Realization of Linear Defuzzified Output via Mixed Fuzzy Logics

Ching-Chang Wong
Department of Electronic Engineering,
Tamkang University, Tamsui, Taipei Hsien,
Taiwan 25137, R.O.C.

Abstract — The realization of the linear defuzzified output of the fuzzy controller is discussed in this paper. Using the mixed fuzzy logics to evaluate the fuzzy control rule, we show that the defuzzified output by appropriate choice of each component can be precisely equivalent to a linear function of the inputs to the fuzzy controller.

I. INTRODUCTION

There are some papers discussing the output of fuzzy controller in a few situations. For the case of two inputs [4], the authors defined a linear fuzzy controller with two fuzzy control inputs and evaluated the fuzzy control rules using different kinds of fuzzy logic operators. The camparison of the consequences for choice of Zadeh, probability and Lukasiewicz logic has been made. They investigated the output of this fuzzy controller for certain t-norm and t-conorm operators, but only the special case of three fuzzy numbers employed to fuzzify each controller input is discussed.

In this paper, the realization of the linear defuzzified output of the fuzzy controller by appropriate choice of each component of the fuzzy controller is discussed. This paper is organized as follows: The fuzzy controller with simplified fuzzy reasoning method is described in Section II. The linear defuzzified output of the fuzzy controller is discussed in Section III. In there, arbitrary numbers of triangular fuzzy numbers are employed to fuzzify the linguistic variables in fuzzy control rules. Using three mixed fuzzy logics, we show that it can be precisely equivalent to a linear function of all the inputs to the fuzzy controller. Finally, we make a brief conclusion in Section IV.

II. STATES DESCRIPTIONS

The fuzzy control rules with two input fuzzy variables is described by

$$R_{(i,j)}: \text{ If } x \text{ is } A_i \text{ and } y \text{ is } B_j$$
$$\text{then } u \text{ is } C_{f(i,j)} \qquad i\in I,\ j\in J \qquad (1)$$

where x, y, u stand for input or output linguistic variables and $I=\{-m,\cdots,m\}$, $J=\{-n,\cdots,n\}$ denote the index sets. A_i, B_j, $C_{f(i,j)}$ are fuzzy sets characterized by membership functions $A_i(x):X\to[0,1]$, $B_j(y):Y\to[0,1]$, $C_{f(i,j)}(u):U\to[0,1]$ corresponding to the universe discourse of X, Y, U,

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respectively. $f(i,j)$ denotes a constant index function which decides a linguistic value of u.

The fuzzy implication in (1) can be translated into a three dimensional relation R defined on the Cartesian product of universe X×Y×U as

$$R = \bigcup_{i,j} R_{(i,j)}$$

In there, the (i,j)-th rule can be described by a fuzzy relation $R_{(i,j)}$ on a universe of X×Y×U as

$$R_{(i,j)} = (A_i \text{ and } B_j) \rightarrow C_{f(i,j)}$$

and its membership function can be expressed by

$$R_{(i,j)}(x,y,u) = \mathbb{T}[A_i(x),\ B_j(y),\ C_{f(i,j)}(u)]$$

where $\mathbb{T}$ denotes the t-norm operator [1]. So the membership function of the overall fuzzy relation R is

$$R(x,y,u) = \underset{\substack{i\in I\\ j\in J}}{\mathbb{C}}\ R_{(i,j)}(x,y,u)$$

$$= \underset{\substack{i\in I\\ j\in J}}{\mathbb{C}}\ \mathbb{T}[A_i(x),\ B_j(y),\ C_{f(i,j)}(u)]$$

where $\mathbb{C}$ denotes the t-conorm operator.

