Symbol Error Probability for Rectangular *M*-QAM OFDM Transmission over Rayleigh Fading Channels

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Abstract

By using what we call the pre-averaging method, an exact closed form expression for the symbol error probability (SEP) is derived for arbitrary rectangular M-QAM signaling in OFDM systems over frequency-selective Rayleigh fading channels. In addition, the channel capacity for the QAM OFDM transmission over Rayleigh fading environment is obtained.

1. Introduction

A common signaling scheme used for OFDM systems is QAM signaling [1]. There have been numerous research studies to evaluate the error probability performance for QAM transmission in digital communication systems (AWGN channels, multipath fading channels, diversity combining systems). Most of these QAM error probability evaluations are only for square (not rectangular) *M*-QAM cases [2-4]. By far, the mostly discussed fading model is Rayleigh fading. The major difficulty in finding the SEP for QAM in fading is the evaluation of the integral of the squared Gaussian-*Q* function, which many times leads to results containing either hypergeometric functions or unevaluated integrals [3,4]

In this work, by using a pre-averaging method, we successfully avoid the squared Gaussian-Q function integral to obtain exact closed-form SEP (containing no hypergeometric functions nor unevaluated integrals) for OFDM systems employing arbitrary rectangular *M*-QAM over frequency-selective Rayleigh fading channels. We will also obtain the channel capacity for the QAM OFDM system in Rayleigh fading channels.

Section 2 presents the OFDM system model. Section 3 derives the exact closed-form SEP for arbitrary rectangular M-QAM OFDM transmission over frequency-selective Rayleigh fading channels. Section 4 gives simulation results. Then, Section 5 derives the channel capacity for the QAM OFDM transmission over Rayleigh fading channels. Finally, Section 6 draws the conclusion.

2. The OFDM system model

The equivalent channel frequency response at subcarrier frequency $f_k = k/T$ for an OFDM system in frequency-selective fading channels is

$$H_k = \sum_{n=0}^{\upsilon-1} h_n e^{-j2\pi nk/N}, \quad k = 0,1,2,\ldots,N-1, \qquad (1)$$

where T is the duration of a block of N data symbols, h_n is the channel impulse response that spans υ symbols. Assuming independent Rayleigh fading channels, $\{h_n\}$ are independent complex Gaussian RV's with zero means and variances $\{\sigma_n^2\}$ in each real dimension. It is straightforward to verify that $\{H_k\}$ are i.i.d. Gaussian RV's with zero means and variances $\sigma_c^2 = \sum_{n=0}^{\upsilon-1} \sigma_n^2$ in each real dimension for all k. This simply means that OFDM converts a frequency-selective fading channel into flat fading.

The kth subband output of the FFT at the receiver is

$$r_k = H_k X_k + z_k, \qquad (2)$$

where $\{X_k\}$ are the transmitted data symbols and $\{z_k\}$ are i.i.d. complex white Gaussian noise RV's with zero means and variances σ_z^2 in each real dimension. By dividing (2) by H_k, we have the kth subband estimate as

$$\hat{X}_k = \frac{r_k}{H_k} = X_k + \frac{z_k}{H_k} = X_k + e_k. \qquad (3)$$

We shall assume that perfect channel state estimate is available.

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3. SEP for arbitrary rectangular *M*-QAM OFDM transmission over frequency-selective Rayleigh fading channels

To find SEP for fading channels, the usual approach is to first compute the SEP conditioned on a fixed channel realization $|H_k|$ (SEP in AWGN), then average this conditional SEP over channel realization (we call this the post-averaging method as in contrast to our pre-averaging method to be described below) to obtain the final overall SEP. As mentioned earlier, this post-averaging approach will inevitably involve the complicated integration of the squared Gaussian-Q function. What we will do for QAM in OFDM system over Rayleigh fading can avoid this integration. We first average (pre-average) $p(e_{ck}, e_{sk} \| H_k |)$, the joint PDF of e_{ck} and e_{sk} (real and imaginary parts of e_k) conditioned on a given channel realization $|H_k|$ to obtain the joint PDF $p_{cs}(e_{ck}, e_{sk})$, then calculate the average SEP from this joint PDF. Straightforward calculations lead to

$$p_{cs}(e_{ck}, e_{sk}) = \frac{(\sigma_z / \sigma_c)^2}{\pi[(\sigma_z / \sigma_c)^2 + e_{ck}^2 + e_{sk}^2]^2}, \quad -\infty < e_{ck}, e_{sk} < \infty. \quad (4)$$

Assume rectangular M_k-QAM signaling for the *k*th subband with $M_k = 2^{n_k} = M_{ck} M_{sk}$, where M_{ck}-PAM and M_{sk}-PAM are employed respectively for real and imaginary parts of X_k, viz., X_{ck} and X_{sk}. The symbol X_{ck} takes on values from the set $\{(2m_{ck} - 1 - M_{ck})d, m_{kc} = 1,2,..., M_{ck}\}$ with equal probabilities, while X_{sk} takes on values from the set $\{(2m_{sk} - 1 - M_{sk})d, m_{sk} = 1,2,..., M_{sk}\}$ with equal probabilities. Since all subbands have the same $p_{cs}(e_{ck}, e_{sk})$, we will simply drop the subscript *k* for these error variables.

