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## THE SECOND REGULARIZED TRACE FORMULA FOR THE STURM-LIOUVILLE OPERATOR

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*Abstract.* In this paper, the second regularized trace formula for the differential operator with antiperiodic boundary conditions is obtained.

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### 1. INTRODUCTION

In the Hilbert space  $H = L_2[0, \pi]$ , we consider operator  $L$  generated by the differential expression

$$l(y) = -y'' + q(x)y,$$

with the antiperiodic boundary conditions

$$y(0) = -y(\pi), y'(0) = -y'(\pi),$$

where  $q(x) \in C^2[0, \pi]$  is real function and satisfies the condition

$$q(0) = q(\pi). \quad (1.1)$$

It is well known from [13] that the eigenvalues of the operator  $L$  form double series

$$\lambda_{n,j} = \left( 2n - 1 + \frac{c_0}{2n-1} + o\left(\frac{1}{n}\right) \right)^2,$$

where  $c_0 = \frac{1}{2\pi} \int_0^\pi q(x) dx$ ,  $j = 1, 2, o\left(\frac{1}{n}\right)$  include  $j$  and  $n = 1, 2, \dots$ . Let  $L_0$  denote the operator  $L$  with  $q(x) \equiv 0$ . The eigenvalues of operator  $L_0$  are

$$\mu_n = (2n - 1)^2, \quad n = 1, 2, \dots$$

The orthonormal eigenfunctions corresponding to the eigenvalues  $\mu_n$  are

$$\psi_n^1 = \sqrt{\frac{2}{\pi}} \cos(2n - 1)x, \psi_n^2 = \sqrt{\frac{2}{\pi}} \sin(2n - 1)x, n = 1, 2, \dots \quad (1.2)$$

It is natural to define second regularized trace of operator  $L$  as

$$\sum_{n=1}^{\infty} \prime (\lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4), \quad (1.3)$$

where the symbol "prime" means that something in this sum is discarded to provide its convergence. The sum in (1.3) is the main interest of this article.

The regularized trace formula

$$\sum_{n=0}^{\infty} \left( \lambda_n - n^2 - \frac{1}{\pi} \int_0^{\pi} q(x) dx \right) = \frac{q(0) + q(\pi)}{4} - \frac{1}{2\pi} \int_0^{\pi} q(x) dx,$$

of the Sturm-Liouville operator

$$-y'' + q(x)y = \lambda y, \quad y'(0) = 0, \quad y'(\pi) = 0, \quad (1.4)$$

with  $q(x) \in C^1[0, \pi]$  was first studied by Gelfand and Levitan ([6]), where the  $\lambda_n$  are the eigenvalues of the operator in (1.4). Afterwards, trace formulas for different differential operators are studied by several mathematicians (see [1–5, 8, 10–12, 14, 15] and references therein).

Note that the first regularized trace formula

$$\sum_{n=0}^{\infty} [\lambda_{n,1} + \lambda_{n,2} - 2(2n-1)^2 - 2c_0] = 0,$$

was obtained for operator  $L$  with a real potential  $q(x) \in L_2(0, \pi)$ , by [10] and with an arbitrary complex  $q(x) \in L_2(0, \pi)$ , by [15]. The similar formula was obtained by [12] for the operator  $L$  with operator function  $q(x)$ .

Trace formulas are used in inverse problems of spectral analysis of differential equations (see [14]) and for approximate calculation of the first eigenvalues of the related operator [1, 4, 5, 7, 9, 14].

## 2. SOME FORMULAS ABOUT THE OPERATOR $qR_{\lambda}^0$

Let  $R_{\lambda}^0$  and  $R_{\lambda}$  be the resolvents of the operator  $L_0$  and  $L$ , respectively. Then, for any  $\lambda \in \rho(L)$  and  $\mu \in \rho(L_0)$  where  $\rho(\cdot)$  is the resolvent set of an operator,  $R_{\lambda} : H \rightarrow H$  and  $R_{\lambda}^0 : H \rightarrow H$  are trace class operators, that is,  $R_{\lambda} R_{\lambda}^0 \in \sigma_1(H)$ . Therefore,

$$\text{tr}(R_{\lambda} - R_{\lambda}^0) = \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_{n,1} - \lambda} + \frac{1}{\lambda_{n,2} - \lambda} - \frac{2}{(2n-1)^2 - \lambda} \right).$$