If the inputs x and y take the fuzzy sets A' and B', respectively, the output fuzzy set C' can be calculated from antecedents A', B' and fuzzy relation R by compositional rule of inferences as follows:

$$C' = (A' \text{ and } B')\circ R$$

where $\circ$ denotes the sup-t-norm composition. Explicitly, the membership function of the consequence C' is

$$C'(u) = \sup_{\substack{x\in X\\ y\in Y}} \mathbb{T}[A'(x),\ B'(y),\ R(x,y,u)]$$

$$= \sup_{\substack{x\in X\\ y\in Y}} \{\mathbb{T}[A'(x),\ B'(y),\ \underset{\substack{i\in I\\ j\in J}}{\mathbb{C}} \{\mathbb{T}[A_i(x),B_j(y),C_{f(i,j)}(u)]\}]\}$$

Theoretical and experimental studies have indicated that some t-operators may work better than others in some situations [1]. In this paper, three t-norm and one t-conorm operators for evaluation of the fuzzy control rules are considered:

(a) Zadeh AND operator:

$$\mathbb{T}(a,b) = \mathrm{ZAND}(a,b) = \min(a,b)$$

(b) Probability AND operator:

$$\mathbb{T}(a,b) = \mathrm{PAND}(a,b) = a\cdot b$$

(c) Lukasiewicz AND operator:

$$\mathbb{T}(a,b) = \mathrm{LAND}(a,b) = \max(0,\ (a+b)-1)$$

(d) Lukasiewicz OR operator:

$$\mathbb{C}(a,b) = \mathrm{LOR}(a,b) = \min(1,\ a+b)$$

where a and b are grades of membership of an object in fuzzy sets. We can represent C'(u) as

$$C'(u)= \underset{\substack{i\in I\\ j\in J}}{\mathbb{C}}\ \mathbb{T}\{\sup_{x\in X} \mathbb{T}[A'(x),A_i(x)],\ \sup_{y\in Y} \mathbb{T}[B'(y),\ B_j(y)],\ C_{f(i,j)}(u)\}$$

In actual applications the inputs of the controller are some crisp datas. It can be realized by a process called fuzzification, which simply considers the input fuzzy sets A' and B' to be singletons, i.e.,

$$A'(x) = \begin{cases} 1 & \text{if } x = x^* \\ 0 & \text{otherwise} \end{cases}$$

and

$$B'(y) = \begin{cases} 1 & \text{if } y = y^* \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sup_{x\in X} \mathbb{T}[A'(x),\ A_i(x)] = A_i(x^*)$$

and

$$\sup_{y\in Y} \mathbb{T}[B'(y),\ B_j(y)] = B_j(y^*)$$

Therefore

$$C'(u) = \underset{\substack{i\in I \\ j\in J}}{\mathbb{C}}\ \mathbb{T}[A_i(x^*), B_j(y^*), C_{f(i,j)}(u)] \tag{2}$$

We may defuzzify the output fuzzy set C' into a scalar output by computing its fuzzy centroid. In this paper, we consider a simple form of output fuzzy sets, where the output fuzzy sets $C_{f(i,j)}$ are considered as distinct output fuzzy singletons [2], i.e.,

$$C_{f(i,j)}(u) = \begin{cases} 1 & \text{if } u = f(i,j) \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$C'(u) = \begin{cases} \mathbb{T}[A(x^*), B(y^*)] & \text{if } u = f(i,j) \\ 0 & \text{otherwise} \end{cases}$$

So the representative point of C' takes the form

$$u = \frac{\sum_{i=-m}^{m} \sum_{j=-n}^{n} f(i,j)\cdot \mathbb{T}[A_i(x^*),\ B_j(y^*)]}{\sum_{i=-m}^{m} \sum_{j=-n}^{n} \mathbb{T}[A_i(x^*),\ B_j(y^*)]} \tag{3}$$

We find that the defuzzified output is concerned directly with the definition of linguistic membership functions and the type of t-norm operators which are discussed below.

