 is given for the second example. In the literature of channel equalization, it is standard practice to choose channel lengths between 2 and 5 in simulations [14–20]. Usually, these lengths are sufficient to demonstrate the purposes. Longer channel lengths require longer equalizers thus increase simulation time and serve no better purpose, Hence, unless necessary, there is no need to use long length channels for demonstration purposes. In the field of minimum error rate equalization, channels of length 2 have been used in [14,15,19]; channels of length 3 have been adopted in [16–20]; channels of length 4 have been taken in [17,21,22] and length 5 in [14,19].

Figure 2 presents SEP vs. SNR curves for MMSE and MMSE/DFE receivers of finite length, with biased and unbiased decisions for the channel $\mathbf{h} = [1, 0.46]$. Figure 3 presents the same curves for the infinite length counterparts for the same channel. As expected, unbiased receivers always outperform biased receivers, DFE receivers always outperform linear receivers, and infinite length receivers outperform finite length receivers. However, for the MMSE/DFE receivers, the biased and unbiased curves seem to coincide with each other. In fact, the unbiased receiver still is better than the biased counterpart by a very slight amount undetectable by eye. This means the DFE has already improve the performance to a degree that will leave very little room for the unbiased

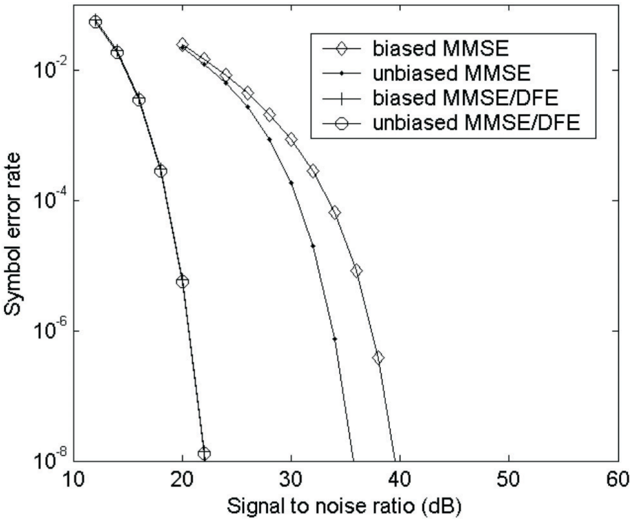

Figure 2. Symbol error rate curves for various finite length MMSE equalizers for 4-PAM transmission in channel **h** = [1, 0.46]. For Linear MMSE, equalizer length = 3. For MMSE/DFE, feedforward filter length = 3 and feedback filter length = 2.

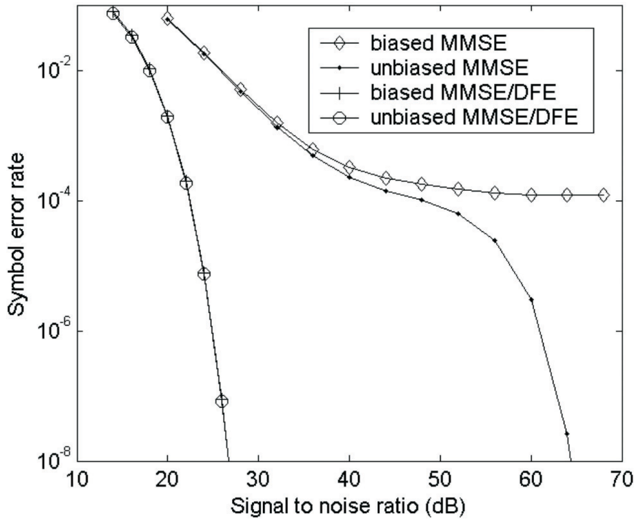

Figure 4. Symbol error rate curves for various finite length MMSE equalizers for 4-PAM transmission in channel **h** = [0.3482, 0.8704, 0.3482]. For linear MMSE, equalizer length = 5. For MMSE/DFE, feedforward filter length = 5 and feedback filter length = 3.

