



INFINITELY MANY WEAK SOLUTIONS FOR A FOURTH-ORDER EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

MOHAMMAD REZA HEIDARI TAVANI AND ABDOLLAH NAZARI

Received 06 June, 2018

Abstract. Existence results of infinitely many solutions for a fourth-order differential equation are established. This equation depends on two real parameters. The approach is based on an infinitely many critical points theorem.

2010 *Mathematics Subject Classification:* 34B15; 58E05

Keywords: infinitely many solutions, fourth-order equation, critical point theory

1. INTRODUCTION

The aim of the present paper is to investigate the existence of infinitely many solutions for the following fourth-order problem

$$\begin{cases} u^{(iv)}(x) = \lambda \alpha(x) f(x, u(x)) + h(x, u(x)), & x \in [0, 1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{cases} \quad (1.1)$$

where λ and μ are positive parameters, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is L^1 -Carathéodory function, $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\alpha \in L^\infty([0, 1])$, $\alpha(x) \geq 0$, for a.e. $x \in \mathbb{R}$, $\alpha \not\equiv 0$ and $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, there exists $0 < L < 1$, such that $h(x, t) \leq L|t|$ for each $x \in [0, 1]$ and $t \in \mathbb{R}$.

The problem (1.1) is related to the deflections of elastic beams based on nonlinear elasticity. In relation with the problem (1.1), there is an interesting physical description.

Suppose an elastic beam of length $d = 1$, which is clamped at its left side $x = 0$, and resting on a kind of elastic bearing at its right side $x = 1$ is given by μg . Along its length, a load $\lambda \alpha f + h$, is added to cause deformations. If $u = u(x)$ denotes the configuration of the deformed beam, then since $u'''(1)$ represents the shear force at $x = 1$, the condition $u'''(1) = \mu g(u(1))$ means that the vertical force is equal to $\mu g(u(1))$, which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u(1)$. In addition, since $u''(1) = 0$ indicates that there is no

bending moment at $x = 1$, the beam is resting on the bearing μg .

Different models and their applications for problems such as (1.1) can be derived from [4]. We refer the reader to the references [14] and [10] for a physical justification of this model. There is increasing interest in studying fourth-order boundary value problems. Because the change of the static form beam or the support of rigid body can be described by a fourth-order equation. Also a model to study travelling waves in suspension bridges can be furnished by nonlinear fourth-order equations (for instance, see [5]).

When the boundary conditions are nonzero or nonlinear, fourth-order equations can model beams resting on elastic bearings located in their extremities (see for instance [1, 3, 6–8, 11–13] and the references therein).

For example, using a variational methods, the existence of three solutions for special cases of problem (1.1) has been established in [13] and [12]. In [7] the author obtained the existence of at least two positive solutions for the problem

$$\begin{cases} u^{(iv)}(x) = f(x, u(x)), & x \in [0, 1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)), \end{cases} \quad (1.2)$$

based on variational methods and maximum principle. It should be noted that the function f is assumed to be continuous. By assuming appropriate conditions on f and g , the author guarantees positive solutions to problem (1.2). Also existence and multiplicity results for this kind of problems were considered in [1, 3, 6]. In all these works the critical point theory is applied.

Moreover in [8] authors, considered numerical solutions for problem (1.2) with nonlinear boundary conditions.

In particular, using a variational methods the existence of non-zero solutions for problem

$$\begin{cases} u^{(iv)}(x) = f(x, u(x)), & x \in [0, 1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)), \end{cases} \quad (1.3)$$

has been established in [1].

In the present paper, using one kind of infinitely many critical points theorem obtained in [2], we establish the existence of infinitely many weak solutions for the problem (1.1). The paper is organized as follows.

In section 2 we establish all the preliminary results that we need, and in section 3 we present our main results.

2. PRELIMINARIES

The following theorem is a smooth version of Theorem 2.1 of [2] which is a more precise version of Ricceri's Variational Principle [9] and it is the main tool of the next section.