Multiplying both sides of the above equality by  $\lambda^2/2\pi i$  then integrating over the circle  $|\lambda| = b_p = (2p-1)^2 + 4p$ ,

$$\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 \text{tr}(R_{\lambda} - R_{\lambda}^0) d\lambda = \sum_{n=1}^p (2(2n-1)^4 - \lambda_{n,1}^2 - \lambda_{n,2}^2) \quad (2.1)$$

is obtained. Taking in the account that  $R_\lambda - R_\lambda^0 = -R_\lambda q R_\lambda^0$ , from equation (2.1),

$$\sum_{n=1}^p (\lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4) = \sum_{j=1}^N M_p^j + M_{pN} \quad (2.2)$$

is obtained. Here

$$M_p^j = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 \operatorname{tr} [R_\lambda^0 (q R_\lambda^0)^j] d\lambda,$$

and

$$M_{pN} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 \operatorname{tr} [R_\lambda (q R_\lambda^0)^{N+1}] d\lambda, \quad (2.3)$$

where  $q = q(x)$  and  $N$  is an integer.

The formula

$$M_p^j = \frac{(-1)^j}{2\pi i j} \int_{|\lambda|=b_p} \lambda \operatorname{tr} (q R_\lambda^0)^j d\lambda, \quad (2.4)$$

can be proved similarly as in Theorem 2 in [12].

From equations (1.2) and (2.4), we write

$$\begin{aligned} M_p^1 &= -\frac{1}{\pi i} \int_{|\lambda|=b_p} \left\{ \lambda \sum_{n=1}^{\infty} [(q R_\lambda^0 \psi_n^1, \psi_n^1) + (q R_\lambda^0 \psi_n^2, \psi_n^2)] \right\} d\lambda \\ &= 2 \sum_{n=1}^{\infty} [(q \psi_n^1, \psi_n^1) + (q \psi_n^2, \psi_n^2)] \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{\lambda - \mu_n} d\lambda \\ &= \frac{4}{\pi} \sum_{n=1}^p (2n-1)^2 \int_0^\pi q(x) [\cos^2(2n-1)x + \sin^2(2n-1)x] dx \\ &= \frac{4}{\pi} \sum_{n=1}^p (2n-1)^2 \int_0^\pi q(x) dx. \end{aligned} \quad (2.5)$$

Now, we shall compute  $M_p^2$ . From equation (2.4),

$$\begin{aligned} M_p^2 &= \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \left\{ \sum_{n=1}^{\infty} [((q R_\lambda^0)^2 \psi_n^1, \psi_n^1) + ((q R_\lambda^0)^2 \psi_n^2, \psi_n^2)] \right\} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \left\{ \sum_{n=1}^{\infty} \frac{1}{\mu_n - \lambda} [(q R_\lambda^0 q \psi_n^1, \psi_n^1) + (q R_\lambda^0 q \psi_n^2, \psi_n^2)] \right\} d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \left\{ \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{(\mu_n - \lambda)(\mu_r - \lambda)} [(q \psi_n^1, \psi_r^1) (q \psi_r^1, \psi_n^1) \right. \\ &\quad \left. + (q \psi_n^1, \psi_r^2) (q \psi_r^2, \psi_n^1) + (q \psi_n^2, \psi_r^1) (q \psi_r^1, \psi_n^2) + (q \psi_n^2, \psi_r^2) (q \psi_r^2, \psi_n^2)] \right\} d\lambda. \end{aligned}$$

For convenience, let

$$q_{nr} = |(q\psi_n^1, \psi_r^1)|^2 + |(q\psi_n^1, \psi_r^2)|^2 + |(q\psi_n^2, \psi_r^1)|^2 + |(q\psi_n^2, \psi_r^2)|^2. \quad (2.6)$$

Then,

$$\begin{aligned} M_p^2 &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &= \sum_{n=1}^p \sum_{r=1}^p q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &\quad + \sum_{n=1}^p \sum_{r=p+1}^{\infty} q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &\quad + \sum_{n=p+1}^{\infty} \sum_{r=1}^p q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &\quad + \sum_{n=p+1}^{\infty} \sum_{r=p+1}^{\infty} q_{nr} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)} d\lambda \\ &= \sum_{n=1}^p \sum_{r=1}^p q_{nr} + 2 \sum_{n=1}^p \sum_{r=p+1}^{\infty} q_{nr} \frac{\mu_n}{\mu_r - \mu_n} \\ &= \sum_{n=1}^p \sum_{r=1}^{\infty} q_{nr} - \sum_{n=1}^p \sum_{r=p+1}^{\infty} \frac{\mu_r + \mu_n}{\mu_r - \mu_n} q_{nr}. \end{aligned}$$

