



ASYMPTOTIC STABILITY OF SOLUTIONS FOR A KIND OF THIRD-ORDER STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAYS

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Received 21 December, 2018

Abstract. This work is devoted to investigate the stochastic asymptotically stability of the zero solution for a kind of third-order stochastic differentials equation with variable and constant delays by a suitable Lyapunov functional. Our results improve and form a complement to some results that can be found in the literature. In the last section, we give an example to illustrate our main result.

2010 Mathematics Subject Classification: 34K20; 34K50; 37B25

Keywords: asymptotic stability, stochastic differential equation with delay, third-order, Lyapunov functional

1. INTRODUCTION

Stochastic delay differential equations (SDDEs) are natural generalizations of stochastic ordinary differential equations (SODEs) by allowing the coefficients to depend on the past values.

Recently, the studies of stochastic differential equations (SDEs) have attracted the considerable attentions of many scholars in the last forty years.

SDEs play an important role in many branches of science and engineering, and there are a large number of books, which provide full details for the background of probability theory and stochastic calculus, see for example, [6, 7, 9, 10, 13–16] and the references therein.

Systems of SDDEs occupy now a place of central importance in many areas of science including medicine, engineering, biology and physics.

Stability theory is one of the main components of SDDEs. The Lyapunov's direct method has been successfully used to investigate stability problems in deterministic SDDEs for more than one hundred years, when there is no analytical expression for solutions.

An apparent advantage of this method is the stability in the large can be obtained without any prior knowledge of solutions. Therefore the method yields stability information directly without solving differential equations.

However, there are many difficulties encountered in the study of stability by means of Lyapunov's direct method. Therefore, in the relevant literature, some authors investigated the stability of solutions for DDEs and SDDEs by using different approaches such as the fixed point method, the inequalities techniques, the perturbation methods, the second method of Lyapunov and so on (see e.g. [19, 22] and reference therein).

In this direction, many authors have proposed different approaches to investigate the stability of solutions of third-order DDEs. We can mention the papers of Ademola et al. [4, 5], Graef and Tunç [8], Mahmoud [11], Omeike [17], Oudjedi et al. [18], Remili et al. [20, 21], Sadek [23], Shekhar et al. [24], Tunç [25–29] and the references cited therein.

Meanwhile, the scarcity of works on stability and boundedness of solutions for third-order SDEs with or without delay were studied very rarely, interesting results are contained, for instance, in [1], [2], [3].

In 2015, Abou-El-Ela et al. [1] considered the stochastic asymptotic stability of the zero solution and the uniform stochastic boundedness of all solutions for the third-order SDE of the form

$$\ddot{x}(t) + a\dot{x}(t) + b\dot{x}(t) + cx(t) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)),$$

where a, b, c and σ are positive constants; $\omega(t) \in \mathbb{R}^m$ is a standard Wiener process, p is a continuous function.

In 2015, Abou-El-Ela et al. [2] investigated the asymptotic stability of the zero solution for the third-order SDDEs given by

$$\begin{aligned} \ddot{x}(t) + a_1\dot{x}(t) + g_1(\dot{x}(t - r_1(t))) + f_1(x(t)) + \sigma_1 x(t)\dot{\omega}(t) &= 0, \\ \ddot{x}(t) + a_2\dot{x}(t) + f_2(x(t))\dot{x}(t) + f_3(x(t - r_2(t))) + \sigma_2 x(t - h(t))\dot{\omega}(t) &= 0, \end{aligned}$$

where a_1, a_2, σ_1 and σ_2 are positive constants; $0 \leq r_1(t) \leq \gamma_1$, $0 \leq r_2(t) \leq \gamma_2$, γ_1 and γ_2 are two positive constants which will be determined later. $0 \leq h(t)$, $\sup h(t) = H$; $\omega(t) \in \mathbb{R}^m$ is a standard Wiener process; g_1, f_1, f_2 and f_3 are continuous functions with $g_1(0) = f_1(0) = f_3(0) = 0$.

In 2017, Ademola [3] studied the problems of stability, boundedness and uniqueness of solutions of a certain third-order SDDE as the following form

$$\ddot{x}(t) + a\dot{x}(t) + b\dot{x}(t) + h(x(t - \tau)) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)),$$

where a, b and σ are positives constants, h, p are nonlinear continuous functions in their respective arguments with $h(0) = 0$, $\tau > 0$ is a constant delay.