Theorem 1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r])} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \rightarrow +\infty} \varphi(r), \quad \delta := \liminf_{r \rightarrow (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r])$ admits a global minimum, which is a critical point (local minimum) of I_λ in X .*

(b) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds:*

either

(b₁) *I_λ possesses a global minimum,*

or

(b₂) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds:*

either

(c₁) *there is a global minimum of Φ which is a local minimum of I_λ ,*

or

(c₂) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ .*

Now, we recall some basic facts and introduce the needed notations.

Definition 1. A function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a Carathéodory function if

(C₁) the function $x \rightarrow f(x, t)$ is measurable for every $t \in \mathbb{R}$;

(C₂) the function $t \rightarrow f(x, t)$ is continuous for a.e. $x \in [0, 1]$.

And $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a L^1 -Carathéodory function if, in addition to conditions (C₁) and (C₂), the following condition is also satisfied :

(C₃) for every $\rho > 0$ there is function $l_\rho \in L^1([0, 1])$ such that $\sup_{|t| \leq \rho} |f(x, t)| \leq l_\rho(x)$ for almost every $x \in [0, 1]$.

Denote

$$X := \{u \in H^2([0, 1]) \mid u(0) = u'(0) = 0\},$$

where $H^2([0, 1])$ is the Sobolev space of all functions $u : [0, 1] \rightarrow \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0, 1])$. X is a Hilbert space with the usual norm

$$\|u\|_X = \left(\int_0^1 (|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2) dx \right)^{1/2}$$

which is equivalent to the norm

$$\|u\| = \left(\int_0^1 |u''(x)|^2 dx \right)^{1/2}.$$

Also the embedding $X \hookrightarrow C^1([0, 1])$ is compact and we have

$$\|u\|_{C^1([0,1])} = \max\{\|u\|_\infty, \|u'\|_\infty\} \leq \|u\| \quad (2.1)$$

for each $u \in X$ (see [12]).

Put

$$F(x, t) = \int_0^t f(x, \xi) d\xi \quad \text{for all } (x, t) \in [0, 1] \times \mathbb{R},$$

$$G(t) = \int_0^t g(\xi) d\xi \quad \text{for all } t \in \mathbb{R},$$

and

$$H(x, t) = \int_0^t h(x, \xi) d\xi \quad \text{for all } (x, t) \in [0, 1] \times \mathbb{R}.$$

Let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2} \int_0^1 |u''(x)|^2 dx - \int_0^1 H(x, u(x)) dx = \frac{1}{2} \|u\|^2 - \int_0^1 H(x, u(x)) dx \quad (2.2)$$

and

$$\Psi(u) = \int_0^1 \alpha(x) F(x, u(x)) dx - \frac{\mu}{\lambda} G(u(1)) \quad (2.3)$$

for every $u \in X$.

Now according to (2.1) we observe that

$$\frac{(1-L)}{2} \|u\|^2 \leq \Phi(u) \leq \frac{(1+L)}{2} \|u\|^2 \quad (2.4)$$

for every $u \in X$. Similar to [[1]-page 3], Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 \alpha(x) f(x, u(x)) v(x) dx - \frac{\mu}{\lambda} g(u(1)) v(1),$$

and, Φ is continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)dx - \int_0^1 h(x, u(x))v(x)dx$$

for every $v \in X$.

Definition 2. A function $u \in X$ is a weak solution to the problem (1.1) if

$$\int_0^1 u''(x)v''(x)dx - \lambda \int_0^1 \alpha(x)f(x, u(x))v(x)dx + \mu g(u(1))v(1) - \int_0^1 h(x, u(x))v(x)dx = 0 \tag{2.5}$$

for every $v \in X$.

3. MAIN RESULTS

Let

$$G_\eta = \min_{|t| \leq \eta} G(t) = \inf_{|t| \leq \eta} G(t) \quad \text{for all } \eta > 0,$$

$$\tau := \frac{1-L}{8\pi^4(\frac{2}{3})^3(1+L)}, \tag{3.1}$$

$$C := \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x)F(x, t)dx}{\xi^2} \tag{3.2}$$

and

$$D := \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x, \xi)dx}{\xi^2}. \tag{3.3}$$

Now we formulate our main result as follows.

