

# Diversity of Traveling Wave Solutions in FitzHugh-Nagumo Type Equations

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## Abstract

In this work we consider the diversity of traveling wave solutions of the FitzHugh-Nagumo type equations

$$u_t = u_{xx} + f(u, w), \quad w_t = \varepsilon g(u, w),$$

where  $f(u, w) = u(u - a(w))(1 - u)$  for some smooth function  $a(w)$  and  $g(u, w) = u - w$ . When  $a(w)$  crosses zero and one, the corresponding profile equation possesses special turning points which result in very rich dynamics. In [26], Liu and Van Vleck examined traveling waves whose slow orbits lie only on two portions of the slow manifold, and obtained the existence results by using the geometric singular perturbation theory. Based on the ideas of their work, we study the co-existence of different traveling waves whose slow orbits could involve all portions of the slow manifold. There are more complicated and richer dynamics of traveling waves than those of [26]. We give a complete classification of all different fronts of traveling waves, and provide an example to support our theoretical analysis.

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# 1 Introduction

The purpose of this work is to investigate the existence of traveling wave solutions of FitzHugh-Nagumo type equations

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) + f(u(x, t), w(x, t)), \\ w_t(x, t) = \varepsilon g(u(x, t), w(x, t)), \end{cases} \quad (1.1)$$

where  $\varepsilon > 0$ ,  $f(u, w) = u(u - a(w))(1 - u)$  for some smooth function  $a(w)$  and  $g(u, w) = u - w$ . The prototype of FitzHugh-Nagumo equation is of (1.1) but with  $f(u, w) = u(1-u)(u-\alpha)-w$  and  $g(u, w) = u-\gamma w$  for some constants  $\alpha$  and  $\gamma$ , which can be considered as a simplification of the Hodgkin-Huxley equation that describes the propagation of action potentials in the nerve axon of the squid, cf. [14]. The dynamics of such specific equations, especially the traveling wave solutions, have been widely studied in the past, see [8, 15, 17, 19, 22, 31] and the references therein.

Recently, Liu and Van Vleck [26] considered the co-existence of different traveling wave fronts of (1.1) by allowing  $a(w)$  to cross 0 and 1, then the profile equations with respect to (1.1) can be reduced as a singularly perturbed system with turning points. Those special turning points exhibit the so-called delay of stability loss. Applying the geometric singular perturbation (GSP) theory and the Exchange Lemma for turning points (cf. [24]), Liu and Van Vleck show the existence of various types of traveling wave solutions which possess a special set of turning points. The slow manifold  $M$  for such singularly perturbed system consists of three parts, by  $M = M_0 \cup M_a \cup M_1$  (see Section 2.1). They studied traveling wave solutions whose slow orbits lie only on the portions  $M_0$  and  $M_1$  of the slow manifold, and gave a complete classification of traveling wave solutions.

Motivated by the work of [26], in this paper we reexamine their results to the cases of traveling waves of (1.1) which involves all the portions  $M_0, M_a, M_1$  of the slow manifold. The main difficulties in applying the (GSP) theory to our problem is to investigate the transversality of invariant manifolds by computing the Melnikov functions. In [26], the slow orbits lie only on the portions  $M_0$  and  $M_1$ , then the Melnikov functions is not zero obviously. However, due to the consideration of  $M_a$ , the computation of Melnikov functions become more complicated. Using the exact formulas for the heteroclinic orbits of fast limiting dynamics (see Remark 2.1), we successfully derive the exact formula of Melnikov functions (first and second order) represented by Beta or Gamma functions. Thus we can apply the Exchange lemma to track the evolution of invariant manifolds as they pass the vicinity of the slow manifold. Under the consideration of  $M_a$ , there are more complicated and richer dynamics of traveling wave solutions than those of [26]. In this article, we give a complete classification of all different fronts of traveling waves.

This paper is organized as follows. In Section 2, we formulate the traveling profiles equations of equation (1.1) from the viewpoint of dynamical systems,

which can be treated as a singularly perturbed problem. Under some assumptions of  $a(w)$ , detailed analysis for the non-normal hyperbolicity of slow manifold (with turning points) are carried out. Then we establish the Exchange lemma of the slow manifold with (and without) turning points, and illustrate some admissible conditions to guarantee that the singular orbits can be shadowed by true orbits even in the presence of turning points. The main theorems are stated in Section 3. In Section 4, we first investigate the Melnikov function of connecting orbits to detect the transversality of invariant manifolds. Then we prove the main theorems by GSP theory. In the last section we provide an example to support our theoretical analysis.

## 2 Formulation of GSP problems

In this section, we consider the traveling wave solutions of system (1.1) by assuming  $u(x, t) = u(x + ct) = u(\xi)$  and  $w(x, t) = w(x + ct) = w(\xi)$  for some real constant  $c > 0$ , which is the speed of traveling waves. Under such assumptions, the profile equations of (1.1) yield to

$$\begin{cases} cu'(\xi) = u''(\xi) + f(u(\xi), w(\xi)), \\ cw'(\xi) = \varepsilon g(u(\xi), w(\xi)). \end{cases} \quad (2.2)$$

Introducing  $v = u'$ , then (2.2) can be rewritten as

$$\begin{cases} u'(\xi) = v(\xi), \\ v'(\xi) = cv(\xi) - f(u(\xi), w(\xi)), \\ cw'(\xi) = \varepsilon g(u(\xi), w(\xi)). \end{cases} \quad (2.3)$$

In terms of the slow variable  $\eta := \varepsilon\xi$ , we have

$$\begin{cases} \varepsilon\dot{u}(\eta) = v(\eta), \\ \varepsilon\dot{v}(\eta) = cv(\eta) - f(u(\eta), w(\eta)), \\ \dot{w}(\eta) = c^{-1}g(u(\eta), w(\eta)), \end{cases} \quad (2.4)$$

here “ $\cdot$ ” means  $\frac{d}{d\eta}$ . Systems (2.3) and (2.4) are equivalent which give the standard singularly perturbed system in fast and slow scales respectively. Assume that  $E := \{w \mid w = a(w)\}$  is a non-empty set, then system (2.3) or (2.4) has equilibria:  $(0, 0, 0)$ ,  $(1, 0, 1)$  and  $(a(w_0), 0, w_0)$  with  $w_0 \in E$ . We are interest in traveling wave solutions related to such equilibria.

The main application of geometric singular perturbation theory to the problem is to lift limiting singular orbits to traveling wave solutions. In the following we examine the limiting slow and fast dynamics of (2.4) and (2.3) respectively.

## 2.1 Dynamics for the limiting slow system

The limiting slow dynamics is governed by

$$0 = v, \quad 0 = cv - f(u, w), \quad \dot{w} = c^{-1}g(u, w). \quad (2.5)$$

Thus the slow manifold  $M$  consists of three parts by  $M := M_0 \cup M_a \cup M_1$ , where

$$M_0 := \{u = v = 0\}, \quad M_a := \{u = a(w), v = 0\}, \quad M_1 := \{u = 1, v = 0\}.$$

It is easy to see that  $M_0$  and  $M_1$  are invariant with respect to the flow (2.3) for all  $\varepsilon$ , and equilibrium  $(0, 0, 0)$  or  $(1, 0, 1)$  attracts all solutions of (2.5) on  $M_0$  or  $M_1$  respectively. If we allow  $a(w)$  crossing 0 and 1, then there exists a special type of turning points on  $M_0$  and  $M_1$ . We will see that the invariance of  $M_0$  and  $M_1$  plays a crucial role when we consider the limiting slow orbits pass through the turning points.

## 2.2 Dynamics for the limiting fast system

The limiting fast dynamics is governed by

$$u' = v, \quad v' = cv - f(u, w), \quad w' = 0, \quad (2.6)$$

According to (2.5), the slow manifold  $M$  consists of equilibria of (2.6). From above equations, we know that each plane  $\{w = \text{const}\}$  is invariant, and there exist three equilibria of system (2.6):

$$E_0 := (0, 0, w) \in M_0, \quad E_a(w) := (a(w), 0, w) \in M_a \text{ and } E_1 := (1, 0, w) \in M_1.$$

Let  $\lambda_0^\pm(w, c)$ ,  $\lambda_a^\pm(w, c)$  and  $\lambda_1^\pm(w, c)$ , be the linearized eigenvalues of system (2.6) with respect to  $E_0$ ,  $E_a$  and  $E_1$  respectively. Then we have

$$\lambda_0^\pm(w, c) = \frac{c \pm \sqrt{c^2 + 4a(w)}}{2}, \quad (2.7)$$

$$\lambda_a^\pm(w, c) = \frac{c \pm \sqrt{c^2 + 4a(w)(a(w) - 1)}}{2}, \quad (2.8)$$

$$\lambda_1^\pm(w, c) = \frac{c \pm \sqrt{c^2 + 4(1 - a(w))}}{2}. \quad (2.9)$$

If  $c \geq 1$ , then  $\lambda_a^\pm(w, c)$  are real. If  $c < 1$  then the sign of above real eigenvalues with respect to the range of  $a(w)$  can be classified in Figure 1.

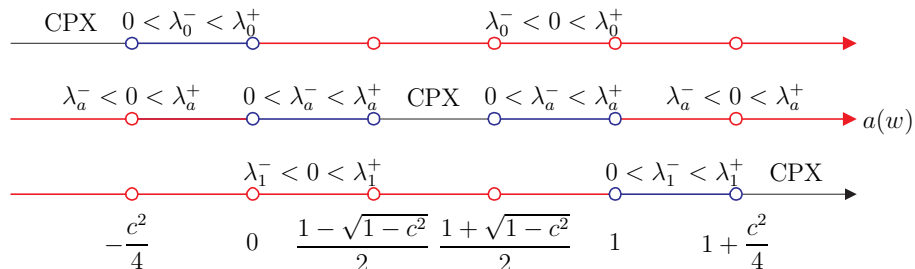


Figure 1: Sign of linearized eigenvalues with respect to the range of  $a(w)$ , where CPX means that the eigenvalues are conjugate complex numbers in the range of  $a(w)$ .