Thus,

$$M_p^2 = \sum_{n=1}^p \sum_{r=1}^{\infty} q_{nr} - \alpha_p, \quad (2.7)$$

where

$$\alpha_p = \sum_{n=1}^p \sum_{r=p+1}^{\infty} \frac{\mu_r + \mu_n}{\mu_r - \mu_n} q_{nr}.$$

By using equations (1.2) and (2.6), we have

$$\alpha_p = \alpha_{p1} + \alpha_{p2} + \alpha_{p3}, \quad (2.8)$$

where

$$\alpha_{p1} = 2\pi^{-2} \sum_{n=1}^p \sum_{r=1}^{\infty} \frac{(2r-1)^2 + (2n-1)^2}{(2r-1)^2 - (2n-1)^2} \left\{ \left| \int_0^{\pi} q(x) \cos 2(n-r)x dx \right|^2 + \right.$$

$$\begin{aligned}
& + \left| \int_0^\pi q(x) \sin 2(n-r)x dx \right|^2 \Big\}, \\
\alpha_{p2} &= 2\pi^{-2} \sum_{n=1}^p \sum_{r=1}^\infty \frac{(2r-1)^2 + (2n-1)^2}{(2r-1)^2 - (2n-1)^2} \left| \int_0^\pi q(x) \cos 2(n+r-1)x dx \right|^2, \\
\alpha_{p3} &= 2\pi^{-2} \sum_{n=1}^p \sum_{r=1}^\infty \frac{(2r-1)^2 + (2n-1)^2}{(2r-1)^2 - (2n-1)^2} \left| \int_0^\pi q(x) \sin 2(n+r-1)x dx \right|^2.
\end{aligned}$$

The formula of  $\alpha_{p1}$  can be written as

$$\alpha_{p1} = 2\pi^{-2} \sum_{i=1}^\infty \sum_{\substack{r-n=i \\ n \leq p \\ r > p}} \left( 1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) \quad (2.9)$$

$$\left\{ \left| \int_0^\pi q(x) \cos 2ix dx \right|^2 + \left| \int_0^\pi q(x) \sin 2ix dx \right|^2 \right\}. \quad (2.10)$$

For  $i \leq p$

$$\begin{aligned}
\sum_{\substack{cr-n=i \\ n \leq p \\ r > p}} \frac{(2n-1)^2}{(2r-1)^2 - (2n-1)^2} &= \sum_{j=0}^{i-1} \frac{(2(p-j)-1)^2}{(2(p-j+i)-1)^2 - (2(p-j)-1)^2} \\
&= \sum_{j=0}^{i-1} \frac{(2(p-j)-1+i)^2 - 2(2(p-j)-1+i)i + i^2}{4i(2(p-j)-1+i)} \\
&= \frac{2p-1}{4} + \frac{1-2i}{4} + \sum_{j=0}^{i-1} \frac{i}{4(2p-1-2j+i)}.
\end{aligned} \quad (2.11)$$

For any integers  $p$  and  $i$ , let

$$E = \{(r, n) : r, n \in \mathbb{N}; r - n = i; n \leq p; r > p\}.$$

Then using (2.11), we write

$$\begin{aligned}
\sum_{n, r \in E} \left( 1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) &= i + 2 \sum_{\substack{cr-n=i \\ n \leq p \\ r > p}} \frac{(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \\
&= p + \frac{i}{2} \sum_{j=p-i+1}^p \frac{1}{2j-1+i}, \quad (i \leq p).
\end{aligned} \quad (2.12)$$

It is easy to see that

$$\frac{i}{2} \sum_{j=p-i+1}^p \frac{1}{2j-1+i} < \frac{i^2}{p}.$$

Using this inequality and equation (2.12),

$$\sum_{n,r \in E} \left( 1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) = p + i^2 O(p^{-1}), \quad (i \leq p) \quad (2.13)$$

is obtained.

