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Maximal, potential and singular operators in the local "complementary" variable exponent Morrey type spaces

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ABSTRACT

We consider local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ in which the p-means of function are controlled over $\Omega \setminus B(x_0, r)$ instead of $B(x_0, r)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $p(x)$ is a variable exponent, and no monotonicity type condition is imposed onto the function $\omega(r)$ defining the "complementary" Morrey-type norm. In the case where ω is a power function, we reveal the relation of these spaces to weighted Lebesgue spaces. In the general case we prove the boundedness of the Hardy–Littlewood maximal operator and Calderon–Zygmund singular operators with standard kernel, in such spaces. We also prove a Sobolev type ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega) \to {}^{\complement}\mathcal{M}^{q(\cdot),\omega}_{\{x_0\}}(\Omega)$-theorem for the potential operators $I^{\alpha(\cdot)}$, also of variable order. In all the cases the conditions for the boundedness are given it terms of Zygmund-type integral inequalities on $\omega(r)$, which do not assume any assumption on monotonicity of $\omega(r)$.

1. Introduction

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ play an important role, see [1]. Introduced by Morrey [2] in 1938, they are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_{x,r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))}, \tag{1.1}$$

where $0 \le \lambda < n,\ 1 \le p < \infty$.

We refer in particular to [3] for the classical Morrey spaces. As is known, in the last two decades there has been increasing interest in the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [4–7] and the recent book [8] on the progress in this field, see also references therein.

The spaces defined by the norm (1.1) are sometimes called *global* Morrey spaces, in contrast to *local* Morrey spaces defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}_{\{x_0\}}} := \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))}. \tag{1.2}$$

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Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [9,10] in the Euclidean setting and in [11] in the setting of metric measure spaces, in case of bounded sets Ω. In [9] the boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ was proved and a Sobolev–Adams type $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to \mathcal{L}^{q(\cdot),\lambda(\cdot)}$-theorem for potential operators of variable order $\alpha(x)$. In the case of constant α, the $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to BMO$-boundedness in the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha}$ was also proved. In [10] the maximal operator and potential operators were considered in a somewhat more general space, but under more restrictive conditions on $p(x)$. Hästö in [12] used his "local-to-global" approach to extend the result of [9] on the maximal operator to the case of the whole space $\mathbb{R}^n$.

In [11] the boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ was proved in the general setting of metric measure spaces. In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderon–Zygmund singular operators go back to [13,14], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [15]; for further results in the case of constant p and λ see for instance [16–19].

In [20] we studied the boundedness of the classical integral operators in the generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ over an open bounded set $\Omega \subset \mathbb{R}^n$. Generalized Morrey spaces of such a kind in the case of constant p, with the norm (1.1) replaced by

$$\|f\|_{\mathcal{M}^{p,\varphi}} := \sup_{x,r>0} \frac{r^{-\frac{n}{p}}}{\varphi(r)} \|f\|_{L^p(B(x,r))}, \tag{1.3}$$

under some assumptions on φ were studied by many authors, see for instance [21–26] and references therein. Results of [20] were extended in [27] to the case of the generalized Morrey spaces $\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ (where the L^∞-norm in r in the definition of the Morrey space is replaced by the Lebesgue L^θ-norm), we refer to [28] for such spaces in the case of constant exponents.

In [29] (see, also [30]) local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ were introduced and studied, with constant $p \in [1,\infty)$, the space of all functions $f \in L^p_{\text{loc}}(\Omega \setminus \{x_0\})$ with the finite norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)} = \sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L^p(\Omega\setminus B(x_0,r))}.$$

For the particular case when ω is a power function [29,30], see also [31,32], we find it convenient to keep the traditional notation ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)$ for the space defined by the norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^p(\mathbb{R}^n\setminus B(x_0,r))} < \infty, \quad x_0 \in \Omega,\ 0 \le \lambda < n. \tag{1.4}$$

Obviously, we recover the space ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)$ from ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$ under the choice $\omega(r) = r^{\frac{n-\lambda}{p'}}$.

In contrast to the Morrey space, where one measures the regularity of a function f near the point x_0 (in the case of local Morrey spaces) and near all points $x \in \Omega$ (in the case of global Morrey spaces), the norm (1.4) is aimed to measure a "bad" behavior of f near the point x_0 in terms of the possible growth of $\|f\|_{L^p(\Omega\setminus B(x_0,r))}$ as $r \to 0$. Correspondingly, one admits $\varphi(0) = 0$ in (1.3) and $\omega(0) = \infty$ in (1.4).

In this paper we consider local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ with variable exponent $p(\cdot)$, see Definition 4.1. However, we start with the case of constant p and in this case reveal an intimate connection of the complementary spaces with weighted Lebesgue spaces. In the case where $\omega(r)$ is a power function, we show that the space ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)$ is embedded between the weighted space $L^p(\Omega, |x-x_0|^{\lambda(p-1)})$ and its weak version, but does not coincide with either of them, which elucidates the nature of these spaces.