Therefore, all the linearized eigenvalues are real in the region  $\Omega$  defined by

$$\Omega := \{(w, c) \in [0, 1] \times \mathbb{R}^+ : a(w) \in [-\frac{c^2}{4}, 1 + \frac{c^2}{4}] \text{ for } c > 1; \text{ or } \\ a(w) \in [-\frac{c^2}{4}, \infty) \setminus (\frac{1 - \sqrt{1 - c^2}}{2}, \frac{1 + \sqrt{1 - c^2}}{2}) \text{ for } c < 1\}.$$

Now we consider the dynamics of (2.6). On each plane  $\{w = \text{const}\}$ , the limiting system is that for a prototype of Nagumo equations with specific cubic nonlinearity. The existence of heteroclinic orbits on the plane is well understood, cf. [4]. To classify all the possible heteroclinic orbits of (2.6), we first introduce the following notations:

$$\begin{aligned} a_1(c) &:= \max\{0, \frac{1 - \sqrt{2}c}{2}\}, \quad a_2(c) := \min\{1, \frac{1 + \sqrt{2}c}{2}\}, \quad a_3(c) := 2 + \sqrt{2}c; \\ a_4(c) &:= -1 - \sqrt{2}c, \quad a_5(c) := \max\{1, 2 - \sqrt{2}c\}, \quad a_6(c) := \min\{0, -1 + \sqrt{2}c\}; \\ H_i(c) &:= \{w \in (0, 1) : a(w) = a_i(c), \quad a'(w) \neq 0\}, \quad i = 1, \dots, 6, \\ G_1(c) &:= \{w \in (0, 1) : a(w) \leq a_6(c)\}, \quad G_2(c) := \{w \in (0, 1) : a(w) \geq a_5(c)\}, \\ G_3(c) &:= \{w \in (0, 1) : 0 > a(w) > a_4(c)\}, \\ G_4(c) &:= \{w \in (0, 1) : 1 < a(w) < a_3(c)\}, \\ G_5(c) &:= \{w \in (0, 1) : 0 < a(w) < a_2(c)\}, \\ G_6(c) &:= \{w \in (0, 1) : 1 > a(w) > a_1(c)\}. \end{aligned}$$

Furthermore, for any fixed  $w \in [0, 1]$  we denote  $r \rightarrow s$  to be the heteroclinic orbit connecting  $(r, 0, w)$  to  $(s, 0, w)$ , where  $r \neq s$  and  $r, s \in \{0, a(w), 1\}$ . According to the results of [4] and phase plane analysis, various types of heteroclinic orbits with respect to different regions of the parameters can be classified in Table 1.

Type of Orbit	Admissible Parameter Condition	Region
$0 \rightarrow 1$	$a(w) = a_1(c)$ or $a(w) \leq a_6(w)$	$w \in H_1 \cup G_1$
$1 \rightarrow 0$	$a(w) = a_2(c)$ or $a(w) \geq a_5(w)$	$w \in H_2 \cup G_2$
$0 \rightarrow a(w)$	$a(w) = a_3(c)$ or $a_4(c) < a(w) < 0$	$w \in H_3 \cup G_3$
$1 \rightarrow a(w)$	$a(w) = a_4(c)$ or $1 < a(w) < a_3(c)$	$w \in H_4 \cup G_4$
$a(w) \rightarrow 0$	$a(w) = a_5(c)$ or $0 < a(w) < a_2(c)$	$w \in H_5 \cup G_5$
$a(w) \rightarrow 1$	$a(w) = a_6(c)$ or $a_1(c) < a(w) < 1$	$w \in H_6 \cup G_6$

Table 1: Classification of admissible heteroclinic orbits.

Note that the linearized eigenvalues are real in the region  $\Omega$ . Throughout this work, we redefine sets  $H_i$  and  $G_i$  in Table 1 by  $H_i \cap \Omega$  and  $G_i \cap \Omega$ . With a slight abusing the notation, we keep the same notations. The regions of  $\Omega$ ,  $H_i$  and  $G_i$  are illustrated in Figure 2.

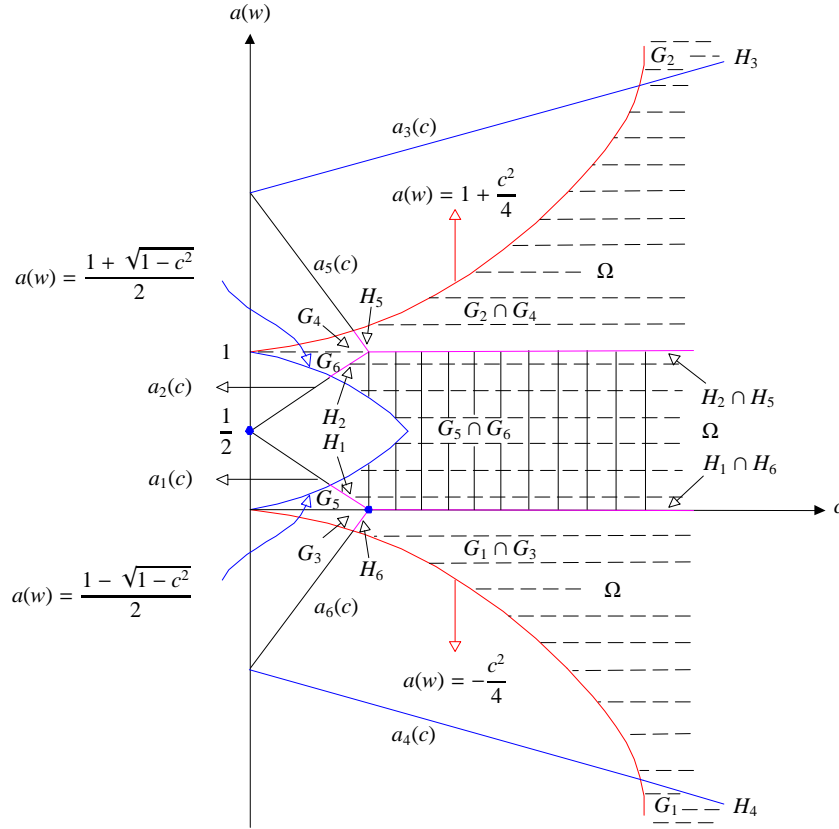


Figure 2: Regions of  $\Omega$ ,  $G_i$  and  $H_i$ .

**Remark 2.1.**

- (1) As shown in [4], if  $w = w_0 \in H_i(c)$ ,  $i = 3, 4, 5, 6$ , then the exact formulas for the heteroclinic orbits  $(u(t; w_0), v(t; w_0))$  of (2.6) can be expressed as follows:

$$u(t; w_0) = \begin{cases} a(w_0) - a(w_0)(1 + e^{a(w_0)t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_3(c); \\ a(w_0) + (1 - a(w_0))(1 + e^{(1-a(w_0))t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_4(c); \\ a(w_0)(1 + e^{a(w_0)t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_5(c); \\ 1 - (1 - a(w_0))(1 + e^{(1-a(w_0))t/\sqrt{2}})^{-1}, & \text{if } w_0 \in H_6(c). \end{cases}$$

Based on the above formulas, the Melnikov functions (first and second order) for invariant manifolds of connecting orbits can be derived explicitly by Beta or Gamma functions, for details see Section 4.

- (2) In [26], they examined traveling waves whose slow orbits lie only on the portions  $M_0$  and  $M_1$  of the slow manifold, thus only regions  $H_1, H_2, G_1, G_2$  are considered (see dashed line paths of Figure 3). To generalize their work to traveling waves whose slow orbits lie on all portions of  $M$ , we need to consider some additional regions than those of [26] (see the non-dash path of Figure 3).

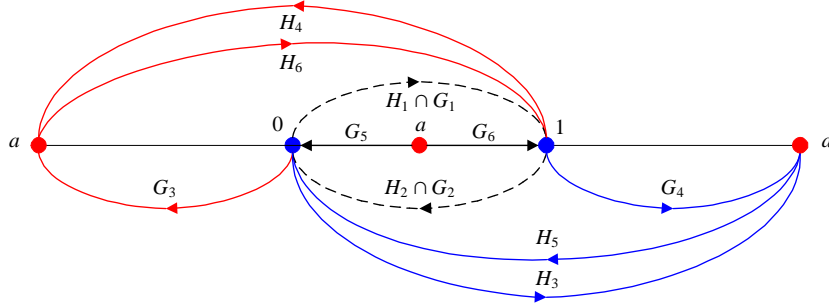


Figure 3: Admissible heteroclinic orbits with respect to regions.

Next, we investigate the normal hyperbolicity of the slow manifolds. The normal hyperbolicity of the slow manifold of  $M_0$  or  $M_1$  is determined by the eigenvalues  $\lambda_0^\pm(w, c)$  or  $\lambda_1^\pm(w, c)$ , respectively. If  $(0, 0, w) \in M_0$  at which  $a(w) = 0$ , then  $\lambda_0^-(w, c) = 0$  and the slow manifold  $M_0$  loses normal hyperbolicity at this point. Similarly, the slow manifold  $M_0$  loses normal hyperbolicity at points  $(1, 0, w) \in M_1$  satisfying  $a(w) = 1$ . All such points are called *turning points*. Since  $M_0$  and  $M_1$  are invariant, the existence of turning points on them can cause the phenomena of delay of stability loss, see [24]. To describe the results for delay of stability loss, Exchange lemma with turning points and our main theorems, in this article we assume that the curve  $u = a(w)$  crosses  $u = 0$  and  $u = 1$ , and satisfies the following assumption:

(H) there exist (increasing) ordered sets  $\{T_0^i\}_{i=1}^p, \{T_1^j\}_{j=1}^q \subseteq [0, 1]$  such that

$$a(T_0^i) = 0, \quad a(T_1^j) = 1, \quad a'(T_0^i) \neq 0, \quad a'(T_1^j) \neq 0,$$

for all  $1 \leq i \leq p$  and  $1 \leq j \leq q$ .

By (H), the sets of points  $\{(0, 0, T_0^i)\}_{i=1}^p$  and  $\{(1, 0, T_1^j)\}_{j=1}^q$  are turning points on the slow manifold  $M_0$  and  $M_1$  respectively. For the position of equilibria and turning points, dynamics on the slow manifold and heteroclinic orbit for fast dynamics, see Figure 4.

From Table 1 and the hyperbolicity of slow manifold for the limiting system (2.4), we plan to construct singular orbits (unions of slow and fast orbits) as candidates for limits of traveling wave solutions. Then we can obtain the existence of traveling wave solutions of (2.2) by applying the geometric singular perturbation theorem to lift singular orbits to the true orbits.