III. MAIN RESULTS

If the triangular fuzzy number is chosen for each linguistic variable as:

$$A_i(x) = \begin{cases} 1-|x-i|, & \text{if } |x-i|\le 1 \\ 0, & \text{otherwise} \end{cases} \qquad i\in I \tag{4}$$

$$B_j(y) = \begin{cases} 1-|y-j|, & \text{if } |y-j|\le 1 \\ 0, & \text{otherwise} \end{cases} \qquad j\in J \tag{5}$$

Given a value for x^* in $[-m,m]$ and a value for y^* in $[-n,n]$, we can represent x^* and y^* as

$$x^* = i'+s, \qquad (0\le s<1) \tag{6}$$

$$y^* = j'+p, \qquad (0\le p<1) \tag{7}$$

where i' and j' are integers. We know that only two of the membership functions $A_i(x^*)$, $B_j(y^*)$ may be positive and that is when $i=i'$, $i'+1$ and $j=j'$, $j'+1$. We obtain

$$A_i(i'+s) = \begin{cases} 1-s, & i=i' \\ s, & i=i'+1 \\ 0, & \text{otherwise} \end{cases} \qquad (0\le s<1) \tag{8}$$

$$B_j(j'+p) = \begin{cases} 1-p, & j=j' \\ p, & j=j'+1 \\ 0, & \text{otherwise} \end{cases} \qquad (0\le p<1) \tag{9}$$

Therefore there are only four rules fired in the set of complete control rules:

$R_{(i',j')}$: If x is $A_{i'}$ and y is $B_{j'}$ then u is $f(i',j')$

$R_{(i',j'+1)}$: If x is $A_{i'}$ and y is $B_{j'+1}$ then u is $f(i',j'+1)$

$R_{(i'+1,j')}$: If x is $A_{i'+1}$ and y is $B_{j'}$ then u is $f(i'+1,j')$

$R_{(i'+1,j'+1)}$: If x is $A_{i'+1}$ and y is $B_{j'+1}$ then u is $f(i'+1,j'+1)$

The defuzzified algorithm (3) can be represented by

$$u=\frac{\sum_{i=i'}^{i'+1}\sum_{j=j'}^{j'+1} f(i,j)\cdot \mathbb{T}[A_i(x^*),\ B_j(y^*)]}{\sum_{i=i'}^{i'+1}\sum_{j=j'}^{j'+1} \mathbb{T}[A_i(x^*),\ B_j(y^*)]} \tag{10}$$

Now, we have the following theorem:

Theorem 1:

Suppose the triangular fuzzy numbers (4) and (5) are used to define the linguistic variables x and y in the fuzzy controller with the rule base:

$R_{(i,j)}$: If x is A_i and y is B_j

then u is $f(i,j)=a\cdot i+b\cdot j$, $i\in I$, $j\in J$

Let

$$w_{00} = \mathbb{T}_1[A_{i'}(x^*),\ B_{j'}(y^*)] \tag{11}$$

$$w_{10} = \mathbb{T}_2[A_{i'+1}(x^*),\ B_{j'}(y^*)] \tag{12}$$

$$w_{01} = \mathbb{T}_3[A_{i'}(x^*),\ B_{j'+1}(y^*)] \tag{13}$$

$$w_{11} = \mathbb{T}_4[A_{i'+1}(x^*),\ B_{j'+1}(y^*)] \tag{14}$$

For any inputs $x^*=i'+s$ and $y^*=j'+p$, if we use $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$ and $\mathbb{T}_4$ operators for the AND clauses of rule $R(i',j')$, $R(i'+1,j')$, $R(i',j'+1)$ and $R(i'+1,j'+1)$, respectively to determine w_{00}, w_{10}, w_{01}, w_{11} such that $w_{00}+w_{10}+w_{01}+w_{11}=1$ and $a\cdot w_{10}+b\cdot w_{01}+(a+b)\cdot w_{11}=a\cdot s+b\cdot p$, then the defuzzified output is

$$u = a\cdot x^* + b\cdot y^*$$

Proof:

For $x^*=i'+s$ and $y^*=j'+p$, if we use $\mathbb{T}_1$ operator for the AND clauses of rule $R(i',j')$, $\mathbb{T}_2$ operator for the AND clauses of rules $R(i'+1,j')$, $\mathbb{T}_3$ operator for the AND clauses of rules $R(i',j'+1)$, and $\mathbb{T}_4$ operator for the AND clauses of rule $R(i'+1,j'+1)$, the defuzzified output (10) is

$$u = a\cdot i' + b\cdot j' + \frac{a\cdot w_{10}+b\cdot w_{01}+(a+b)\cdot w_{11}}{w_{00}+w_{10}+w_{01}+w_{11}}$$

So if $w_{00} + w_{10} + w_{01} + w_{11} = 1$ and $a\cdot w_{10}+b\cdot w_{01}+(a+b)\cdot w_{11}=a\cdot s+b\cdot p$, the defuzzified output becomes

$$u = a\cdot x^* + b\cdot y^*$$

This completes the proof. Q.E.D.