The main purpose of this work is to establish new criteria for the stochastic asymptotic stability of the zero solution for a kind of third-order nonlinear SDE with variable and constant delays as the following form

$$\ddot{x}(t) + a\dot{x}(t) + \phi(\dot{x}(t - r(t))) + \psi(x(t - r(t))) + \sigma x(t - h)\dot{\omega}(t) = 0, \quad (1.1)$$

where a, σ and h are positive constants, $r(t)$ is a continuously differentiable function with $0 \leq r(t) \leq \gamma_1$, γ_1 is a positive constant which will be determined later, ϕ, ψ are two nonlinear continuous functions in their respective arguments with $\phi(0) = \psi(0) = 0$, $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t)) \in \mathbb{R}^m$ is m -dimensional standard Brownian motion, defined on the probability space. The functions ϕ and ψ are also differentiable throughout this work.

In this paper, by constructing a suitable Lyapunov functional, sufficient conditions for the stochastically asymptotically stability of the zero solution of (1.1) are established. Our result includes and improves the former results that can be found in the literature.

The remainder of this work is organized as follows. In section 2, we give a theorem, which deals with stochastic asymptotically stability of the zero solution for (1.1). In section 3, we introduced the proof of the main theorem. In the last section, we gave an example to verify the analysis made in this work.

2. STABILITY RESULT

Let $\omega(t) = (\omega_1(t), \dots, \omega_m(t))$ be an m -dimensional Brownian motion defined on the probability space. Consider an n -dimensional SDE

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t) \text{ on } t \geq 0, \tag{2.1}$$

with initial value $x(0) = x_0 \in \mathbb{R}^n$. As a standing condition, we assume that $f : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ satisfy the local Lipschitzian condition and the linear growth condition for the existence and uniqueness of solutions for equation (2.1) (see for example, [12,30]). It is therefore known that equation (2.1) has a unique continuous solution on $t \geq 0$, which is denoted by $x(t; x_0)$ in this work. Assume furthermore that $f(t, 0) = 0$ and $g(t, 0) = 0$, for all $t \geq 0$. Hence the stochastic differential equation admits the zero solution $x(t; 0) \equiv 0$.

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ denote the family of non-negative functions $V(t, x)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, which are once continuously differentiable in t and twice continuously differentiable in x .

Define the differential operator \mathcal{L} associated with equation (2.1) by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^n f_i(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n [g(t, x) g^T(t, x)]_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

If \mathcal{L} acts on a function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$, then

$$\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x) \cdot f(t, x) + \frac{1}{2} \text{trace}[g^T(t, x) V_{xx}(t, x) g(t, x)], \tag{2.2}$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})$ and

$$V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{n \times n} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{pmatrix}.$$

Moreover, let \mathcal{K} denote the family of all continuous nondecreasing functions $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\theta(0) = 0$ and $\theta(r) > 0$, if $r > 0$.

Lemma 1 ([13]). *Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\theta \in \mathcal{K}$ such that*

$$V(t, 0) = 0, \quad \theta(|x|) \leq V(t, x), \quad \text{and} \\ \mathcal{L}V(t, x) \leq 0, \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then the zero solution of the stochastic differential equation (2.1) is stochastically stable.

Lemma 2 ([13]). *Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\theta_1, \theta_2, \theta_3 \in \mathcal{K}$ such that*

$$\theta_1(|x|) \leq V(t, x) \leq \theta_2(|x|), \quad \text{and} \\ \mathcal{L}V(t, x) \leq -\theta_3(|x|), \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then the zero solution of the stochastic differential equation (2.1) is stochastically asymptotically stable.

Now we present the main stability result of (1.1).