Here  $O(p^{-1})$  depends on  $p$  and  $i$ , and satisfies the inequality

$$|O(p^{-1})| < \text{const.} \cdot p^{-1}.$$

In a similar form, for  $i \geq p$ , it can be shown that

$$\sum_{n,r \in E} \left( 1 + \frac{2(2n-1)^2}{(2r-1)^2 - (2n-1)^2} \right) = O(p). \quad (2.14)$$

Here  $O(p)$  depends on  $p$  and  $i$ , and satisfies the inequality

$$|O(p)| < \text{const.} \cdot p.$$

From (2.9), (2.13) and (2.14), we obtain

$$\begin{aligned} \alpha_{p1} &= 2\pi^{-2} p \sum_{i=1}^{\infty} \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \\ &\quad + \sum_{i=1}^p i^2 O(p^{-1}) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \\ &\quad + \sum_{i=p+1}^{\infty} O(p) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \\ &= \frac{p}{\pi} \int_0^{\pi} |q(x)|^2 dx - \frac{p}{\pi^2} \left| \int_0^{\pi} q(x) dx \right|^2 + \alpha_{p1}^{(1)} + \alpha_{p1}^{(2)}. \end{aligned} \quad (2.15)$$

Here, since  $q(0) = q(\pi)$ ,

$$\begin{aligned} |\alpha_{p1}^{(1)}| &= \left| \sum_{i=1}^p i^2 O(p^{-1}) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \right| \\ &\leq \text{const.} \cdot p^{-1} \int_0^{\pi} |q'(x)|^2 dx, \end{aligned} \quad (2.16)$$

and

$$|\alpha_{p1}^{(2)}| = \left| \sum_{i=p+1}^{\infty} O(p) \left\{ \left| \int_0^{\pi} q(x) \cos 2ix dx \right|^2 + \left| \int_0^{\pi} q(x) \sin 2ix dx \right|^2 \right\} \right| \quad (2.17)$$

$$\leq \text{const.} \cdot p^{-1} \int_0^{\pi} |q'(x)|^2 dx$$

is obtained. From (2.15)-(2.17), we get

$$\alpha_{p1} = \frac{p}{\pi} \int_0^{\pi} q^2(x) dx - \frac{p}{\pi^2} \left| \int_0^{\pi} q(x) dx \right|^2 + O(p^{-1}). \quad (2.18)$$

Since  $q(x)$  satisfies the condition in equation (1.1), it can be shown that

$$|\alpha_{pj}| \leq \text{const.} \cdot (p^{-1}), \quad (j = 2, 3). \quad (2.19)$$

From equation (2.6), we have

$$\sum_{n=1}^p \sum_{r=1}^{\infty} q_{nr} = \sum_{n=1}^p \{ \|q\psi_n^1\|^2 + \|q\psi_n^2\|^2 \} = \frac{2p}{\pi} \int_0^{\pi} q^2(x) dx. \quad (2.20)$$

From equations (2.7), (2.8) and (2.18)-(2.20),

$$M_p^2 = \frac{p}{\pi} \int_0^{\pi} q(x)^2 dx + \frac{p}{\pi^2} \left( \int_0^{\pi} q(x) dx \right)^2 + O(p^{-1}),$$

is obtained.

### 3. THE SECOND REGULARIZED TRACE FORMULA

In this section we obtain the second regularized trace formula for the operator  $L$ . To do this, we will first show that the formulas

$$\lim_{p \rightarrow \infty} M_p^j = 0, \quad j \geq 3, \quad (3.1)$$

$$\lim_{p \rightarrow \infty} M_p^N = 0, \quad N \geq 6, \quad (3.2)$$

are satisfied.

The inequalities

$$\|qR_{\lambda}^0\|_{\sigma_1(H)} < C, \|R_{\lambda}^0\| < Cp^{-1}, \|R_{\lambda}\| < Cp^{-1}, \quad (|\lambda| = b_p) \quad (3.3)$$

are true (see [12]). Here  $C > 0$  is a constant. From (2.4) and (3.3), we have

$$|M_p^j| = \frac{1}{\pi j} \left| \int_{|\lambda|=b_p} \lambda \text{tr}(qR_{\lambda}^0)^j d\lambda \right|$$

$$\leq \frac{b_p}{\pi j} \int_{|\lambda|=b_p} \| (qR_{\lambda}^0)^j \|_{\sigma_1(H)} |d\lambda|$$

$$\begin{aligned}
&\leq \frac{b_p}{\pi j} \int_{|\lambda|=b_p} \|(qR_\lambda^0)\|_{\sigma_1(H)} \|(qR_\lambda^0)^{j-1}\| |d\lambda| \\
&\leq \frac{Cb_p}{\pi j} \int_{|\lambda|=b_p} \|q\|^{j-1} \|(R_\lambda^0)\|^{j-1} |d\lambda| \\
&< C_1 p^{5-j} (C_1 > 0).
\end{aligned}$$