Theorem 2. Assume that

$$(A_1) \int_0^{\frac{3}{4}} \alpha(x)F(x, t)dx \geq 0 \quad \text{for all } t \geq 0.$$

$$(A_2) C < \tau D, \text{ where } \tau, C \text{ and } D \text{ are given by (3.1), (3.2) and (3.3)}.$$

Then, setting

$$\lambda_1 := \frac{4\pi^4(\frac{2}{3})^3(1+L)}{D}, \quad \lambda_2 := \frac{1-L}{2C},$$

for each $\lambda \in (\lambda_1, \lambda_2)$, for every arbitrary continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ whose potential

$$G(t) = \int_0^t g(\xi) d\xi \quad \text{for all } t \in \mathbb{R},$$

is a non-positive function satisfying the condition

$$G^* := -\frac{2}{1-L} \lim_{\xi \rightarrow +\infty} \frac{G_\xi}{\xi^2} < +\infty \quad (3.4)$$

and for every $\mu \in (0, \mu_{g,\lambda})$ where $\mu_{g,\lambda} = \frac{1}{G^*} \left(1 - \frac{\lambda}{\lambda_2}\right)$, the problem (1.1) has unbounded sequence of weak solution in X .

Proof. From (A_2) we see that $\lambda_1 < \lambda_2$. Our aim is to apply Theorem 1, to problem (1.1). Fix $\bar{\lambda} \in (\lambda_1, \lambda_2)$ and let g be a function satisfies the condition (3.4). Since $\bar{\lambda} < \lambda_2$, we have $\mu_{g,\bar{\lambda}} = \frac{1}{G^*} \left(1 - \frac{\bar{\lambda}}{\lambda_2}\right) > 0$. Now fix $\bar{\mu} \in (0, \mu_{g,\bar{\lambda}})$ and put $z_1 = \lambda_1$ and $z_2 = \frac{\lambda_2}{1 + \frac{\bar{\mu}}{\lambda} \lambda_2 G^*}$. If $G^* = 0$, clearly, $z_1 = \lambda_1$, $z_2 = \lambda_2$ and $\bar{\lambda} \in (z_1, z_2)$. If $G^* \neq 0$, since $\bar{\mu} < \mu_{g,\bar{\lambda}}$, we obtain $\frac{\bar{\lambda}}{\lambda_2} + \bar{\mu} G^* < 1$, and so $\frac{\lambda_2}{1 + \frac{\bar{\mu}}{\lambda} \lambda_2 G^*} > \bar{\lambda}$, namely, $\bar{\lambda} < z_2$. Hence, since $\bar{\lambda} > \lambda_1 = z_1$, one has $\bar{\lambda} \in (z_1, z_2)$.

Take X , Φ and Ψ as in the previous section.

For each $u \in X$, we let the functional $I_{\bar{\lambda}} : X \rightarrow \mathbb{R}$ be defined

$$I_{\bar{\lambda}}(u) = \Phi(u) - \bar{\lambda} \Psi(u).$$

Φ is a sequentially weakly lower semicontinuous on X . Indeed, consider an arbitrary $u \in X$ and $\{u_n\}_{n=1}^\infty \subset X$ such that $u_n \rightharpoonup u$ in X . Due to the compact embedding X into $C([0, 1])$, we have that $u_n \rightarrow u$ in $C([0, 1])$. This implies

$$\int_0^1 H(x, u_n(x)) dx \rightarrow \int_0^1 H(x, u(x)) dx, \quad (3.5)$$

and the weakly sequentially lower semicontinuous property of the $\|\cdot\|$ implies

$$\liminf_{n \rightarrow +\infty} \|u_n\|^2 \geq \|u\|^2. \quad (3.6)$$

From (3.5)-(3.6) we have

$$\liminf_{n \rightarrow +\infty} \Phi(u_n) \geq \Phi(u).$$

Since

$$\Phi(u) \geq \left(\frac{1-L}{2}\right) \|u\|^2,$$

taking the condition $0 < L < 1$ into account we observe Φ is coercive. Moreover, Φ is strongly continuous. On the other hand, the compact embedding X into $C[0, 1]$

implies that the functional Ψ is continuously differentiable and with compact derivative. Hence Ψ is sequentially weakly upper semicontinuous (see [15, Corollary 41.9]). Therefore we observe that the regularity assumptions of Theorem 1 on Φ and Ψ are satisfied. Note that the weak solution of problem (1.1) are exactly the critical points of $I_{\bar{\lambda}}$ (in particular, see [12, Lemma 2.1]).

