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GENERALIZED CORE INVERSES OF MATRICES

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Abstract. In this paper, we introduce two new generalized inverses of matrices, namely, the $\langle i,m\rangle$-core inverse and the (j,m)-core inverse. The $\langle i,m\rangle$-core inverse of a complex matrix extends the notions of the core inverse defined by Baksalary and Trenkler [1] and the core-EP inverse defined by Manjunatha Prasad and Mohana [10]. The (j,m)-core inverse of a complex matrix extends the notions of the core inverse and the DMP-inverse defined by Malik and Thome [9]. Moreover, the formulae and properties of these two new concepts are investigated by using matrix decompositions and matrix powers.

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1. INTRODUCTION

Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ complex matrices. Let A^*, $\mathcal{R}(A)$ and $\mathrm{rk}(A)$ denote the conjugate transpose, column space, and rank of $A\in\mathbb{C}^{m\times n}$, respectively. For $A\in\mathbb{C}^{m\times n}$, if $X\in\mathbb{C}^{n\times m}$ satisfies $AXA=A$, $XAX=X$, $(AX)^*=AX$, and $(XA)^*=XA$, then X is called a *Moore-Penrose inverse* of A. This matrix X is unique and denoted by $A^\dagger$. A matrix $X\in\mathbb{C}^{n\times m}$ is called an *outer inverse* of A if it satisfies $XAX=X$; is called a $\{2,3\}$-*inverse* of A if it satisfies $XAX=X$ and $(AX)^*=AX$; is called a $\{1,3\}$-*inverse* of A if it satisfies $AXA=A$ and $(AX)^*=AX$; is called a $\{1,2,3\}$-*inverse* of A if it satisfies $AXA=A$, $XAX=X$ and $(AX)^*=AX$.

The core inverse of a complex matrix was introduced by Baksalary and Trenkler [1]. Let $A\in\mathbb{C}^{n\times n}$. A matrix $X\in\mathbb{C}^{n\times n}$ is called a *core inverse* of A, if it satisfies $AX=P_A$ and $\mathcal{R}(X)\subseteq\mathcal{R}(A)$, here P_A denotes the orthogonal projector onto $\mathcal{R}(A)$. If such a matrix exists, then it is unique and denoted by $A^{\#\!\!\!\bigcirc}$. For a square complex matrix A, one has that A is core invertible, A is group invertible, and $\mathrm{rk}(A)=\mathrm{rk}(A^2)$ are three equivalent conditions (see [2]). We denote $\mathbb{C}_n^{CM}=\{A\in\mathbb{C}^{n\times n}\mid \mathrm{rk}(A)=\mathrm{rk}(A^2)\}$.

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Let $A \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ such that $XA^{k+1} = A^k$, $XAX = X$ and $AX = XA$ is called the *Drazin inverse* of A and denoted by A^D. The Drazin inverse of a square matrix always exists and it is unique. Such integer k is called the Drazin index of A, denoted by $\operatorname{ind}(A)$. If $\operatorname{ind}(A) \le 1$, then the Drazin inverse of A is called the *group inverse* and denoted by $A^{\#}$.

The DMP-inverse for a complex matrix was introduced by Malik and Thome [9]. Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a *DMP-inverse* of A, if it satisfies $XAX = X$, $XA = A^D A$ and $A^k X = A^k A^{\dagger}$. It is unique and denoted by $A^{D,\dagger}$. Malik and Thome gave several characterizations of the DMP-inverse by using the decomposition of Hartwig and Spindelböck [8].

The notion of the core-EP inverse for a complex matrix was introduced by Manjunatha Prasad and Mohana [10]. A matrix $X \in \mathbb{C}^{n\times n}$ is a *core-EP inverse* of $A \in \mathbb{C}^{n\times n}$ if X is an outer inverse of A satisfying $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, where k is the index of A. The core-EP inverse is unique and denoted by $A^{\textcircled{\dagger}}$.

In addition, $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the $n \times 1$ column vectors all of whose components are 1 and 0, respectively. $0_{m\times n}$ (abbr. 0) denotes the zero matrix of size $m \times n$. If $\mathcal{S}$ is a subspace of $\mathbb{C}^n$, then $P_{\mathcal{S}}$ stands for the *orthogonal projector* onto the subspace $\mathcal{S}$. A matrix $A \in \mathbb{C}^{n\times n}$ is called an *EP* matrix if $\mathcal{R}(A) = \mathcal{R}(A^*)$, A is called *Hermitian* if $A^* = A$ and A is *unitary* if $AA^* = I_n$, where I_n denote the *identity matrix* of size n. Let $\mathbb{N}$ denote the set of positive integers.

2. Preliminaries

A related decomposition of the matrix decomposition of Hartwig and Spindelböck [8] was given in [2, Theorem 2.1] by Benítez, in [3] it can be found a simpler proof of this decomposition. Let us start this section with the concept of principal angles.

Definition 1 ([12])**.** Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two nontrivial subspaces of $\mathbb{C}^n$. We define the *principal angles* $\theta_1, \ldots, \theta_r \in [0, \pi/2]$ between $\mathcal{S}_1$ and $\mathcal{S}_2$ by

$$\cos\theta_i = \sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}),$$

for $i = 1, \ldots, r$, where $r = \min\{\dim \mathcal{S}_1, \dim \mathcal{S}_2\}$. The real numbers $\sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}) \ge 0$ are the singular values of $P_{\mathcal{S}_1} P_{\mathcal{S}_2}$.

The following theorem can be found in [2, Theorem 2.1].

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$, $r = \operatorname{rk}(A)$, and let $\theta_1, \ldots, \theta_p$ be the principal angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ belonging to $]0, \pi/2[$. Denote by x and y the multiplicities of the angles 0 and $\pi/2$ as a canonical angle between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. There exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*, \tag{2.1}$$

where $M \in \mathbb{C}^{r\times r}$ is nonsingular,

$$C = \operatorname{diag}(\mathbf{0}_y, \cos\theta_1, \ldots, \cos\theta_p, \mathbf{1}_x),$$

$$S = \begin{bmatrix} \operatorname{diag}(\mathbf{1}_y, \sin\theta_1, \ldots, \sin\theta_p) & 0_{p+y,n-(r+p+y)} \\ 0_{x,p+y} & 0_{x,n-(r+p+y)} \end{bmatrix},$$

and $r = y+p+x$. Furthermore, x and $y+n-r$ are the multiplicities of the singular values 1 and 0 in $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}$, respectively.

In this decomposition, one has $C^2+SS^* = I_r$. Recall that $A^\dagger$ always exists. We have that $A^\#$ exists if and only if C is nonsingular in view of [2, Theorem 3.7]. The following equalities hold

$$A^\dagger = U\begin{bmatrix} CM^{-1} & 0 \\ S^*M^{-1} & 0 \end{bmatrix}U^*, \quad A^\# = U\begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix}U^*.$$

By [3, Theorem 2], we have that

$$A^D = U\begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix}U^*. \tag{2.2}$$

We also have

$$AA^\dagger = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{2.3}$$

$$A^{\textcircled{\#}} = A^\#AA^\dagger = U\begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{2.4}$$

Lemma 1 ([13, Theorem 3.1])**.** *Let $A \in \mathbb{C}^{n\times n}$. Then A is core invertible if and only if there exists $X \in \mathbb{C}^{n\times n}$ such that $(AX)^* = AX$, $XA^2 = A$ and $AX^2 = X$. In this situation, we have $A^{\textcircled{\#}} = X$.*

Lemma 2. *Let $A \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$ for some $k \in \mathbb{N}$, then for $m \in \mathbb{N}$ we have*

(1) $A^k = X^mA^{k+m}$;
(2) $X^k = A^mX^{k+m}$;
(3) $A^kX^k = A^{k+m}X^{k+m}$;
(4) $X^kA^k = X^{k+m}A^{k+m}$;
(5) $A^k = A^mX^mA^k$;
(6) $X^k = X^mA^mX^k$.