This implies that

$$\lim_{p \rightarrow \infty} M_p^j = 0, \quad (j \geq 6),$$

but we claim that it is also true for  $j = 3, 4, 5$ . Now let we prove the formula (3.1) for  $j = 3$ .

$$\begin{aligned}
M_p^3 &= -\frac{1}{3\pi i} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\mu_n - \lambda)(\mu_r - \lambda)(\mu_k - \lambda)} \\
&\quad * \{ (q\psi_n^1, \psi_r^1)[(q\psi_r^1, \psi_k^1)\psi_k^1 + (q\psi_r^1, \psi_k^2)\psi_k^2] \\
&\quad + (q\psi_n^1, \psi_r^2)[(q\psi_r^2, \psi_k^1)\psi_k^1 + (q\psi_r^2, \psi_k^2)\psi_k^2], \psi_n^1 \} \\
&\quad + \{ (q\psi_n^2, \psi_r^1)[(q\psi_r^1, \psi_k^1)\psi_k^1 + (q\psi_r^1, \psi_k^2)\psi_k^2] \\
&\quad + (q\psi_n^2, \psi_r^2)[(q\psi_r^2, \psi_k^1)\psi_k^1 + (q\psi_r^2, \psi_k^2)\psi_k^2], \psi_n^2 \} \\
&= -\frac{1}{3\pi i} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\mu_n - \lambda)(\mu_r - \lambda)(\mu_k - \lambda)} F(n, r, k),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
F(n, r, k) &= (q\psi_n^1, \psi_r^1)(q\psi_r^1, \psi_k^1)(q\psi_k^1, \psi_n^1) + (q\psi_n^1, \psi_r^1)(q\psi_r^1, \psi_k^2)(q\psi_k^2, \psi_n^1) \\
&\quad + (q\psi_n^1, \psi_r^2)(q\psi_r^2, \psi_k^1)(q\psi_k^1, \psi_n^1) + (q\psi_n^1, \psi_r^2)(q\psi_r^2, \psi_k^2)(q\psi_k^2, \psi_n^1) \\
&\quad + (q\psi_n^2, \psi_r^1)(q\psi_r^1, \psi_k^1)(q\psi_k^1, \psi_n^2) + (q\psi_n^2, \psi_r^1)(q\psi_r^1, \psi_k^2)(q\psi_k^2, \psi_n^2) \\
&\quad + (q\psi_n^2, \psi_r^2)(q\psi_r^2, \psi_k^1)(q\psi_k^1, \psi_n^2) + (q\psi_n^2, \psi_r^2)(q\psi_r^2, \psi_k^2)(\psi_k^2, \psi_n^2).
\end{aligned}$$



Since  $F(n, r, k) = F(r, n, k) = F(k, n, r) = F(k, r, n) = F(n, k, r) = F(r, k, n)$ , from equation (3.4)

$$\begin{aligned}
M_p^3 &= \frac{1}{\pi i} \sum_{n=1}^p \sum_{r=p+1}^{\infty} \sum_{k=p+1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)(\lambda - \mu_k)} F(n, r, k) \\
&+ \frac{1}{\pi i} \sum_{n=1}^p \sum_{\substack{r=1 \\ n \neq r}}^p \sum_{k=p+1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\lambda - \mu_n)(\lambda - \mu_r)(\lambda - \mu_k)} F(n, r, k) \\
&+ \frac{1}{\pi i} \sum_{n=1}^p \sum_{k=p+1}^{\infty} \int_{|\lambda|=b_p} \frac{\lambda d\lambda}{(\lambda - \mu_n)(\lambda - \mu_k)} F(n, n, k) \\
&= 2 \sum_{n=1}^p \sum_{r=p+1}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F(n, r, k) \\
&+ 4 \sum_{n=1}^p \sum_{\substack{r=1 \\ n \neq r}}^p \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_n - \mu_r)(\mu_n - \mu_k)} F(n, r, k) \\
&- 2 \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)^2} F(n, n, k)
\end{aligned} \tag{3.5}$$