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu} G_{\xi_n}}{\bar{\lambda} \xi_n^2} \right) \\ &= \liminf_{\xi \rightarrow +\infty} \left(\frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x) F(x, t) dx}{\xi^2} - \frac{\bar{\mu} G_{\xi}}{\bar{\lambda} \xi^2} \right). \end{aligned} \tag{3.7}$$

According to (A_2) and (3.4), it is obvious that the limit on the right-hand side in (3) is finite and therefore we have

$$\lim_{n \rightarrow +\infty} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu} G_{\xi_n}}{\bar{\lambda} \xi_n^2} \right) < +\infty. \tag{3.8}$$

Now for all $n \in \mathbb{N}$, put $r_n = \left(\frac{1-L}{2}\right)\xi_n^2$. Since $\left(\frac{1-L}{2}\right)\|u\|^2 \leq \Phi(u)$, for each $u \in X$ and bearing (2.1) in mind, we see that

$$\begin{aligned} \Phi^{-1}([-\infty, r_n]) &= \{u \in X; \Phi(u) < r_n\} \\ &\subseteq \left\{ u \in X; \left(\frac{1-L}{2}\right)\|u\|^2 < r_n \right\} \\ &\subseteq \{u \in X; |u(x)| \leq \xi_n \text{ for each } x \in [0, 1]\}. \end{aligned}$$

Note that $\Phi(0) = \Psi(0) = 0$. Hence, for all $n \in \mathbb{N}$, one has

$$\begin{aligned} \varphi(r_n) &= \inf_{u \in \Phi^{-1}([-\infty, r_n])} \frac{\sup_{v \in \Phi^{-1}([-\infty, r_n])} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\ &\leq \frac{\sup_{v \in \Phi^{-1}([-\infty, r_n])} \Psi(v)}{r_n} \\ &= \frac{\sup_{v \in \Phi^{-1}([-\infty, r_n])} \left(\int_0^1 \alpha(x) F(x, v(x)) dx - \frac{\bar{\mu}}{\bar{\lambda}} G(v(1)) \right)}{r_n} \leq \end{aligned}$$

$$\begin{aligned} & \frac{\left(\sup_{v \in \Phi^{-1}([- \infty, r_n])} \int_0^1 \alpha(x) F(x, v(x)) dx \right) - \frac{\bar{\mu}}{\bar{\lambda}} \left(\inf_{v \in \Phi^{-1}([- \infty, r_n])} G(v(1)) \right)}{r_n} \leq \\ & \frac{\left(\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx \right) - \frac{\bar{\mu}}{\bar{\lambda}} \left(\inf_{|t| \leq \xi_n} G(t) \right)}{r_n} = \\ & \frac{2}{1-L} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right). \end{aligned}$$

Therefore, from (3.8) we obtain

$$\gamma \leq \liminf_{n \rightarrow +\infty} \varphi(r_n) \leq \lim_{n \rightarrow +\infty} \frac{2}{1-L} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right) < +\infty. \quad (3.9)$$

Now we show that $\bar{\lambda} < \frac{1}{\gamma}$. Since $\bar{\mu} \in (0, \mu_{g, \bar{\lambda}})$ then $\bar{\mu} < \frac{1}{G^*} (1 - \frac{\bar{\lambda}}{\lambda_2}) = \frac{1}{G^*} (1 - \frac{2C\bar{\lambda}}{1-L})$. Hence from (3) and (3.9) we have