In the general case, for the spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ over bounded sets $\Omega \subset \mathbb{R}^n$ we consider the following operators:

(1) the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{\widetilde{B}(x,r)} |f(y)|dy$$

(2) variable order potential type operators

$$I^{\alpha(x)}f(x) = \int_\Omega \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}},$$

(3) variable order fractional maximal operator

$$M^{\alpha(x)}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha(x)}{n}}} \int_{\widetilde{B}(x,r)} |f(y)|dy,$$

where $0 < \inf \alpha(x) \le \sup \alpha(x) < n$, and

(4) Calderon–Zygmund type singular operator

$$Tf(x) = \int_{\Omega} K(x, y)f(y)dy$$

with a "standard" singular kernel in the sense of R. Coifman and Y. Meyer, see for instance [33, p. 99].

We find conditions on the pair of functions $\omega_1(r)$ and $\omega_2(r)$ for the $p(\cdot) \to p(\cdot)$-boundedness of the operators M and T from a variable exponent local "complementary" generalized Morrey space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ into another one ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)$, and for the corresponding Sobolev $p(\cdot) \to q(\cdot)$-boundedness for the potential operators $I^{\alpha(\cdot)}$, under the log-condition on $p(\cdot)$.

The paper is organized as follows.

In Section 2 we start with the case of the spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)$ with constant p and show their relation to the weighted Lebesgue space $L^p(\Omega, |x - x_0|^{\lambda(p-1)})$. The main statements are given in Theorem 2.1. In Section 3 we provide necessary preliminaries on variable exponent Lebesgue and Morrey spaces. In Section 4 we introduce the local "complementary" generalized Morrey spaces with variable exponents and recall some facts known for generalized Morrey spaces with constant p. In Section 5 we deal with the maximal operator, while potential operators are studied in Section 6. In Section 7 we treat Calderon–Zygmund singular operators.

The main results are given in Theorems 5.2, 6.2 and 7.2. We emphasize that the results we obtain for generalized Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\omega(r)$.

Notation:

$\mathbb{R}^n$ is the n-dimensional Euclidean space,
$\Omega \subset \mathbb{R}^n$ is an open set, $\ell = \operatorname{diam} \Omega$;
$\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$;
$B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $\widetilde{B}(x, r) = B(x, r) \cap \Omega$;

by c, C, c_1, c_2 etc., we denote various absolute positive constants, which may have different values even in the same line.

2. Relations of the "complementary" Morrey spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)$ with weighted Lebesgue spaces; the case of constant *p*

We use the standard notation $L^p(\Omega, \varrho) = \left\{f : \int_{\Omega} \varrho(y)|f(y)|^p\, dy < \infty\right\}$, where ϱ is a weight function. For the space ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)$ defined in (1.4), the following statement holds.

Theorem 2.1. *Let Ω be a bounded open set, $1 \le p < \infty$, $0 \le \lambda \le n$ and $A > \ell$. Then*

$$L^p(\Omega, |y - x_0|^{\lambda(p-1)}) \hookrightarrow {}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega) \hookrightarrow \bigcap_{\varepsilon>0} L^p\left(\Omega, \frac{|y - x_0|^{\lambda(p-1)}}{\left(\ln \frac{A}{|y-x_0|}\right)^{1+\varepsilon}}\right) \tag{2.1}$$

where both the embeddings are strict, with the counterexamples $f(x) = \frac{1}{|x-x_0|^{\frac{n}{p}+\frac{\lambda}{p'}}}$ *and* $g(x) = \frac{\ln\left(\ln \frac{B}{|x-x_0|}\right)}{|x-x_0|^{\frac{n}{p}+\frac{\lambda}{p'}}}$, $B > \ell e^e$:

$$f \in {}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega), \quad \text{but } f \notin L^p(\Omega, |y - x_0|^{\lambda(p-1)}),$$

and

$$g \in \bigcap_{\varepsilon>0} L^p\left(\Omega, \frac{|y - x_0|^{\lambda(p-1)}}{\left(\ln \frac{A}{|y-x_0|}\right)^{1+\varepsilon}}\right), \quad \text{but } g \notin {}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega).$$

Proof.

1^0. *The left-hand side embedding.*

Denote $\nu = \lambda(p - 1)$. For all $0 < r < \ell$ we have

$$\left(\int_{\Omega} |y - x_0|^{\nu}|f(y)|^p\, dy\right)^{\frac{1}{p}} \ge \left(\int_{\Omega\setminus\widetilde{B}(x_0,r)} |y - x_0|^{\nu}|f(y)|^p\, dy\right)^{\frac{1}{p}} \ge r^{\frac{\nu}{p}} \left(\int_{\Omega\setminus\widetilde{B}(x_0,r)} |f(y)|^p\, dy\right)^{\frac{1}{p}}. \tag{2.2}$$

Thus $\|f\|_{L^p(\Omega,|y-x_0|^\nu)} \geq r^{\frac{\lambda}{p'}} \|f\|_{L^p\Omega\setminus\widetilde{B}(x_0,r)}$ and then

$$\|f\|_{L^p(\Omega,|y-x_0|^\nu)} \geq \|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)}.$$

2^0. *The right-hand side embedding.*

We take $x_0 = 0$ for simplicity and denote $w_\varepsilon(|y|) = \frac{|y|^{\lambda(p-1)}}{\left(\ln \frac{A}{|y|}\right)^{1+\varepsilon}}$. We have

$$\int_{\widetilde{B}(x_0,t)} |f(y)|^p w_\varepsilon(|y|)dy = \int_{\widetilde{B}(x_0,t)} |f(y)|^p \left(\int_0^{|y|} \frac{d}{ds} w_\varepsilon(s)ds\right) dy, \tag{2.3}$$

with

$$w'_\varepsilon(t) = t^{\lambda(p-1)-1}\left[\frac{\lambda(p-1)}{\left(\ln \frac{A}{t}\right)^{1+\varepsilon}} + \frac{(1+\varepsilon)}{\left(\ln \frac{A}{t}\right)^{2+\varepsilon}}\right] \geq 0.$$

