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BASIC AND FRACTIONAL q-CALCULUS AND ASSOCIATED FEKETE-SZEGŐ PROBLEM FOR p-VALENTLY q-STARLIKE FUNCTIONS AND p-VALENTLY q-CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract. In this paper, we introduce and study some subclasses of p-valently analytic functions in the open unit disk $\mathbb{U}$ by applying the q-derivative operator and the fractional q-derivative operator in conjunction with the principle of subordination between analytic functions. For the Taylor-Maclaurin coefficients $\{a_k\}_{k=p+1}^{\infty}$ of each of these subclasses of p-valently analytic functions, we derive sharp bounds for the Fekete-Szegő functional given by

$$\left|a_{p+2}-\mu a_{p+1}^2\right|.$$

Relevant connections of the results presented in this paper with those derived in earlier works are also considered.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The theory of the basic and the fractional quantum calculus (that is, the basic q-calculus and the fractional q-calculus) plays important roles in many diverse areas of the mathematical, physical and engineering sciences (see, for example, [5, 9, 16] and [21]). Our main objective in this paper is to introduce and study some subclasses of p-valently analytic functions in the open unit disk $\mathbb{U}$ by applying the q-derivative operator and the fractional q-derivative operator in conjunction with the principle of subordination between analytic functions (see, for details, [12]).

We begin by denoting by $\mathcal{A}(p)$ the class of functions $f(z)$ of the form:

$$f(z)=z^p+\sum_{k=p+1}^{\infty} a_k z^k \quad (p\in\mathbb{N}=\{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In particular, we write

$$\mathcal{A}(1) = \mathcal{A}.$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of p-valently starlike of order α in $\mathbb{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (0 \leqq \alpha < p;\ z \in \mathbb{U}). \tag{1.2}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{C}_p(\alpha)$ of p-valently convex of order α in $\mathbb{U}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (0 \leqq \alpha < p;\ z \in \mathbb{U}). \tag{1.3}$$

The p-valent function classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha)$ were studied by Owa [13]. From (1.2) and (1.3), it follows that

$$f(z) \in \mathcal{C}_p(\alpha) \iff \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha). \tag{1.4}$$

We now recall some basic definitions and concept details of the q-calculus which are used in this paper (see, for details, [7] and [8]; see also [5] and [21]).

Definition 1. Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ and the q-factorial $[n]_q!$ by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \displaystyle\sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}) \end{cases} \tag{1.5}$$

and

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \displaystyle\prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}), \end{cases} \tag{1.6}$$

respectively.

Definition 2. The q-derivative (or the q-difference) $D_q f(z)$ of a function $f(z)$ is defined in a given subset of $\mathbb{C}$ by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{1.7}$$

provided that $f'(0)$ exists.

We note from Definition 2 that

$$\lim_{q \to 1-} D_q f(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z) \tag{1.8}$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1.1) and (1.7) that

$$D_q f(z) = [p]_q \, z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q \, a_k z^{k-1} \; (z \neq 0), \tag{1.9}$$

where $[\lambda]_q$ is given by (1.5) and the function $f(z) \in \mathcal{A}(p)$ is given by (1.1).

Making use of the q-derivative operator D_q given by (1.7), we introduce the subclass $\mathcal{S}_q^*(p,\alpha)$ of p-valently q-starlike functions of order α in $\mathbb{U}$ and the subclass $\mathcal{C}_q(p,\alpha)$ of p-valently q-convex functions of order α in $\mathbb{U}$ as follows:

$$f(z) \in \mathcal{S}_q^*(p,\alpha) \iff \Re\left(\frac{1}{[p]_q} \frac{z D_q f(z)}{f(z)}\right) > \alpha \tag{1.10}$$
$$(0 < q < 1;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U})$$

and

$$f(z) \in \mathcal{C}_q(p,\alpha) \iff \Re\left(\frac{1}{[p]_q} \frac{D_q\big(z D_q f(z)\big)}{D_q f(z)}\right) > \alpha \tag{1.11}$$
$$(0 < q < 1;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U}),$$

respectively. From (1.10) and (1.11), it follows that

$$f(z) \in \mathcal{C}_q(p,\alpha) \iff \frac{z D_q f(z)}{[p]_q} \in \mathcal{S}_q^*(p,\alpha). \tag{1.12}$$

We note also that

$$\lim_{q \to 1-} \mathcal{S}_q^*(p,\alpha) = \mathcal{S}_p^*(\alpha) \qquad \text{and} \qquad \lim_{q \to 1-} \mathcal{C}_q(p,\alpha) = \mathcal{C}_p(\alpha).$$

We next introduce the familiar principle of subordination between analytic functions. Indeed, for two given functions $f(z)$ and $g(z)$ which are analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to the function $g(z)$ and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1$$

such that

$$f(z) = g\big(w(z)\big).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then it follows that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Throughout our present investigation, let the function $\varphi(z)$ be analytic with positive real part in $\mathbb{U}$ and satisfy the following conditions:

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi'(0) > 0,$$

which maps $\mathbb{U}$ onto a region which is starlike with respect to 1 and symmetric with respect to the real axis. Suppose now that $\mathcal{S}^*_{b,p}(\varphi)$ denotes the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\,\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z) \qquad (b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\};\ z \in \mathbb{U}). \tag{1.13}$$

Also let $\mathcal{C}_{b,p}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \qquad (b \in \mathbb{C}^*;\ z \in \mathbb{U}). \tag{1.14}$$

The p-valent function classes $\mathcal{S}^*_{b,p}(\varphi)$ and $\mathcal{C}_{b,p}(\varphi)$ were introduced and studied by Ali *et al.* [2]. We note that each of the following function classes:

$$\mathcal{S}^*_{1,1}(\varphi) =: \mathcal{S}^*(\varphi) \qquad \text{and} \qquad \mathcal{C}_{1,1}(\varphi) =: \mathcal{C}(\varphi)$$

was introduced and studied by Ma and Minda [11]. In fact, the widely-investigated function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are the special cases of $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$, respectively, when

$$\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}).$$

Finally, for $b \in \mathbb{C}^*$, $0 < q < 1$, $0 \leqq \lambda \leqq 1$ and $p \in \mathbb{N}$, we define the subclass $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ of the p-valently analytic function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1) and satisfying the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_q f(z) + \lambda q z^2 D_q\big(D_q f(z)\big)}{(1-\lambda) f(z) + \lambda z D_q f(z)} - 1\right) \prec \varphi(z), \tag{1.15}$$

where D_q denotes the q-derivative operator given by Definition 2.