$$\begin{aligned} \gamma \leq \liminf_{n \rightarrow +\infty} \varphi(r_n) & \leq \lim_{n \rightarrow +\infty} \frac{2}{1-L} \left(\frac{\int_0^1 \sup_{|t| \leq \xi_n} \alpha(x) F(x, t) dx}{\xi_n^2} - \frac{\bar{\mu}}{\bar{\lambda}} \frac{G_{\xi_n}}{\xi_n^2} \right) = \\ & \liminf_{\xi \rightarrow +\infty} \frac{2 \left[\int_0^1 \sup_{|t| \leq \xi} \alpha(x) F(x, t) dx - \frac{\bar{\mu}}{\bar{\lambda}} G_{\xi} \right]}{(1-L)\xi^2} < \\ & \frac{2C}{1-L} - \lim_{\xi \rightarrow +\infty} \frac{2(1 - \frac{2C\bar{\lambda}}{1-L})G_{\xi}}{\bar{\lambda}(1-L)G^*\xi^2} = \frac{2C}{1-L} + \frac{1 - \frac{2C\bar{\lambda}}{1-L}}{\bar{\lambda}} = \frac{1}{\bar{\lambda}}. \end{aligned}$$

This implies that $\bar{\lambda} < \frac{1}{\gamma}$. So we proved that

$$\bar{\lambda} \in (z_1, z_2) \subseteq (\lambda_1, \lambda_2) \subseteq \left(0, \frac{1}{\gamma} \right).$$

For the fixed $\bar{\lambda}$, the inequality (3.9) concludes that the condition (b) of Theorem 1 can be applied and either $I_{\bar{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\lim_{n \rightarrow +\infty} \|u_n\| = +\infty$.

The other step is to prove that for the fixed $\bar{\lambda}$ the functional $I_{\bar{\lambda}}$ has no global minimum. Let us show that the functional $I_{\bar{\lambda}}$ is unbounded from below. Since $\bar{\lambda} > \lambda_1$ then we have

$$\frac{1}{\bar{\lambda}} < \frac{1}{4\pi^4(\frac{2}{3})^3(1+L)} \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x,\xi)dx}{\xi^2}$$

and so there exists a sequence $\{\eta_n\}$ of positive numbers and a constant θ such that $\eta_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\frac{1}{\bar{\lambda}} < \theta < \frac{1}{4\pi^4(\frac{2}{3})^3(1+L)} \frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x,\eta_n)dx}{\eta_n^2} \tag{3.10}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define

$$w_n(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{3}{8}] \\ \eta_n \cos^2(\frac{4\pi x}{3}) & \text{if } x \in]\frac{3}{8}, \frac{3}{4}] \\ \eta_n & \text{if } x \in]\frac{3}{4}, 1]. \end{cases}$$

We observe that $w_n \in X$ and $\|w_n\|^2 = 8\pi^4(\frac{2}{3})^3\eta_n^2$ and so from (2.4) we see that

$$\Phi(w_n) \leq \frac{(1+L)}{2} \|w_n\|^2 = 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2. \tag{3.11}$$

On the other hand, bearing (A_1) in mind and since G is non-positive, we have

$$\Psi(w_n) \geq \int_{\frac{3}{4}}^1 \alpha(x)F(x,\eta_n)dx. \tag{3.12}$$

It follows from (3.10)-(3.12) that

$$\begin{aligned} I_{\bar{\lambda}}(w_n) &= \Phi(w_n) - \bar{\lambda}\Psi(w_n) \\ &\leq 4(1+L)\pi^4(\frac{2}{3})^3\eta_n^2 - \bar{\lambda} \int_{\frac{3}{4}}^1 \alpha(x)F(x,\eta_n)dx \\ &< 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2 - 4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2\bar{\lambda}\theta = \\ &4\pi^4(\frac{2}{3})^3(1+L)\eta_n^2(1-\bar{\lambda}\theta) \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\bar{\lambda}\theta > 1$ and $\eta_n \rightarrow +\infty$ as $n \rightarrow +\infty$, we have

$$\lim_{n \rightarrow +\infty} I_{\bar{\lambda}}(w_n) = -\infty$$

and it follows that $I_{\bar{\lambda}}$ has no global minimum. Therefore, taking the fact

$$\Phi(u) \leq \frac{1+L}{2} \|u\|^2$$

into account, by (b) from Theorem 1, there exist a sequence $\{u_n\}$ of critical points of $I_{\bar{\lambda}}$ such that $\lim_{n \rightarrow \infty} \|u_n\| = +\infty$, and the conclusion is achieved. \square

Now we present the following example to illustrate the result.