Proof. (1). For $m = 1$, it is clear by the hypotheses. If the formula is true for $m \in \mathbb{N}$, then $X^{m+1}A^{k+m+1} = XX^mA^{k+m}A = XA^kA = XA^{k+1} = A^k$.

(3). It is easy to check that $A^kX^k = A^{k+1}X^{k+1}$ by $AX^{k+1} = X^k$. It is not difficult to check the equality $A^kX^k = A^{k+m}X^{k+m}$ by induction.

(5). From (1) we have $A^k = X^kA^{2k}$. Thus by $AX^{k+1} = X^k$, we have $A^k = X^kA^{2k} = AX^{k+1}A^{2k} = AX^kXA^{2k} = A(AX^{k+1})XA^{2k} = A^2X^{k+2}A^{2k} = A^2X^2X^kA^{2k} = \cdots = A^mX^mX^kA^{2k} = A^mX^mA^k$.

The proofs of (2), (4), and (6) are similar to the proofs of (1), (3), (5), respectively. □

Lemma 3. *Let $A \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$ for some $k \in \mathbb{N}$, then $A^D = X^{k+1}A^k$.*

Proof. Since A is Drazin invertible. We will check that $A^D = X^{k+1}A^k$. Have in mind, $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$, thus

$$A(X^{k+1}A^k) = X^kA^k = X^k(XA^{k+1}) = X^{k+1}A^kA. \tag{2.5}$$

That is, $X^{k+1}A^k$ and A commute. Then by (1) and (4) in Lemma 2, we have that

$$\begin{aligned}(X^{k+1}A^k)A(X^{k+1}A^k) &= X^{k+1}A^{k+1}X^{k+1}A^k = X^kA^k(X^{k+1}A^k)\\ &= X^kX^{k+1}A^kA^k = X^{k+1}X^kA^{2k} = X^{k+1}A^k.\end{aligned} \tag{2.6}$$

From (1) in Lemma 2, we have that

$$(X^{k+1}A^k)A^{k+1} = X(X^kA^{2k})A = XA^{k+1} = A^k. \tag{2.7}$$

Thus we have $A^D = X^{k+1}A^k$ by the definition of the Drazin inverse and in view of (2.5), (2.6), and (2.7). □

Remark 1. From the proofs of Lemma 2 and Lemma 3, it is obvious that Lemma 2 and Lemma 3 are valid for rings. Moreover, we can get that for an element $a \in R$, a is Drazin invertible if and only if there exist $x \in R$ and $k \in \mathbb{N}$ such that $ax^{k+1} = x^k$ and $xa^{k+1} = a^k$, where R is a ring.

The following lemma is similar to [9, Theorem 2.5].

Lemma 4. *Let $A \in \mathbb{C}^{n\times n}$ be the form (2.1). Then*

$$A^{D,\dagger} = U\begin{bmatrix} (MC)^D & 0\\ 0 & 0\end{bmatrix}U^*. \tag{2.8}$$

Proof. By (2.2), (2.3) and the definition of DMP-inverse we have

$$\begin{aligned}A^{D,\dagger} &= A^DAA^\dagger\\ &= U\begin{bmatrix} (MC)^D & [(MC)^D]^2MS\\ 0 & 0\end{bmatrix}\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix}U^* = U\begin{bmatrix} (MC)^D & 0\\ 0 & 0\end{bmatrix}U^*.\end{aligned}$$

□

Lemma 5 ([11, Corollary 3.3])**.** *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k. Then $AA^{\textcircled{\dagger}} = A^k(A^k)^\dagger$.*

3. $\langle i,m\rangle$-CORE INVERSE

Let us start this section by introducing the definition of the $\langle i,m\rangle$-core inverse.

Definition 2. Let $A \in \mathbb{C}^{n\times n}$ and $m,i \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called an $\langle i,m\rangle$-core inverse of A, if it satisfies

$$X = A^D AX \quad \text{and} \quad A^m X = A^i (A^i)^\dagger. \tag{3.1}$$

It will be proved that if X exists, then it is unique and denoted by $A^{\oplus}_{i,m}$.

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$. If exists $X \in \mathbb{C}^{n\times n}$ such that (3.1) holds, then X is unique.*

Proof. Assume that X satisfies the system in (3.1), that is $X = A^D AX$ and $A^m X = A^i (A^i)^\dagger$. Thus $X = A^D AX = (A^D)^m A^m X = (A^D)^m A^i (A^i)^\dagger$. Therefore, X is unique by the uniqueness of A^D and $A^i (A^i)^\dagger$. □

Theorem 3. *The system in (3.1) is consistent if and only if $i \ge \mathrm{ind}(A)$. In this case, the solution of (3.1) is $X = (A^D)^m A^i (A^i)^\dagger$.*

Proof. Assume that $i \ge \mathrm{ind}(A)$. Let $X = (A^D)^m A^i (A^i)^\dagger$. We have

$$A^D AX = A^D A(A^D)^m A^i (A^i)^\dagger = (A^D)^m A^D AA^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger = X$$
$$A^m X = A^m (A^D)^m A^i (A^i)^\dagger = A^D AA^i (A^i)^\dagger = A^i (A^i)^\dagger.$$

Thus, the system in (3.1) is consistent and the solution of (3.1) is $X = (A^D)^m A^i (A^i)^\dagger$.

If the system in (3.1) is consistent, then exists X_0 such that $X_0 = A^D AX_0$ and $A^m X_0 = A^i (A^i)^\dagger$. Then $X_0 = A^D AX_0 = (A^D)^m A^m X_0 = (A^D)^m A^i (A^i)^\dagger$ and $A^i (A^i)^\dagger = A^m X_0 = A^m (A^D)^m A^i (A^i)^\dagger = AA^D A^i (A^i)^\dagger$. Hence $A^i = A^i (A^i)^\dagger A^i = AA^D A^i (A^i)^\dagger A^i = AA^D A^i$, that is $i \ge \mathrm{ind}(A)$. □

Example 1. We will give an example that shows if $i < \mathrm{ind}(A)$, then the system in (3.1) is not consistent. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to get $\mathrm{ind}(A) = 2$ and $A^D = 0$. Let $i = 1$ and suppose that X is the solution of system in (3.1), then $X = A^D AX = 0$, which gives $AA^\dagger = A^m X = 0$, thus $A = AA^\dagger A = 0$, this is a contradiction.

Remark 2. If $i \ge \mathrm{ind}(A)$, then $A^{\oplus}_{i,m+1} = A^D A^{\oplus}_{i,m}$.

Remark 3. The $\langle i,m\rangle$-core inverse is a generalization of the core inverse and the core-EP inverse. More precisely, we have the following statements:

(1) If $m = i = \mathrm{ind}(A) = 1$, then the $\langle 1,1\rangle$-core inverse coincides with the core inverse;
(2) If $m = 1$ and $i = \mathrm{ind}(A)$, then the $\langle i,1\rangle$-core inverse coincides with the core-EP inverse.

For the convenience of the readers, in the following, we give some notes of (1) and (2) in Remark 3.