The following function classes are included in the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ of p-valently q-starlike functions of *complex* order b in $\mathbb{U}$ (see also [1]):

(i) $\lim_{q\to 1-} \mathcal{S}_{0,q,b}(p,\varphi) =: \mathcal{S}^{*}_{b,p}(\varphi)$ and $\lim_{q\to 1-} \mathcal{S}_{1,q,b}(p,\varphi) =: \mathcal{C}_{b,p}(\varphi)$
(see Ali *et al.* [2]);

(ii) $\lim_{q\to 1-} \mathcal{S}_{\lambda,q,b}(p,\varphi) =: \mathcal{S}_{\lambda,b,p}(\varphi)$
(see Aouf *et al.* [3]) *with*

$$g(z) = \frac{z^p}{1-z};$$

(iii) $\mathcal{S}_{0,q,b}(1,\varphi) =: \mathcal{S}_{q,b}(\varphi)$ and $\mathcal{S}_{1,q,b}(1,\varphi) =: \mathcal{C}_{q,b}(\varphi)$
(see Seoudy and Aouf [19]);

(iv) $\lim_{q\to 1-} \mathcal{S}_{0,q,b}(1,\varphi) =: \mathcal{S}^{*}_{b}(\varphi)$ and $\lim_{q\to 1-} \mathcal{S}_{1,q,b}(1,\varphi) =: \mathcal{C}_{b}(\varphi)$
(see Ravichandran *et al.* [17]).

We also note here that
(i) $\mathcal{S}_{\lambda,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha}\cos\alpha}(p,\varphi) = \mathcal{S}^{\alpha,\gamma}_{q,p,\lambda}(\varphi)$

$$= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{zD_qf(z)+\lambda qz^2 D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}\right) - \gamma\cos\alpha - i\,[p]_q\,\sin\alpha}{([p]_q-\gamma)\cos\alpha} \prec \varphi(z) \right.$$

$$\left. \left(|\alpha| < \frac{\pi}{2};\ 0 \leqq \gamma < [p]_q\,;\ 0 \leqq \lambda \leqq 1\right) \right\};$$

(ii) $\mathcal{S}_{0,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha}\cos\alpha}(p,\varphi) = \mathcal{S}^{\alpha}_{q,p,\gamma}(\varphi)$

$$= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{zD_qf(z)}{f(z)}\right) - \gamma\cos\alpha - i\,[p]_q\sin\alpha}{([p]_q-\gamma)\cos\alpha} \prec \varphi(z) \right.$$

$$\left. \left(|\alpha| < \frac{\pi}{2};\ 0 \leqq \gamma < [p]_q\right) \right\};$$

(iii) $\mathcal{S}_{0,q,\left(1-\frac{\gamma}{[p]_q}\right)e^{-i\alpha}\cos\alpha}(p,\varphi) = \mathcal{C}^{\alpha}_{q,p,\gamma}(\varphi)$

$$= \left\{ f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\right) - \gamma\cos\alpha - i\,[p]_q\sin\alpha}{([p]_q-\gamma)\cos\alpha} \prec \varphi(z) \right.$$

$$\left. \left(|\alpha| < \frac{\pi}{2};\ 0 \leqq \gamma < [p]_q\right) \right\}.$$

In Geometric Function Theory, various subclasses of the normalized analytic function class $\mathcal{A}$ as well as the normalized p-valently analytic function class $\mathcal{A}(p)$ have been studied from different viewpoints. The above-defined q-calculus provides important tools that have been used in order to investigate various subclasses of $\mathcal{A}$ and $\mathcal{A}(p)$. Historically speaking, even though the q-derivative operator D_q was first applied by Ismail *et al.* [6] to study a q-extension of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [20, pp. 347 *et seq.*]).

2. Fekete-Szegő problem for the function class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$

Let Ω be the class of functions $w(z)$ of the form:

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \qquad (z \in \mathbb{U}), \tag{2.1}$$

which satisfy the following inequality:

$$|w(z)| < 1 \qquad (z \in \mathbb{U}).$$

Each of the following lemmas will be needed in our present investigation of the Fekete-Szegő problem for the function class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ which we have introduced by using the subordination condition (1.15) (see, for example, [4]; see also [22] and [24]).

Lemma 1 ([10])**.** *Let the function $w(z) \in \Omega$ be given by (2.1). Then*

$$\left|w_2 - \tau w_1^2\right| \leqq \max\{1, |\tau|\} \qquad (\tau \in \mathbb{C}).$$

The result is sharp for the function given by

$$w(z) = z \qquad or \qquad w(z) = z^2 \qquad (z \in \mathbb{U}).$$

Lemma 2 ([2], [11])**.** *Let the function $w(z) \in \Omega$ be given by (2.1). Then*

$$\left|w_2 - \kappa w_1^2\right| \leqq \begin{cases} -\kappa & (\kappa \leqq -1) \\ 1 & (-1 \leqq \kappa \leqq 1) \\ \kappa & (\kappa \geqq 1). \end{cases} \tag{2.2}$$