is obtained. Let

$$\begin{aligned}
F_1(n, r, k) &= \pi^{-3} \int_0^\pi q(x) \cos 2(n-r)x dx \\
&\int_0^\pi q(x) \cos 2(r-k)x dx \int_0^\pi q(x) \cos 2(k-n)x dx,
\end{aligned} \tag{3.6}$$

$$F_2(n, r, k) = F(n, r, k) - F_1(n, r, k), \tag{3.7}$$

$$A_{pi} = \sum_{n=1}^p \sum_{r=p+1}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_i(n, r, k), \tag{3.8}$$

$$B_{pi} = \sum_{n=1}^p \sum_{\substack{r=1 \\ n \neq r}}^p \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_n - \mu_r)(\mu_n - \mu_k)} F_i(n, r, k), \tag{3.9}$$

$$C_{pi} = \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)^2} F_i(n, n, k), \quad (i = 1, 2). \tag{3.10}$$

From equation (3.6), we obtain

$$F_1(n, r, k) = F_1(n, k, r), \quad F_1(n, n, k) = F_1(n, k, k).$$

Using these equalities and equation (3.8), we have

$$\begin{aligned} A_{p1} &= \sum_{n=1}^p \sum_{\substack{cr=p+1 \\ r>k}}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k) \\ &+ \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_k - \mu_n)^2} F_1(n, n, k). \end{aligned} \quad (3.11)$$

In similar way, it can be shown that,

$$B_{p1} = - \sum_{n=1}^p \sum_{\substack{cr=1 \\ n<r}}^p \sum_{k=p+1}^{\infty} \frac{\mu_k}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k). \quad (3.12)$$

Let

$$A_p^1 = \sum_{n=1}^p \sum_{\substack{cr=p+1 \\ r>k}}^{\infty} \sum_{k=p+1}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k), \quad (3.13)$$

$$B_p^1 = -B_{p1}, \quad (3.14)$$

$$C_p^1 = \sum_{n=1}^p \sum_{k=p+1}^{\infty} \frac{1}{\mu_k - \mu_n} F_1(n, n, k). \quad (3.15)$$

Hence, by using equations (3.5)-(3.15), we obtain

$$M_p^3 = 4A_p^1 - 4B_p^1 - 2C_p^1 + 2A_{p2} + 4B_{p2} - 2C_{p2}. \quad (3.16)$$

Let us find a formula for  $A_p^1$ .

Let

$$E_1 = \{(n, r, k) : n, r, k \in N; r - n = i; k - n = j; n \leq p; r, k > p\},$$

where  $p, i$  and  $j$  are integers such that  $p \geq j, i \geq j$ , then

$$\begin{aligned} A_p^1 &= \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^{\infty} \sum_{\substack{k=p+1 \\ k-n \leq p}}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k) \\ &+ \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^{\infty} \sum_{\substack{k=p+1 \\ k-n > p}}^{\infty} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k) \\ &= \pi^{-3} \sum_{\substack{i=2 \\ i>j}}^p \sum_{j=1}^p \left[ \sum_{n, r, k \in E_1} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} \int_0^\pi q(x) \cos 2ix dx \right] \end{aligned}$$

$$\begin{aligned} & \times \int_0^\pi q(x) \cos 2(i-j)x dx \int_0^\pi q(x) \cos 2jx dx \Big] \\ & + \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^\infty \sum_{\substack{k=p+1 \\ k-n>p}}^\infty \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k). \end{aligned}$$

Let

$$\beta_{ij} = \pi^{-3} \int_0^\pi q(x) \cos 2ix dx \int_0^\pi q(x) \cos 2(i-j)x dx \int_0^\pi q(x) \cos 2jx dx, \quad (3.17)$$

and

$$\begin{aligned} A_p^{11} &= \sum_{i=2}^\infty \sum_{\substack{j=1 \\ i>j}}^p \left[ \left( \sum_{\substack{n,r,k \in E_1 \\ i>j}} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} \right) \beta_{ij} \right], \\ A_p^{12} &= \sum_{n=1}^p \sum_{\substack{r=p+1 \\ r>k}}^\infty \sum_{\substack{k=p+1 \\ k-n>p}}^\infty \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} F_1(n, r, k), \end{aligned} \quad (3.18)$$