For the 4 corner symbol points of the M_k-QAM constellation, due to constellation symmetry, each point has the identical correct probability given by

$$P_{c1} = \int_{-d}^{\infty}\int_{-d}^{\infty} p_{cs}(e_c, e_s) de_c de_s = \frac{1}{4} + \frac{a}{2\sqrt{1+a^2}} + \frac{a}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}, \quad (5)$$

where $a = (\sigma_c / \sigma_z)d$.

For the $2(M_{ck} + M_{sk} - 4)$ border points, not including the corner points, the correct probability is

$$P_{c2} = \int_{-d}^{d}\int_{-d}^{\infty} p_{cs}(e_c, e_s) de_c de_s = \frac{a}{2\sqrt{1+a^2}} + \frac{2a}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}. \quad (6)$$

Then, for the $M_k - 2(M_{ck} + M_{sk}) + 4$ inner points, the correct probability is

$$P_{c3} = \int_{-d}^{d}\int_{-d}^{d} p_{cs}(e_c, e_s) de_c de_s = \frac{4a}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}. \quad (7)$$

The overall average SEP can now be obtained as

$$\begin{aligned} P_{M_k} &= \frac{1}{M_k}[4(1-P_{c1}) + 2(M_{ck} + M_{sk} - 4)(1-P_{c2}) + (M_k - 2M_{ck} - 2M_{sk} + 4)(1-P_{c3})] \\ &= \frac{1}{M_k}[M_k - 1 - \frac{a}{\sqrt{1+a^2}}(M_{ck} + M_{sk} - 2) - \frac{4a}{\pi\sqrt{1+a^2}}(M_k - M_{ck} - M_{sk} + 1)\tan^{-1}\frac{a}{\sqrt{1+a^2}}] \end{aligned} \quad (8)$$

If square QAM is used, $M_{ck} = M_{sk} = \sqrt{M_k}$, (8) becomes

$$P_{M_k} = \frac{1}{M_k}[(M_k - 1) - \frac{2a(\sqrt{M_k} - 1)}{\sqrt{1+a^2}} - \frac{4a(\sqrt{M_k} - 1)^2}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}]. \quad (9)$$

Setting $M_{ck} = M_k = 2$ and $M_{cs} = 1$, (8) reduces to the well known result for M_k-PAM [5].

It can be readily shown that

$$a = \frac{\sigma_c d}{\sigma_\eta} = \sqrt{\frac{3\bar{\gamma}_k}{M_{ck}^2 + M_{sk}^2 - 2}}. \quad (10)$$

Replacing a by $\bar{\gamma}_k$ using (10), we can get the SEP expressions in terms of SNR as is usually preferred.

The block or frame error probability is simply given by

$$P_B = 1 - \prod_{k=0}^{N-1}(1 - P_{M_k}). \quad (11)$$

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Further, if Gray coding is used for each group (subband), we can approximate the average bit error probability by

$$P_b \cong \frac{1}{N}\sum_{k=0}^{N-1}\frac{P_{M_k}}{n_k}. \qquad (12)$$

At this point, we must note that the noise or error term $e_k = z_k / H_k$ in (3) is not Gaussian. We need to show that the minimum distance detector used here is optimum.

Let the complex received data symbol and noise samples out of the correlation demodulator be $s_m = s_{mc} + js_{ms}$, $m = 1,2,...,M$, and $e = e_c + je_s$ respectively subscript *k* dropped). Then the total received data sample is

$$r_t = r_c + jr_s = (s_{mc} + e_c) + j(s_{ms} + e_s). \qquad (13)$$

Assume a priori symbol probabilities $\{ p_m(s_m) \}$ are equal for all *m*. Then, if the joint density function $p_{cs}(e_c, e_s)$ is monotonically decreasing with $|e| = \sqrt{e_c^2 + e_s^2}$, it is readily shown that a minimum distance detector is equivalent to the optimum maximum a posteriori probability (MAP) detector or the maximum-likelihood (ML) detector. Now, applying the above fact, since $p_{cs}(e_{ck}, e_{sk})$ given by (4) is monotonically decreasing with $e_{ck}^2 + e_{sk}^2$, we conclude that our minimum distance detector is indeed optimum.

4. Simulation results

Figure 1 presents the plots of P_{M_k} vs. $\bar{\gamma}_k$ for various combinations of $M_{ck}M_{sk} = M_k = 256$. It is seen that, for a given M_k , the best choice is to use square QAM, and when rectangular QAM is used, then as the difference between M_{ck} and M_{sk} gets larger, the performance gets worse. This can also be readily proven analytically by taking the derivative of P_{M_k} of (8) with respect to M_{ck} and setting the result to zero, meanwhile fixing M_k and $\bar{\gamma}_k$. Also included in Fig. 1 is a curve obtained by Monte Carlo simulations for the square 256-QAM case. It is seen that this curve is in excellent agreement with the theoretical curve.