(1) If $m = i = \operatorname{ind}(A) = 1$, then A is group invertible and $A^D = A^\#$ and (3.1) is equivalent to $X = A^\# AX$ and $AX = AA^\dagger$. Thus $X = A^\# AX = A^\# AA^\dagger$, $(AX)^* = (AA^\dagger)^* = AA^\dagger = AX$, $AX^2 = AA^\# AA^\dagger A^\# AA^\dagger = AA^\# AA^\dagger AA^\# A^\dagger = AA^\# A^\dagger = X$ and $XA^2 = A^\# AA^\dagger A^2 = A^\# A^2 = A$. Hence, $\langle 1,1\rangle$-core inverse coincides with the core inverse by Lemma 1. Note that if A is group invertible, then we have that X is the core inverse of A if and only if $X = A^\# AX$ and $AX = AA^\dagger$.

(2) If $m = 1$ and $i = \operatorname{ind}(A)$, then by Theorem 3.3, $A^{\oplus}_{i,1}$ exists and $A^{\oplus}_{i,1} = A^D A^i (A^i)^\dagger$. Let us denote $X = A^{\oplus}_{i,1} = A^D A^i (A^i)^\dagger$. Observe that $AX = A^i (A^i)^\dagger$ is Hermitian. Now,

$$XAX = A^D A^i (A^i)^\dagger A^i (A^i)^\dagger = A^D A^i (A^i)^\dagger = X,$$

that is X is an outer inverse of A. From

$$A^i = A^D A^{i+1} = A^D A^i (A^i)^\dagger A^i A = XA^{i+1}$$

we get $\mathcal{R}(A^i) \subseteq \mathcal{R}(X)$. Also,

$$\begin{aligned} AX^2 = (AX)X &= A^i (A^i)^\dagger A^D A^i (A^i)^\dagger \\ &= A^i (A^i)^\dagger A^i A^D (A^i)^\dagger = A^D A^i (A^i)^\dagger = X, \end{aligned}$$

which implies $X = (AX)^* X \in \mathcal{R}(X^*)$, therefore, $\mathcal{R}(X) \subseteq \mathcal{R}(X^*)$. Finally, $X^* = [A^D A^i (A^i)^\dagger]^* = A^i (A^i)^\dagger (A^D)^*$ implies $\mathcal{R}(X^*) \subseteq \mathcal{R}(A^i)$. Hence $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^i)$. Therefore, the $\langle i,1\rangle$-core inverse coincides with the core-EP inverse by the definition of the core-EP inverse.

From the above statement, we have the following theorem.

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$ with $i = \operatorname{ind}(A)$. Then X is the core-EP inverse of A if and only if $X = A^D AX$ and $AX = A^i (A^i)^\dagger$.*

Corollary 1. *Let $A \in \mathbb{C}^{n\times n}$ with $1 = \operatorname{ind}(A)$. Then X is the core inverse of A if and only if $X = A^\# AX$ and $AX = AA^\dagger$.*

For any $A \in \mathbb{C}^{n\times n}$, either $A^l = 0$ for some $l \in \mathbb{N}$, or $A^l \neq 0$ for all positive integers. Moreover, if $\operatorname{ind}(A) = k$, then $G_k B_k$ is nonsingular (see [5–7]), where $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$, $l = 1,\ldots,k-1$. When $A^k \neq 0$, then it can be written as

$$A^k = \prod_{l=1}^{k} B_l \prod_{l=1}^{k} G_{k+1-l}. \tag{3.2}$$

We have the following results, (see [5, Theorem 4] or [4, Theorem 7.8.2]):

$$\operatorname{ind}(A) = \begin{cases} k & \text{when } G_k B_k \text{ is nonsingular,} \\ k+1 & \text{when } G_k B_k = 0. \end{cases}$$

and

$$A^D = \begin{cases} \prod_{l=1}^{k} B_l (G_k B_k)^{-k-1} \prod_{l=1}^{k} G_{k+1-l} & \text{when } G_k B_k \text{ is nonsingular,} \\ 0 & \text{when } G_k B_k = 0. \end{cases} \tag{3.3}$$

In the sequel, we always assume that $A^k \neq 0$.

It is well-known that if $A = EF$ is a full rank factorization of A, where $r = \operatorname{rk}(A)$, $E \in \mathbb{C}^{n\times r}$ and $F \in \mathbb{C}^{r\times n}$, then (see [4, Theorem 1.3.2])

$$A^\dagger = F^*(FF^*)^{-1}(E^*E)^{-1}E^*. \tag{3.4}$$

Remark 4. The notations and results in above paragraph will be used many times in the sequel.

We will investigate the $\langle i,m\rangle$-core inverse of a matrix $A \in \mathbb{C}^{n\times n}$ by using Remark 4.

Theorem 5. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $i \geq k$, then $A^{\oplus}_{i,m} = A^{\oplus}_{k,m}$.*

Proof. Since $\operatorname{ind}(A) = k$, we have $\mathcal{R}(A^k) = \mathcal{R}(A^i)$ for any $i \geq k$, and therefore, $A^k(A^k)^\dagger = A^i(A^i)^\dagger$. Now, the conclusion follows from Theorem 3. □

Remark 5. The proof of Theorem 5 also can be proved as follows. Since the proof in this remark will be used several times in the sequel, we write this proof here.

Proof. If A is nilpotent, then $A^D = 0$, hence by Theorem 3, one has $A^{\oplus}_{i,m} = A^{\oplus}_{k,m} = 0$. Therefore, we can assume that $A^k \neq 0$. By equality (3.2), we have

$$A^k = \prod_{l=1}^{k} B_l \prod_{l=1}^{k} G_{k+1-l}. \tag{3.5}$$

where $A = B_1G_1$ is a full rank factorization of A and $G_lB_l = B_{l+1}G_{l+1}$ is a full rank factorization of G_lB_l, $l = 1,\dots,k-1$. Let $M = \prod_{l=1}^{k} B_l$, $N = \prod_{l=1}^{k} G_{k+1-l}$ and $L = G_kB_k$. Now, we will show that

$$A^i = \prod_{l=1}^{k} B_l (G_kB_k)^{i-k} \prod_{l=1}^{k} G_{k+1-l} = ML^{i-k}N.$$

In fact,

$$\begin{aligned} A^i &= \prod_{l=1}^{i} B_l \prod_{l=1}^{i} G_{k+1-l} \\ &= B_1 \cdots B_i G_i \cdots G_1 \\ &= B_1 \cdots B_{i-1} (B_i G_i) G_{i-1} \cdots G_1 \\ &= B_1 \cdots B_{i-1} (G_{i-1} B_{i-1}) G_{i-1} \cdots G_1 \\ &= B_1 \cdots B_{i-2} (G_{i-2} B_{i-2})^2 G_{i-2} \cdots G_1 \\ &= \cdots \\ &= B_1 \cdots B_k (G_k B_k)^{i-k} G_k \cdots G_1 = M L^{i-k} N. \end{aligned} \tag{3.6}$$