Example 1. Let $F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be the function defined as

$$F(x, t) := \begin{cases} e^{x^2} t^3 (1 - \cos(\ln(|t|))) & \text{if } (x, t) \in [0, 1] \times (\mathbb{R} - \{0\}) \\ 0 & \text{if } (x, t) \in [0, 1] \times \{0\} \end{cases}$$

and

$$\alpha(x) := \begin{cases} \frac{1}{4} & \text{if } x \in [0, \frac{1}{4}] \\ 0 & \text{if } x \in (\frac{1}{4}, 1]. \end{cases}$$

Consider the problem

$$\left\{ \begin{array}{l} u^{(iv)} = \lambda \alpha(x) e^{x^2} \left(3u^2 (1 - \cos(\ln(|u|))) + u^2 \sin(\ln(|u|)) \right) + \frac{1}{2} \sin u, \\ \hspace{20em} x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)). \end{array} \right. \tag{3.13}$$

Let $f(x, u) = e^{x^2} \left(3u^2 (1 - \cos(\ln(|u|))) + u^2 \sin(\ln(|u|)) \right)$, $h(x, u) = \frac{1}{2} \sin u$ with $L = \frac{1}{2}$, $g(u) = -u$.

We observe that

$$C := \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x) F(x, t) dx}{\xi^2} = 0$$

$$D := \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x) F(x, \xi) dx}{\xi^2} = +\infty$$

and

$$G^* := -\frac{2}{1-L} \lim_{\xi \rightarrow +\infty} \frac{G_\xi}{\xi^2} = 2.$$

Hence, by Theorem 2, for every $\lambda \in (0, +\infty)$ and $\mu \in (0, \frac{1}{2})$ the problem (3.13) has a sequence of generalized solutions which is unbounded in X .

Theorem 3. Assume that assumption (A_1) in Theorem 2 holds and

$$D > 4\pi^4\left(\frac{2}{3}\right)^3(1+L) \quad \text{and} \quad C < \frac{1-L}{2}$$

and for every arbitrary continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ whose potential

$G(t) = \int_0^t g(\xi)d\xi$ for all $t \in \mathbb{R}$, is a non-positive function, satisfying the condition

$\lim_{\xi \rightarrow +\infty} \frac{G_\xi}{\xi^2} = 0$. Then, the problem

$$\begin{cases} u^{iv} = f(x,u) + h(x,u), & x \in (0,1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{cases} \quad (3.14)$$

for every $\mu \in (0, +\infty)$ has an unbounded sequence of weak solution in X .

Proof. Theorem is an immediately consequence of Theorem 2 when $\lambda = 1$. \square

Remark 1. In Theorem 2, if we assume that the function f is non-negative, the assumption (A_2) can be written as

$$\liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \alpha(x)F(x,\xi)dx}{\xi^2} < \tau \limsup_{\xi \rightarrow +\infty} \frac{\int_{\frac{3}{4}}^1 \alpha(x)F(x,\xi)dx}{\xi^2}$$

as well as $\mu_{g,\lambda} = \frac{1}{G^*} \left(1 - \frac{2\lambda}{1-L} \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \alpha(x)F(x,\xi)dx}{\xi^2} \right)$. Moreover, in the

autonomous case, putting $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$, and $\alpha(x) = 1$ for a.e. $x \in \mathbb{R}$, the assumption (A_2) assumes the form

$$\liminf_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2} < \frac{\tau}{4} \limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2},$$

and in this case, we have

$$\lambda_1 = \frac{16\pi^4\left(\frac{2}{3}\right)^3(1+L)}{\limsup_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2}} \quad \text{and} \quad \lambda_2 = \frac{1-L}{2 \liminf_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2}}$$

and $\mu_{g,\lambda} = \frac{1}{G^*} \left(1 - \frac{2\lambda}{1-L} \liminf_{\xi \rightarrow +\infty} \frac{F(\xi)}{\xi^2} \right)$.