Theorem 1. *In additions to the basic assumptions imposed on the functions ϕ and ψ appearing in (1.1), suppose that there exists positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, c_1, L$ and M such that:*

- (i) $\alpha_1 \leq \frac{\psi(x)}{x} \leq \alpha_2$ and $\psi(x) \operatorname{sgn} x > 0$, for all $x \neq 0$.
- (ii) $\sup\{\psi'(x)\} = \frac{c_1}{2}$ and $|\psi'(x)| \leq L$, for all x .
- (iii) $\beta_1 \leq \frac{\phi(y)}{y} \leq \beta_2$, for all $y \neq 0$ and $|\phi'(y)| \leq M$, for all y .
- (iv) $0 \leq r(t) \leq \gamma_1$ and $r'(t) \leq \gamma_2$, such that $0 < \gamma_2 < 1$.
- (v) $a\beta_1 - c_1 > 2\beta_1 + 6$.
- (vi) $\sigma^2 < 2\alpha_1 - a - \beta_1 - 2$.

Then the zero solution of (1.1) is stochastically asymptotically stable, provided that

$$\gamma_1 < \min \left\{ \frac{2\alpha_1 - \sigma^2 - a - \beta_1 - 2}{2(L + M)}, \frac{(a\beta_1 - c_1 - 2\beta_1 - 6)(1 - \gamma_2)}{4\lambda(L + M)(1 - \gamma_2) + 4L(\lambda + 2)}, \frac{(a\beta_1 - c_1 - 2\beta_1)(1 - \gamma_2)}{4\beta_1(L + M)(1 - \gamma_2) + 4\beta_1(\lambda + 2)M} \right\},$$

with $\lambda = \frac{a\beta_1 + c_1}{4\beta_1}$.

3. PROOF OF THEOREM 1

The equation (1.1) can be written in the following equivalent system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -az - \phi(y) - \psi(x) + \int_{t-r(t)}^t \phi'(y(s))z(s)ds + \int_{t-r(t)}^t \psi'(x(s))y(s)ds \\ &\quad - \sigma x(t-h)\dot{\omega}(t). \end{aligned} \tag{3.1}$$

Define the Lyapunov functional $V(t, X_t)$, where $X_t = (x_t, y_t, z_t)$, as the following

$$\begin{aligned} V(t, X_t) &= \lambda \int_0^x \psi(\xi)d\xi + \psi(x)y + \frac{1}{2}\lambda ay^2 + \int_0^y \phi(\eta)d\eta + \lambda yz + \frac{1}{2}z^2 \\ &\quad + xz + x^2 + \mu \int_{-r(t)}^0 \int_{t+s}^t y^2(\vartheta)d\vartheta ds \\ &\quad + \nu \int_{-r(t)}^0 \int_{t+s}^t z^2(\vartheta)d\vartheta ds + \frac{1}{2}\sigma^2 \int_{t-h}^t x^2(s)ds, \end{aligned} \tag{3.2}$$

where μ and ν are two positive constants, which will be determined later.

Our target here is to show that the Lyapunov functional $V(t, X_t)$ satisfies the conditions of Lemma 2.

Thus from (3.2), (3.1) and by using Itô formula (2.2), we get

$$\begin{aligned} \mathcal{L}V(t, X_t) &= \psi'(x)y^2 + \lambda z^2 - \lambda\phi(y)y - az^2 + yz - axz - x\phi(y) - x\psi(x) + 2xy \\ &\quad + (x + \lambda y + z) \left(\int_{t-r(t)}^t \phi'(y(s))z(s)ds + \int_{t-r(t)}^t \psi'(x(s))y(s)ds \right) \\ &\quad + \mu r(t)y^2 - \mu(1-r'(t)) \int_{t-r(t)}^t y^2(\vartheta)d\vartheta \\ &\quad + \nu r(t)z^2 - \nu(1-r'(t)) \int_{t-r(t)}^t z^2(\vartheta)d\vartheta + \frac{1}{2}\sigma^2 x^2. \end{aligned}$$

In view the assumptions (i) – (iv) of Theorem 1, we obtain

$$\begin{aligned} \mathcal{L}V(t, X_t) &\leq \frac{c_1}{2}y^2 + \lambda z^2 - \lambda\beta_1 y^2 - az^2 + yz - axz - \beta_1 xy - \alpha_1 x^2 + 2xy + \frac{1}{2}\sigma^2 x^2 \\ &\quad + (x + \lambda y + z) \left(M \int_{t-r(t)}^t z(s)ds + L \int_{t-r(t)}^t y(s)ds \right) \\ &\quad + \mu\gamma_1 y^2 + \nu\gamma_1 z^2 - (1-\gamma_2) \left(\mu \int_{t-r(t)}^t y^2(\vartheta)d\vartheta + \nu \int_{t-r(t)}^t z^2(\vartheta)d\vartheta \right). \end{aligned}$$