Therefore,

$$\begin{aligned}\int_{\widetilde{B}(x_0,t)} |f(y)|^p w_\varepsilon(|y|)dy &= \int_0^t w'_\varepsilon(s) \left(\int_{\{y\in\Omega: s<|x_0-y|<t\}} |f(y)|^p dy\right) ds \\ &\leq \int_0^\ell w'_\varepsilon(s) \|f\|^p_{L^p(\Omega\setminus\widetilde{B}(x_0,s))} ds \leq \|f\|^p_{{}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)} \int_0^\ell s^{\lambda(p-1)} w'_\varepsilon(s)\, ds\end{aligned}$$

where the last integral converges when $\varepsilon > 0$ since $s^{\lambda(p-1)} w'_\varepsilon(s) \leq \frac{C}{s\left(\ln \frac{A}{s}\right)^{1+\varepsilon}}$. This completes the proof. □

3^0. *The strictness of the embeddings.*

Calculations for the function f are obvious. In case of the function g, take $x_0 = 0$ for simplicity and denote $w_\varepsilon(|x|) = \frac{|x|^{\lambda(p-1)}}{\left(\ln \frac{A}{|x|}\right)^{1+\varepsilon}}$. We have

$$\|f\|^p_{L^p(\Omega,w_\varepsilon)} = \int_\Omega \frac{\ln^p\left(\ln \frac{B}{|x|}\right)}{|x|^n \left(\ln \frac{A}{|x|}\right)^{1+\varepsilon}} dx \leq C \int_0^\ell \frac{\ln^p\left(\ln \frac{B}{t}\right)}{t\left(\ln \frac{A}{t}\right)^{1+\varepsilon}} dt < \infty$$

for every $\varepsilon > 0$. However, for small $r \in \left(0, \frac{\delta}{2}\right)$, where $\delta = \mathrm{dist}(0, \partial\Omega)$, we obtain

$$\begin{aligned}r^{\frac{\lambda}{p'}} \|f\|_{L^p(\Omega\setminus B(0,r))} &= \left(r^{\lambda(n-1)} \int_{x\in\Omega:\ |x|>r} \frac{\ln^p\left(\ln \frac{B}{|x|}\right) dx}{|x|^{n+\lambda(p-1)}}\right)^{\frac{1}{p}} \\ &\geq \left(r^{\lambda(p-1)} \int_{x\in\Omega:\ r<|x|<\delta} \frac{\ln^p\left(\ln \frac{B}{|x|}\right) dx}{|x|^{n+\lambda(p-1)}}\right)^{\frac{1}{p}} = \left(r^{\lambda(p-1)} |\mathbb{S}^{n-1}| \int_r^\delta \frac{\ln^p\left(\ln \frac{B}{t}\right) dt}{t^{1+\lambda(p-1)}}\right)^{\frac{1}{p}}.\end{aligned}$$

But

$$\int_r^\delta \frac{\ln^p\left(\ln \frac{B}{t}\right) dt}{t^{1+\lambda(p-1)}} \geq \int_r^{2r} \frac{\ln^p\left(\ln \frac{B}{t}\right) dt}{t^{1+\lambda(p-1)}} \geq \ln^p\left(\ln \frac{B}{2r}\right) \int_r^{2r} \frac{dt}{t^{1+\lambda(p-1)}} = C \ln^p\left(\ln \frac{B}{r}\right) r^{-\lambda(p-1)},$$

so that

$$r^{\frac{\lambda}{p'}} \|f\|_{L^p(\Omega\setminus B(0,r))} \geq C \ln\left(\ln \frac{B}{2r}\right) \to \infty \quad \text{as } r \to 0,$$

which completes the proof of the lemma.

Remark 2.2. The arguments similar to those in (2.2) show that the left-hand side embedding in (2.1) may be extended to the case of more general spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$:

$$L^p(\Omega, \rho(|y-x_0|)) \hookrightarrow {}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$$

where ρ is a positive increasing (or almost increasing) function such that $\inf_{r>0} \frac{\rho(r)\omega^p(r)}{r^{n(p-1)}} > 0$.

3. Preliminaries on variable exponent Lebesgue and Morrey spaces

We refer to the book [8] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $[1, \infty)$. An open set Ω is assumed to be bounded throughout the whole paper. We mainly suppose that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{3.1}$$

where $p_- := \text{ess inf}_{x\in\Omega} p(x),\ p_+ := \text{ess sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

By $WL(\Omega)$ (weak Lipschitz) we denote the class of functions defined on Ω satisfying the log-condition

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2}x,\ y \in \Omega, \tag{3.2}$$

where $A = A(p) > 0$ does not depend on x, y.

Theorem 3.1 (*[34]*)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy condition (3.1). Then the maximal operator M is bounded in $L^{p(\cdot)}(\Omega)$.*

The following theorem was proved in [35] under the condition that the maximal operator is bounded in $L^{p(\cdot)}(\Omega)$, which became an unconditional result after the result of Theorem 3.1.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $p, \alpha \in WL(\Omega)$ satisfy assumption (3.1) and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \qquad \sup_{x\in\Omega} \alpha(x)p(x) < n. \tag{3.3}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ with $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$.