If we let $M_1 = ML^{i-k}$, then $A^i = ML^{i-k}N = M_1 N$ is a full rank factorization of A^i (see [7, p.183]). Thus

$$(A^i)^{\dagger} = N^*(NN^*)^{-1}(M_1^* M_1)^{-1} M_1^*. \tag{3.7}$$

Note that $NM = \prod_{l=1}^{k} G_{k+1-l} \prod_{l=1}^{k} B_l = L^k$. By Theorem 3, (3.3) and (3.7) we have

$$\begin{aligned} A^{\oplus}_{i,1} &= A^D A^i (A^i)^{\dagger} \\ &= ML^{-k-1} NML^{i-k} N (A^i)^{\dagger} \\ &= ML^{-k-1} NML^{i-k} NN^*(NN^*)^{-1}(M_1^* M_1)^{-1} M_1^* \\ &= ML^{i-k-1} NN^*(NN^*)^{-1}(M_1^* M_1)^{-1} M_1^* \\ &= ML^{i-k-1} (M_1^* M_1)^{-1} M_1^* \\ &= ML^{i-k-1} [(L^{i-k})^* M^* M L^{i-k}]^{-1} (L^{i-k})^* M^* \\ &= ML^{i-k-1} L^{k-i} (M^*M)^{-1} [(L^{i-k})^*]^{-1} (L^{i-k})^* M^* \\ &= ML^{-1} (M^*M)^{-1} M^*. \end{aligned} \tag{3.8}$$

The last expression does not depend on i, then $A^{\oplus}_{i,1} = A^{\oplus}_{k,1}$. Thus, by Remark 2, we have $A^{\oplus}_{i,m} = A^D A^{\oplus}_{i,m-1} = A^D (A^D A^{\oplus}_{i,m-2}) = (A^D)^2 A^{\oplus}_{i,m-2} = \cdots = (A^D)^{m-1} A^{\oplus}_{i,1} = (A^D)^{m-1} A^{\oplus}_{k,1} = A^{\oplus}_{k,m}$. □

Remark 6. By Theorem 5, it is enough to investigate the $i = \operatorname{ind}(A) = k$ case, when we discuss the $\langle i, m \rangle$-core inverse of a matrix $A \in \mathbb{C}^{n\times n}$. That is, the Theorem 5 is a key theorem.

Theorem 6. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$ *and $k, m \in \mathbb{N}$. If $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$,*

$l = 1,\dots,k-1$, *then* $A^{\oplus}_{k,m} = ML^{-m}M^{\dagger}$, *where* $M = \prod_{l=1}^{k} B_l$, $N = \prod_{l=1}^{k} G_{k+1-l}$ *and* $L = G_k B_k$.

Proof. By the proof of Remark 5, we have $A^{\oplus}_{k,1} = ML^{-1}(M^*M)^{-1}M^*$ and $NM = L^k$. Now, we will prove $(A^D)^s A^{\oplus}_{k,1} = ML^{-s-1}(M^*M)^{-1}M^*$ for any $s \in \mathbb{N}$. By (3.3) we have $A^D = \prod_{l=1}^{k} B_l (G_k B_k)^{-k-1} \prod_{l=1}^{k} G_{k+1-l} = ML^{-k-1}N$. When $s = 1$, we have

$$\begin{aligned} A^D A^{\oplus}_{k,1} &= ML^{-k-1}NML^{-1}(M^*M)^{-1}M^* = ML^{-k-1}(NM)L^{-1}(M^*M)^{-1}M^* \\ &= ML^{-k-1}L^kL^{-1}(M^*M)^{-1}M^* = ML^{-2}(M^*M)^{-1}M^*. \end{aligned}$$

Assume that $(A^D)^{s-1}A^{\oplus}_{k,1} = ML^{-s}(M^*M)^{-1}M^*$. Then

$$\begin{aligned} (A^D)^s A^{\oplus}_{k,1} &= A^D (A^D)^{s-1} A^{\oplus}_{k,1} = A^D ML^{-s}(M^*M)^{-1}M^* \\ &= ML^{-k-1}NML^{-s}(M^*M)^{-1}M^* \\ &= ML^{-k-1}L^kL^{-s}(M^*M)^{-1}M^* \\ &= ML^{-s-1}(M^*M)^{-1}M^*. \end{aligned}$$

Thus by Remark 2, we have

$$A^{\oplus}_{k,m} = (A^D)^{m-1}A^{\oplus}_{k,1} = ML^{-m}(M^*M)^{-1}M^* = ML^{-m}M^{\dagger}.$$

□

In the following theorem, we will give a canonical form for the $\langle k,m\rangle$-core inverse of a matrix $A \in \mathbb{C}^{n\times n}$ by using the matrix decomposition in Theorem 1. We will also use the following simple fact: Let $X \in \mathbb{C}^{n\times m}$ and $\mathbf{b} \in \mathbb{C}^n$. If $\mathbf{y} \in \mathbb{C}^m$ satisfies $X^*X\mathbf{y} = X^*\mathbf{b}$, then $XX^{\dagger}\mathbf{b} = X\mathbf{y}$.

Theorem 7. *Let* $A \in \mathbb{C}^{n\times n}$ *have the form (2.1) with* $\operatorname{ind}(A) = k$ *and* $m \in \mathbb{N}$. *Then*

$$A^{\oplus}_{k,m} = U \begin{bmatrix} (MC)^{\oplus}_{k-1,m} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{3.9}$$

Proof. Let r be the rank of A. By Theorem 3 we have

$$A^{\oplus}_{k,m} = (A^D)^m A^k (A^k)^{\dagger}. \tag{3.10}$$

Since A has the form given in Theorem 1 we have

$$A^k = U \begin{bmatrix} (MC)^k & (MC)^{k-1}MS \\ 0 & 0 \end{bmatrix} U^*. \tag{3.11}$$

Let $\mathbf{b} \in \mathbb{C}^n$ be arbitrary and let us decompose $\mathbf{b} = U\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$, where $\mathbf{b}_1 \in \mathbb{C}^r$. Let $\mathbf{x}_0 \in \mathbb{C}^n$ satisfy $(A^k)^*A^k\mathbf{x}_0 = (A^k)^*\mathbf{b}$ [this $\mathbf{x}_0$ always exists because the normal equations always have a solution]. We can decompose $\mathbf{x}_0$ by writing $\mathbf{x}_0 = U\left[\begin{smallmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{smallmatrix}\right]$, where $\mathbf{x}_1 \in \mathbb{C}^r$. Let us denote $N = (MC)^{k-1}M$. Using (3.11),

$$U\begin{bmatrix} CN^* & 0 \\ S^*N^* & 0 \end{bmatrix}\begin{bmatrix} NC & NS \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = U\begin{bmatrix} CN^* & 0 \\ S^*N^* & 0 \end{bmatrix}\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

Therefore,

$$CN^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = CN^*\mathbf{b}_1 \quad \text{and} \quad S^*N^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = S^*N^*\mathbf{b}_1.$$

Premultiplying the first equality by C and the second equality by S and after, adding them, we get $N^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = N^*\mathbf{b}_1$, and hence, $N(C\mathbf{x}_1 + S\mathbf{x}_2) = NN^\dagger\mathbf{b}_1$. Now,

$$\begin{aligned} A^k(A^k)^\dagger\mathbf{b} = A^k\mathbf{x}_0 &= U\begin{bmatrix} NC & NS \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\ &= U\begin{bmatrix} NC\mathbf{x}_1 + NS\mathbf{x}_2 \\ \mathbf{0} \end{bmatrix} = U\begin{bmatrix} NN^\dagger\mathbf{b}_1 \\ \mathbf{0} \end{bmatrix} = U\begin{bmatrix} NN^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^*\mathbf{b}. \end{aligned}$$

Since $\mathbf{b}$ is arbitrary,

$$A^k(A^k)^\dagger = U\begin{bmatrix} NN^\dagger & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Now we will prove $NN^\dagger = (MC)^{k-1}[(MC)^{k-1}]^\dagger$. Recall that we have $N = (MC)^{k-1}M$, and so, $\mathcal{R}(N) \subseteq \mathcal{R}((MC)^{k-1})$. Since M is nonsingular, $\text{rk}(N) = \text{rk}((MC)^{k-1})$, and thus, $\mathcal{R}(N) = \mathcal{R}((MC)^{k-1})$. Since $(MC)^{k-1}[(MC)^{k-1}]^\dagger$ and $NN^\dagger$ are the orthogonal projectors onto $\mathcal{R}((MC)^{k-1})$ and $\mathcal{R}(N)$, respectively, we get $NN^\dagger = (MC)^{k-1}[(MC)^{k-1}]^\dagger$.