Remark 1:

When there are some rules with the same consequence $f(i,j)$, we can combine they with one and the formula (3) can be rewritted by

$$u = \frac{\sum_{\substack{t=f(i,j)\\ i\in I,\ j\in J}} t\cdot \underset{t}{\mathrm{LOR}}\ \mathbb{T}[A_i(x^*),\ B_j(y^*)]}{\sum_{\substack{t=f(i,j)\\ i\in I,\ j\in J}} \underset{t}{\mathrm{LOR}}\ \mathbb{T}[A_i(x^*),\ B_j(y^*)]}$$

Under this situation, we can also derive the same result of Theorem 1.

From (11), (12), (13) and (14), we see that these values w_{00}, w_{10}, w_{01} and w_{11} are determined by the choice of the logical AND operators T_1, T_2, T_3 and T_4, respectively. Basded on this, we now evaluate the rules using the following appropriate t-norm operators such that $w_{00}+w_{10}+w_{01}+w_{11}=1$ and $a\cdot w_{10}+b\cdot w_{01}+(a+b)\cdot w_{11}=a\cdot s+b\cdot p$. First, we use the Zadeh AND logic for the T_1 and T_4 operators and the Lukasiewicz AND logic for the T_2 and T_3 operators. Next, we use Lukasiewicz AND logic for the T_1 and T_4 operators and Zadeh AND logic for the T_2 and T_3 operators. Finally, we use probability AND logic for the T operators of the fired rules. We have the following theorem:

Theorem 2:

Suppose the triangular fuzzy numbers (4) and (5) are used to define the linguistic variables x and y in the fuzzy controller with the rule base:

$R_{(i,j)}$: If x is A_i and y is B_j then u is $a\cdot i+b\cdot j$, $i\in I$, $j\in J$.

For any inputs $x^*=i'+s$ and $y^*=j'+p$, if we use T_1, T_2, T_3 and T_4 operators for the AND clauses of rule $R(i',j')$, $R(i'+1,j')$, $R(i',j'+1)$ and $R(i'+1,j'+1)$, respectively and the following three cases of the appropriate operators are chosen as

(Case 1): $T_1=T_4$=ZAND, $T_2=T_3$=LAND,

(Case 2): $T_1=T_4$=LAND, $T_2=T_3$=ZAND, and

(Case 3): $T_1=T_2=T_3$=PAND,

then the defuzzified output is

$$u = a\cdot x^* + b\cdot y^*$$

Proof:

Substituting the results of (8) and (9) into the three cases, we have:

In Case 1, since the logic operators are chosed as $T_1=T_4$=ZAND and $T_2=T_3$=LAND, from (11), (12), (13) and (14), we obtain

$$w_{00} = \min(1-s,\ 1-p) = \begin{cases} 1-s & \text{if } s\ge p \\ 1-p & \text{if } s<p \end{cases}$$

$$w_{10} = \max(0,\ s-p) = \begin{cases} s-p & \text{if } s\ge p \\ 0 & \text{if } s<p \end{cases}$$

$$w_{01} = \max(0,\ p-s) = \begin{cases} 0 & \text{if } s\ge p \\ p-s & \text{if } s<p \end{cases}$$

$$w_{11} = \min(s,p) = \begin{cases} p & \text{if } s\ge p \\ s & \text{if } s<p \end{cases}$$