Singular operators within the framework of the spaces with variable exponents were studied in [36]. From Theorem 4.8 and Remark 4.6 of [36] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition on $p(x)$ at infinity.

Theorem 3.3 (*[36]*)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p \in WL(\Omega)$ satisfy condition (3.1). Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

The estimate provided by the following lemma (see [35, Corollary to Lemma 3.22]) is crucial for our further proofs.

Lemma 3.4. *Let Ω be a bounded domain and p satisfy the condition (3.2) and $1 \le p_- \le p(x) \le p_+ < \infty$. Let also $\sup_{x\in\Omega} \nu(x) < \infty$ and $\sup_{x\in\Omega}[n + \nu(x)p(x)] < 0$. Then*

$$\| |x - \cdot|^{\nu(x)} \chi_{\Omega\setminus\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le Cr^{\nu(x)+\frac{n}{p(x)}}, \quad x \in \Omega,\ 0 < r < \ell = \text{diam}\,\Omega, \tag{3.4}$$

where C does not depend on x and r.

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ introduced in [9], is defined as the set of integrable functions f on Ω with the finite norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

The following statements are known.

Theorem 3.5 (*[9]*)**.** *Let Ω be bounded and $p \in WL(\Omega)$ satisfy condition (3.1) and let a measurable function λ satisfy the conditions $0 \le \lambda(x)$, $\sup_{x\in\Omega} \lambda(x) < n$. Then the maximal operator M is bounded in $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.*

Theorem 3.5 was extended to unbounded domains in [12]. Note that the boundedness of the maximal operator in Morrey spaces with variable $p(x)$ was studied in [11] in the more general setting of quasimetric measure spaces.

The known statements for the potential operators read as follows.

Theorem 3.6 *([9]; Spanne-Type Result).* *Let Ω be bounded, $p, \alpha \in WL(\Omega)$ and p satisfy condition (3.1). Let also $\lambda(x) \geq 0$, the conditions (3.3) be fulfilled and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$. Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathcal{L}^{q(\cdot),\mu(\cdot)}(\Omega)$, where $\frac{\mu(x)}{q(x)} = \frac{\lambda(x)}{p(x)}$.*

Theorem 3.7 *([9]; Adams-Type Result).* *Let Ω be bounded, $p, \alpha \in WL(\Omega)$ and p satisfy condition (3.1). Let also $\lambda(x) \geq 0$ and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \qquad \sup_{x\in\Omega}[\lambda(x) + \alpha(x)p(x)] < n \tag{3.5}$$

hold. Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathcal{L}^{q(\cdot),\lambda(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n-\lambda(x)}$.

4. Variable exponent local "complementary" generalized Morrey spaces

Everywhere in the sequel the functions $\omega(r)$, $\omega_1(r)$ and $\omega_2(r)$ used in the body of the paper, are non-negative measurable function on $(0, \ell)$, $\ell = \operatorname{diam}\Omega$. Without loss of generality we may assume that they are bounded beyond any small neighborhood $(0, \delta)$ of the origin.

The local generalized Morrey space $\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ and global generalized Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ with variable exponent are defined (see [20]) by the norms

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}} = \sup_{r>0} \frac{r^{-\frac{n}{p(x_0)}}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x_0,r))},$$

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

where $x_0 \in \Omega$ and $1 \leq p(x) \leq p_+ < \infty$ for all $x \in \Omega$.

We find it convenient to introduce the variable exponent version of the local "complementary" space as follows (compare with (1.4)).

Definition 4.1. Let $x_0 \in \Omega$, $1 \leq p(x) \leq p_+ < \infty$. The *variable exponent generalized local "complementary" Morrey space* ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ is defined by the norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}} = \sup_{r>0} \frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}.$$

Similarly to the notation in (1.4), we introduce the following particular case of the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$, defined by the norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda}_{\{x_0\}}(\Omega)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^{p(\cdot)}(\Omega\setminus B(x_0,r))} < \infty, \quad x_0 \in \Omega,\ 0 \leq \lambda < n. \tag{4.1}$$

Everywhere in the sequel we assume that

$$\sup_{0<r<\ell} \frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} < \infty, \quad \ell = \operatorname{diam}\Omega, \tag{4.2}$$

which makes the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ non-trivial, since it contains $L^{p(\cdot)}(\Omega)$ in this case.

Remark 4.2. Suppose that

$$\inf_{\delta<r<\ell} \frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} > 0$$

for every $\delta > 0$. Then

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)} \sim \|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(B(x_0,\delta))} + \|f\|_{L^{p(\cdot)}(\Omega\setminus B(x_0,\delta))}$$

(with the constants in the above equivalence depending on δ). Since $\delta > 0$ is arbitrarily small, the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ may be interpreted as the space of functions *whose restrictions onto an arbitrarily small neighborhood $B(x_0, \delta)$ are in local* *"complementary" Morrey space* ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(B(x_0,\delta))$ *with the exponent* $p(\cdot)$ *close to the constant value* $p(x_0)$ *and the restrictions onto the exterior* $\Omega\setminus B(x_0,\delta)$ *are in the variable exponent Lebesgue space* $L^{p(\cdot)}$.