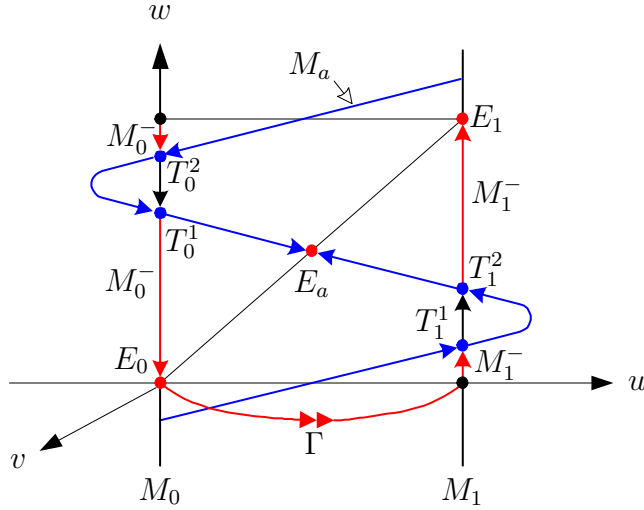


Figure 4: Equilibria, turning points, dynamics on the slow manifold and heteroclinic orbit  $\Gamma$  of the fast dynamics which connects  $E_0$  and  $(1, 0, 0)$  when  $a(0) = a_1(c)$  or  $a(0) \leq a_6(0)$ . The red segments  $M_{0,1}^-$  on  $M_{0,1}$  are defined by  $M_0^- = \{(0, 0, w) \in M_0 \mid \lambda_0^-(w, c) < 0\}$  and  $M_1^- = \{(1, 0, w) \in M_1 \mid \lambda_1^-(w, c) < 0\}$ .

### 2.3 Delay of stability loss and Exchange lemmas

In this section, we recall and reformulate the results in [24, 26] about the delay of stability loss and Exchange lemma for turning points. For any fixed  $c > 0$ , let us denote

$$M_0^- := \{(0, 0, w) \in M_0 \mid \lambda_0^-(w, c) < 0\} \text{ and } M_1^- := \{(1, 0, w) \in M_1 \mid \lambda_1^-(w, c) < 0\}.$$

If the above sets are non-empty, then we define two maps  $P_0$  and  $P_1$  on such sets as following.

$$(P_0) \text{ Let } P_0 : M_0^- \rightarrow M_0 \text{ be defined by } P_0(0, 0, w) = \begin{cases} (0, 0, \bar{w}), & \text{if } \bar{w} \text{ exists,} \\ (0, 0, 0), & \text{otherwise,} \end{cases}$$

where  $\bar{w} \in (0, w)$  is the first value such that  $\int_{\bar{w}}^w \frac{\lambda_0^-(\eta, c)}{g(0, \eta)} d\eta = 0$ .

$$(P_1) \text{ Let } P_1 : M_1^- \rightarrow M_1 \text{ be defined by } P_1(1, 0, w) = \begin{cases} (1, 0, \bar{w}), & \text{if } \bar{w} \text{ exists,} \\ (1, 0, 1), & \text{otherwise,} \end{cases}$$

where  $\bar{w} \in (w, 1)$  is the first value such that  $\int_w^{\bar{w}} \frac{\lambda_1^-(\eta, c)}{g(1, \eta)} d\eta = 0$ .

Based on the above two maps, Liu and Van Vleck [26] reformulated the Exchange lemma on  $M_0$  and  $M_1$  for system (2.3) with an extra equation  $c' = 0$ , that is

$$u'(\xi) = v, \quad v'(\xi) = cv - f(u, w), \quad w'(\xi) = \varepsilon c^{-1} g(u, w), \quad c' = 0. \quad (2.10)$$



To guarantee the existence of unstable manifold  $W_0^u(K)$  and center manifold  $W_0^c(K)$  for any set  $K \subset M_0 \cup M_1$ , we restrict  $c$  belonging to the following set

$$S := \{c > 0 : a(w) \in [-\frac{c^2}{4}, 1 + \frac{c^2}{4}] \text{ for all } w \in [0, 1]\}.$$

Denote

$$\begin{aligned} M_1^\delta(w) &:= \{(1, 0, \bar{w}) \in M_1 : \bar{w} \in (w - \delta, w + \delta)\}, \\ M_0^\delta(w) &:= \{(0, 0, \bar{w}) \in M_0 : \bar{w} \in (w - \delta, w + \delta)\}, \\ M_a^\delta(w) &:= \{(a(\bar{w}), 0, \bar{w}) \in M_a : \bar{w} \in (w - \delta, w + \delta)\}, \end{aligned}$$

for any small  $\delta > 0$  and any  $w \in [0, 1]$ . The Exchange lemma for  $M_1$  with turning points is stated as follows.

**Proposition 2.2.** (Exchange lemma with Turning point, cf. [24, 26]) Let  $M^\varepsilon$  be a two-dimensional invariant manifold of system (2.10) which is smooth in  $\varepsilon$ . For  $\varepsilon = 0$ , suppose that  $M^0$  intersects  $W_0^c(M_1 \times (c_1, c_2))$  transversally. Let  $N$  be the intersection. Then  $\dim N=1$ . Suppose that  $\omega(N) = \{(1, 0, w_1, c^*)\}$  and let  $w_2 \in (w_1, 1)$  be any number. We have

- (1) If  $w_2 < P_1(w_1)$ , then for  $\varepsilon > 0$  small, a portion of  $M^\varepsilon$  will approach  $(1, 0, w_1, c^*)$ , follow the slow orbit from  $(1, 0, w_1, c^*)$  to  $(1, 0, w_2, c^*)$ , leave the vicinity of  $M_1 \times (c_1, c_2)$ , and upon leaving, it is  $C^1$   $O(\varepsilon)$ -close to the unstable manifold  $W^u(M_1^\delta(w_2) \times \{c^*\})$  for some  $\delta > 0$  independent of  $\varepsilon$  (see Figure 5).
- (2) If  $w_2 = P_1(w_1) \notin \{T_1^1, T_1^2, \dots, T_1^q\}$ , then for  $\varepsilon > 0$  small, a portion of  $M^\varepsilon$  will approach  $(1, 0, w_1, c^*)$ , follow the slow orbit from  $(1, 0, w_1, c^*)$  to  $(1, 0, w_2, c^*)$ , leave the vicinity of  $M_1 \times (c_1, c_2)$ , and upon leaving, it is  $C^1$   $O(\varepsilon)$ -close to the center-unstable manifold  $W^{cu}(1, 0, w_2, c^*)$  (see Figure 6).
- (3) If  $w_2 > P_1(w_1)$ , then for  $\varepsilon > 0$  small, there is no portion of  $M^\varepsilon$  that approaches  $(1, 0, w_1, c^*)$ , follows the slow orbit from  $(1, 0, w_1, c^*)$ , leave the vicinity of  $M_1$  in a neighborhood of  $(1, 0, w_2, c^*)$ .

For singular orbits passing no turning point, we use the following Exchange lemma without turning points.

**Proposition 2.3.** (Exchange lemma without Turning point, cf. [20, 23, 29]) Let  $M^\varepsilon$  be a two-dimensional invariant manifold of system (2.10) which is smooth in  $\varepsilon$ . For  $\varepsilon = 0$ , suppose that  $M^0$  intersects  $W_0^c(M_a \times \{c^*\})$  transversally. Let  $N$  be the intersection. Then  $\dim N=1$ . Suppose that  $\omega(N) = \{(a(w_1), 0, w_1, c^*)\}$ . Let  $w_2$  be any number such that  $a(w) \neq 0$  or  $1$ , for all  $w$  between  $w_1$  and  $w_2$ . then for  $\varepsilon > 0$  small, a portion of  $M^\varepsilon$  will approach  $(a(w_1), 0, w_1, c^*)$ , follow the slow orbit from  $(a(w_1), 0, w_1, c^*)$  to  $(a(w_2), 0, w_2, c^*)$ , leave the vicinity of  $M_a \times \{c^*\}$ , and upon leaving, it is  $C^1$   $O(\varepsilon)$ -close to the unstable manifold  $W^u(M_a^\delta(w_2) \times \{c^*\})$  for some  $\delta > 0$  independent of  $\varepsilon$ .

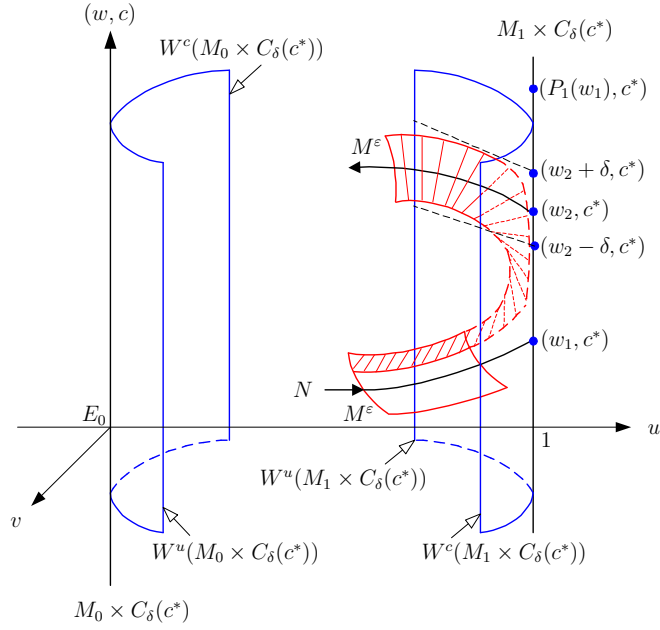


Figure 5: Part (1) of Proposition 2.2. In this graph, we denote  $C_\delta(c) := (c - \delta, c + \delta)$  for some  $\delta > 0$  and identify the  $(w, c)$ -plane with the vertical axis.

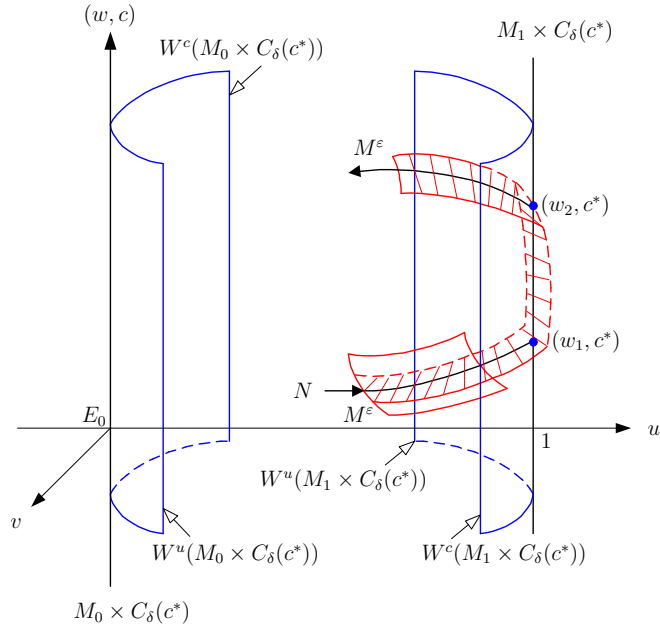


Figure 6: Part (2) of Proposition 2.2. In this graph, we denote  $C_\delta(c) := (c - \delta, c + \delta)$  for some  $\delta > 0$  and identify the  $(w, c)$ -plane with the vertical axis.