Remark 2. Replacing $\xi \rightarrow +\infty$ with $\xi \rightarrow 0^+$ and also put

$$C := \liminf_{\xi \rightarrow 0^+} \frac{\int_0^1 \sup_{|t| \leq \xi} \alpha(x)F(x,t)dx}{\xi^2} \quad (3.15)$$

and

$$D := \limsup_{\xi \rightarrow 0^+} \frac{\int_{\frac{3}{4}}^1 \alpha(x) F(x, \xi) dx}{\xi^2} \quad (3.16)$$

and

$$G^* := -\frac{2}{1-L} \lim_{\xi \rightarrow 0^+} \frac{G_\xi}{\xi^2} < +\infty \quad (3.17)$$

in Theorem 2, by the same way as in the proof of Theorem 2 but using conclusion (c) of Theorem 2 instead of (b), we can obtain a sequence of pairwise distinct weak solutions to the problem (1.1) which converges uniformly to zero.

We present the following example to illustrate Remark 2.

Example 2. Let $\beta > 1847$ be a real number and $F : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by putting

$$F(t) := \begin{cases} t^2 \left(1 + \beta \sin^2\left(\frac{1}{t}\right)\right) & \text{if } t \in]0, +\infty[\\ 0 & \text{if } t \in]-\infty, 0]. \end{cases}$$

We see that $F(t) = \int_0^t f(\xi) d\xi$, where $f(t) = 2t + 2\beta t \sin^2\left(\frac{1}{t}\right) - \beta \sin\left(\frac{2}{t}\right)$.

Put $g(t) = -t^3$, $\alpha(x) = 1$ and $h(x, t) = \frac{1}{3}xt \sin t$ for every $x \in (0, 1)$ and $t \in \mathbb{R}$. We can consider $L = \frac{1}{3}$ and in this case $\tau = \frac{27}{128\pi^4}$. Let $a_n = \frac{1}{n\pi}$, $b_n = \frac{1}{\frac{2n+1}{2}\pi}$. For every $n \in \mathbb{N}$, we have

$$C := \liminf_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} \leq \lim_{n \rightarrow \infty} \frac{F(a_n)}{a_n^2} = 1, \quad (3.18)$$

and

$$D := \limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} \geq \lim_{n \rightarrow \infty} \frac{F(b_n)}{b_n^2} = \beta + 1, \quad (3.19)$$

and by (3.18) and (3.19) one has

$$\liminf_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} < \frac{\tau}{4} \limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2},$$

as well as

$$\left(\frac{512\pi^4}{81(\alpha+1)}, \frac{1}{3}\right) \subset \left(\frac{16\pi^4\left(\frac{2}{3}\right)^3(1+L)}{\limsup_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2}}, \frac{1-L}{2 \liminf_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2}}\right).$$

Therefore, by applying Remark 1 and Remark 2, for every $\beta > 1847$, and every $\lambda \in \left(\frac{512\pi^4}{81(\alpha+1)}, \frac{1}{3}\right)$ and $\mu \in (0, +\infty)$ the problem

$$\begin{cases} u^{iv} = \lambda \left(2u + 2\beta u \sin^2\left(\frac{1}{u}\right) - \beta \sin\left(\frac{2}{u}\right)\right) + \frac{1}{3}x u \sin u, & x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{cases} \quad (3.20)$$

has a sequence of pairwise distinct weak solutions which converges uniformly to zero.

ACKNOWLEDGEMENT

The authors are very thankful for the many helpful suggestions and corrections given by the referees who reviewed this paper.