Then by using the inequality $2uv \leq u^2 + v^2$, with the condition $r(t) \leq \gamma_1$ in Theorem 1, we have

$$\begin{aligned} \mathcal{L}V(t, X_t) \leq & - \left\{ \alpha_1 - \frac{1}{2}(\sigma^2 + a + \beta_1 + 2) - \frac{1}{2}(L + M)\gamma_1 \right\} x^2 \\ & - \left\{ \lambda\beta_1 - \frac{1}{2}(c_1 + \beta_1 + 3) - \frac{1}{2}\lambda(L + M)\gamma_1 - \mu\gamma_1 \right\} y^2 \\ & - \left\{ \frac{1}{2}(a - 1) - \frac{1}{2}(L + M)\gamma_1 - \lambda - v\gamma_1 \right\} z^2 \\ & + \left\{ L + \frac{\lambda L}{2} - \mu(1 - \gamma_2) \right\} \int_{t-r(t)}^t y^2(\vartheta) d\vartheta \\ & + \left\{ M + \frac{\lambda M}{2} - v(1 - \gamma_2) \right\} \int_{t-r(t)}^t z^2(\vartheta) d\vartheta. \end{aligned}$$

If we take

$$\mu = \frac{L(\lambda + 2)}{2(1 - \gamma_2)} > 0 \quad \text{and} \quad v = \frac{M(\lambda + 2)}{2(1 - \gamma_2)} > 0,$$

it follows that

$$\begin{aligned} \mathcal{L}V(t, X_t) \leq & - \left\{ \alpha_1 - \frac{1}{2}(\sigma^2 + a + \beta_1 + 2) - \frac{1}{2}(L + M)\gamma_1 \right\} x^2 \\ & - \left\{ \lambda\beta_1 - \frac{1}{2}(c_1 + \beta_1 + 3) - \frac{1}{2}\lambda(L + M)\gamma_1 - \frac{L(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} y^2 \\ & - \left\{ \frac{1}{2}(a - 1) - \frac{1}{2}(L + M)\gamma_1 - \lambda - \frac{M(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} z^2. \end{aligned}$$

In view of

$$\lambda\beta_1 - \frac{1}{2}c_1 = \frac{a\beta_1 - c_1}{4} > 0 \quad \text{and} \quad \frac{a}{2} - \lambda = \frac{a\beta_1 - c_1}{4\beta_1} > 0,$$

We have

$$\begin{aligned} \mathcal{L}V(t, X_t) \leq & - \left\{ \alpha_1 - \frac{1}{2}(\sigma^2 + a + \beta_1 + 2) - \frac{L + M}{2}\gamma_1 \right\} x^2 \\ & - \left\{ \frac{1}{4}(a\beta_1 - c_1 - 2\beta_1 - 6) - \frac{\lambda(L + M)(1 - \gamma_2) + L(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} y^2 \\ & - \left\{ \frac{a\beta_1 - c_1 - 2\beta_1}{4\beta_1} - \frac{(L + M)(1 - \gamma_2) + M(\lambda + 2)}{2(1 - \gamma_2)}\gamma_1 \right\} z^2. \end{aligned} \tag{3.3}$$

Thus, in view of (3.3), one can conclude that $\mathcal{L}V(t, X_t)$ satisfies the condition (ii) of Lemma 2 as:

$$\mathcal{L}V(t, X_t) \leq -D_1(x^2 + y^2 + z^2), \quad \text{for some } D_1 > 0, \tag{3.4}$$

provided that

$$\gamma_1 < \min \left\{ \frac{2\alpha_1 - \sigma^2 - a - \beta_1 - 2}{2(L + M)}, \frac{(a\beta_1 - c_1 - 2\beta_1 - 6)(1 - \gamma_2)}{4\lambda(L + M)(1 - \gamma_2) + 4L(\lambda + 2)}, \frac{(a\beta_1 - c_1 - 2\beta_1)(1 - \gamma_2)}{4\beta_1(L + M)(1 - \gamma_2) + 4\beta_1(\lambda + 2)M} \right\}.$$

Next, we shall show that the assumption (i) of Lemma 2 is satisfied.