then we write

$$A_p^1 = A_p^{11} + A_p^{12}. \quad (3.19)$$

By similar proof of (2.14), it can be shown that

$$\begin{aligned} & \sum_{n,r,k \in E_1} \frac{\mu_n}{(\mu_r - \mu_n)(\mu_k - \mu_n)} = \\ &= \sum_{n,r,k \in E_1} \frac{(2n-1)^2}{((2r-1)^2 - (2n-1)^2)((2k-1)^2 - (2n-1)^2)} \\ &= \frac{1}{16i} + \frac{j}{p} O(1), \end{aligned} \quad (3.20)$$

where  $O(1)$  satisfies the condition

$$|O(1)| < \text{const.}$$

and depends on  $p, i$  and  $j$ . Moreover, if  $q(x)$  has a continuous derivative of second order at  $[0, \pi]$  and satisfies the condition in (1.1), then it can be shown that

$$\beta_{ij} = \frac{\text{const.}}{i^2 j^2}. \quad (3.21)$$

From (3.18), (3.20) and (3.21),

$$A_p^{11} = \sum_{i=2}^\infty \sum_{\substack{j=1 \\ i>j}}^p \left[ \frac{\beta_{ij}}{16i} + \frac{O(1)}{pi^2 j} \right],$$

is obtained.

Since

$$\left| \sum_{i=2}^{\infty} \sum_{j=1}^p \frac{O(1)}{pi^2 j} \right| \leq \text{const.} p^{-1} \left( \sum_{i=1}^{\infty} i^{-2} \right) \sum_{j=1}^p j^{-1} < \text{const.} p^{-1} \ln p,$$

we find

$$A_p^{11} = \sum_{i=2}^{\infty} \sum_{\substack{j=1 \\ i>j}}^p \frac{\beta_{ij}}{16i} + o(1). \quad (3.22)$$

Here  $o(1)$  is an expression which satisfies the condition

$$\lim_{p \rightarrow \infty} o(1) = 0,$$

and depends on  $p$ . From 3.19 and 3.22, we obtain

$$A_p^1 = \sum_{i=2}^{\infty} \sum_{\substack{j=1 \\ i>j}}^p \frac{\beta_{ij}}{16i} + A_p^{12} + o(1). \quad (3.23)$$

Now, to find the formula for  $B_p^1$ , let

$$B_p^{11} = \sum_{n=1}^p \sum_{r=1}^p \sum_{\substack{k=p+1 \\ n<r, k-n \leq p}}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} F_1(n, r, k),$$

and

$$B_p^{12} = \sum_{n=1}^p \sum_{r=1}^p \sum_{\substack{k=p+1 \\ n<r, k-n > p}}^{\infty} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} F_1(n, r, k),$$

then from (3.12) and (3.14), we have

$$B_p^1 = B_p^{11} + B_p^{12}. \quad (3.24)$$

By using equations in (3.6) and (3.7),  $B_p^{11}$  can be written as

$$B_p^{11} = \sum_{j=2}^p \sum_{\substack{i=1 \\ j>i}}^{p-1} \left( \sum_{n,r,k \in E_2} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} \right) \beta_{ij}, \quad (3.25)$$

where  $E_2$  is a set defined by

$$E_2 = \{(n, r, k) : n, r, k \in N; r - n = i; k - n = j; n, r \leq p; k > p\}$$

for  $i < j \leq p$ . Moreover it can be shown that

$$\sum_{n,r,k \in E_2} \frac{\mu_k}{(\mu_k - \mu_n)(\mu_k - \mu_r)} = \frac{1}{16j} + \frac{i}{p} O(1). \quad (3.26)$$

From (3.21), (3.25) and (3.26), we obtain

$$B_p^{11} = \sum_{j=2}^p \sum_{\substack{i=1 \\ j>i}}^{p-1} \left[ \frac{\beta_{ij}}{16j} + \frac{O(1)}{pj^2i} \right].$$

and, since  $\beta_{ij} = \beta_{ji}$ , we write

$$B_p^{11} = \sum_{j=2}^p \sum_{\substack{i=1 \\ j>i}}^{p-1} \frac{\beta_{ij}}{16j} + O(1).$$

By using above equation and 3.24, we have

$$B_p^1 = \sum_{i=2}^p \sum_{\substack{j=1 \\ i>j}}^{p-1} \frac{\beta_{ij}}{16i} + B_p^{12} + o(1). \quad (3.27)$$