REFERENCES

- [1] G. Bonanno, A. Chinni, and S. Tersian, “Existence results for a two point boundary value problem involving a fourth-order equation.” *Electron. j. Qual. Theory Differ. Equ.*, vol. 2015, no. 33, pp. 1–9, 2015, doi: [10.14232/ejqtde.2015.1.33](https://doi.org/10.14232/ejqtde.2015.1.33).
- [2] G. Bonanno and G. Molica Bisci, “Infinitely many solutions for a boundary value problem with discontinuous nonlinearities.” *Bound. Value Probl.*, vol. 2009, pp. 1–20, 2009, doi: [10.1155/2009/670675](https://doi.org/10.1155/2009/670675).
- [3] M. Grossinhol and S. A.Tersian, “The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation.” *Nonlinear Anal., Theory Methods Appl., Ser.A, Theory Methods*, vol. 41, no. 3-4, pp. 417–431, 2000, doi: [10.1016/S0362-546X\(98\)00285-5](https://doi.org/10.1016/S0362-546X(98)00285-5).
- [4] W. W. Jr., S. Timoshenko, and D. Young, *Vibrations Problems in Engineering*. New York: Wiley, 1990.
- [5] A. Lazer and P.J.McKenna, “Large- amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis.” *SIAM Rev.*, vol. 32, no. 4, pp. 537–578, 1990, doi: [10.1137/1032120](https://doi.org/10.1137/1032120).
- [6] T. F. Ma, “Existence results and numerical solutions for a beam equation with nonlinear boundary conditions.” *Appl. Numer. Math.*, vol. 47, no. 2, pp. 189–196, 2003, doi: [10.1016/S0168-9274\(03\)00065-5](https://doi.org/10.1016/S0168-9274(03)00065-5).
- [7] T. F. Ma, “Positive solutions for a beam equation on a nonlinear elastic foundation.” *Math. Comput. Modelling.*, vol. 39, no. 11-12, pp. 1195–1201, 2004, doi: [10.1016/j.mcm.2004.06.001](https://doi.org/10.1016/j.mcm.2004.06.001).
- [8] T. F. Ma and J. da Silva, “Iterative solutions for a beam equation with nonlinear boundary conditions of third order.” *Appl. Math. Comput.*, vol. 159, no. 1, pp. 11–18, 2004, doi: [10.1016/j.amc.2003.08.088](https://doi.org/10.1016/j.amc.2003.08.088).
- [9] B. Ricceri, “A general variational principle and some of its applications.” *J. Comput. Appl. Math.*, vol. 113, no. 1-2, pp. 401–410, 2000, doi: [10.1016/S0377-0427\(99\)00269-1](https://doi.org/10.1016/S0377-0427(99)00269-1).
- [10] D. Russel, *Mathematical models for the elastic beam and their control- theoretic implications*. . Longman, Essex, 1986, pp. 177–216.
- [11] X. Wang, “Infinitely many solutions for a fourth-order differential equation on a nonlinear elastic foundation.” *Boundl. Value Probl.*, vol. 2013, no. 9, pp. 1–10, 2013, doi: [10.1186/1687-2770-2013-258](https://doi.org/10.1186/1687-2770-2013-258).

- [12] L. Yang, H. Chen, and X. Yang, “The multiplicity of solutions for fourth-order equations generated from a boundary condition.” *Appl. Math. Lett.*, vol. 24, no. 9, pp. 1599–1603, 2011, doi: [10.1016/j.aml.2011.04.008](https://doi.org/10.1016/j.aml.2011.04.008).
- [13] A. Cabada and S. Tersian, “Multiplicity of solutions of a two point boundary value problem for a fourth-order equation.” *Appl. Math. Comput.*, vol. 219, no. 10, pp. 5261–5267, 2011, doi: [10.1016/j.amc.2012.11.066](https://doi.org/10.1016/j.amc.2012.11.066).
- [14] E. Feireisl, “Non-zero time periodic solutions to an equation of Petrovsky type with nonlinear boundary conditions: slow oscillations of beams on elastic bearings.” *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, vol. 20, no. 1, pp. 133–146, 1993.
- [15] E. Zeidler, *Nonlinear functional analysis and its applications*. New York: Springer, 1989.

Authors' addresses

Mohammad Reza Heidari Tavani

Department of Mathematics, Ramhormoz branch, Islamic Azad University, Ramhormoz, Iran

E-mail address: m.reza.h56@gmail.com

Abdollah Nazari

Department of Mathematics, Kazerun Branch, Islamic Azad University, Kazerun, Iran

E-mail address: nazari_mat@yahoo.com