## 2.4 Admissible conditions for singular orbits

In view of the results of Exchange lemma with turning points, not all singular orbits are shadowed by true orbits. To guarantee the shadowing property, we introduce some admissible conditions for the construction of singular orbits.

Let  $w = (w_1, w_2, \dots, w_n)$  with  $w_i \in [0, 1]$  and  $s = (s_1, s_2, \dots, s_{n+1})$  with  $s_1 = 1$ ,  $s_i \in \{0, a, 1\}$ ,  $s_i \neq s_{i+1}$  and  $s_{n+1} \in \{0, 1\}$ . For any two words  $w$  and  $s$ , we denote the singular orbit starting from 0 to  $s_{n+1}$  by  $0 \rightarrow s_1 \rightarrow \dots \rightarrow s_{n+1}$  such that the local path  $s_i \rightarrow s_{i+1}$  (part of the orbit from  $s_i$  to  $s_{i+1}$ ) occurring at the plane  $w = w_i$ . Since the manifold  $M_a$  does not persist for all  $\varepsilon > 0$ , the Exchange lemma can not be applied to our problem directly. Therefore, we only focus on the cases with  $w \in (w_{i-1}, w_i)$ ,  $a(w) > 1$  if  $w_i \in H_5(c^*)$ , and  $a(w) < 0$  if  $w_i \in H_6(c^*)$ . It could be seen that the singular orbit on the manifold  $M_a$  will not pass the turning point.

In the following we say that  $w$  is  $s$ -admissible with respect to some  $c^* > 0$  if

$$\left\{ \begin{array}{ll} w_i \in H_1(c^*) \cup G_1(c^*) \setminus \{T_0^1, T_0^2, \dots, T_0^p\}, & \text{when } s_i s_{i+1} = 01, \\ w_i \in H_2(c^*) \cup G_2(c^*) \setminus \{T_1^1, T_1^2, \dots, T_1^q\}, & \text{when } s_i s_{i+1} = 10, \\ w_i \in H_3(c^*) \cup G_3(c^*) \setminus \{T_0^1, T_0^2, \dots, T_0^p\}, & \text{when } s_i s_{i+1} = 0a, \\ w_i \in H_4(c^*) \cup G_4(c^*) \setminus \{T_1^1, T_1^2, \dots, T_1^q\}, & \text{when } s_i s_{i+1} = 1a, \\ w_i \in H_5(c^*), & \text{when } s_i s_{i+1} = a0, \\ w_i \in H_6(c^*), & \text{when } s_i s_{i+1} = a1, \end{array} \right. \quad (2.11)$$

for  $i = 1, \dots, n$  and the following conditions (A1)~(A3) hold:

$$(A1) \quad P_1(0) > w_1 \text{ and } \begin{cases} a(w) < 1, \forall w \in [w_n, 1], & \text{if } s_{n+1} = 1, \\ a(w) > 0, \forall w \in [0, w_n], & \text{if } s_{n+1} = 0; \end{cases}$$

(A2) For  $s_i = 0$ ,  $w_{i-1} > w_i$  and

$$P_0(w_{i-1}) \begin{cases} < w_i, & \text{when } w_i \in H_1(c^*) \cup H_3(c^*), \\ = w_i, & \text{when } w_i \in G_1(c^*) \cup G_3(c^*) \setminus \{T_0^1, T_0^2, \dots, T_0^p\}; \end{cases}$$

(A3) For  $s_i = 1$ ,  $w_{i-1} < w_i$  and

$$P_1(w_{i-1}) \begin{cases} > w_i, & \text{when } w_i \in H_2(c^*) \cup H_4(c^*), \\ = w_i, & \text{when } w_i \in G_2(c^*) \cup G_4(c^*) \setminus \{T_1^1, T_1^2, \dots, T_1^q\}. \end{cases}$$

Furthermore, we say that a word  $w = (w_1, w_2, \dots, w_n)$  is *admissible with respect to  $c^*$*  if there is a word  $s = (s_1, s_2, \dots, s_{n+1})$  with  $s_1 = 1$ ,  $s_i \in \{0, a, 1\}$ ,  $s_i \neq s_{i+1}$  and  $s_{n+1} \in \{0, 1\}$ , such that  $w$  is  $s$ -admissible with respect to  $c^*$ .

## 3 Main results

According to the Exchange lemma and the admissible conditions defined in previous section, we state the main theorems in this section and prove them in

next section. For a description of the statement of our main results, we give the following definition.

**Definition 3.1.** *Let  $\mathcal{O}$  be a singular orbit for some fixed  $c^* > 0$ . The singular orbit  $\mathcal{O}$  is “weakly shadowed” if for any neighborhood  $\mathcal{U}$  of the singular orbit,  $\mathcal{O}$ , there is an  $\varepsilon_0 > 0$  such that, for all  $0 < \varepsilon \leq \varepsilon_0$ , there is a true orbit  $\mathcal{O}(\varepsilon) \in \mathcal{U}$  of system (2.3) with  $c = c(\varepsilon)$  and  $(\mathcal{O}(\varepsilon), c(\varepsilon)) \rightarrow (\mathcal{O}, c^*)$  as  $\varepsilon \rightarrow 0$  with respect to the Hausdorff distance of sets. Furthermore, if  $c(\varepsilon) = c^*$  for all  $0 < \varepsilon \leq \varepsilon_0$ , then we say the singular orbit  $\mathcal{O}$  is “strongly shadowed”.*

First, we consider the traveling wave solutions connecting  $(0,0,0)$  to  $(1,0,1)$ . From Table 1, we know that such kind of traveling wave solutions exist only if  $s_1 = 1$  or  $s_1 = a(0)$ . If  $s_1 = 1$  then it is required that  $\lambda_1^-(0; c) < 0$  and  $\lambda_1^-(1; c) < 0$  to guarantee the first and last connection. It is easy to see that these two conditions are equivalent to  $a(0) < 1$  and  $a(1) < 1$  respectively. In addition, it could be seen that the structures of traveling wave solutions are dramatically different for different sign of  $a(0)$ . Therefore, we will consider two situations  $a(0) > 0$  and  $a(0) < 0$  separately.

If  $a(0) > 0$ , then there exists a unique  $c^*$  with  $a_1(c^*) = a(0)$  (in fact  $c^* = (1 - 2a(0))/\sqrt{2}$ ) such that system (2.6) has a heteroclinic orbit connecting from  $(0,0,0)$  to  $(1,0,0)$  approaching  $(0,0,0)$  backward along the eigenvector associated to  $\lambda_0^+(0, c^*)$ .

**Theorem 3.2.** *Assume that  $0 < a(0) < 1$ ,  $a(1) < 1$  and  $c^* \in S$  is the unique value such that  $a_1(c^*) = a(0)$ .*

- (1) *If  $w = (w_1, w_2, \dots, w_n)$  is admissible with respect to  $c^*$ , then the associated singular orbit is weakly shadowed.*
- (2) *If  $w = (w_1, w_2, \dots, w_n)$  is not admissible with respect to  $c^*$ , then the associated singular orbit is not weakly shadowed.*

If  $a(0) \leq 0$ , from Table 1, system (2.6) possess a heteroclinic orbit connecting from  $(0,0,0)$  to  $(1,0,0)$  only if  $a(0) \leq a_6(c)$ , or equivalent to  $c \in \Lambda := \{c : c \geq (1 + a(0))/\sqrt{2}\}$ .

**Theorem 3.3.** *Assume that  $a(0) < 0$ ,  $a(1) < 1$  and  $c^* \in \Lambda \cap S$ .*

- (1) *If  $w = (w_1, w_2, \dots, w_n)$  is admissible with respect to  $c^*$ , then the associated singular orbit is strongly shadowed.*
- (2) *If  $w = (w_1, w_2, \dots, w_n)$  is not admissible with respect to  $c^*$ , then the associated singular orbit is not weakly shadowed.*

Next, we consider the traveling wave solutions connecting  $(0,0,0)$  to  $(0,0,0)$ , i.e. traveling pulse solutions. By Table 1, we know that such kind of traveling wave solutions exist only if  $s_1 = 1$  or  $s_1 = a(0)$ . If  $s_1 = 1$ , then it is required

that  $\lambda_1^-(0; c) < 0$  and  $\lambda_0^-(0; c) < 0$  to guarantee the first and last connection. Both conditions are equivalent to  $0 < a(0) < 1$ . If  $0 < a(0) < 1$  then there exists a unique  $c^*$  with  $a_1(c^*) = a(0)$  (in fact  $c^* = (1 - 2a(0))/\sqrt{2}$ ) such that system (2.6) has a heteroclinic orbit from  $(0,0,0)$  to  $(1,0,0)$  approaching  $(0,0,0)$  backward along the eigenvector associated to  $\lambda_0^+(0, c^*)$ .

**Theorem 3.4.** *Assume that  $0 < a(0) < 1$  and  $c^* \in S$  is the unique value such that  $a_1(c^*) = a(0)$ .*

- (1) *If  $w = (w_1, w_2, \dots, w_n)$  is admissible with respect to  $c^*$ , then the associated singular orbit is weakly shadowed.*
- (2) *If  $w = (w_1, w_2, \dots, w_n)$  is not admissible with respect to  $c^*$ , then the associated singular orbit is not weakly shadowed.*

**Remark 3.5.** Theorems 3.2~3.4 present results on traveling fronts from  $(0, 0, 0)$  to  $(1, 0, 1)$ , and traveling pulses to  $(0, 0, 0)$ . Following the similar arguments, the existence of traveling waves involving the equilibria  $(a(w_0), 0, w_0)$  for  $w_0 \in E$  can also be investigated in the same way.

## 4 Proof of the main results

To prove the main results in this section, we first detect the transversality of invariant manifolds for connecting orbits by investigating the Melnikov function.

### 4.1 Melnikov function and transversality of manifolds

First, we recall the results for the formula of Melnikov function, [5, 16, 20, 27].