If also $\inf_{0<r<\ell}\frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} > 0$, then ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega) = L^{p(\cdot)}(\Omega)$. Therefore, to guarantee that the "complementary" space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ is strictly larger than $L^{p(\cdot)}(\Omega)$, one should be interested in the cases where

$$\lim_{r\to 0}\frac{r^{\frac{n}{p'(x_0)}}}{\omega(r)} = 0. \tag{4.3}$$

Clearly, the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ may contain functions with a non-integrable singularity at the point x_0, if no additional assumptions are introduced. To study the operators in ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$, we need its embedding into $L^1(\Omega)$. The corollary below shows that the Dini condition on ω is sufficient for such an embedding.

Lemma 4.3. *Let* $f\in L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))$ *for every* $s\in(0.\ell)$ *and* $\gamma\in\mathbb{R}$. *Then*

$$\int_{\widetilde{B}(x_0,t)}|y-x_0|^{\gamma}|f(y)|dy \le C\int_0^t s^{\gamma+\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds \tag{4.4}$$

for every $t\in(0,\ell)$, *where* C *does not depend on* f, t *and* x_0.

Proof. We use the following trick, where the parameter $\beta>0$ which will be chosen later:

$$\begin{aligned}\int_{\widetilde{B}(x_0,t)}|y-x_0|^{\gamma}|f(y)|dy &= \beta\int_{\widetilde{B}(x_0,t)}|x_0-y|^{\gamma-\beta}|f(y)|\left(\int_0^{|x_0-y|}s^{\beta-1}ds\right)dy\\ &= \beta\int_0^t s^{\beta-1}\left(\int_{\{y\in\Omega:s<|x_0-y|<t\}}|x_0-y|^{\gamma-\beta}\,|f(y)|dy\right)ds.\end{aligned} \tag{4.5}$$

Hence

$$\int_{\widetilde{B}(x_0,t)}|f(y)|dy \le C\int_0^t s^{\beta-1}\,\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}\,\||x_0-y|^{\gamma-\beta}\|_{L^{p'(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds.$$

Now we make use of Lemma 3.4 which is possible if we choose $\beta>\max\left(0,\frac{n}{p'_-}+\gamma\right)$ and then arrive at (4.4). □

Corollary 4.4. *The following embeddings hold*

$$L^{p(\cdot)}(\Omega)\hookrightarrow {}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)\hookrightarrow L^1(\Omega) \tag{4.6}$$

where the left-hand side embedding is guaranteed by the condition (4.2) *and the right-hand side one by the condition*

$$\int_0^{\ell}\frac{\omega(r)\,dr}{r}<\infty. \tag{4.7}$$

Proof. The statement for the left-hand side embedding is obvious. The right-hand side follows from Lemma 4.3 with $\gamma=0$ and the inequality

$$\int_0^{\ell} r^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}dr \le \|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)}\int_0^{\ell}\omega(r)\frac{dr}{r}. \quad \square \tag{4.8}$$

Remark 4.5. Note that similarly to the arguments in the proof of Corollary 4.4, we can see that the condition $\int_0^{\ell}\omega(r)r^{\gamma-1}\,dr<\infty$, $\gamma\in\mathbb{R}$, guarantees the embedding ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)\hookrightarrow L^1(\Omega,|y-x_0|^{\gamma})$ into the weighted space; note that only the values $\gamma>-n/p'(x_0)$ may be of interest for us, because the above condition with $\gamma\le -n/p'(x_0)$ is not compatible with the condition (4.2) of the non-triviality of the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$.

Remark 4.6. We also find it convenient to give a condition for $f\in L^1(\Omega)$ in the form as follows:

$$\int_0^{\ell} t^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}dt<\infty \Longrightarrow f\in L^1(\Omega), \tag{4.9}$$

for which it suffices to refer to (4.4).

In the sequel all the operators under consideration (maximal, singular and potential operators) will be considered on function f either satisfying the condition of the existence of the integral in (4.9), or belonging to ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ with ω satisfying the condition (4.7). Such functions are therefore integrable on Ω in both the cases, and consequently all the studied operators exist on such functions a.e.

Note that the statements on the boundedness of the maximal, singular and potential operators in the "complementary" Morrey spaces known for the case of the constant exponent p, obtained in [29], read as follows. Note that the theorems below do not assume no monotonicity type conditions on the functions ω, ω_1 and ω_2.

Theorem 4.7 *([29, Theorem 1.4.6]). Let $1 < p < \infty$, $x_0 \in \mathbb{R}^n$ and $\omega_1(r)$ and $\omega_2(r)$ be positive measurable functions satisfying the condition*

$$\int_0^r \omega_1(t)\frac{dt}{t} \le c\,\omega_2(r)$$

with $c > 0$ not depending on $r > 0$. Then the operators M and T are bounded from ${}^{\complement}\mathcal{M}^{p,\omega_1}_{\{x_0\}}(\mathbb{R}^n)$ to ${}^{\complement}\mathcal{M}^{p,\omega_2}_{\{x_0\}}(\mathbb{R}^n)$.