For $\kappa < -1$ or $\kappa > 1$, the equality holds true in (2.2) if and only if $w(z) = z$ or one of its rotations. If $-1 < \kappa < 1$, then the equality holds true in (2.2) if and only if $w(z) = z^2$ or one of its rotations. If $\kappa = -1$, then the equality holds true in (2.2) if and only if

$$w(z) = \frac{z(z+\eta)}{1+\eta z} \qquad (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $\kappa = 1$, *then the equality holds true in* (2.2) *if and only if*

$$w(z) = -\frac{z(z+\eta)}{1+\eta z} \qquad (0 \leqq \eta \leqq 1)$$

or one of its rotations. The upper bound in (2.2) *is sharp and it can be improved as follows when* $-1 < \kappa < 1$:

$$\left|w_2 - \kappa w_1^2\right| + (\kappa+1)\left|w_1\right|^2 \leqq 1 \qquad (-1 < \kappa \leqq 0)$$

and

$$\left|w_2 - \kappa w_1^2\right| + (1-\kappa)\left|w_1\right|^2 \leqq 1 \qquad (0 < \kappa < 1).$$

Lemma 3 ([14]). *Let the function* $w(z) \in \Omega$ *be given by* (2.1). *Then, for any real numbers* q_1 *and* q_2, *the following sharp estimates hold true*:

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \leqq H(q_1, q_2), \tag{2.3}$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \left((q_1,q_2) \in D_1 \cup D_2\right) \\ |q_2| & \left((q_1,q_2) \in \bigcup_{k=3}^{7} D_k\right) \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \left((q_1,q_2) \in D_8 \cup D_9\right) \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \left((q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}\right) \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \left((q_1,q_2) \in D_{12}\right). \end{cases}$$

The extremal functions, up to rotations, are of the form given by

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{z\left[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1\right]z - \varepsilon_1\varepsilon_2 z}{1 - \left[(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2\right]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \quad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z}, \quad |\varepsilon_1| = |\varepsilon_2| = 1,$$

$$\varepsilon_1 = t_0 - e^{-i\left(\frac{\theta_0}{2}\right)}(a \mp b), \quad \varepsilon_2 = -e^{-i\left(\frac{\theta_0}{2}\right)}(ia \pm b),$$

$$a = t_0 \cos\left(\frac{\theta_0}{2}\right), \quad b = \sqrt{1 - t_0^2 \sin^2\left(\frac{\theta_0}{2}\right)}, \quad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left(\frac{2q_2(q_1^2+2)-3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right)^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} \text{ and } \cos\left(\frac{\theta_0}{2}\right) = \frac{q_1}{2}\left(\frac{q_2(q_1^2+8)-2(q_1^2+2)}{2q_2(q_1^2+2)-3q_1^2}\right).$$

The sets D_k $(k = 1, \cdots, 12)$ are defined as follows:

$$D_1 := \left\{(q_1, q_2) : |q_1| \leqq \frac{1}{2} \text{ and } |q_2| \leqq 1\right\},$$

$$D_2 := \left\{(q_1, q_2) : \frac{1}{2} \leqq |q_1| \leqq 2 \text{ and } \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \leqq q_2 \leqq 1\right\},$$

$$D_3 := \left\{(q_1, q_2) : |q_1| \leqq \frac{1}{2} \text{ and } q_2 \leqq -1\right\},$$

$$D_4 := \left\{(q_1, q_2) : |q_1| \geqq \frac{1}{2} \text{ and } q_2 \leqq -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5 := \{(q_1, q_2) : |q_1| \leqq 2 \text{ and } q_2 \geqq 1\},$$

$$D_6 := \left\{(q_1, q_2) : 2 \leqq |q_1| \leqq 4 \text{ and } q_2 \geqq \frac{1}{12}(q_1^2+8)\right\},$$

$$D_7 := \left\{(q_1, q_2) : |q_1| \geqq 4 \text{ and } q_2 \geqq \frac{2}{3}(|q_1|-1)\right\},$$

$$D_8 := \left\{(q_1, q_2) : \frac{1}{2} \leqq |q_1| \leqq 2 \text{ and } -\frac{2}{3}(|q_1|+1) \leqq q_2 \leqq \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\right\},$$

$$D_9 := \left\{(q_1, q_2) : |q_1| \geqq 2 \text{ and } -\frac{2}{3}(|q_1|+1) \leqq q_2 \leqq \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10} := \left\{(q_1, q_2) : 2 \leqq |q_1| \leqq 4 \text{ and } \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqq q_2 \leqq \frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11} := \left\{(q_1, q_2) : |q_1| \geqq 4 \text{ and } \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqq q_2 \leqq \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\}$$

and

$$D_{12} := \left\{ (q_1, q_2) : |q_1| \geqq 4 \ \text{and} \ \frac{2|q_1|(|q_1|-1)}{q_1^2 - 2|q_1| + 4} \leqq q_2 \leqq \frac{2}{3}(|q_1|-1) \right\}.$$

Remark 1. Unless otherwise mentioned, we assume throughout this paper that

$$b \in \mathbb{C}^*, \quad 0 \leqq \lambda \leqq 1, \quad 0 < q < 1 \qquad \text{and} \qquad p \in \mathbb{N}.$$