Since $\int_{-r(t)}^0 \int_{t+s}^t y^2(\vartheta) d\vartheta ds$ and $\int_{-r(t)}^0 \int_{t+s}^t z^2(\vartheta) d\vartheta ds$ are non-negative and by using the assumption (iii) of Theorem 1, we obtain

$$\begin{aligned} V(t, X_t) &\geq \lambda \int_0^x \psi(\xi) d\xi + \psi(x)y + \frac{1}{2}\lambda ay^2 + \frac{1}{2}\beta_1 y^2 + \lambda yz + \frac{1}{2}z^2 + xz + x^2 \\ &= \frac{1}{2\beta_1}(\beta_1 y + \psi(x))^2 + (\lambda y + \frac{z}{2})^2 + (x + \frac{z}{2})^2 + \frac{1}{2}\lambda(a - 2\lambda)y^2 \\ &\quad + \frac{2}{\beta_1 y^2} \int_0^x \psi(\xi) \left\{ \int_0^y (\lambda\beta_1 - \psi'(\xi))\eta d\eta \right\} d\xi; \quad y \neq 0. \end{aligned} \tag{3.5}$$

Now we recall that:

$$a - 2\lambda = \frac{a\beta_1 - c_1}{2\beta_1} > 0,$$

and

$$\lambda\beta_1 - \psi'(\xi) \geq \frac{a\beta_1 + c_1}{4} - \frac{c_1}{2} = \frac{a\beta_1 - c_1}{4} > 0; \text{ by condition (ii) of Theorem 1.}$$

Then, we get

$$\begin{aligned} &\frac{2}{\beta_1 y^2} \int_0^x \psi(\xi) \left\{ \int_0^y (\lambda\beta_1 - \psi'(\xi))\eta d\eta \right\} d\xi \\ &\geq \frac{a\beta_1 - c_1}{4\beta_1} \int_0^y \psi(\xi) d\xi, \end{aligned}$$

which together with (3.5), implies the following inequality

$$\begin{aligned} V(t, X_t) &\geq \frac{1}{2\beta_1}(\beta_1 y + \psi(x))^2 + (x + \frac{z}{2})^2 + \frac{1}{2}\lambda \left(\frac{a\beta_1 - c_1}{2\beta_1} \right) y^2 \\ &\quad + (\lambda y + \frac{z}{2})^2 + \frac{a\beta_1 - c_1}{4\beta_1} \int_0^y \psi(\xi) d\xi. \end{aligned}$$

Hence, we can see that

$$V(t, X_t) \geq D_2(x^2 + y^2 + z^2), \quad \text{for some } D_2 > 0. \tag{3.6}$$

In view of the assumptions $\psi(x) \leq \alpha_2 x$, $\psi(y) \leq \beta_2 y$ from the conditions (i) and (iii) of Theorem 1 respectively; and the inequality $uv \leq \frac{1}{2}(u^2 + v^2)$, then we can

write from (3.2) that

$$\begin{aligned} V(t, X_t) \leq & \lambda \int_0^x \alpha_1 \xi d\xi + \frac{\alpha_2}{2}(x^2 + y^2) + \frac{1}{2}\lambda a y^2 + \int_0^y \beta_2 \eta d\eta + \frac{\lambda}{2}(y^2 + z^2) \\ & + \frac{1}{2}z^2 + \frac{1}{2}(x^2 + z^2) + x^2 + \mu \int_{t-r(t)}^t (\vartheta - t + r(t)) y^2(\vartheta) d\vartheta \\ & + \nu \int_{t-r(t)}^t (\vartheta - t + r(t)) z^2(\vartheta) d\vartheta + \frac{1}{2}\sigma^2 \int_{t-h}^t x^2(s) ds. \end{aligned}$$