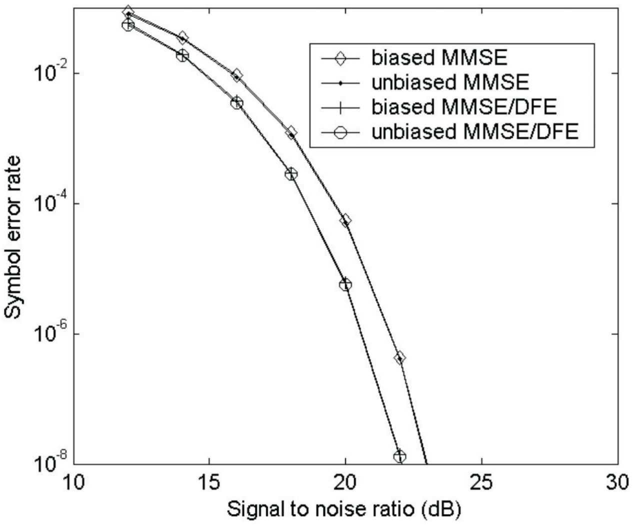

Figure 3. Symbol error probability curves for various MMSE equalizers for 4-PAM transmission in channel **h** = [1, 0.46].

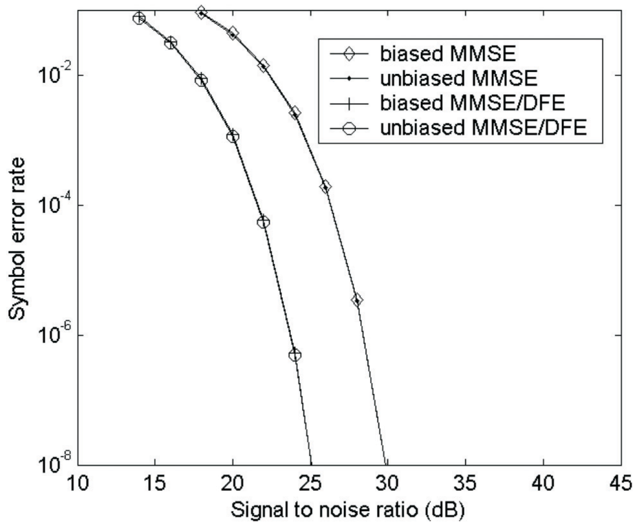

Figure 5. Symbol error probability curves for various MMSE equalizers for 4-PAM transmission in channel **h** = [0.3482, 0.8704, 0.3482].

decision operation to improve further. Then, Figure 4 and Figure 5 show various MMSE receivers of finite and infinite length respectively for a longer channel **h** = [0.3482, 0.8704, 0.3482]. For this longer channel, we observe a somewhat unusual phenomenon in the finite length linear MMSE cases of Figure 4. We deliberately provide these cases to show this unusualness as it is unseen ever before. For the biased linear MMSE receiver, the SEP stops to fall sharply after SNR increases beyond about 40 dB, while for the unbiased linear MMSE receiver, the curve also turns around at about 40 dB but then again falls sharply after about 50 dB. This is because the chosen equalizer lengths for both are not long enough. As a result, the equalizers cannot eliminate ISI efficiently. The best a biased linear MMSE receiver can do is to get the SEP down to about 10^{-4}. However, an unbiased linear MMSE receiver will overcome the difficulty caused by the short equalizer length again around 50 dB SNR. We have tried to use longer equalizers for this second channel. Then, this unusual phenomenon disappears.

6. Conclusion

We analyze biased and unbiased MMSE equalizers of finite length as well as infinite length, with and without decision feedback sections. For all these equalizers, new closed-form expressions of optimum equalizer weights, the MMSE, and SEP solely in terms of channel information and noise power are derived. Then, SEP performance curves for all receivers are presented and compared. The results meet all expectations. Moreover, we have reached two interesting conclusions: (a) The biased and unbiased MMSE equalizers have the same optimum equalizer weights. This is true whether the equalizers are of finite or infinite length, with or without decision feedback sections; (b) When the equalizer length approaches infinity, an unbiased MMSE receiver, with or without decision feedback section, approaches the optimum minimum error probability receiver. SEP expressions best suited to computer simulations for various presented equalizers of infinite length are also provided.

References

[1] Proakis, J. G., *Digital Communications*, 4th ed. McGraw-Hill, New York (2001).

[2] Gitlin, R. D., Hayes, J. F. and Weinstein, S. B., *Data Communication Principles,* Plenum (1992).

[3] Haykin, S., *Adaptive Filter Theory,* 4th ed. Prentice-Hall, Upper Saddle River, NJ (2002).

[4] Saltzburg, B. R., "Intersymbol Interference Error Bounds with Application to Ideal Bandlimited Signaling," *IEEE Trans. Inform. Theory,* Vol. IT-14, pp. 563–568 (1968).