Theorem 1. *Let the function $\varphi(z)$ be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 > 0).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ and $\mu \in \mathbb{C}$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqq \frac{[p]_q B_1 |b|}{[p+2]_q - [p]_q} \left(\frac{1 + \lambda\big([p]_q - 1\big)}{1 + \lambda\big([p+2]_q - 1\big)} \right)$$
$$\cdot \max\left\{ 1, \left| \frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right.\right.$$
$$\left.\left. \cdot \left(1 - \mu \frac{[p+2]_q - [p]_q}{[p+1]_q - [p]_q} \frac{\big[1 + \lambda\big([p+2]_q - 1\big)\big]\big[1 + \lambda\big([p]_q - 1\big)\big]}{\big[1 + \lambda\big([p+1]_q - 1\big)\big]^2} \right) \right| \right\} \tag{2.4}$$

and

$$\left|a_{p+3}\right| \leqq \frac{[p]_q \ B_1 |b|}{[p+3]_q - [p]_q} \left(\frac{1 + \lambda\big([p]_q - 1\big)}{1 + \lambda\big([p+3]_q - 1\big)} \right) H(q_1, q_2), \tag{2.5}$$

where

$$q_1 = \frac{2B_2}{B_1} - \frac{[p]_q \big(2[p]_q - [p+1]_q - [p+2]_q\big) B_1 b}{\big([p+1]_q - [p]_q\big)\big([p+2]_q - [p]_q\big)} \tag{2.6}$$

and

$$q_2 = \frac{B_3}{B_1} - \left(\frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right)^2 - \frac{[p]_q \big(2[p]_q - [p+1]_q - [p+2]_q\big) B_1 b}{\big([p+1]_q - [p]_q\big)\big([p+2]_q - [p]_q\big)}$$
$$\cdot \left(\frac{B_2}{B_1} + \frac{[p]_q B_1 b}{[p+1]_q - [p]_q} \right). \tag{2.7}$$

The result is sharp.

Proof. If $f(z) \in \mathcal{S}_{\lambda,q,b}(p,\varphi)$, then there is a Schwarz function $w(z) \in \Omega$ given by

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$$

such that

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right)=\varphi\big(w(z)\big).$$

Since

$$\begin{aligned}
&\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}\\
&=1+\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}z\\
&+\left[\frac{1+\lambda\left([p+2]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(\frac{[p+2]_q}{[p]_q}-1\right)a_{p+2}-\left(\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\right)^2\right.\\
&\left.\cdot\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}^2\right]z^2+\left[\frac{1+\lambda\left([p+3]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(\frac{[p+3]_q}{[p]_q}-1\right)a_{p+3}\right.\\
&+\left(\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\right)^3\left(\frac{[p+1]_q}{[p]_q}-1\right)a_{p+1}^3+\frac{1+\lambda\left([p+1]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\\
&\left.\cdot\frac{1+\lambda\left([p+2]_q-1\right)}{1+\lambda\left([p]_q-1\right)}\left(2-\frac{[p+1]_q}{[p]_q}-\frac{[p+2]_q}{[p]_q}\right)a_{p+1}a_{p+2}\right]z^3+\cdots
\end{aligned}$$

and

$$\varphi\big(w(z)\big)=1+w_1B_1z+(w_1^2B_2+w_2B_1)z^2+(w_3B_1+w_1^3B_3+2w_1w_2B_2)z^3+\cdots,$$

we observe that

$$a_{p+1}=\frac{[p]_q\,B_1bw_1}{[p+1]_q-[p]_q}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+1]_q-1\right)}\right), \tag{2.8}$$

$$\begin{aligned}
a_{p+2}=&\frac{[p]_q\,B_1b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\right)\\
&\cdot\left[w_2+w_1^2\left(\frac{B_2}{B_1}+\frac{[p]_q\,B_1b}{[p+1]_q-[p]_q}\right)\right]
\end{aligned} \tag{2.9}$$

and

$$a_{p+3}=\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+3]_q-1\right)}\frac{[p]_q\,B_1b}{[p+3]_q-[p]_q}\left\{w_3+\left[\frac{2B_2}{B_1}\right.\right.$$

$$-\frac{[p]_q\left(2[p]_q-[p+1]_q-[p+2]_q\right)B_1b}{\left([p+1]_q-[p]_q\right)\left([p+2]_q-[p]_q\right)}\Bigg]w_1w_2+\left[\frac{B_3}{B_1}-\left(\frac{[p]_q B_1 b}{[p+1]_q-[p]_q}\right)^2\right.$$
$$\left.-\frac{[p]_q\left(2[p]_q-[p+1]_q-[p+2]_q\right)B_1b}{\left([p+1]_q-[p]_q\right)\left([p+2]_q-[p]_q\right)}\left(\frac{B_2}{B_1}+\frac{[p]_q B_1 b}{[p+1]_q-[p]_q}\right)\right]w_1^3\Bigg\}. \quad (2.10)$$

Therefore, we have

$$a_{p+2}-\mu a_{p+1}^2=\frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\left[w_2-\upsilon w_1^2\right],$$

where

$$\upsilon=\frac{[p]_q B_1|b|}{[p+1]_q-[p]_q}\cdot\left(\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}-1\right)-\frac{B_2}{B_1}. \quad (2.11)$$

The result (2.4) follows by an application of Lemma 1 and the result (2.5) follows by an application of Lemma 3. These results are sharp for the functions given by

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right)=\varphi(z^2)$$

and

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)+\lambda qz^2D_q(D_qf(z))}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right)=\varphi(z),$$

respectively. This completes the proof of Theorem 1. □

By using Lemma 2, we can obtain the following theorem.

Theorem 2. *Let $b>0$ and let the function $\varphi(z)$ be given by*

$$\varphi(z)=1+B_1z+B_2z^2+\cdots(B_k>0;\ k\in\{1,2\}).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$ and $\mu\in\mathbb{R}$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leqq$$

$$\begin{cases} \frac{[p]_q b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\right) \\ \cdot\left\{B_2+\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left(1-\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right)\right\} \\ (\mu \leqq \sigma_1) \\ \frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\right) \\ (\sigma_1 \leqq \mu \leqq \sigma_2) \\ \frac{[p]_q b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\right)\cdot \\ \cdot\left\{-B_2+\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left(\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}-1\right)\right\} \\ (\mu \geqq \sigma_2), \end{cases}$$

where

$$\sigma_1 = \frac{\left([p+1]_q-[p]_q\right)\left[(B_2-B_1)\left([p+1]_q-[p]_q\right)+[p]_q B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b}$$

and

$$\sigma_2 = \frac{\left([p+1]_q-[p]_q\right)\left[(B_2+B_1)\left([p+1]_q-[p]_q\right)+[p]_q B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b}.$$

The result is sharp.