Corollary 4.8 *([29]). Let $1 < p < \infty$, $x_0 \in \mathbb{R}^n$ and $0 \le \lambda < n$. Then the operators M and T are bounded in the space ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$.*

Theorem 4.9 *([29, Theorem 1.3.9]). Let $0 < \alpha < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $x_0 \in \mathbb{R}^n$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition*

$$r^{\alpha}\int_0^r \omega_1(t)\frac{dt}{t} \le c\,\omega_2(r),$$

with $c > 0$ not depending on $r > 0$. Then the operators M^{α} and I^{α} are bounded from ${}^{\complement}\mathcal{M}^{p,\omega_1}_{\{x_0\}}(\mathbb{R}^n)$ to ${}^{\complement}\mathcal{M}^{q,\omega_2}_{\{x_0\}}(\mathbb{R}^n)$.

Corollary 4.10 *([29]). Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $x_0 \in \mathbb{R}^n$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p'} = \frac{\mu}{q'}$. Then the operators M^{α} and I^{α} are bounded from ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$ to ${}^{\complement}\mathcal{L}^{q,\mu}_{\{x_0\}}(\mathbb{R}^n)$.*

Remark 4.11. The introduction of global "complementary" Morrey-type spaces has no big sense, neither in case of constant exponents, nor in the case of variable exponents. In the case of constant exponents this was noted in [37, pp. 19–20]; in this case the global space defined by the norm

$$\sup_{x\in\Omega, r>0}\frac{r^{\frac{n}{p}}}{\omega(r)}\|f\|_{L^p(\Omega\setminus\widetilde{B}(x,r))}$$

reduces to $L^p(\Omega)$ under the assumption (4.2). In the case of variable exponents there happens the same. In general, to make it clear, note that for instance under the assumption (4.3) if we admit that $\sup_{r>0}\frac{r^{\frac{n}{p(\cdot)}}}{\omega(r)}\|f\|_{L^p(\Omega\setminus\widetilde{B}(x,r))}$ for two different points $x = x_0$ and $x = x_1$, $x_0 \ne x_1$, this would immediately imply that $f \in L^{p(\cdot)}$ in a neighborhood of both the points x_0 and x_1.

5. The maximal operator in the spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$

The proof of the main result of this section presented in Theorem 5.2 is based on the estimate given in the following preliminary theorem.

Theorem 5.1. *Let Ω be bounded, $p \in WL(\Omega)$ satisfy the condition (3.1) and $f \in L^{p(\cdot)}(\Omega \setminus \widetilde{B}(x_0, r))$ for every $r \in (0, \ell)$. If the integral*

$$\int_0^{\ell} r^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}dr \tag{5.1}$$

converges, then

$$\|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \le Ct^{-\frac{n}{p'(x_0)}}\int_0^t r^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}dr \tag{5.2}$$

for every $t \in (0, \ell)$, where C does not depend on f, t and x_0.

Proof. We represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x_0,t)}(y) \qquad f_2(y) = f(y)\chi_{\widetilde{B}(x_0,t)}(y). \tag{5.3}$$

1°. *Estimation of* Mf_1. This case is easier, being treated by means of Theorem 3.1. Obviously $f_1 \in L^{p(\cdot)}(\Omega)$ so that by Theorem 3.1

$$\|Mf_1\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \le \|Mf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}. \tag{5.4}$$

By the monotonicity of the norm $\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}$ with respect to r we have

$$\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \le Ct^{-\frac{n}{p'(x_0)}}\int_0^t r^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}dr \tag{5.5}$$

and then

$$\|Mf_1\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \le Ct^{-\frac{n}{p'(x_0)}}\int_0^t r^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}dr.$$

2°. *Estimation of* Mf_2. This case needs the application of Lemma 4.3. To estimate $Mf_2(z)$ by means of (4.4), we observe that for $z \in \Omega\setminus\widetilde{B}(x_0,2t)$ we have

$$\begin{aligned} Mf_2(z) &= \sup_{r>0}|B(z,r)|^{-1}\int_{\widetilde{B}(z,r)}|f_2(y)|dy \\ &\le \sup_{r\ge t}\int_{\widetilde{B}(x_0,t)\cap B(z,r)}|y-z|^{-n}\,|f(y)|dy \\ &\le 2^n\,|x_0-z|^{-n}\int_{\widetilde{B}(x_0,t)}|f(y)|dy. \end{aligned}$$

Then by (4.4)

$$Mf_2(z) \le C|x_0-z|^{-n}\int_0^t s^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds. \tag{5.6}$$

Therefore

$$\begin{aligned} \|Mf_2\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} &\le C\int_0^t s^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds\,\left\||x_0-z|^{-n}\right\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \\ &\le Ct^{-\frac{n}{p'(x_0)}}\int_0^t s^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds. \end{aligned} \tag{5.7}$$

Since $\|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \le \|Mf_1\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} + \|Mf_2\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}$, from (5.6) and (5.7)we arrive at (5.2) with $\|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}$ on the left-hand side and then (5.2) obviously holds also for $\|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}$. □

The following theorem for the complementary Morrey spaces is, in a sense, a counterpart to Theorem 3.5 formulated in Section 3 for the usual Morrey spaces.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $p \in WL(\Omega)$ satisfy the assumption (3.1) and the functions $\omega_1(t)$ and $\omega_2(t)$ satisfy the condition*

$$\int_0^t \omega_1(r)\frac{dr}{r} \le C\,\omega_2(t), \tag{5.8}$$

where C does not depend on t. Then the maximal operator M is bounded from the space ${}^{\complement}\mathcal{M}_{\{x_0\}}^{p(\cdot),\omega_1}(\Omega)$ to the space ${}^{\complement}\mathcal{M}_{\{x_0\}}^{p(\cdot),\omega_2}(\Omega)$.