Since $r(t) \leq \gamma_1$, then it follows that

$$\begin{aligned} V(t, X_t) \leq & \frac{1}{2} \left\{ (\lambda + 1)\alpha_2 + 3 + \sigma^2 h \right\} \|x\|^2 + \frac{1}{2} \left\{ \alpha_2 + \beta_2 + \lambda(a + 1) + \mu\gamma_1^2 \right\} \|y\|^2 \\ & + \frac{1}{2} \left\{ \lambda + 2 + \nu\gamma_1^2 \right\} \|z\|^2. \end{aligned}$$

Then there exists a positive constant D_3 such that

$$V(t, X_t) \leq D_3(x^2 + y^2 + z^2), \quad D_3 > 0. \quad (3.7)$$

Therefore from (3.6) and (3.7), we note that $V(t, X_t)$ satisfies condition (i) of Lemma 2.

Thus all the assumptions of Lemma 2 are satisfied, so the zero solution of (1.1) is stochastically asymptotically stable.

This completes the proof of Theorem 1.

4. EXAMPLE

In this section, we give an example to show the applicability of the result obtained and for illustrations.

As an application of Theorem 1, we consider the third-order stochastic delay differential equation as the following form:

$$\begin{aligned} \ddot{x}(t) + 12\dot{x}(t) + 8\dot{x}(t - r(t)) + \sin(\dot{x}(t - r(t))) + 24x(t - r(t)) \\ + \frac{x(t - r(t))}{1 + x^2(t - r(t))} + 3x(t)\dot{\omega}(t) = 0. \end{aligned} \quad (4.1)$$

Its equivalent system is given as:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -12z - (8y + \sin y) - \left(24x + \frac{x}{1 + x^2}\right) + \int_{t-r(t)}^t \{8 + \cos y(s)\} z(s) ds \\ &+ \int_{t-r(t)}^t \left\{24 + \frac{1 - x^2(s)}{(1 + x^2(s))^2}\right\} y(s) ds - 2x(t)\dot{\omega}(t). \end{aligned}$$

By comparing the above differential system (3.1) and taking into account the assumptions of Theorem 1.

The path of the function $\sin y$ is shown in Figure 1. It follows that

$$a = 12, \phi(y) = 8y + \sin y, \frac{\phi(y)}{y} - 8 = \frac{\sin y}{y},$$

then we find

$$-1 \leq \frac{\phi(y)}{y} - 8 \leq 1.$$

Hence, we have

$$\beta_1 = 7, \beta_2 = 9, \text{ also } |\phi'(y)| = |8 + \cos y| \leq 9 = M.$$

The behaviour of $\frac{x}{1+x^2}$ is shown in Figure 2. Therefore we obtain

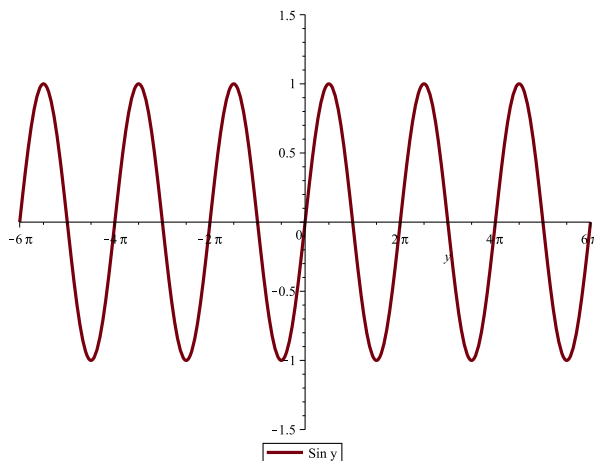


FIGURE 1. The behaviour of the function $\sin y$

$$\psi(x) = 24x + \frac{x}{1+x^2}, \frac{\psi(x)}{x} - 24 = \frac{1}{1+x^2}, \text{ then } 0 \leq \frac{\psi(x)}{x} - 24 \leq 1.$$

It tends to

$$\alpha_1 = 24, \alpha_2 = 25, \psi'(x) = 24 + \frac{1-x^2}{(1+x^2)^2}, \text{ since}$$

$$\left| \frac{1-x^2}{(1+x^2)^2} \right| \leq 1, \text{ then } |\psi'(x)| \leq 25 = L.$$

Therefore $\sup\{|\psi'(x)|\} = 25$, then we obtain $c_1 = 50$, and $\lambda = \frac{a\beta_1+c_1}{4\beta_1} = \frac{67}{14}$.