[5] Lee, E. A. and Messerschmitt, D. G., *Digital Communications,* 2nd ed. Kluwer, Boston, MA (1994).

[6] Cioffi, J. M., Dudevoir, G. P., Eyuboglu, M. V. and Forney, Jr., G. D., "MMSE Decision-Feedback Equalizera and Coding-Part I: Equalization Results," *IEEE Trans. Commun.,* Vol. 43, pp. 2582–2594 (1995).

[7] Lugannani, R., "Intersymbol Interference and Probability of Error in Digital Systems," *IEEE Trans. Inform. Theory,* Vol. IT-15, pp. 682–688 (1969).

[8] Ho, E. Y. and Yeh, Y. S., "A New for Evaluating the Error Probability in the Presence of Intersymbol Interference and Additive Gaussian Noise," *Bell Syst. Tech. J.,* Vol. 49, pp. 2249–2265 (1970).

[9] Shimbo, O. and Celebiler, M., "The Probability of Error Due to Intersymbol Interference and Gaussian Noise in Digital Communication Systems," *IEEE Trans. Commun. Technol.,* Vol. COM-19, pp. 113–119 (1971).

[10] Glave, F. E., "An Upper Bound on the Probability of Error Due to Intersymbol Interference for Correlated Digital Signals," *IEEE Trans. Inform. Theory,* Vol. IT-18, pp. 356–363 (1972).

[11] Yao, K., "On Minimum Average Probability of Error Expression for Binary Pulse-Communication System with Intersymbol Interference," *IEEE Trans. Inform. Theory,* Vol. IT-18, pp. 528–531 (1972).

[12] Yao, K. and Tobin, R. M., "Moment Space Upper and Lower Error Bounds Digital Systems with Intersymbol Interference," *IEEE Trans. Inform. Theory,* Vol. IT-22, pp. 65–74 (1976).

[13] Salz, J., "Optimum Mean-Square Decision-Feedback Equalization," *Bell Syst. Tech. J.,* Vol. 52, pp. 1341–1373 (1973).

[14] Chen, S., Mulgrew, B., Chng, E. S. and Gibson, G. J., "Space Translation Properties and the Minimum-BER Linear-Combiner DFE," *IEE Proc.-Commun.,* Vol. 145, pp. 316–322 (1998).

[15] Chen, S. and Mulgrew, B., "Minimum-SER Linear-Combiner Decision Feedback Equalizer," *IEE Proc. - Commun.,* Vol. 146, pp. 347–353 (1999).

[16] Chen, S., Samingan, A. K., Mulgrew, B. and Hanzo, L., "Adaptive Minimum-BER Linear Multiuser Detection for DS-CDMA Signals in Multipath Channels," *IEEE Trans. Signal Process.,* Vol. 49, pp. 1240–1247 (2001).

[17] Chen, S., Hanzo, L. and Mulgrew, B., "Adaptive Minimum Symbol-Error-Rate Decision Feedback Equalization for Multilevel Pulse-Amplitude Modulation," *IEEE Trans. Signal Process.,* Vol. 52, pp. 2092–2101

(2004).

[18] Yeh, C.-C. and Barry, J. R., “Adaptive Minimum Bit-Error Rate Equalization for Binary Signaling,” *IEEE Trans. Commun.,* Vol. 48, pp. 1226–1235 (2000).

[19] ____, “Adaptive Minimum Symbol-Error Rate Equalization for Quadrature-Amplitude Modulation,” *IEEE Trans. Signal Process.,* Vol. 51, pp. 3263–3269 (2003).

[20] Liu, H. Y. and Yen, R. Y., “Channel Equalization Using Biased and Unbiased Minimum Error Rate Criteria,” *IEICE Trans. Fundamentals,* Vol. E87-A, pp. 605–609 (2004).

[21] Yen, R. Y., Liu, H. Y., Li, C. W. and Cheng, W. C., “The Adaptive MSINR Algorithm to Improve Error Rate for Channel Equalization,” *Signal Processing,* Vol. 86, pp. 1984–1991 (2006).

[22] Yen, R. Y., “Stochstic Unbiased Minimum Mean Error Rate Algorithm for Decision Feedback Equalizers,” *IEEE Trans. Signal Processing,* Vol. 55, pp. 4758–4766 (2007).

Manuscript Received: Apr. 26, 2007
Accepted: Jun. 3, 2008