Proof. For $f \in {}^{\complement}\mathcal{M}_{\{x_0\}}^{p(\cdot),\omega_1}(\Omega)$ we have

$$\|Mf\|_{{}^{\complement}\mathcal{M}_{\{x_0\}}^{p(\cdot),\omega_2}(\Omega)} = \sup_{t\in(0,\ell)}\frac{t^{\frac{n}{p'(x_0)}}}{\omega_2(t)}\|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))},$$

where Theorem 5.1 is applicable to the norm $\|Mf\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}$. Indeed from (5.8) it follows that the integral $\int_0^t \frac{\omega_1(r)}{r}dr$ converges. This implies that for $f \in {}^{\complement}\mathcal{M}_{\{x_0\}}^{p(\cdot),\omega_1}(\Omega)$ the assumption of the convergence of the integral of type (5.1) is fulfilled by (4.8). Then by Theorem 5.1 we obtain

$$\|Mf\|_{{}^{\complement}\mathcal{M}_{\{x_0\}}^{p(\cdot),\omega_2}(\Omega)} \le C\sup_{0<t\le\ell}\omega_2^{-1}(t)\int_0^t r^{-\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}dr.$$

Hence

$$\|Mf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)} \le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)} \sup_{t\in(0,\ell)} \frac{1}{\omega_2(t)} \int_0^t \omega_1(r)\frac{dr}{r} \le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)}$$

by (5.8), which completes the proof. □

Corollary 5.3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $x_0 \in \Omega$, $0 \le \lambda < n$, $\lambda \le \mu \le n$ and let $p \in WL(\Omega)$ satisfy the assumption (3.1). Then the operator M is bounded from the space ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda}_{\{x_0\}}(\Omega)$ to ${}^{\complement}\mathcal{L}^{p(\cdot),\mu}_{\{x_0\}}(\Omega)$.*

6. Riesz potential operator in the spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$

In this section we extend Theorem 4.9 to the variable exponent setting. Note that Theorems 6.1 and 6.2 in the case of constant exponent p were proved in [29, Theorems 1.3.2 and 1.3.9] (see also [30, p. 112,129]).

Theorem 6.1. *Let (3.1) be fulfilled and $p(\cdot), \alpha(\cdot) \in WL(\Omega)$ satisfy the conditions in (3.3). If f is such that the integral (5.1) converges, then*

$$\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \le Ct^{-\frac{n}{p'(x_0)}} \int_0^t s^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds \tag{6.1}$$

for every $f \in L^{p(\cdot)}(\Omega \setminus \widetilde{B}(x_0, t))$, where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} \tag{6.2}$$

and C does not depend on f, x_0 and $t \in (0, \ell)$.

Proof. We represent the function f in the form $f = f_1 + f_2$ as in (5.3) so that

$$\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \le \|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} + \|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}.$$

Since $f_1 \in L^{p(\cdot)}(\Omega)$, by Theorem 3.2 we have

$$\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \le \|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}$$

and then

$$\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \le Ct^{-\frac{n}{p'(x_0)}} \int_0^t s^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds \tag{6.3}$$

in view of (5.5).

To estimate

$$\|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} = \left\| \int_{\widetilde{B}(x_0,t)} |z-y|^{\alpha(z)-n} f(y)dy \right\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))},$$

we observe that for $z \in \Omega \setminus \widetilde{B}(x_0, 2t)$ and $y \in \widetilde{B}(x_0, t)$ we have $\frac{1}{2}|x_0 - z| \le |z - y| \le \frac{3}{2}|x_0 - z|$, so that

$$\|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \le C \int_{\widetilde{B}(x_0,t)} |f(y)|dy \, \left\| |x_0 - z|^{\alpha(z)-n} \right\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}.$$

From the log-condition for $\alpha(\cdot)$ it follows that

$$c_1|x_0 - z|^{\alpha(x_0)-n} \le |x_0 - z|^{\alpha(z)-n} \le c_2|x_0 - z|^{\alpha(x_0)-n}.$$

Therefore,

$$\|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \le C \int_{\widetilde{B}(x_0,t)} |f(y)|dy \, \left\| |x_0 - z|^{\alpha(x_0)-n} \right\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}. \tag{6.4}$$

The norm in the integral on the right-hand side is estimated by means of Lemma 3.4, which yields

$$\|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))} \le Ct^{-\frac{n}{p'(x_0)}} \int_{\widetilde{B}(x_0,t)} |f(y)|dy.$$

It remains to make use of (4.4) and obtain

$$\|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \le Ct^{-\frac{n}{p'(x_0)}} \int_0^t s^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))} ds. \tag{6.5}$$

From (6.3) and (6.5) we arrive at (6.1). □

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $x_0 \in \Omega$ and $p(\cdot), \alpha(\cdot) \in WL(\Omega)$ satisfy assumptions (3.1) and (3.3), $q(x)$ given by (6.2) and the functions $\omega_1(r)$ and $\omega_2(r)$ fulfill the condition*

$$t^{\alpha(x_0)} \int_0^t \omega_1(r) \frac{dr}{r} \le C\, \omega_2(t), \tag{6.6}$$

where C does not depend on t. Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ to ${}^{\complement}\mathcal{M}^{q(\cdot),\omega_2}_{\{x_0\}}(\Omega)$.