**Lemma 4.1.** *Consider the plane system:*

$$y' = R_0(y) + \bar{\varepsilon}R_1(y, \bar{\varepsilon}), \quad (4.12)$$

where  $\bar{\varepsilon} \geq 0$  and  $R_0, R_1 \in C^r$  with  $r \geq 2$ . Suppose that  $y_0^1$  and  $y_0^2$  are two different hyperbolic saddle points of  $(4.12)|_{\bar{\varepsilon}=0}$ , and there exists a heteroclinic orbit  $y_0(t)$  of  $(4.12)|_{\bar{\varepsilon}=0}$  connecting from  $y_0^1$  to  $y_0^2$ . Then the Melnikov function of (4.12) is

$$\mathcal{M}(y_0) = \int_{-\infty}^{\infty} e^{-\int_0^t \sigma(s) ds} D(t) dt \quad (4.13)$$

where  $\sigma(t) = \text{tr} \frac{\partial R_0}{\partial y}(y_0(t))$  and  $D(t) = R_0(y_0(t)) \wedge R_1(y_0(t), 0)$ .

According to formula (4.13), we can compute the Melnikov function of system (2.6) in the following.

**Lemma 4.2.** *Suppose, for some  $c_0$  and  $w_0$ , system (2.6) has a heteroclinic orbit  $\Gamma := r(t; w_0) = (u_0(t; w_0), v_0(t; w_0))$ .*

- (1) For fixed  $w = w_0$  and varying  $c$ , the Melnikov function with respect to the heteroclinic orbit  $\Gamma$  is given by

$$\mathcal{M}(c_0) = \int_{-\infty}^{\infty} e^{-c_0 t} v_0^2(t; w_0) dt.$$

In particular,  $\mathcal{M}(c_0) \neq 0$ .

- (2) For fixed  $c = c_0$  and varying  $w$ , the Melnikov function is given by

$$\mathcal{M}(w_0) = a'(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0) (1 - u_0(t; w_0)) dt. \quad (4.14)$$

*Proof.* (1) For fixed  $w = w_0$ , let  $F(u, v; c)$  be the vector field of system (2.6), i.e.,  $F(u, v; c) = (v, cv - u(u - a(w_0)(1 - u)))$  and define  $F_c(u, v; c) := (0, v)$ . Applying Lemma 4.1 by taking  $\bar{\varepsilon} = c$ , the Melnikov function is

$$\begin{aligned} \mathcal{M}(c_0) &= \int_{-\infty}^{\infty} e^{-\int_0^t \text{tr} DF(r(s; w_0); c_0) ds} (F(r(t; w_0); c_0) \wedge F_c(r(t; w_0); c_0)) dt \\ &= \int_{-\infty}^{\infty} e^{-c_0 t} v_0^2(t; w_0) dt \neq 0. \end{aligned}$$

(2) For fixed  $c = c_0$ , we have  $F(u, v; w) = (v, c_0 v - u(u - a(w)(1 - u)))$ . Denote  $F_w(u, v; w) := (0, u(1 - u)a'(w))$ . Applying Lemma 4.1 by taking  $\bar{\varepsilon} = w$ , the Melnikov function is

$$\begin{aligned} \mathcal{M}(w_0) &= \int_{-\infty}^{\infty} e^{-\int_0^t \text{tr} DF(r(s; w_0); w_0) ds} (F(r(t; w_0); w_0) \wedge F_w(r(t; w_0); w_0)) dt \\ &= a'(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0) (1 - u_0(t; w_0)) dt. \end{aligned}$$

The proof is complete.  $\square$

Based on the results of Lemma 4.2, we now compute the first order Melnikov function of system (2.6) when  $w$  varies in different parameter regions.

**Lemma 4.3.** *Assume that  $a'(w_0) \neq 0$ , then  $\mathcal{M}(w_0) \neq 0$  for any  $c_0 \in (0, 1/\sqrt{2})$  and  $w_0 \in H_i(c_0)$ ,  $i = 1 \dots 6$ .*

*Proof.* (1) If  $w_0 \in H_1(c_0) \cup H_2(c_0)$  then  $u_0(t; w_0) \in (0, 1)$  for all  $t$ . By Lemma 4.2, we have

$$\mathcal{M}(w_0) = a'(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0) (1 - u_0(t; w_0)) dt \neq 0.$$

(2) If  $w_0 \in H_5(c_0)$  then  $a(w_0) = 2 - \sqrt{2}c_0 \in (1, 2)$ . According to Remark 2.1, the heteroclinic orbit  $(u_0(t; w_0), v_0(t; w_0))$  can be represented explicitly in the following:

$$u_0(t; w_0) = a(w_0)(1 + e^{a(w_0)t/\sqrt{2}})^{-1} \quad \text{and} \quad v_0(t; w_0) = u_0'(t; w_0).$$

Thus

$$e^{-c_0 t} = (a(w_0) - u_0(t; w_0))^\ell u_0^{-\ell}(t; w_0), \quad \text{where } 0 < \ell := 1 - \frac{2}{a(w_0)} < 1/2.$$

We can compute equation (4.14) by

$$\begin{aligned} \frac{\mathcal{M}(w_0)}{a'(w_0)} &= \int_{a(w_0)}^0 u^{1-\ell} (a(w_0) - u)^\ell (1 - u) du \\ &= \int_0^{a(w_0)} u^{2-\ell} (a(w_0) - u)^\ell du - \int_0^{a(w_0)} u^{1-\ell} (a(w_0) - u)^\ell du \\ &= a^3(w_0) \int_0^1 t^{2-\ell} (1 - t)^\ell dt - a^2(w_0) \int_0^1 t^{1-\ell} (1 - t)^\ell dt \\ &= a^3(w_0) B(1 + \ell, 3 - \ell) - a^2(w_0) B(1 + \ell, 2 - \ell) \\ &= a^3(w_0) (\Gamma(1 + \ell) \Gamma(3 - \ell) / \Gamma(4)) - a^2(w_0) (\Gamma(1 + \ell) \Gamma(2 - \ell) / \Gamma(3)) \\ &= a^2(w_0) (a(w_0) - 1) \Gamma(1 + \ell) \Gamma(2 - \ell) / \Gamma(4) > 0, \end{aligned}$$

where  $B(x, y)$  and  $\Gamma(x)$  are the Beta function and the Gamma function respectively. Note that  $B(x, y) = \Gamma(x) \Gamma(y) / \Gamma(x + y)$ .

(3) If  $w_0 \in H_3(c_0)$  then  $a(w_0) = 2 + \sqrt{2}c_0 \in (2, 3)$ . Similar to the proof of part (2), the heteroclinic orbit  $(u_0(t; w_0), v_0(t; w_0))$  satisfies

$$\begin{aligned} u_0(t; w_0) &= a(w_0) - a(w_0)(1 + e^{a(w_0)t/\sqrt{2}})^{-1}, \\ e^{-c_0 t} &= (a(w_0) - u_0(t; w_0))^\ell u_0^{-\ell}(t; w_0), \quad \text{where } 0 < \ell < 1/3. \end{aligned}$$

After simple computation, we can obtain

$$\mathcal{M}(w_0) = a'(w_0) a^2(w_0) (1 - a(w_0)) \Gamma(1 + \ell) \Gamma(2 - \ell) / \Gamma(4) < 0.$$

(4) Similarly, if  $w_0 \in H_4(c_0)$  then  $a(w_0) = -1 - \sqrt{2}c_0 \in (-2, -1)$  and the heteroclinic orbit  $(u_0(t; w_0), v_0(t; w_0))$  satisfies

$$\begin{aligned} u_0(t; w_0) &= a(w_0) + (1 - a(w_0))(1 + e^{(1-a(w_0))t/\sqrt{2}})^{-1}, \\ e^{-c_0 t} &= (u_0(t; w_0) - a(w_0))^\gamma (1 - u_0(t; w_0))^{-\gamma}, \end{aligned}$$

where  $\gamma := 1 - 2(1 - a(w_0))^{-1}$ . Therefore  $0 < \gamma < 1/3$  and

$$\mathcal{M}(w_0) = -a'(w_0) a(w_0) (1 - a(w_0))^2 \Gamma(1 + \gamma) \Gamma(2 - \gamma) / \Gamma(4) > 0.$$

(5) Finally, if  $w_0 \in H_6(c_0)$  then  $a(w_0) = -1 + \sqrt{2}c_0 \in (-1, 0)$  and the heteroclinic orbit  $(u_0(t; w_0), v_0(t; w_0))$  satisfies

$$\begin{aligned} u_0(t; w_0) &= 1 - (1 - a(w_0))(1 + e^{(1-a(w_0))t/\sqrt{2}})^{-1}, \\ e^{-c_0 t} &= (u_0(t; w_0) - a(w_0))^\gamma (1 - u_0(t; w_0))^{-\gamma}, \end{aligned}$$

where  $-1 < \gamma < 0$ . Then we have

$$\mathcal{M}(w_0) = a'(w_0) a(w_0) (1 - a(w_0))^2 \Gamma(1 + \gamma) \Gamma(2 - \gamma) / \Gamma(4) < 0.$$

The proof is complete.  $\square$

However, if  $a'(w_0) = 0$  in Lemma 4.3 then  $\mathcal{M}(w_0) = 0$ . Therefore we need to compute the higher order term of Melnikov function to detect the transversality of the invariant manifolds. In the following we only investigate the second-order term of Melnikov function  $\mathcal{M}_2(w_0)$ .