5. Channel capacity

The channel capacity for a Rayleigh fading channel has been solved by Lee [6]. For average SNR $\bar{\gamma} > 2$, the capacity can be expressed as

$$C = B\log_2 e \cdot [e^{-1/\bar{\gamma}}(\ln\bar{\gamma} + \frac{1}{\bar{\gamma}} - E)] \text{ bits/sec}, \qquad (14)$$

where $E = 0.5772157$ is Euler constant. For OFDM systems, with a very small sub bandwidth $\Delta f = 1/T$, the overall channel capacity can be written as

$$C = \sum_{k=0}^{N-1} C_k = \Delta f \sum_{k=0}^{N-1}\log_2 e \cdot [e^{-1/\bar{\gamma}_k}(\ln\bar{\gamma}_k + \frac{1}{\bar{\gamma}_k} - E)] . \qquad (15)$$

As $\Delta f \to 0$, (15) can be written as

$$C = (\log_2 e)\int_0^W e^{-1/\bar{\gamma}(f)}(\ln\bar{\gamma}(f) + \frac{1}{\bar{\gamma}(f)} - E)df . \qquad (16)$$

With AWGN, we want to find the optimum $\bar{\gamma}(f)$ to get maximum *C* subject to the power constraint that

$$\int_0^W \bar{\gamma}(f)df = \text{constant}. \qquad (17)$$

The maximization is obtained by maximizing the integral

$$\int_0^W [e^{-1/\bar{\gamma}(f)}(\ln\bar{\gamma}(f) + \frac{1}{\bar{\gamma}(f)} - E) + \lambda\bar{\gamma}(f)]df , \qquad (18)$$

where λ is a Lagrange multiplier. By use of the calculus of variations, we differentiate the integrand with respect to $\bar{\gamma}(f)$ and then set the result to zero. We get

$$\lambda\bar{\gamma}^3(f)e^{1/\bar{\gamma}(f)} + \bar{\gamma}^2(f) + \bar{\gamma}(f)[\ln\bar{\gamma}(f) - 1 - E] + 1 = 0. \qquad (19)$$

This transcendental equation is hard to solve. Fortunately, we need not solve it as evidently by its look, the solution for $\bar{\gamma}(f)$, if exists, will not be a function of *f*. It will be a value depending only on λ which is again dictated by the constrained power. We thus conclude that the best choice for $\bar{\gamma}_k$ is constant over all subbands.

Now, let the available average power of the transmitter be P_{av}. Then, each subband has the same average power P_{av}/N . Then,

$$\bar{\gamma}_k = \frac{\sigma_0^2 P_{av} T}{\sigma_n^2 N^2}. \quad (20)$$

Substituting (20) into (19), we find the average channel capacity in terms of the transmitted signal power as

$$C = \frac{W}{\ln 2}[\exp(-\frac{\sigma_n^2 N^2}{\sigma_0^2 P_{av} T})(\ln\frac{\sigma_0^2 P_{av} T}{\sigma_n^2 N^2} + \frac{\sigma_n^2 N^2}{\sigma_0^2 P_{av} T} - E)]. \quad (21)$$

It must be noted here that the received SNR $\bar{\gamma}_k$ is an overall average value. For frequency-selective channels, the received SNR during each symbol interval denoted by γ_k is different for different subbands and will vary from one symbol interval to another. Therefore, if one wants to achieve channel capacity by whatever coding means, one must go through the painful process of optimizing signal power distribution by water pouring principle during every symbol interval.

6. Conclusion

By using what we call the pre-averaging method, we derive an exact closed-form SEP expression for arbitrary rectangular *M*-QAM OFDM transmission over Rayleigh fading channels. Monte Carlo simulations are performed to check with the theoretical results. By using the pre-averaging technique, we successfully evade the need for integrating the squared Gaussian-*Q* function which is unavoidable if using post-averaging method adopted by most researchers. As a result, our SEP expression contains no hypergemetric functions nor unevaluated integrals, hence can be easily computed by the computer. We have also obtained the channel capacity in terms of the transmitted signal power for the QAM OFDM transmission over Rayleigh fading channels.

7. References

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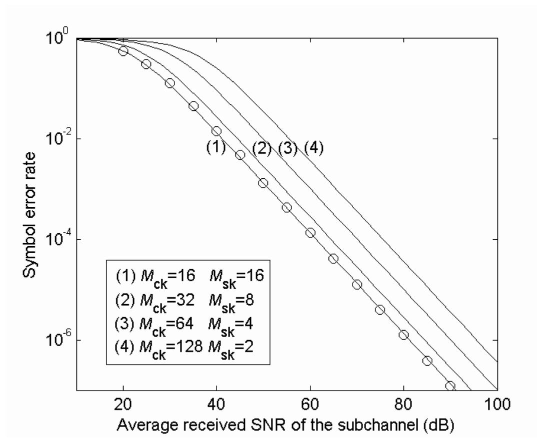

Fig. 1 Rectangular -QAM performance for the OFDM system over Rayleigh fading channels for $M_k = 256$. Monte Carlo simulation curve (marked with O) for square 256-QAM is also incorporated.

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