It is obvious that

$$a\beta_1 - c_1 = 34 \text{ and } 2\beta_1 + 6 = 20.$$

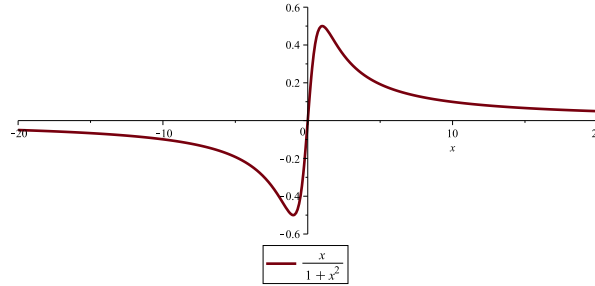


FIGURE 2. The behaviour of the function $\frac{x}{1+x^2}$

Therefore

$$a\beta_1 - c_1 > 2\beta_1 + 6, \text{ and } 2\alpha_1 - a - \beta_1 - 2 = 27 > \sigma^2 = 9 \text{ (since } \sigma = 3\text{)}.$$

Thus, the above estimates show all the assumptions of Theorem 1 hold, so we can prove that

$$\begin{aligned} \mathcal{L}V(t, X_t) &\leq (9 - 17\gamma_1)x^2 - \left\{ \frac{14}{4} - \frac{1}{2}(163 + 2\mu)\gamma_1 \right\} y^2 \\ &\quad - \left\{ \frac{10}{14} - (17 + \nu)\gamma_1 \right\} z^2 + \left\{ 85 - \mu(1 - \gamma_2) \right\} \int_{t-r(t)}^t y^2(\vartheta) d\vartheta \\ &\quad + \left\{ 30.5 - \nu(1 - \gamma_2) \right\} \int_{t-r(t)}^t z^2(\vartheta) d\vartheta. \end{aligned}$$

Let us choose

$$\mu = \frac{85}{1 - \gamma_2} > 0 \text{ and } \nu = \frac{30.5}{1 - \gamma_2} > 0,$$

where $0 < \gamma_2 < 1$.

Consequently, it follows for a positive constant ρ_1 that

$$\mathcal{L}V(t, X_t) \leq -\rho_1(x^2 + y^2 + z^2), \quad (4.2)$$

provided that

$$\gamma_1 < \min \left\{ \frac{9}{34}, \frac{7(1 - \gamma_2)}{326(1 - \gamma_2) + 340}, \frac{5(1 - \gamma_2)}{14\{17(1 - \gamma_2) + 30.5\}} \right\}.$$

Also, we can see that

$$\begin{aligned} V(t, X_t) &\geq \frac{1}{14} \left(7y + \psi(x) \right)^2 + \left(\frac{67}{14}y + \frac{z}{2} \right)^2 + \left(x + \frac{z}{2} \right)^2 + 5.8 y^2 \\ &\quad + \frac{17}{14} \int_0^y \psi(\xi) d\xi. \end{aligned}$$

Then, there exists a positive constant ρ_2 such that

$$V(t, X_t) \geq \rho_2(x^2 + y^2 + z^2). \quad (4.3)$$

As well it can be shown that:

$$V(t, X_t) \leq \frac{157}{2} \|x\|^2 + \frac{1}{2} \left\{ 96 + \frac{25(\lambda + 2)}{2(1 - \gamma_2)} \gamma_1^2 \right\} \|y\|^2 + \frac{1}{2} \left\{ \frac{95}{14} + \frac{9(\lambda + 2)}{2(1 - \gamma_2)} \gamma_1^2 \right\} \|z\|^2.$$

Hence there exists a positive constant ρ_3 satisfying

$$V(t, X_t) \leq \rho_3(x^2 + y^2 + z^2). \quad (4.4)$$

Now from the results (4.2), (4.3) and (4.4), we note that all the conditions of Lemma 2 are satisfied, then the zero solution of (4.1) is stochastically asymptotically stable.

5. ACKNOWLEDGEMENT

The authors of this paper would like to express their sincere appreciation to the anonymous referee for his/her valuable comments and suggestions which have led to an improvement in the presentation of the paper.

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