By (2.2) we have

$$A^D = U\begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix}U^*. \tag{3.12}$$

Thus, we have

$$(A^D)^m = U\begin{bmatrix} [(MC)^D]^m & [(MC)^D]^{m+1}MS \\ 0 & 0 \end{bmatrix}U^*.$$

Since $\text{ind}(A) = k$, we have $A^DA^{k+1} = A^k$. By using the above representations of A^D and A^k given in (3.11) and (3.12), respectively,

$$\begin{aligned} &\begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (MC)^{k+1} & (MC)^kMS \\ 0 & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} (MC)^k & (MC)^{k-1}MS \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$(MC)^D(MC)^kM[C \mid S] = (MC)^{k-1}M[C \mid S]. \tag{3.13}$$

Have in mind that we have $C^2 + SS^* = I_r$. Thus, postmultiplying (3.13) by $\left[\begin{smallmatrix} C \\ S^* \end{smallmatrix}\right]$ gives us $(MC)^D(MC)^k M = (MC)^{k-1}M$ and from the nonsingularity of M we obtain $(MC)^D(MC)^k = (MC)^{k-1}$, and so, $\operatorname{ind}(MC) \le k-1$. Therefore we have

$$\begin{aligned} A^{\oplus}_{k,m} &= (A^D)^m A^k (A^k)^\dagger \\ &= U \begin{bmatrix} [(MC)^D]^m & [(MC)^D]^{m+1}MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (MC)^{k-1}((MC)^{k-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} [(MC)^D]^m (MC)^{k-1}((MC)^{k-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)^{\oplus}_{k-1,m} & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

□

Remark 7. If we use the decomposition of Hartwig and Spindelböck in [8, Corollary 6], then an expression of the $\langle k,m\rangle$-core inverse of A is

$$A^{\oplus}_{k,m} = U \begin{bmatrix} (\Sigma K)^{\oplus}_{k-1,m} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

which is similar to the expression of $A^{\oplus}_{k,m}$ in Theorem 7. Since the proof of this result can be proved as the proof of Theorem 7, we omit this proof.

Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The Jordan Canonical form of A is $P^{-1}AP = J$, where $P \in \mathbb{C}^{n\times n}$ is nonsingular and $J \in \mathbb{C}^{n\times n}$ is a block diagonal matrix composed of Jordan blocks. In the following theorem, we will compute the $\langle k,m\rangle$-core inverse by using the Jordan Canonical form of A.

Theorem 8. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$, then $A^{\oplus}_{k,m} = P_1 D^{-m} P_1^\dagger$, where $A = P\left[\begin{smallmatrix} D & 0 \\ 0 & N \end{smallmatrix}\right]P^{-1}$ with $D \in \mathbb{C}^{r\times r}$ is nonsingular, N is nilpotent and $P = [P_1 \mid P_2]$ with $P_1 \in \mathbb{C}^{n\times r}$.*

Proof. The Jordan Canonical form of A is $P^{-1}AP = J$, where $P \in \mathbb{C}^{n\times n}$ is nonsingular and $J \in \mathbb{C}^{n\times n}$ is a block diagonal matrix. Rearrange the elements of J such that $A = P\left[\begin{smallmatrix} D & 0 \\ 0 & N \end{smallmatrix}\right]P^{-1}$, where D is nonsingular and N is nilpotent. It is well-known that $A^D = P\left[\begin{smallmatrix} D^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]P^{-1}$ and $A^k = P\left[\begin{smallmatrix} D^k & 0 \\ 0 & 0 \end{smallmatrix}\right]P^{-1}$. If we let $P = [P_1 \mid P_2]$ and $P^{-1} = \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right]$, then

$$(A^D)^m A^k = [P_1 \mid P_2] \begin{bmatrix} (D^{-1})^m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = P_1 D^{k-m} Q_1.$$

Observe that $A^k = (P_1 D^k) Q_1$ is a full rank factorization of A^k. Hence by (3.4) we have

$$\begin{aligned}(A^k)^\dagger &= (P_1 D^k Q_1)^\dagger \\ &= Q_1^*(Q_1 Q_1^*)^{-1}[(P_1 D^k)^* P_1 D^k]^{-1}(P_1 D^k)^* \\ &= Q_1^*(Q_1 Q_1^*)^{-1} D^{-k}(P_1^* P_1)^{-1}[(D^k)^*]^{-1}(D^k)^* P_1^* \\ &= Q_1^*(Q_1 Q_1^*)^{-1} D^{-k}(P_1^* P_1)^{-1} P_1^* \\ &= Q_1^\dagger D^{-k} P_1^\dagger.\end{aligned}$$

By Theorem 3, we have $A^{\oplus}_{k,m} = (A^D)^m A^k (A^k)^\dagger$. Thus we have

$$\begin{aligned}A^{\oplus}_{k,m} &= (A^D)^m A^k (A^k)^\dagger = P_1 D^{k-m} Q_1 Q_1^\dagger D^{-k} P_1^\dagger \\ &= P_1 D^{k-m} Q_1 Q_1^*(Q_1 Q_1^*)^{-1} D^{-k} P_1^\dagger = P_1 D^{-m} D^k D^{-k} P_1^\dagger = P_1 D^{-m} P_1^\dagger.\end{aligned}$$

□

Proposition 1. *Let $A \in \mathbb{C}^{n\times n}$. If $i \geq \operatorname{ind}(A)$, then $A^m A^{\oplus}_{i,m}$ is the projector onto $\mathcal{R}(A^i)$ along $\mathcal{R}(A^i)^\perp$.*

Proof. It is trivial. □

In the following proposition, we will investigate some properties of the $\langle i,m\rangle$-core inverse.

Proposition 2. *Let $A \in \mathbb{C}^{n\times n}$, $m, i \in \mathbb{N}$. If $i \geq \operatorname{ind}(A)$, then*

(1) *$A^{\oplus}_{i,m}$ is a $\{2,3\}$-inverse of A^m;*

(2) *$A^{\oplus}_{i,m} = (A^D)^m P_{A^i}$;*

(3) *$(A^{\oplus}_{i,m})^n = (A^D)^{m(n-1)} A^{\oplus}_{i,m} = (A^D)^{mn} P_{A^i}$;*

(4) *$A^i A^{\oplus}_{i,m} = A^{\oplus}_{i,m} A^i$ if and only if $\mathcal{R}(A^i)^\perp \subseteq \mathcal{N}(A^i)$;*

(5) *$A^{\oplus}_{i,m} = A$ implies that A is EP.*

Proof. (1). By Theorem 3 we have $A^{\oplus}_{i,m} = (A^D)^m A^i (A^i)^\dagger$, thus

$$\begin{aligned}A^{\oplus}_{i,m} A^m A^{\oplus}_{i,m} &= (A^D)^m A^i (A^i)^\dagger A^m (A^D)^m A^i (A^i)^\dagger \\ &= (A^D)^m A^i (A^i)^\dagger A^i A^m (A^D)^m (A^i)^\dagger \\ &= (A^D)^m A^i A^m (A^D)^m (A^i)^\dagger = (A^D)^m A^m (A^D)^m A^i (A^i)^\dagger \\ &= A^D A (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger = A^{\oplus}_{i,m}.\end{aligned}$$

Thus $A^{\oplus}_{i,m}$ is a $\{2,3\}$-inverse of A^m in view of $A^m A^{\oplus}_{i,m} = A^i (A^i)^\dagger$.