From (3.24), we get

$$\begin{aligned} \left| \sum_{i=2}^p \frac{\beta_{ip}}{i} \right| &\leq \sum_{i=2}^p \frac{1}{i^3 p^2} = o(1), \\ \left| \sum_{i=p+1}^{\infty} \sum_{j=2}^p \frac{\beta_{ij}}{i} \right| &\leq \sum_{i=p+1}^{\infty} \sum_{j=2}^p \frac{1}{i^3 j^2} = \left( \sum_{i=p+1}^{\infty} \frac{1}{i^3} \right) \left( \sum_{j=1}^p \frac{1}{j^2} \right) = o(1). \end{aligned}$$

Therefore, by (3.27), we have

$$B_p^1 = \sum_{i=2}^p \sum_{\substack{j=1 \\ i>j}}^p \frac{\beta_{ij}}{16i} + B_p^{12} + o(1). \quad (3.28)$$

From (3.16), (3.23) and (3.28), we obtain

$$M_p^3 = 4A_p^{12} - 4B_p^{12} - 2C_p^1 + 2A_{p2} + 4B_{p2} - 2C_{p2} + o(1). \quad (3.29)$$

Here, it can be easily seen that,

$$\lim_{p \rightarrow \infty} A_p^{12} = \lim_{p \rightarrow \infty} B_p^{12} = \lim_{p \rightarrow \infty} C_p^1 = \lim_{p \rightarrow \infty} A_{p2} = \lim_{p \rightarrow \infty} B_{p2} = \lim_{p \rightarrow \infty} C_{p2} = 0. \quad (3.30)$$

From (3.29) and (3.30), we obtain

$$\lim_{p \rightarrow \infty} M_p^3 = 0.$$

In a similar way, it can be proved that

$$\lim_{p \rightarrow \infty} M_p^4 = \lim_{p \rightarrow \infty} M_p^5 = 0.$$

Now, let us prove the expression given in (3.2). By employing (2.2) and (3.3), we obtain

$$\begin{aligned} |M_{pN}| &= \frac{1}{2\pi} \left| \int_{|\lambda|=b_p} \lambda^2 \operatorname{tr} \left[ R_\lambda (qR_\lambda^0)^{N+1} \right] d\lambda \right| \\ &\leq \frac{b_p^2}{2\pi} \int_{|\lambda|=b_p} \left\| R_\lambda (qR_\lambda^0)^{N+1} \right\|_{\sigma_1(H)} |d\lambda| \\ &\leq b_p^2 \int_{|\lambda|=b_p} \|R_\lambda\| \left\| (qR_\lambda^0)^{N+1} \right\|_{\sigma_1(H)} |d\lambda| \\ &\leq C b_p^2 p^{-1} \int_{|\lambda|=b_p} \|qR_\lambda^0\|^N \left\| (qR_\lambda^0) \right\|_{\sigma_1(H)} |d\lambda| \\ &\leq \operatorname{const} \cdot p^{5-N}. \end{aligned}$$

This shows that

$$\lim_{p \rightarrow \infty} M_{pN} = 0, \quad N \geq 6.$$

By using the equations (2.2), (2.5), (3.1) and (3.2), we find

$$\begin{aligned} \sum_{n=1}^p (\lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4) &= \frac{4}{\pi} \sum_{n=1}^p (2n-1)^2 \int_0^\pi q(x) dx \\ &\quad + \frac{p}{\pi} \int_0^\pi q^2(x) dx + \frac{p}{\pi^2} \left( \int_0^\pi q(x) dx \right)^2 + o(1). \end{aligned}$$

As a result, we get

$$\begin{aligned} \sum_{n=1}^p \left[ \lambda_{n,1}^2 + \lambda_{n,2}^2 - 2(2n-1)^4 - \frac{4}{\pi} (2n-1)^2 \int_0^\pi q(x) dx \right. \\ \left. - \frac{1}{\pi} \int_0^\pi q^2(x) dx - \frac{1}{\pi^2} \left( \int_0^\pi q(x) dx \right)^2 \right] = 0. \end{aligned} \tag{3.31}$$

The left hand-side of this equality is called the second regularized trace of the operator  $L$ . Thus we have proven the main result of this article given by the following theorem.

**Theorem 1.** *If  $q(x) \in C^2[0, \pi]$  is a real function which satisfies the condition (1.1), then the equality obtained in (3.31) holds for the second regularized trace of the operator  $L$ .*

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