Further, if we set

$$\sigma_3 = \frac{\left([p+1]_q-[p]_q\right)\left[\left([p+1]_q-[p]_q\right)B_2+[p]_q B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b},$$

then each of the following assertions holds true:
(i) *If* $\sigma_1 \leqq \mu \leqq \sigma_3$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2 b}\,\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}$$
$$\cdot\left\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left[1-\mu\,\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right.\right.$$
$$\left.\left.\cdot\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\right]\right\}\left|a_{p+1}\right|^2 \leqq \frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda\,[p]_q-1}{1+\lambda\left([p+2]_q-1\right)}\right).$$

(ii) *If* $\sigma_3 \leqq \mu \leqq \sigma_2$, *then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right|+\frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2b}\,\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}$$
$$\cdot\left\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\left[\mu\,\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right.\right.$$
$$\left.\left.\cdot\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}-1\right]\right\}\left|a_{p+1}\right|^2\leqq\frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\left(\frac{1+\lambda[p]_q-1}{1+\lambda\left([p+2]_q-1\right)}\right).$$

Proof. By applying Lemma 2 to (2.8), (2.9) and (2.11), we obtain the required results asserted by Theorem 2. In order to show that the bounds are sharp, we define the functions $K_{\varphi n}$ $(n \in \mathbb{N}\setminus\{1\})$ by

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_qK_{\varphi n}(z)+\lambda qz^2D_q(D_qK_{\varphi n}(z))}{(1-\lambda)K_{\varphi n}(z)+\lambda zD_qK_{\varphi n}(z)}-1\right)=\varphi(z^{n-1})$$
$$\left(z^{1-p}\,K_{\varphi n}(z)\big|_{z=0}=0=z^{1-p}\,K'_{\varphi n}(z)\big|_{z=0}-p\right)$$

and the functions F_β and G_β $(0 \leqq \beta \leqq 1)$ by

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_qF_\beta(z)+\lambda qz^2D_q(D_qF_\beta(z))}{(1-\lambda)F_\beta(z)+\lambda zD_qF_\beta(z)}-1\right)=\varphi\left(\frac{z(z+\beta)}{1+\beta z}\right)$$
$$\left(z^{1-p}\,F_\beta(z)\big|_{z=0}=0=z^{1-p}\,F'_\beta(z)\big|_{z=0}-p\right)$$

and

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\,\frac{zD_qG_\beta(z)+\lambda qz^2D_q(D_qG_\beta(z))}{(1-\lambda)G_\beta(z)+\lambda zD_qG_\beta(z)}-1\right)=\varphi\left(-\frac{z(z+\beta)}{1+\beta z}\right).$$
$$\left(z^{1-p}\,G_\beta(z)\big|_{z=0}=0=z^{1-p}\,G'_\beta(z)\big|_{z=0}-p\right).$$

Clearly, the functions $K_{\varphi n}$, F_β and G_β are in the class $\mathcal{S}_{\lambda,q,b}(p,\varphi)$. We also write

$$K_\varphi = K_{\varphi 2}.$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds true if and only if the function f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds true if the function f is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds true if and only if the function f is F_β or one of its rotations. If $\mu = \sigma_2$, then the equality holds true if and only if the function f is G_β or one of its rotations. This completes the proof of Theorem 2. □

Remark 2. For different choices of the parameters q, b and λ in Theorem 1 and Theorem 2, we can deduce the corresponding results derived earlier by Ali *et al.* [2], Aouf *et al.* [3] and Seoudy and Aouf [19].

3. Applications to functions defined by the fractional q-derivative operator

We first recall some definitions of the *fractional* q-calculus which we will be used in this section.

First of all, for $0 < q < 1$ and $\lambda, \mu \in \mathbb{C}$, the basic (or q-) shifted factorial $(\lambda;q)_\mu$ is defined by (see, for example, [5, 21] and [23])

$$(\lambda;q)_\mu = \prod_{j=0}^{\infty}\left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right) \qquad (0<q<1;\ \lambda,\mu\in\mathbb{C}),$$

so that

$$(\lambda;q)_n := \begin{cases} 1 & (n=0) \\ \prod_{j=0}^{n-1}\left(1-\lambda q^j\right) & (n\in\mathbb{N}) \end{cases} \tag{3.1}$$

and

$$(\lambda;q)_\infty := \prod_{j=0}^{\infty}\left(1-\lambda q^j\right) \qquad (0<q<1;\ \lambda\in\mathbb{C}), \tag{3.2}$$

Furthermore, in terms of the basic (or q-) Gamma function $\Gamma_q(z)$ defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}\,(1-q)^{1-z} \qquad (0<q<1;\ z\in\mathbb{C}), \tag{3.3}$$

so that

$$\lim_{q\to 1-}\{\Gamma_q(z)\} = \Gamma(z)$$

for the familiar (Euler's) Gamma function $\Gamma(z)$, we find from (3.1) that

$$\left(q^\alpha;q\right)_n = \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}\,(1-q)^n \qquad (n\in\mathbb{N};\ \alpha\in\mathbb{C}).$$

For $0 < q < 1$, the (Jackson's) q-integral is defined (in general) by

$$\int_0^x f(t)\,\mathrm{d}_q t = (1-q)x\sum_{k=0}^{\infty} f(xq^k)q^k \qquad (x>0), \tag{3.4}$$

provided that the series on the right-hand side converges absolutely. In the limit case when $q \to 1-$, the q-integral in (3.4) reduces to

$$\int_0^x f(t)\,\mathrm{d}t.$$

In general, for any closed interval $[a,b]$ $(0 \leqq a < b)$, we write

$$\int_a^b f(t)\mathrm{d}_q t = \int_0^b f(t)\mathrm{d}_q t - \int_0^a f(t)\mathrm{d}_q t. \tag{3.5}$$

In order to introduce the subclasses

$$\mathcal{S}^*_{q,b,\delta}(p,\varphi) \qquad \text{and} \qquad \mathcal{C}_{q,b,\delta}(p,\varphi),$$

we need the following definitions.