(2) is trivial.

(3). By

$$\begin{aligned}(A^{\oplus}_{i,m})^n &= (A^D)^m A^i (A^i)^\dagger (A^D)^m A^i (A^i)^\dagger (A^{\oplus}_{i,m})^{n-2}\\ &= (A^D)^m (A^D)^m A^i (A^i)^\dagger (A^{\oplus}_{i,m})^{n-2}\\ &= (A^D)^m A^{\oplus}_{i,m} (A^{\oplus}_{i,m})^{n-2} = (A^D)^m (A^{\oplus}_{i,m})^{n-1},\end{aligned}$$

it is easy to check (3).

(4). By $\mathcal{R}[(I_n - A^i(A^i)^\dagger] = \mathcal{N}[(A^i)^\dagger]$, we have

$$\begin{aligned}A^i A^{\oplus}_{i,m} = A^{\oplus}_{i,m} A^i &\Leftrightarrow A^i (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger A^i\\ &\Leftrightarrow A^i (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i\\ &\Leftrightarrow A^i (A^D)^m (I_n - A^i (A^i)^\dagger) = 0\\ &\Leftrightarrow \mathcal{R}[I_n - A^i(A^i)^\dagger] \subseteq \mathcal{N}[A^i (A^D)^m]\\ &\Leftrightarrow \mathcal{N}[(A^i)^\dagger] \subseteq \mathcal{N}[(A^D)^m]\\ &\Leftrightarrow \mathcal{N}[(A^i)^*] \subseteq \mathcal{N}[(A^D)^m]\\ &\Leftrightarrow \mathcal{R}(A^i)^\perp \subseteq \mathcal{N}[(A^D)^m]\\ &\Leftrightarrow \mathcal{R}(A^i)^\perp \subseteq \mathcal{N}[A^i].\end{aligned}$$

(5). Let A be written in the form (2.1). We have $A^{\oplus}_{i,m} = U \begin{bmatrix} (MC)^{\oplus}_{i-1,m} & 0 \\ 0 & 0 \end{bmatrix} U^*$ by Theorem 7. Thus, $A^{\oplus}_{i,m} = A$ implies $MS = 0$. From the nonsingularity of M, we have $S = 0$, which is equivalent to say that A is EP in view of [2, Theorem 3.7]. □

4. (j,m)-CORE INVERSE

Let us start this section by introducing the definition of the (j,m)-core inverse.

Definition 3. Let $A \in \mathbb{C}^{n\times n}$ and $m, j \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a (j,m)-core inverse of A, if it satisfies

$$X = A^D AX \quad \text{and} \quad A^m X = A^m (A^j)^\dagger. \tag{4.1}$$

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$. If the system in (4.1) is consistent, then the solution is unique.*

Proof. Assume that X satisfies (4.1), that is $X = A^D AX$ and $A^m X = A^m (A^j)^\dagger$. Then $X = A^D AX = (A^D)^m A^m X = (A^D)^m A^m (A^j)^\dagger = A^D A (A^j)^\dagger$. Thus X is unique. □

By Theorem 9 if X exists, then it is unique and denoted by $A^{\ominus}_{j,m}$.

Theorem 10. *Let $A \in \mathbb{C}^{n\times n}$ and $m, j \in \mathbb{N}$.*

(1) *If $m \geq \mathrm{ind}(A)$, then the system in (4.1) is consistent and the solution is $X = A^D A (A^j)^\dagger$.*

(2) *If the system in (4.1) is consistent, then* $\mathrm{ind}(A) \le \max\{j, m\}$.

Proof. (1). Let $X = A^D A (A^j)^\dagger$. We have $A^D A X = A^D A A^D A (A^j)^\dagger = A^D A (A^j)^\dagger = X$ and $A^m X = A^m A^D A (A^j)^\dagger = A^D A A^m (A^j)^\dagger = A^m (A^j)^\dagger$.

(2). If the system in (4.1) is consistent, then exits $X_0 \in \mathbb{C}^{n\times n}$ such that $X_0 = A^D A X_0 = (A^D)^m A^m X_0 = (A^D)^m A^m (A^j)^\dagger = A^D A (A^j)^\dagger$ and $A^m (A^j)^\dagger = A^m X_0 = A^m A^D A (A^j)^\dagger = A^m (A^D)^j A^j (A^j)^\dagger$. Thus

$$A^m (A^j)^\dagger A^j = A^m (A^D)^j A^j (A^j)^\dagger A^j = A^m (A^D)^j A^j = A^m A^D A.$$

If $m \ge j$, then $A^m A^D A = A^m (A^j)^\dagger A^j = A^{m-j} A^j (A^j)^\dagger A^j = A^{m-j} A^j = A^m$. That is, $\mathrm{ind}(A) \le m$. If $j > m$, then $A^j = A^j (A^j)^\dagger A^j = A^{j-m} A^m (A^j)^\dagger A^j = A^{j-m} A^m A^D A = A^j A^D A$. That is, $\mathrm{ind}(A) \le j$. Thus, $\mathrm{ind}(A) \le \max\{j, m\}$. □

Example 2. We will give an example that shows if $m < \mathrm{ind}(A)$, then the system in (4.1) is not consistent. Let A be the same matrix in Example 1. It is easy to get $\mathrm{ind}(A) = 2$ and $A^D = 0$. Let $m = j = 1$ and suppose that X is the solution of system in (4.1), then $X = A^D A X = 0$, which gives $AA^\dagger = AX = 0$, thus $A = AA^\dagger A = 0$, this is a contradiction.

Example 3. The converse of Theorem 10 (1) is not true. Let $m = 1$ and $j = 3$. If we let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\mathrm{ind}(A) = 3$ and $A^3 = 0$. Hence $X = 0$ is a solution of (4.1), but $m < \mathrm{ind}(A)$.

Example 4. If $\mathrm{ind}(A) \le \max\{j, m\}$, then the system in (4.1) may be not consistent. If we let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$A^3 = A^2 = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$A^D = A^2$ and $\mathrm{ind}(A) = 2$. Let $m = 1$ and $j = 2$, then $\mathrm{ind}(A) \le \max\{j, m\}$. It is easy to check that

$$(A^2)^\dagger = \frac{1}{15} \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix}.$$

If the system in (4.1) has a solution X_0, then $X_0 = A^D A X_0 = A^D A(A^2)^\dagger$ and $A(A^2)^\dagger = AX_0 = AA^D A(A^2)^\dagger = A^4(A^2)^\dagger = A^2(A^2)^\dagger$ would hold. But

$$A(A^2)^\dagger = \frac{1}{15}\begin{bmatrix} 10 & -5 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \frac{1}{15}\begin{bmatrix} 12 & -6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^2(A^2)^\dagger.$$

Thus, the system in (4.1) is not consistent.

Remark 8. If $m \geq \operatorname{ind}(A) = k$, it is not difficult to see that $A^{\ominus}_{j,m} = A^{\ominus}_{j,m+1}$. That is to say, the (j,m)-core inverse of A coincides with the $(j,m+1)$-core inverse of A. Thus, in the sequel, we only discuss the $m = \operatorname{ind}(A)$ case.