**Lemma 4.4.** *Suppose, for some small  $c_0$  and  $w_0$ ,  $a'(w_0) = 0$  and system (2.6) has a heteroclinic orbit  $\Gamma_0 := (u_0(t; w_0), v_0(t; w_0))$ . For fixed  $c = c_0$  and varying parameter  $w$ , the second order Melnikov function is given by*

$$\mathcal{M}_2(w_0) = a''(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0) (1 - u_0(t; w_0)) dt.$$

*Proof.* Without loss of generality, we may assume  $w_0 = 0$ . For such fixed  $w$  near  $w_0$ , let us write the system (2.6) in the following vector form

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} u(t; w) \\ v(t; w) \end{pmatrix} &= \begin{pmatrix} v(t; w) \\ c_0 v(t; w) - u(t; w)(1 - u(t; w))(u(t; w) - a(w)) \end{pmatrix} \quad (4.15) \\ &= R_0(r(t; w)) + R_1(r(t; w))w + R_2(r(t; w))w^2 + O(w^3), \end{aligned}$$

where  $r(t; w) := (u(t; w), v(t; w))^T$ ,

$$\begin{aligned} R_0(r(t; w)) &= \begin{pmatrix} v(t; w) \\ c_0 v(t; w) + u(t; w)^3 - (a(0) + 1)u(t; w)^2 + a(0)u(t; w) \end{pmatrix}, \\ R_1(r(t; w)) &= \begin{pmatrix} 0 \\ a'(0)(u(t; w) - u^2(t; w)) \end{pmatrix}, \\ R_2(r(t; w)) &= \begin{pmatrix} 0 \\ a''(0)(u(t; w) - u^2(t; w))/2 \end{pmatrix}. \end{aligned}$$

As  $w = 0$ , system (2.6) has a heteroclinic orbit  $r(t; w)$  connecting two equilibria,  $E_0^1$  and  $E_0^2$ . Let  $L$  be a line segment transversal to  $r(t; 0)$  at  $r(0; 0)$ . For sufficiently small  $w$ , there exists a unique bounded solution  $r^u(t; w)$  for  $t \leq 0$  such that  $r^u(t; w)$  in the unstable manifold of one equilibrium  $E_w^1$  and  $r^u(0; w) \in L$ . For  $t \leq 0$ , let us define

$$\begin{aligned} z^u(t) &:= \frac{\partial}{\partial w} r^u(t; w)|_{w=0}, & \Delta^u(t) &:= z^u(t) \wedge R_0(r^u(t; 0)), \\ y^u(t) &:= \frac{\partial^2}{\partial^2 w} r^u(t; w)|_{w=0}, & \square^u(t) &:= y^u(t) \wedge R_0(r^u(t; 0)). \end{aligned}$$

Differentiating equation (4.15) with respect to  $w$ , we have

$$\begin{aligned} \frac{d}{dt} \frac{\partial}{\partial w} r^u(t; w) &= \frac{\partial R_0}{\partial r}(r^u(t; w)) \frac{\partial}{\partial w} r^u(t; w) + \frac{\partial R_1}{\partial r}(r^u(t; w)) \frac{\partial}{\partial w} r^u(t; w) w + R_1(r^u(t; w)) \\ &\quad + \frac{\partial R_2}{\partial r}(r^u(t; w)) \frac{\partial}{\partial w} r^u(t; w) w^2 + 2R_2(r^u(t; w))w + O(w^2). \end{aligned}$$



Thus

$$\begin{aligned}
\frac{d}{dt}y^u(t) &= \frac{\partial R_0}{\partial r}(r^u(t; 0))y^u(t) + \left(\frac{\partial}{\partial w} \frac{\partial R_0}{\partial r}(r^u(t; w))\Big|_{w=0}\right)z^u(t) \\
&\quad + 2\frac{\partial R_1}{\partial r}(r^u(t; 0))z^u(t) + 2R_2(r^u(t; 0)), \\
\frac{d}{dt}\square^u(t) &= \left(\frac{d}{dt}y^u(t)\right) \wedge R_0(r^u(t; 0)) + y^u(t) \wedge \left(\frac{\partial R_0}{\partial r}(r^u(t; 0))R_0(r^u(t; 0))\right) \\
&= tr\frac{\partial R_0}{\partial r}\square^u(t) + 2R_2 \wedge R_0 + \left(\frac{\partial}{\partial w} \frac{\partial R_0}{\partial r}(r^u(t; w))\Big|_{w=0}\right)z^u(t) \wedge R_0(r^u(t; 0)) \\
&\quad + 2\frac{\partial R_1}{\partial r}(r^u(t; 0))z^u(t) \wedge R_0(r^u(t; 0)) \\
&= \sigma(t)\square^u(t) + D_2(t),
\end{aligned}$$

where  $\sigma(t) = tr\frac{\partial R_0}{\partial r}$  and

$$D_2(t) = v_0u_0(1 - u_0)a''(0) + 2v_0[3u_0 - a(0) - 1]\left(\frac{\partial u}{\partial w}\Big|_{w=0}\right)^2 = v_0u_0(1 - u_0)a''(0),$$

since  $\frac{\partial u}{\partial w}\Big|_{w=0} = \frac{\partial u}{\partial a}a'(0) = 0$ . Our purpose is to compute  $\square^u(0)$ . By the variation of constant formula,

$$\square^u(t) = e^{\int_0^t \sigma(\tau)d\tau} \left\{ \square^u(0) + \int_0^t e^{-\int_0^\tau \sigma(s)ds} D_2(\tau)d\tau \right\}, \quad \text{for } t < 0.$$

Since

$$\lim_{t \rightarrow -\infty} e^{-\int_0^t \sigma(\tau)d\tau} \square^u(t) = 0,$$

we have

$$\square^u(0) = \int_{-\infty}^0 e^{-\int_0^\tau \sigma(s)ds} D_2(\tau)d\tau.$$

Similarly, there exists a unique bounded solution  $r^s(t; w)$  for  $t \geq 0$  such that  $r^s(t; w)$  in the unstable manifold of the other equilibrium  $E_w^2$  and  $r^s(0; w) \in L$ . For  $t \geq 0$ , let us define

$$y^s(t) := \frac{\partial^2}{\partial^2 w} r^s(t; w)\Big|_{w=0} \quad \text{and} \quad \square^s(t) := y^s(t) \wedge R_0(r^s(t; 0)).$$

By the similar computation, we have

$$\square^s(0) = \int_{\infty}^0 e^{-\int_0^\tau \sigma(s)ds} D_2(\tau)d\tau.$$

Hence

$$\mathcal{M}_2(0) = \square^u(0) - \square^s(0) = a''(0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t) u_0(t) (1 - u_0(t)) dt.$$

The proof is complete.  $\square$

By the proof of Lemma 4.3 and Lemma 4.4, we have the following corollary.

**Corollary 4.5.** *Under the same assumptions as stated in Lemma 4.4, if  $a''(w_0) \neq 0$ , then  $\mathcal{M}_2(w_0) \neq 0$  for any  $c_0 \in (0, 1/\sqrt{2})$  and  $w_0 \in H_i(c_0)$ ,  $i = 1 \cdots 6$ .*

For more higher order terms of Melnikov function, the computation is similar but more complicated. In the following we only state the general result, and skip the proof.

**Lemma 4.6.** *Suppose, for some small  $c_0$  and  $w_0$ ,  $a^{(i)}(w_0) = 0$  for all  $1 \leq i < k$ , where  $k$  is a given positive integer, and system (2.6) has a heteroclinic orbit  $\Gamma_0 := (u_0(t; w_0), v_0(t; w_0))$ . For fixed  $c = c_0$  and varying parameter  $w$ , the  $k$ th order Melnikov function is given by*

$$\mathcal{M}_k(w_0) = a^{(k)}(w_0) \int_{-\infty}^{\infty} e^{-c_0 t} v_0(t; w_0) u_0(t; w_0) (1 - u_0(t; w_0)) dt.$$

As a consequence of previous lemmas and corollary, we have the following conclusions for the transversality of invariant manifolds.

**Lemma 4.7.** *Let  $M \pitchfork N$  be in the sense that manifolds  $M$  and  $N$  intersect transversally, and  $C_\delta(c) := (c - \delta, c + \delta)$ .*

- (1) *Consider system (2.6) with  $c = c_0 \in (0, 1/\sqrt{2})$ . The transversality of various invariant manifolds of (2.6) along  $\Gamma(w)$  are illustrated in Table 2.*

Region of $w$	transversality of manifolds along $\Gamma(w)$
$w \in H_1(c_0)$	$W_0^u(M_0^\delta(w)) \pitchfork W_0^c(M_1^\delta(w))$
$w \in H_2(c_0)$	$W_0^u(M_1^\delta(w)) \pitchfork W_0^c(M_0^\delta(w))$
$w \in H_3(c_0)$	$W_0^u(M_a^\delta(w)) \pitchfork W_0^c(M_0^\delta(w))$
$w \in H_4(c_0)$	$W_0^u(M_1^\delta(w)) \pitchfork W_0^c(M_a^\delta(w))$
$w \in H_5(c_0)$	$W_0^u(M_a^\delta(w)) \pitchfork W_0^c(M_0^\delta(w))$
$w \in H_6(c_0)$	$W_0^u(M_a^\delta(w)) \pitchfork W_0^c(M_1^\delta(w))$
$w \in G_1(c_0)$	$W_0^{cu}(0, 0, w) \pitchfork W_0^c(M_1^\delta(w))$
$w \in G_2(c_0)$	$W_0^{cu}(1, 0, w) \pitchfork W_0^c(M_0^\delta(w))$
$w \in G_3(c_0)$	$W_0^{cu}(0, 0, w) \pitchfork W_0^c(M_a^\delta(w))$
$w \in G_4(c_0)$	$W_0^{cu}(1, 0, w) \pitchfork W_0^c(M_a^\delta(w))$

Table 2: Transversalities of manifolds  $W_0^c(M_{0,1,a}^\delta(w))$ ,  $W_0^u(M_{0,1,a}^\delta(w))$ ,  $W_0^{cu}(0, 0, w)$  and  $W_0^{cu}(1, 0, w)$ .

- (2) *Consider (2.10) with  $c = c_0 \in (0, 1/\sqrt{2})$ . The transversality of various invariant manifolds of (2.10) along  $\Gamma(w) \times \{c_0\}$  are illustrated in Table 3.*

Region of $w$	transversality of manifolds along $\Gamma(w) \times \{c_0\}$
$w \in H_1(c_0)$	$W_0^u(M_0^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_1^\delta(w) \times C_\delta(c_0))$
$w \in H_2(c_0)$	$W_0^u(M_1^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_0^\delta(w) \times C_\delta(c_0))$
$w \in H_3(c_0)$	$W_0^u(M_0^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$
$w \in H_4(c_0)$	$W_0^u(M_1^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$
$w \in H_5(c_0)$	$W_0^u(M_a^\delta(w) \times \{c_0\}) \pitchfork W_0^c(M_0^\delta(w) \times C_\delta(c_0))$
$w \in H_6(c_0)$	$W_0^u(M_a^\delta(w)) \times \{c_0\} \pitchfork W_0^c(M_1^\delta(w) \times C_\delta(c_0))$
$w \in G_1(c_0)$	$W_0^{cu}(0, 0, w, c_0) \pitchfork W_0^c(M_1^\delta(w) \times C_\delta(c_0))$
$w \in G_2(c_0)$	$W_0^{cu}(1, 0, w, c_0) \pitchfork W_0^c(M_0^\delta(w) \times C_\delta(c_0))$
$w \in G_3(c_0)$	$W_0^{cu}(0, 0, w, c_0) \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$
$w \in G_4(c_0)$	$W_0^{cu}(1, 0, w, c_0) \pitchfork W_0^c(M_a^\delta(w) \times C_\delta(c_0))$

Table 3: *Transversalities of manifolds  $W_0^c(M_{0,1,a}^\delta(w) \times C_\delta(c_0))$ ,  $W_0^u(M_{0,1,a}^\delta(w)) \times \{c_0\}$ ,  $W_0^{cu}(0, 0, w, c_0)$  and  $W_0^{cu}(1, 0, w, c_0)$ .*

## 4.2 Proof of the theorems

Now we begin the proof of the main theorems.