Definition 3 ([15], [16] and see also the references cited therein)**.** The *fractional* q-integral operator $\mathfrak{I}^{\delta}_{q,z}$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{I}^{\delta}_{q,z} f(z) = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z-tq)^{\delta-1}_q \; f(t)\mathrm{d}_q t \qquad (\delta > 0), \tag{3.6}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the q-binomial $(\kappa+z)^n_q$ is defined by (see, for example, [21, p. 486])

$$(\kappa+z)^{\mu}_q = \sum_{k=0}^{\infty} \begin{bmatrix} \mu \\ k \end{bmatrix}_q q^{\binom{k}{2}} \, \kappa^{\mu-k} \, z^k = \kappa^{\mu} \; {}_1\Phi_0 \left[\begin{matrix} q^{-\mu}; \\ \text{---}; \end{matrix} \; q, -\frac{zq^{\mu}}{\kappa} \right] \tag{3.7}$$

in which the generalized basic (or q-) binomial coefficient $\left[\begin{smallmatrix} \lambda \\ n \end{smallmatrix}\right]_q$ is defined by

$$\begin{bmatrix} \mu \\ n \end{bmatrix}_q = \frac{(q^{-\mu};q)_n}{(q;q)_n} \left(-q^{\mu}\right)^n q^{-\binom{n}{2}} \qquad (0<q<1;\ \mu \in \mathbb{C};\ n \in \mathbb{N}_0) \tag{3.8}$$

and ${}_r\Phi_s$ denotes the generalized q-hypergeometric function with r numerator and s denominator parameters (see, for example, [23, p. 347, Eq. 9.4 (272)]).

Remark 3. It follows from Definition 3 that

$$(z-tq)^{\delta-1}_q = z^{\delta-1} \; {}_1\Phi_0 \left[\begin{matrix} q^{1-\delta}; \\ \text{---}; \end{matrix} \; q, \frac{tq^{\delta}}{z} \right]. \tag{3.9}$$

Since the series

$${}_1\Phi_0 \left[\begin{matrix} \lambda; \\ \text{---}; \end{matrix} \; q, z \right]$$

is single-valued when (see, for details, [5])

$$|\arg(-z)| < \pi \qquad \text{and} \qquad |z| < 1,$$

the q-binomial $(z-tq)^{\delta-1}_q$ in (3.9) is single-valued when

$$\left|\arg\left(-\frac{tq^{\delta}}{z}\right)\right| < \pi, \quad \left|\frac{tq^{\delta}}{z}\right| < 1 \qquad \text{and} \qquad |\arg(z)| < \pi.$$

Definition 4 ([15], [16] and see also the references cited therein)**.** The *fractional* q-derivative operator $\mathfrak{D}_{q,z}^{\delta}$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{D}_{q,z}^{\delta} f(z) = \mathfrak{D}_{q,z}^{\delta}\, \mathfrak{I}_{q,z}^{1-\delta} f(z) = \frac{1}{\Gamma_q(1-\delta)}\, \mathfrak{D}_{q,z} \int_0^z (z-tq)_q^{-\delta} f(t)\, \mathrm{d}_q t \tag{3.10}$$

$$(0 \leqq \delta < 1),$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of the q-binomial $(z-tq)_q^{-\delta}$ is removed as in Definition 3.

Definition 5 ([15], [16] and see also the references cited therein)**.** Under the hypotheses of Definition 4, the *fractional* q-derivative operator $\mathfrak{D}_{q,z}^{\delta}$ of order δ is defined, for a function $f(z)$, by

$$\mathfrak{D}_{q,z}^{\delta} f(z) = \mathfrak{D}_{q,z}^{m}\, \mathfrak{I}_{q,z}^{m-\delta}\, f(z) \qquad (m-1 \leqq \delta < 1;\; m \in \mathbb{N}_0 = \mathbb{N} \setminus \{0\}). \tag{3.11}$$

Definition 6 ([18])**.** In terms of the *fractional* q-derivative operator $\mathfrak{D}_{q,z}^{\delta}$ given by Definition 3.13, the *fractional* q-derivative operator

$$\Omega_{q,p}^{\delta} : \mathcal{A}(p) \to \mathcal{A}(p)$$

is defined as follows:

$$\begin{aligned} \Omega_{q,p}^{\delta} f(z) &= \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)}\, z^{\delta}\, \mathfrak{D}_{q,z}^{\delta}\, f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\, a_k z^k, \end{aligned} \tag{3.12}$$

where the function $f(z) \in \mathcal{A}(p)$ is given by (1.1).

Thus, for $b \in \mathbb{C}^*$, $0 < q < 1$, $0 \leqq \lambda \leqq 1$, $0 \leqq \delta < 1$ and $p \in \mathbb{N}$, we now let $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ be the subclass of the normalized p -valently analytic function class $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1) and satisfying the following subordination condition:

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\, \frac{z D_q\left(\Omega_{q,p}^{\delta} f(z)\right) + \lambda q z^2 D_q^2\left(\Omega_{q,p}^{\delta} f(z)\right)}{(1-\lambda)\Omega_{q,p}^{\delta} f(z) + \lambda z D_q\left(\Omega_{q,p}^{\delta} f(z)\right)} - 1 \right) \prec \varphi(z). \tag{3.13}$$

Remark 4. By using arguments and analysis to those in the proofs of Theorem 1 and Theorem 2, we can analogously derive Theorem 3 and Theorem 4 below. The details involved are being left as an exercise for the interested reader.