Theorem 11. *Let $A, X \in \mathbb{C}^{n\times n}$, $k, j \in \mathbb{N}$. If $\operatorname{ind}(A) = k$ and X is the (j,k)-core inverse of A, then we have $X^j A^j X^j = (A^D)^{j(j-1)} X^j$ and $XA^j = A^D A$.*

Proof. By the definition of the (j,k)-core inverse, we have $X = A^D AX$ and $A^k X = A^k(A^j)^\dagger$. By $X = A^D A(A^j)^\dagger$, it is easy to check that $X^{n+1} = (A^D)^j X^n$ for arbitrary $n \in \mathbb{N}$, which gives that $X^j = (A^D)^{j(j-1)} X$.

$$XA^j = A^D A(A^j)^\dagger A^j = (A^D)^j A^j (A^j)^\dagger A^j = (A^D)^j A^j = A^D A;$$
$$X^j A^j X^j = (A^D)^{j(j-1)} XA^j X^j = (A^D)^{j(j-1)} A^D AX^j = (A^D)^{j(j-1)} X^j.$$

□

Corollary 2. *Let $A, X \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. If X is the $(1,k)$-core inverse of A, then we have $XAX = X$ and $XA = A^D A$.*

The (j,m)-core inverse is a generalization of the core inverse and the DMP-inverse in view of Theorem 11.

Remark 9. When $j = m = 1 = \operatorname{ind}(A)$, the equations in (4.1) are equivalent to $XAX = X$, $XA = A^\# A$, and $AX = AA^\dagger$. Thus $AX = AA^\dagger$ implies that $(AX)^* = AX$; $XA = A^\# A$ gives that $XA^2 = A$ and $AXA = A$; and $X = XAX = A^\# AX = AA^\# X$, which means that $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, then $X = AY$ for some $Y \in \mathbb{C}^{n\times n}$, thus $X = AY = AXAY = AX^2$. Therefore, we have $A^{\textcircled{\#}} = X$ by Lemma 1. In a word, the $(1,1)$-core inverse coincides with the usual core inverse.

Remark 10. If we let $j = 1$ and $m = \operatorname{ind}(A)$, then the equations in (4.1) are equivalent to $XAX = X$, $XA = A^D A$, and $A^k X = A^k A^\dagger$ by Theorem 11. Thus $(1,k)$-core inverse coincides with the DMP-inverse.

From Remark 10, Theorem 11 and the definition of the (j,k)-core inverse, we have the following theorem, which says that the conditions $XAX = X$, and $XA = A^D A$ in the definition of the DMP-inverse can be replaced by $X = A^D AX$.

Theorem 12. *Let $A \in \mathbb{C}^{n\times n}$ with $k = \operatorname{ind}(A)$. Then $X \in \mathbb{C}^{n\times n}$ is the DMP-inverse of A if and only if $X = A^D AX$ and $A^k X = A^k A^\dagger$.*

In the following theorem, we will give a canonical form for the (j,k)-core inverse of a matrix $A \in \mathbb{C}^{n\times n}$ by using the matrix decomposition in Theorem 1.

Theorem 13. *Let $A \in \mathbb{C}^{n\times n}$ have the form (2.1) with $\text{ind}(A) = k$ and $j \in \mathbb{N}$. Then*

$$A^{\ominus}_{j,k} = U\begin{bmatrix} (MC)^D(MC)^{\ominus}_{j-1,k} & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Proof. By Theorem 10 and the idempotency of A^DA we have

$$A^{\ominus}_{j,k} = A^DA(A^j)^{\dagger} = (A^D)^jA^j(A^j)^{\dagger}. \tag{4.2}$$

From the proof of Theorem 7, we have

$$A^j(A^j)^{\dagger} = U\begin{bmatrix} (MC)^{j-1}((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{4.3}$$

By (2.2) we have

$$(A^D)^j = U\begin{bmatrix} [(MC)^D]^j & [(MC)^D]^{j+1}MS \\ 0 & 0 \end{bmatrix}U^*. \tag{4.4}$$

By the proof of Theorem 7, we have $\text{ind}(MC) \le k-1 < k$. From (4.2), (4.3) and (4.4), we have

$$\begin{aligned} A^{\ominus}_{j,k} &= (A^D)^jA^j(A^j)^{\dagger} \\ &= U\begin{bmatrix} [(MC)^D]^j & [(MC)^D]^{j+1}MS \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (MC)^{j-1}((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} [(MC)^D]^j(MC)^{j-1}((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (MC)^D[(MC)^D]^{j-1}(MC)^{j-1}((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (MC)^D(MC)^DMC((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}U^* \\ &= U\begin{bmatrix} (MC)^D(MC)^{\ominus}_{j-1,k} & 0 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned}$$

□

Remark 11. If we use the decomposition of Hartwig and Spindelböck in [8, Corollary 6], then an expression of the (j,k)-core inverse of A is

$$A^{\ominus}_{j,k} = U\begin{bmatrix} (\Sigma K)^D(\Sigma K)^{\ominus}_{j-1,k} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

which is similar to the expression of $A^{\ominus}_{j,k}$ in Theorem 13. Since the proof of this result can be proved like the proof of Theorem 13, we omit this proof.

Theorem 14. *Let $A \in \mathbb{C}^{n\times n}$ and $\operatorname{ind}(A) = k$. If $(A^k X^k)^* = A^k X^k$, $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$, then A is (k,k)-core invertible and $A^{\ominus}_{k,k} = X^k$.*

Proof. By Lemma 2 and Lemma 3, we have $A^k X^k A^k = A^k$, $X^k A^k X^k = X^k$, $A^k = X^k A^{2k}$, and $A^D = X^{k+1} A^k$. Equalities $(A^k X^k)^* = A^k X^k$ and $A^k X^k A^k = A^k$ imply that X^k is a $\{1,3\}$-inverse of A^k. From $A^D = X^{k+1} A^k$, we can obtain $(A^D)^k = X^{k-1} A^D$ by induction. Thus

$$\begin{aligned} A^{\ominus}_{k,k} &= A^D A (A^k)^{\dagger} = (A^D)^k A^k (A^k)^{\dagger} = (A^D)^k A^k (A^k)^{(1,3)} \\ &= (A^D)^k A^k X^k = (X^{k+1} A^k)^k A^k X^k = X^{k-1} X^{k+1} A^k A^k X^k \\ &= X^{2k} A^{2k} X^k = X^k (X^k A^{2k}) X^k = X^k A^k X^k = X^k. \end{aligned}$$

□

Proposition 3. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with $j \geq \operatorname{ind}(A) = k$. If A is (j,k)-core invertible, then $A^j A^{\ominus}_{j,k}$ is the projector onto $\mathcal{R}(A^j)$ along $\mathcal{R}(A^j)^{\perp}$.*

Proof. It is trivial. □

In the following proposition, we will investigate some properties of the (j,k)-core inverse.