### Proof of Theorem 3.2.

We only prove the first part of the theorem. The proof for the second part of the theorem is similar to the proof of Theorem 2.3 of [26] and omitted.

Assume that  $w = (w_1, \dots, w_n)$  is  $s$ -admissible with respect to  $c^*$ , where  $s = (s_1, \dots, s_{n+1})$ . We first claim that the singular local orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$  is weakly shadowed by a true local orbit.

According to Table 1 and the assumption  $a(0) = a_1(c^*)$ , we know that  $0 \in H_1(c^*)$  and there exists a singular local orbit  $0 \rightarrow 1$  at  $w = 0$ . For the following local path  $1 \rightarrow s_2$ , the admissible conditions lead to  $s_2 = 0$  or  $a$ . Therefore, there exists a singular local orbit  $1 \rightarrow s_2$  at  $w = w_1$  if  $w_1 \in H_2(c^*) \cup G_2(c^*)$  or  $H_4(c^*) \cup G_4(c^*)$  (in fact,  $w_1 \in H_2(c^*)$  or  $H_4(c^*)$ ). Take  $M^0 = W_0^u((0, 0, 0) \times C_\delta(c^*))$ . According to Lemma 4.7, we have

$$M^0 \pitchfork W_0^c(M_1^\delta(0) \times C_\delta(c^*)).$$

Let  $N_0$  be their intersection. Since the phase space of system (2.10) is  $\mathbb{R}^4$  and dimensions of  $M^0$  and  $W_0^c(M_1^\delta(0) \times C_\delta(c^*))$  are 2 and 3 respectively, then  $\dim N_0 = 2 + 3 - 4 = 1$ . We now apply the Exchange lemma 2.2 to the vicinity of the slow manifold  $M_1 \times C_\delta(c^*)$  along the slow orbit from  $(1, 0, 0, c^*)$  to  $(1, 0, w_1, c^*)$ . Taking  $M^\varepsilon = W_\varepsilon^u((0, 0, 0) \times C_\delta(c^*))$  be such that  $M^\varepsilon \rightarrow M^0$  as  $\varepsilon \rightarrow 0$ . By condition (A1) and part (1) of Lemma 2.2, a portion  $M_{p_0}^\varepsilon$  of  $M^\varepsilon$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u(M_1^\delta(w_1) \times C_\delta(c^*))$ . Note that  $\dim W_0^u(M_1^\delta(w_1) \times C_\delta(c^*)) = 3$ . Thus, the singular orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$  is weakly shadowed by a true orbit.

Next, we claim that the singular local orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2 \xrightarrow{w=w_2} s_3$  is weakly shadowed by a true local orbit. Two cases for  $s_2 = 0$  or  $a$  are considered. For the case  $s_2 = 0$ , a portion  $M_{p_1}^\varepsilon$  of  $W_\varepsilon^u((a(w_1), 0, w_1) \times C_\delta(c^*))$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_3, 0) \times (w_2 - \delta, w_2 + \delta) \times C_\delta(c^*))$  or  $W_\varepsilon^{cu}((s_3, 0) \times \{w_2\} \times \{c^*\})$ . The details of proof can be found in [26] by using the Exchange lemma with turning point (Lemma 2.2) and omitted. For the case  $s_2 = a$ , the admissible conditions imply  $s_3 = 1$  or  $0$ . Thus there exists a singular local orbit  $s_2 \rightarrow s_3$  at  $w = w_2$  if  $w_2 \in H_5(c^*)$  or  $H_6(c^*)$ . From the admissible condition (A4), there is no turning point between  $w_1$  and  $w_2$ . It is also easy to see that  $W_0^u((1, 0) \times (w_1 - \delta, w_1 + \delta) \times (C_\delta(c^*)))$  and  $W_0^c((a(w_1), 0) \times (w_1 - \delta, w_1 + \delta) \times C_\delta(c^*))$  intersect transversally. Then, by Lemma 2.3, a portion  $M_{p_1}^\varepsilon$  of  $W_\varepsilon^u((a(w_1), 0) \times (w_1 - \delta, w_1 + \delta) \times C_\delta(c^*))$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_3, 0) \times (w_2 - \delta, w_2 + \delta) \times C_\delta(c^*))$ . Note that  $\dim W_0^u(M_1^\delta(w_1) \times C_\delta(c^*)) = \dim W_0^{cu}((s_3, 0) \times \{w_2\} \times C_\delta(c^*)) = 3$ .

According to the above discussions, we can conclude inductively that the singular local orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} \dots \xrightarrow{w=w_{n-1}} s_n$  is weakly shadowed by a true local orbit. Generally, for  $2 < i < (n + 1)$ , we consider the following two cases.

(1) Assume that there exists a turning point between  $w_{i-1}$  and  $w_i$ . By Lemma 2.2, a portion  $M_{p_{i-1}}^\varepsilon$  of  $W_\varepsilon^u((s_i, 0, w_{i-1}) \times C_\delta(c^*))$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta) \times C_\delta(c^*))$  or  $W_\varepsilon^{cu}((s_{i+1}, 0, w_i) \times C_\delta(c^*))$ .

(2) Assume that there is no turning point between  $w_{i-1}$  and  $w_i$ . By Lemma 2.3, a portion  $M_{p_{i-1}}^\varepsilon$  of  $W_\varepsilon^u((s_i, 0, w_{i-1}) \times C_\delta(c^*))$  will leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta) \times C_\delta(c^*))$ .

Furthermore, we have  $\dim W_0^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta) \times C_\delta(c^*)) = 3$  and  $\dim W_0^{cu}((s_{i+1}, 0, w_i) \times C_\delta(c^*)) = 3$ .

Finally, we prove that the true orbits obtained by above arguments are  $C^1$   $O(\varepsilon)$ -close to the unstable manifold  $W_0^u((s_{n+1}, 0, w_n) \times C_\delta(c^*))$ . Since  $s_{n+1} = 1$ , the admissible conditions lead to  $w_n \in H_1(c^*) \cup G_1(c^*)$  or  $H_6(c^*)$ . Thus, there exists a singular local orbit  $s_n \rightarrow 1$  at  $w = w_n$ . By condition (A1), we have  $a(w) < 1$  for all  $w \in [w_n, 1]$  and the singular orbit will approach to  $(1, 0, 1)$  as time goes infinity. Moreover,  $W_0^u((s_n, 0, w_n) \times \{c^*\})$  intersects  $W_0^c(M_1^\delta(w_n) \times C_\delta(c^*))$  transversally. As a result, the true orbit will approach a neighborhood of  $(1, 0, 1, c^*)$ , near the singular orbit and  $C^1$   $O(\varepsilon)$ -close to the unstable manifold  $W_0^u((s_{n+1}, 0, w_n) \times C_\delta(c^*))$ . The proof of the theorem is complete.

### Proof of Theorem 3.3.

Let  $w = (w_1, \dots, w_n)$  is  $s$ -admissible for  $c = c^*$  and  $s = (s_1, \dots, s_{n+1})$ . We first claim that the singular local orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$  is strongly shadowed by a true orbit.

According to Table 1 and the assumption  $a(0) \leq a_6(c^*)$ , we know that  $0 \in G_1(c^*)$  and there exists a singular local orbit  $0 \rightarrow 1$  at  $w = 0$ . For the following local path  $1 \rightarrow s_2$ , the admissible conditions lead to  $s_2 = 0$  or  $a$ . Therefore, there exists a singular local orbit  $1 \rightarrow s_2$  at  $w = w_1$  if  $w_1 \in H_2(c^*) \cup G_2(c^*)$  or  $H_4(c^*) \cup G_4(c^*)$  (in fact,  $w_1 \in H_2(c^*)$  or  $H_4(c^*)$ ). Take  $M^0 = W_0^u(0, 0, 0)$ . According to Lemma 4.7, we have

$$M^0 \pitchfork W_0^c\{M_1^\delta(0)\}.$$

Let  $N_0$  be their intersection. Since the phase space of system (2.3) is  $\mathbb{R}^3$  and both dimensions of  $M^0$  and  $W_0^c(M_1^\delta(0))$  are 2, then  $\dim N_0 = 2 + 2 - 3 = 1$ . We now apply Exchange lemma 2.2 to the vicinity of the slow manifold  $M_1$  along the slow orbit from  $(1, 0, 0)$  to  $(1, 0, w_1)$ . Taking  $M^\varepsilon = W_\varepsilon^u((0, 0, 0))$  be such that  $M^\varepsilon \rightarrow M^0$  as  $\varepsilon \rightarrow 0$ . By condition (A1) and part (1) of Lemma 2.2, a portion  $M_{p_0}^\varepsilon$  of  $M^\varepsilon$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u(M_1^\delta(w_1))$ . Note that  $\dim W_0^u(M_1^\delta(w_1)) = 2$ . Thus, the singular orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2$  is strongly shadowed by a true orbit.

Next, we claim that the singular local orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} s_2 \xrightarrow{w=w_2} s_3$  is strongly shadowed by a true local orbit. Two cases for  $s_2 = 0$  or  $a$  are considered. For the case  $s_2 = 0$ , a portion  $M_{p_1}^\varepsilon$  of  $W_\varepsilon^u((a(w_1), 0, w_1))$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_3, 0) \times (w_2 - \delta, w_2 + \delta))$  or  $W_\varepsilon^{cu}((s_3, 0) \times \{w_2\})$ . The details of proof can be found in [26] by using the Exchange lemma with turning point (Lemma 2.2) and omitted. For the case  $s_2 = a$ , the admissible conditions imply  $s_3 = 1$  or  $0$ . Thus there exists a singular local orbit  $s_2 \rightarrow s_3$  at  $w = w_2$  if  $w_2 \in H_5(c^*)$  or  $H_6(c^*)$ . From the admissible condition (A4), there is no turning point between  $w_1$  and  $w_2$ . It is also easy to see that  $W_0^u((1, 0) \times (w_1 - \delta, w_1 + \delta))$  and  $W_0^c((a(w_1), 0) \times (w_1 - \delta, w_1 + \delta))$  intersect transversally. Then, by Lemma 2.3, a portion  $M_{p_1}^\varepsilon$  of  $W_\varepsilon^u((a(w_1), 0) \times (w_1 - \delta, w_1 + \delta))$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_3, 0) \times (w_2 - \delta, w_2 + \delta))$ . Note that  $\dim W_0^u(M_1^\delta(w_1)) = \dim W_0^{cu}((s_3, 0) \times \{w_2\}) = 2$ .