Theorem 3. *Let the function $\varphi(z)$ be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 > 0).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ and $\mu \in \mathbb{C}$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqq \frac{[p]_q\, B_1\,|b|}{[p+2]_q-[p]_q}\;\frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}\right)$$
$$\cdot \max\left\{1, \left|\frac{B_2}{B_1}+\frac{[p]_q\, B_1 b}{[p+1]_q-[p]_q}\left(1-\mu\,\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\right.\right.\right.$$
$$\left.\left.\left.\cdot\frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})}\;\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right)\right|\right\}$$

and

$$\left|a_{p+3}\right| \leqq \frac{[p]_q\, B_1\,|b|}{[p+3]_q-[p]_q}\;\frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})(1-q^{p-\delta+3})}{(1-q^{p+1})(1-q^{p+2})(1-q^{p+3})}$$
$$\cdot\left(\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+3]_q-1\right)}\right)H(q_1,q_2),$$

where q_1 and q_2 are defined by (2.6) and (2.7). The result is sharp.

Theorem 4. *Let the function $\varphi(z)$ be given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_k > 0;\ k \in \{1,2\}).$$

If the function $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_{\lambda,q,b,\delta}(p,\varphi)$ and $\mu \in \mathbb{R}$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqq$$

$$\begin{cases} \dfrac{[p]_q\, b}{[p+2]_q-[p]_q}\, \dfrac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\, \dfrac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)} \\ \quad \cdot\left\{B_2+\dfrac{[p]_q\, B_1^2 b}{[p+1]_q-[p]_q}\left(1-\mu\, \dfrac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\, \dfrac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})}\right.\right. \\ \quad \left.\left.\cdot\dfrac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\right)\right\} \\ (\mu \leqq \sigma_1^*) \\ \\ \dfrac{[p]_q\, B_1 b}{[p+2]_q-[p]_q}\, \dfrac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\, \dfrac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)} \\ (\sigma_1^* \leqq \mu \leqq \sigma_2^*) \\ \\ \dfrac{[p]_q\, b}{[p+2]_q-[p]_q}\, \dfrac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\, \dfrac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)} \\ \quad \cdot\left\{B_2+\dfrac{[p]_q\, B_1^2 b}{[p+1]_q-[p]_q}\left(\mu\, \dfrac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\, \dfrac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})}\right.\right. \\ \quad \left.\left.\cdot\dfrac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}-1\right)\right\} \\ (\mu \leqq \sigma_2^*), \end{cases}$$

where

$$\sigma_1^* = \frac{\left([p+1]_q-[p]_q\right)\left[(B_2-B_1)\left([p+1]_q-[p]_q\right)+[p]_q\, B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b} \cdot\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}$$

and

$$\sigma_2^* = \frac{\left([p+1]_q-[p]_q\right)\left[(B_2+B_1)\left([p+1]_q-[p]_q\right)+[p]_q\, B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b} \cdot\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}.$$

The result is sharp.

Further, if we set

$$\sigma_3^* = \frac{\left([p+1]_q-[p]_q\right)\left[\left([p+1]_q-[p]_q\right)B_2+[p]_q\, B_1^2 b\right]\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{[p]_q\left([p+2]_q-[p]_q\right)\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]B_1^2 b} \cdot\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}.$$

then each of the following assertions holds true:

(i) If $\sigma_1^* \leqq \mu \leqq \sigma_3^*$, *then*

$$\begin{aligned}
&\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2 b}\,\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}\\
&\quad\cdot\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}\Bigg\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\\
&\quad\cdot\Bigg(1-\mu\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\,\frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})}\\
&\quad\cdot\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}\Bigg)\Bigg\}\left|a_{p+1}\right|^2\\
&\leqq \frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\,\frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\,\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}
\end{aligned}$$

(ii) If $\sigma_3^* \leqq \mu \leqq \sigma_2^*$, *then*

$$\begin{aligned}
&\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{\left([p+1]_q-[p]_q\right)^2}{[p]_q\left([p+2]_q-[p]_q\right)B_1^2 b}\,\frac{(1-q^{p+1})(1-q^{p-\delta+2})}{(1-q^{p+2})(1-q^{p-\delta+1})}\\
&\quad\cdot\frac{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}\Bigg\{(B_1-B_2)-\frac{[p]_q B_1^2 b}{[p+1]_q-[p]_q}\\
&\quad\cdot\Bigg(\mu\,\frac{[p+2]_q-[p]_q}{[p+1]_q-[p]_q}\,\frac{(1-q^{p+2})(1-q^{p-\delta+1})}{(1-q^{p+1})(1-q^{p-\delta+2})}\\
&\quad\cdot\frac{\left[1+\lambda\left([p+2]_q-1\right)\right]\left[1+\lambda\left([p]_q-1\right)\right]}{\left[1+\lambda\left([p+1]_q-1\right)\right]^2}-1\Bigg)\Bigg\}\left|a_{p+1}\right|^2\\
&\leqq \frac{[p]_q B_1 b}{[p+2]_q-[p]_q}\,\frac{(1-q^{p-\delta+1})(1-q^{p-\delta+2})}{(1-q^{p+1})(1-q^{p+2})}\,\frac{1+\lambda\left([p]_q-1\right)}{1+\lambda\left([p+2]_q-1\right)}.
\end{aligned}$$

Remark 5. For different choices of the parameters p, q and λ in Theorem 3 and Theorem 4, we can obtain new results for each of the following p-valently analytic function classes:

$$\mathcal{S}_{b,p}^*(\varphi),\quad \mathcal{C}_{b,p}(\varphi),\quad \mathcal{S}_{\lambda,b,p}(\varphi),\quad \mathcal{S}_{q,b}(\varphi),\quad \mathcal{C}_{q,b}(\varphi),\quad \mathcal{S}_b^*(\varphi)\quad \text{and}\quad \mathcal{C}_b(\varphi),$$

which are defined in Section 1.