In Case 2, since the operators are chosed as $T_1=T_4$=LAND and $T_2=T_3$=ZAND, from (11), (12), (13) and (14), we get

$$w_{00} = \mathrm{LAND}(1-s,1-p) = \max(0,\ 1-s-p) \quad (15)$$

$$w_{10} = \mathrm{ZAND}(s,1-p) = \min(s,1-p) \quad (16)$$

$$w_{01} = \mathrm{ZAND}(1-s,p) = \min(1-s,p) \quad (17)$$

$$w_{11} = \mathrm{LAND}(s,p) = \max(0,s+p-1) \quad (18)$$

From Figure 1, we can partition the unit square into eight regions as follows:

(a) $0\le 1-p\le s\le 1-s\le p\le 1$,

(b) $0\le 1-p\le 1-s\le s\le p\le 1$,

(c) $0\le 1-s\le 1-p\le p\le s\le 1$,

(d) $0\le 1-s\le p\le 1-p\le s\le 1$,

(e) $0\le p\le 1-s\le s\le 1-p\le 1$,

(f) $0\le p\le s\le 1-s\le 1-p\le 1$.

(g) $0\le s\le p\le 1-p\le 1-s\le 1$,

(h) $0\le s\le 1-p\le p\le 1-s\le 1$,

From (15), (16), (17), (18) we can easily list the values of w_{00}, w_{10}, w_{01} and w_{11} for these eight regions as follows:

<Regions a,b,c,d>

$w_{00}=0$, $w_{10}=1-p$, $w_{01}=1-s$, $w_{11}=s+p-1$;

<Regions e,f,g,h>

$w_{00}=1-s-p$, $w_{10}=s$, $w_{01}=p$, $w_{11}=0$.

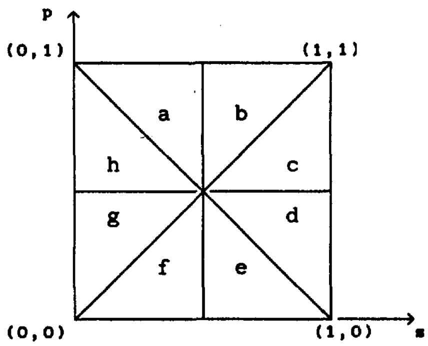

Figure 1. Partition of the unit square into eight regions.

So we obtain

$$w_{00} = \begin{cases} 0 & \text{if } s+p \geq 1 \\ 1-s-p & \text{if } s+p<1 \end{cases}$$

$$w_{10} = \begin{cases} 1-p & \text{if } s+p \geq 1 \\ s & \text{if } s+p<1 \end{cases}$$

$$w_{10} = \begin{cases} 1-s & \text{if } s+p \geq 1 \\ p & \text{if } s+p<1 \end{cases}$$

$$w_{11} = \begin{cases} s+p-1 & \text{if } s+p \geq 1 \\ 0 & \text{if } s+p<1 \end{cases}$$

In Case 3, since the operators are chosed as $T_1=T_2=T_3=T_4=$PAND, from (11), (12), (13) and (14), we get

$$w_{00} = \mathrm{PAND}(1-s,1-p) = 1-s-p+sp$$

$$w_{10} = \mathrm{PAND}(s,1-p) = s(1-p)$$

$$w_{01} = \mathrm{PAND}(1-s,p) = (1-s)p$$

$$w_{11} = \mathrm{PAND}(s,p) = sp$$

We see w_{00}, w_{10}, w_{01} and w_{11} in the three cases all satisfy that $w_{00}+w_{10}+w_{01}+w_{11}=1$ and $a \cdot w_{10}+b \cdot w_{01}+(a+b) \cdot w_{11}=a \cdot s+b \cdot p$. From Theorem 1, we have the result. Q.E.D.

IV. Conclusion

In this paper, the fuzzy controller with a simplified fuzzy reasoning method is considered. We have shown that the defuzzified output of the fuzzy controller can be precisely equivalent to a linear function of all the inputs to the fuzzy controller by using three mixed fuzzy operators. In there, arbitrary numbers of triangular fuzzy numbers are employed to fuzzify the linguistic variables in fuzzy control rules which generalize the investigation only on the the special number of fuzzy numbers for each fuzzy input [3,4,6]. The result also indicates that the linear nonfuzzy controllers are the special cases of the fuzzy controllers.

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