Proposition 4. *Let $A \in \mathbb{C}^{n\times n}$ with $j \geq \operatorname{ind}(A) = k$. If A is (j,k)-core invertible, then*

(1) *$A^{\ominus}_{j,k}$ is a $\{1,2,3\}$-inverse of A^j;*
(2) *$A^{\ominus}_{j,k} = (A^D)^j P_{A^j}$;*
(3) *$(A^{\ominus}_{j,k})^n = \begin{cases} \left[(A^D)^j (A^j)^{\dagger}\right]^{n/2} & \text{if } n \text{ is even}, \\ A^j \left[(A^D)^j (A^j)^{\dagger}\right]^{(n+1)/2} & \text{if } n \text{ is odd}. \end{cases}$*
(4) *$A^{\ominus}_{j,k} A^D = (A^D)^{j+1}$;*
(5) *$A^j A^{\ominus}_{j,k} = A^{\ominus}_{j,k} A^j$ if and only if $\mathcal{R}(A^j)^{\perp} \subseteq \mathcal{N}(A^j)$;*
(6) *$A^{\ominus}_{j,k} = A$ implies that A is EP.*

Proof. (1). By Theorem 10 we have $A^{\ominus}_{j,k} = A^D A (A^j)^{\dagger} = (A^D)^j A^j (A^j)^{\dagger}$, thus

$$\begin{aligned} A^j A^{\ominus}_{j,k} A^j &= A^j (A^D)^j A^j (A^j)^{\dagger} A^j = A^j (A^D)^j A^j = A^j A^D A = A^j; \\ A^{\ominus}_{j,k} A^j A^{\ominus}_{j,k} &= (A^D)^j A^j (A^j)^{\dagger} A^j A^{\ominus}_{j,k} = A^D A A^{\ominus}_{j,k} \\ &= A^D A A^D A (A^j)^{\dagger} = A^D A (A^j)^{\dagger} = A^{\ominus}_{j,k}; \\ A^j A^{\ominus}_{j,k} &= A^j (A^D)^j A^j (A^j)^{\dagger} = A^j (A^j)^{\dagger}. \end{aligned}$$

(2) is trivial.

(3). By $(A^{\circleddash}_{j,k})^2 = (A^D)^j A^j (A^j)^\dagger (A^D)^j A^j (A^j)^\dagger = (A^D)^j (A^j)^\dagger$ and induction it is easy to check (3).

(4). $A^{\circleddash}_{j,k} A^D = (A^D)^j A^j (A^j)^\dagger A^D = (A^D)^j A^j (A^j)^\dagger (A^D)^j A^j A^D = (A^D)^{j+1}$.

(5). By $\mathcal{R}\left[I_n - A^j (A^j)^\dagger\right] = \mathcal{N}\left[(A^j)^\dagger\right]$ and $\mathcal{N}(A^D A) = \mathcal{N}(A^D)$, we have

$$\begin{aligned} A^j A^{\circleddash}_{j,k} = A^{\circleddash}_{j,k} A^j &\Leftrightarrow A^j (A^D)^j A^j (A^j)^\dagger = (A^D)^j A^j (A^j)^\dagger A^j \\ &\Leftrightarrow A^j (A^D)^j A^j (A^j)^\dagger = (A^D)^j A^j \\ &\Leftrightarrow A^j (A^D)^j \left[I_n - A^j (A^j)^\dagger\right] = 0 \\ &\Leftrightarrow \mathcal{R}\left[I_n - A^j (A^j)^\dagger\right] \subseteq \mathcal{N}(A^D A) \\ &\Leftrightarrow \mathcal{N}[(A^j)^\dagger] \subseteq \mathcal{N}(A^D A) \\ &\Leftrightarrow \mathcal{N}[(A^j)^*] \subseteq \mathcal{N}(A^D) \\ &\Leftrightarrow \mathcal{R}(A^j)^\perp \subseteq \mathcal{N}(A^j). \end{aligned}$$

(6). Let A be written in the form (2.1). We have

$$A^{\circleddash}_{j,k} = U \begin{bmatrix} (MC)^D (MC)^{\circleddash}_{j-1,k} & 0 \\ 0 & 0 \end{bmatrix} U^*$$

by Theorem 13. Thus, $A^{\circleddash}_{j,k} = A$ implies $MS = 0$. From the nonsingularity of M, we have $S = 0$, which is equivalent to say that A is EP in view of [2, Theorem 3.7]. □

In the following proposition, we shall give the the relationship between the (j,k)-core inverse and DMP-inverse and core-EP inverse.

Proposition 5. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then*

$$A^{\circleddash}_{k,k} = A^{D,\dagger} (A^D)^{k-1} A A^{\textcircled{\dagger}}.$$

Proof. We have that $A^k (A^k)^\dagger = A A^{\textcircled{\dagger}}$ by Lemma 5 and $A^{D,\dagger} = A^D A A^\dagger$. Thus

$$\begin{aligned} A^{\circleddash}_{k,k} &= A^D A (A^k)^\dagger = (A^D)^k A^k (A^k)^\dagger = A^D A^k (A^D)^{k-1} (A^k)^\dagger \\ &= A^D A A^\dagger A^k (A^D)^{k-1} (A^k)^\dagger = A^{D,\dagger} (A^D)^{k-1} A^k (A^k)^\dagger \\ &= A^{D,\dagger} (A^D)^{k-1} A A^{\textcircled{\dagger}}. \end{aligned}$$

□

In the following theorem, we will give a relationship between the $\langle i,m\rangle$-core inverse and (j,m)-core inverse.

Theorem 15. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then $A^{\oplus}_{k,m} = A^{\circleddash}_{m,k}$ for any $m \geq k$.*

Proof. By Theorem 10, we have $A^{\ominus}_{m,k} = A^D A(A^m)^\dagger = (A^D)^k A^k (A^m)^\dagger$. By the proof of Remark 5, we have $A^k = MN$ and $NM = L^k$, where $M = \prod_{l=1}^{k} B_l$, $N = \prod_{l=1}^{k} G_{k+1-l}$ and $L = G_k B_k$. It is easy to see that $(A^D)^s = ML^{-k-s}N$ for any $s \in \mathbb{N}$ by $NM = L^k$. Thus $(A^D)^k = ML^{-2k}N$ and

$$(A^D)^k A^k = ML^{-2k}NMN = ML^{-2k}L^kN = ML^{-k}N.$$

By the proof of Remark 5, we have $A^m = ML^{m-k}N = M_1N$ is a full rank factorization of A^m, where $M_1 = ML^{m-k}$ and

$$(A^m)^\dagger = N^*(NN^*)^{-1}(M_1^*M_1)^{-1}(M_1)^*.$$

By Theorem 6, we have $A^{\oplus}_{k,m} = ML^{-m}M^\dagger$. In the following steps, we will show that $A^{\ominus}_{m,k} = ML^{-m}M^\dagger$. From $A^{\ominus}_{m,k} = (A^D)^k A^k (A^m)^\dagger$, we have

$$\begin{aligned}
A^{\ominus}_{k,m} &= (A^D)^k A^k (A^m)^\dagger = ML^{-k}NN^*(NN^*)^{-1}(M_1^*M_1)^{-1}(M_1)^* \\
&= ML^{-k}(M_1^*M_1)^{-1}(M_1)^* = ML^{-k}[(L^{m-k})^*M^*ML^{m-k}]^{-1}(L^{m-k})^*M^* \\
&= ML^{-k}L^{k-m}(M^*M)^{-1}[(L^{m-k})^*]^{-1}(L^{m-k})^*M^* \\
&= ML^{-m}(M^*M)^{-1}M^* = ML^{-m}M^\dagger.
\end{aligned}$$

□

Theorem 16. *Let $A \in \mathbb{C}^{n\times n}$ with $i \geq \operatorname{ind}(A) = k$, then $A^{\ominus}_{i,k} = P_1 D^{-i} P_1^\dagger$, where $A = P\left[\begin{smallmatrix} D & 0 \\ 0 & N\end{smallmatrix}\right]P^{-1}$ with $D \in \mathbb{C}^{r\times r}$ is nonsingular, N is nilpotent and $P = [P_1 \mid P_2]$ with $P_1 \in \mathbb{C}^{n\times r}$.*

Proof. It is easy to see that by Theorem 8 and Theorem 15. □

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