Remark 6. For different choices of the parameters b and λ in Theorem 3 and Theorem 4, we can deduce new results for each of the following p-valently analytic function classes:

$$\mathcal{S}^{\alpha,\gamma}_{q,p,\lambda}(\varphi), \quad \mathcal{S}^{\alpha}_{q,p,\lambda}(\varphi) \quad \text{and} \quad \mathcal{C}^{\alpha}_{q,p,\lambda}(\varphi),$$

which are defined in Section 1.

REFERENCES

[1] S. Agrawal and S. K. Sahoo, "A generalization of starlike functions of order α." *Hokkaido Math. J.*, vol. 46, pp. 15–27, 2017.

[2] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Coefficient bounds for p-valent functions." *Appl. Math. Comput.*, vol. 187, pp. 35–46, 2007, doi: 10.1016/j.amc.2006.08.100.

[3] M. K. Aouf, R. M. El-Ashwah, and H. M. Zayed, "Fekete-Szegö inequalities for p-valent starlike and convex functions of complex order." *J. Egyptian Math. Soc.*, vol. 22, pp. 190–196, 2014, doi: 10.1016/j.joems.2013.06.012.

[4] M. Fekete and G. Szegö, "Eine Bemerkung Über Ungerade Schlichte Funktionen." *J. London Math. Soc.*, vol. [s1-8 (2)], pp. 85–89, 1933, doi: 10.1112/jlms/s1-8.2.85.

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*. Cambridge, London and New York: (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Applications, Vol. 35, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne and Sydney, 1990; Second edition, Encyclopedia of Mathematics and Its Applications, Vol. 96; Cambridge University Press, 2004.

[6] M. E. H. Ismail, E. Merkes, and D. Styer, "A generalization of starlike functions." *Complex Variables Theory Appl.*, vol. 14, pp. 77–84, 1990, doi: 10.1080/17476939008814407.

[7] F. H. Jackson, "On q-definite integrals." *Quart. J. Pure Appl. Math.*, vol. 41, pp. 193–203, 1910.

[8] F. H. Jackson, "q-difference equations." *Amer. J. Math.*, vol. 32, pp. 305–314, 1910.

[9] V. G. Kac and P. Cheung, *Quantum Calculus.* New York: Universitext, Springer-Verlag, 2002. doi: 10.1007/978-1-4613-0071-7.

[10] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions." *Proc. Amer. Math. Soc.*, vol. 20, pp. 8–12, 1969, doi: 10.2307/2035949.

[11] W. Ma and D. Minda, "A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis, Z. Li, F. Ren, L. Lang and S. Zhang (Editors), International Press," , pp. 157–169, 1994.

[12] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications.* New York: Series on Monographs and Textbooks in Pure and Appl. Math. No. 255; Marcel Dekker, 2000.

[13] S. Owa, *The quasi-Hadamard products of certain analytic functions*. Singapore, New Jersey, London and Hong Kong: In: *Current Topics in Analytic Function Theory* (H. M. Srivastava and S. Owa, Editors), pp. 234–251; World Scientific Publishing Company, 1992. doi: 10.1142/9789814355896˙0019.

[14] D. V. Prokhorov and J. Szynal, "Inverse coefficients for (α,β)-convex functions." *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, vol. 35, no. 234-261, pp. 125–143, 1981.

[15] S. D. Purohit and R. K. Raina, "Certain subclasses of analytic functions associated with fractional q-calculus operators." *Math. Scand.*, vol. 109, pp. 55–70, 2011, doi: 10.7146/math.scand.a-15177.

[16] P. M. Rajković, S. D. Marinković, and M. S. Stanković, "Fractional integrals and derivatives in q-calculus." *Appl. Anal. Discrete Math.*, vol. 1, pp. 311–323, 2007, doi: 10.2298/AADM0701311R.

[17] V. Ravichandran, Y. Polatoğlu, M. Bolcal, and A. Sen, "Certain subclasses of starlike and convex functions of complex order." *Hacettepe J. Math. Stat.*, vol. 34, pp. 9–15, 2005.

[18] K. A. Selvakumaran, S. D. Purohit, A. Secer, and M. Bayram, "Convexity of certain q-integral operators of p-valent functions." *Abstr. Appl. Anal.*, vol. 2014, pp. 1–7, 2014, doi: 10.1155/2014/925902.

[19] T. M. Seoudy and M. K. Aouf, "Coefficient estimates of new classes of q-starlike and q-convex functions of complex order." *J. Math. Inequal.*, vol. 10, pp. 135–145, 2016, doi: 10.7153/jmi-10-11.

[20] H. M. Srivastava, *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*. New York, Chichester, Brisbane and Toronto: In: *Univalent Functions, Fractional Calculus, and Their Applications* (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329–354; John Wiley and Sons, 1989.

[21] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals.* Amsterdam, London and New York: Elsevier Science Publishers, 2012.

[22] H. M. Srivastava, A. K. Mishra, and M. K. Das, "The Fekete-Szegö problem for a subclass of close-to-convex functions." *Complex Variables Theory Appl.*, vol. 44, pp. 145–163, 2001, doi: 10.1080/17476930108815351.

[23] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series.* New York, Chichester, Brisbane and Toronto: Halsted Press (Ellis Horwood Limited, Chichester); John Wiley and Sons, 1985.

[24] H. Tang, H. M. Srivastava, S. Sivasubramanian, and P. Gurusamy, "The Fekete-Szegö functional problems for some classes of m-fold symmetric bi-univalent functions." *J. Math. Inequal.*, vol. 10, pp. 1063–1092, 2016, doi: 10.7153/jmi-10-85.

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