According to the above discussions, we can conclude inductively that the singular local orbit  $0 \xrightarrow{w=0} 1 \xrightarrow{w=w_1} \dots \xrightarrow{w=w_{n-1}} s_n$  is strongly shadowed by a true local orbit. Generally, for  $2 < i < (n + 1)$ , we consider the following two cases.

(1) Assume that there exists a turning point between  $w_{i-1}$  and  $w_i$ . By Lemma 2.2, a portion  $M_{p_{i-1}}^\varepsilon$  of  $W_\varepsilon^u((s_i, 0, w_{i-1}))$  will proceed near the singular orbit and leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta))$  or  $W_\varepsilon^{cu}((s_{i+1}, 0, w_i))$ .

(2) Assume that there is no turning point between  $w_{i-1}$  and  $w_i$ . By Lemma 2.3, a portion  $M_{p_{i-1}}^\varepsilon$  of  $W_\varepsilon^u((s_i, 0, w_{i-1}))$  will leave the vicinity of slow orbit close to  $W_\varepsilon^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta))$ .

Furthermore, we have  $\dim W_0^u((s_{i+1}, 0) \times (w_i - \delta, w_i + \delta)) = \dim W_0^{cu}((s_{i+1}, 0, w_i)) = 2$ .

Finally, we prove that the true orbits obtained by above arguments are  $C^1$   $O(\varepsilon)$ -close to the unstable manifold  $W_0^u((s_{n+1}, 0, w_n))$ . Since  $s_{n+1} = 1$ , the admissible conditions lead to  $w_n \in H_1(c^*) \cup G_1(c^*)$  or  $H_6(c^*)$ . Thus, there exists a singular local orbit  $s_n \rightarrow 1$  at  $w = w_n$ . By condition (A1), we have  $a(w) < 1$  for all  $w \in [w_n, 1]$  and the singular orbit will approach to  $(1, 0, 1)$  as time goes infinity. Moreover,  $W_0^u((s_n, 0, w_n))$  intersects  $W_0^c(M_1^\delta(w_n))$  transversally. As a result, the true orbit will approach a neighborhood of  $(1, 0, 1)$ , near the singular orbit and  $C^1$   $O(\varepsilon)$ -close to the unstable manifold  $W_0^u((s_{n+1}, 0, w_n))$ . The proof of the theorem is complete.

The results of Theorem 3.4 can also be proved in the same way and omitted.

## 5 Examples

In this section we provide an example to support our main results. Note that  $H_6 \cap \partial\Omega$  at  $c = 2\sqrt{3} - 2\sqrt{2}$ . Assume that  $c \in S \cap (2\sqrt{3} - 2\sqrt{2}, 1/\sqrt{2})$ ,  $0 < \varepsilon < 1/4$  and  $\delta \geq 0$  be fixed numbers. Then  $a_6(c) < 0 < a_1(c) < a_2(c) < 1$ . Let us define  $a(w)$  on  $[0, 1]$  by

$$a(w) = \begin{cases} (\alpha - a_2(c)) \exp\left\{\frac{(w-1)^4}{(w-1)^4 - (1/4)^4}\right\} + a_2(c) + \delta, & \text{if } w \in [\frac{3}{4}, 1], \\ (a_2(c) - \varepsilon) \exp\left\{\frac{(w-(3/4))^4}{(w-(3/4))^4 - \varepsilon^4}\right\} + \varepsilon + \delta, & \text{if } w \in [\frac{3}{4} - \varepsilon, \frac{3}{4}], \\ \varepsilon \exp\left\{\frac{(w+\varepsilon-(3/4))^4}{(w+\varepsilon-(3/4))^4 - ((1/4)-\varepsilon)^4}\right\} + \delta, & \text{if } w \in [\frac{1}{2}, \frac{3}{4} - \varepsilon], \\ \beta(1 - \exp\left\{\frac{(w-(1/2))^4}{(w-(1/2))^4 - (1/2)^4}\right\}) + \delta, & \text{if } w \in [0, \frac{1}{2}], \end{cases}$$

where  $\alpha \in (a_2(c), 1)$  and  $\beta \in (-c^2/4, a_6(c))$ . It is not difficult to verify that  $a(w)$  is a monotonic increasing  $C^2$  function on  $[0, 1]$ .

If  $\delta = 0$ , then  $a(3/4) = a_2(c)$ ,  $a(1/2) = 0$  and there exist  $w^L < 1/2 < w^R < 3/4$  such that  $a(w^L) = a_6(c)$  and  $a(w^R) = a_1(c)$ , see the left part of Figure . According to Table 1, there exists orbits of (2.5) connecting from  $a$  to 1, 1 to 0 and 0 to 1 at levels  $w = w^L$ ,  $w = 3/4$ , and  $w = w^R$  respectively. Now we estimate the following integrals:

$$\begin{aligned} I_1 &:= \int_{\frac{1}{2}}^{w^L} \frac{\lambda_0^-(\eta; c)}{g(0, \eta)} d\eta = \int_{\frac{1}{2}}^{w^L} \frac{\sqrt{c^2 - 4a(\eta)} + c}{2\eta} d\eta > 0; \\ I_2 &:= \int_{\frac{3}{4} - \varepsilon}^{\frac{1}{2}} \frac{\lambda_0^-(\eta; c)}{g(0, \eta)} d\eta = \int_{\frac{3}{4} - \varepsilon}^{\frac{1}{2}} \frac{\sqrt{c^2 + 4a(\eta)} - c}{2\eta} d\eta > K_1\varepsilon; \\ I_3 &:= \int_{\frac{3}{4}}^{\frac{3}{4} - \varepsilon} \frac{\lambda_0^-(\eta; c)}{g(0, \eta)} d\eta = \int_{\frac{3}{4}}^{\frac{3}{4} - \varepsilon} \frac{\sqrt{c^2 + 4a(\eta)} - c}{2\eta} d\eta > K_2\varepsilon, \end{aligned}$$

where  $K_1$  and  $K_2$  are negative constants. Since  $I_1$  is independent of  $\varepsilon$ , if  $\varepsilon$  is small enough then

$$\int_{\frac{3}{4}}^{w^L} \frac{\lambda_0^-(\eta; c)}{g(0, \eta)} d\eta = I_1 + I_2 + I_3 > 0.$$

Thus there exist  $\bar{w} \in (w^L, 1/2)$  such that

$$\int_{\frac{3}{4}}^{\bar{w}} \frac{\lambda_0^-(\eta; c)}{g(0, \eta)} d\eta = 0,$$

and orbit of (2.5) connecting from 0 to  $a$  at level  $w = \bar{w}$ . Since  $a(1/2) = 0$  and  $a'(1/2) = 0$ ,  $w = 1/2$  is a degenerate turning point such that Theorem 3.3 can not be applied directly to obtain the traveling wave solutions. To avoid the degeneracy of turning point, in the following we consider the case of  $a(w)$  but with  $\delta > 0$ .

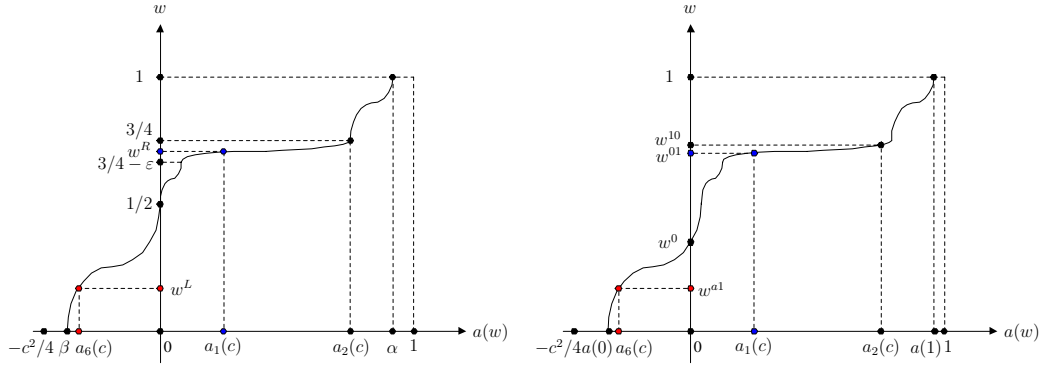


Figure 7: Graph of  $a(w)$  with  $\delta = 0$  (left part) and  $\delta > 0$  (right part).

For  $\delta > 0$ , it is obvious that the graph of  $a(w)$  is a shift of left part of Figure 7, see the right part of Figure 7. By continuity, if  $\delta$  is small enough then there exists  $w^{a1} < w^{0a} < w^0 < w^{01} < w^{10}$  such that

- (1)  $a(1) \in (a_2(c), 1)$  and  $a(0) \in (-c^2/4, a_6(c))$ ;
- (2)  $a(w^{a1}) = a_6(c)$ ,  $a(w^0) = 0$ ,  $a'(w^0) \neq 0$ ,  $a(w^{01}) = a_1(c)$  and  $a(w^{10}) = a_2(c)$ ;
- (3)  $\int_{w^{10}}^{w^{a1}} \frac{\lambda_0^-(\eta; c)}{g(0, \eta)} d\eta > 0$  and  $\int_{w^{10}}^{w^{0a}} \frac{\lambda_0^-(\eta; c)}{g(0, \eta)} d\eta = 0$ .

Thus there exist orbits of (2.5) connecting from  $a$  to 1, 0 to  $a$ , 0 to 1 and 1 to 0 at levels  $w = w^{a1}$ ,  $w = w^{0a}$ ,  $w = w^{01}$ , and  $w = w^{10}$  respectively. Since  $a'(w^0) \neq 0$ , by the admissible conditions, the word  $w = (w^{10}, w^{0a}, w^{a1}, w^{10}, w^{01})$

is  $s$ -admissible with respect to  $c$  with  $s = (1, 0, a, 1, 0, 1)$ . Furthermore, by repeating the local paths, the singular orbits along the path

$$01 \xrightarrow{H_2} \underbrace{01 \cdots 01}_{n_1 \ 01s} \xrightarrow{H_2} \underbrace{0a1 \cdots 0a1}_{n_2 \ 0a1s} \xrightarrow{H_2} \underbrace{01 \cdots 01}_{n_3 \ 01s}, \quad n_1, n_2, n_3 \in \mathbb{Z}^+ \cup \{0\},$$

or any copy of the such path can be weakly shadowed by true orbits. Since all  $n_i$  are arbitrary, by Theorem 3.3, such kind of  $a(w)$  with small  $\varepsilon > 0$  and  $\delta > 0$  provide us the multiplicity of traveling wave solutions.

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