Proof. It suffices to prove the boundedness of the operator $I^{\alpha(\cdot)}$, since $M^{\alpha(\cdot)}f(x) \le CI^{\alpha(\cdot)}|f|(x)$.

Let $f \in {}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$. We have

$$\|I^{\alpha(\cdot)}f\|_{{}^{\complement}\mathcal{M}^{\{x_0\}}_{q(\cdot),\omega_2}(\Omega)} = \sup_{t>0} \frac{t^{\frac{n}{q'(x_0)}}}{\omega_2(t)} \|\chi_{\Omega\backslash\widetilde{B}(x_0,t)} I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\Omega)}. \tag{6.7}$$

We estimate $\|\chi_{\Omega\backslash\widetilde{B}(x_0,t)} I^{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\Omega)}$ in (6.7) by means of Theorem 6.1. This theorem is applicable since the integral (5.1) with $\omega = \omega_1$ converges by (4.8). We obtain

$$\begin{aligned}\|I^{\alpha(\cdot)}f\|_{{}^{\complement}\mathcal{M}^{\{x_0\}}_{q(\cdot),\omega_2}(\Omega)} &\le C \sup_{t>0} \frac{t^{-\frac{n}{p'(x_0)}+\frac{n}{q'(x_0)}}}{\omega_2(t)} \int_0^t r^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr \\ &\le C\|f\|_{{}^{\complement}\mathcal{M}^{\{x_0\}}_{p(\cdot),\omega_1}(\Omega)} \sup_{t>0} \frac{t^{\alpha(x_0)}}{\omega_2(t)} \int_0^t \frac{\omega_1(r)}{r} dr.\end{aligned}$$

It remains to make use of the condition (6.6). □

Corollary 6.3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p(\cdot), \alpha(\cdot) \in WL(\Omega)$ satisfy assumptions (3.1) and (3.3), $q(x)$ given by (6.2), $x_0 \in \Omega$ and $\frac{\lambda}{p'(x_0)} \le \frac{\mu}{q'(x_0)}$. Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda}_{\{x_0\}}(\Omega)$ to ${}^{\complement}\mathcal{L}^{q(\cdot),\mu}_{\{x_0\}}(\Omega)$.*

7. Singular integral operators in the spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$

Theorems 7.1 and 7.2 proved below, in the case of the constant exponent p were proved in [29, Theorems 1.4.2 and 1.4.6] (see also [30, p. 132,135]).

Theorem 7.1. *Let Ω be an open bounded set, $p \in WL(\Omega)$ satisfy condition (3.1) and $f \in L^{p(\cdot)}(\Omega \setminus \widetilde{B}(x_0, t))$ for every $t \in (0, \ell)$. If the integral*

$$\int_0^{\ell} r^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr$$

converges, then

$$\|Tf\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le Ct^{-\frac{n}{p'(x_0)}} \int_0^{2t} r^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr,$$

where C does not depend on f, x_0 and $t \in (0, \ell)$.

Proof. We split the function f in the form $f_1 + f_2$ as in (5.3) and have

$$\|Tf\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \le \|Tf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} + \|Tf_2\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))}.$$

Taking into account that $f_1 \in L^{p(\cdot)}(\Omega)$, by Theorem 3.3 we have

$$\|Tf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \le \|Tf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))}.$$

Then in view of (5.5)

$$\|Tf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le Ct^{-\frac{n}{p'(x_0)}} \int_0^t r^{\frac{n}{p'(x_0)}-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr. \tag{7.1}$$

To estimate $\|Tf_2\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}$, note that $\frac{1}{2}|x_0-z|\le|z-y|\le\frac{3}{2}|x_0-z|$ for $z\in\Omega\setminus\widetilde{B}(x_0,2t)$ and $y\in\widetilde{B}(x_0,t)$, so that

$$\|Tf_2\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}\le C\left\|\int_{\widetilde{B}(x_0,t)}|z-y|^{-n}f(y)dy\right\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}$$
$$\le C\int_{\widetilde{B}(x_0,t)}|f(y)|dy\|\,|x_0-z|^{-n}\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}.$$

Therefore, with the aid of the estimate (3.4) and inequality (4.4), we get

$$\|Tf_2\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,2t))}\le Ct^{-\frac{n}{p'(x_0)}}\int_0^t s^{\frac{n}{p'(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))}ds,$$

which together with (7.1) yields (7.1). □

Theorem 7.2. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded set, $x_0\in\Omega$, $p\in WL(\Omega)$ satisfy condition (3.1) and $\omega_1(t)$ and $\omega_2(t)$ fulfill condition (5.8). Then the singular integral operator T is bounded from the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ to the space ${}^{\complement}\mathcal{M}^{\{x_0\}}_{p(\cdot),\omega_2}(\Omega)$.*

Proof. Let $f\in{}^{\complement}\mathcal{M}^{\{x_0\}}_{p(\cdot),\omega_1}(\Omega)$. We follow the procedure already used in the proof of Theorems 5.2 and 6.2: in the norm

$$\|Tf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)}=\sup_{t>0}\frac{t^{\frac{n}{p'(x_0)}}}{\omega_2(t)}\|Tf\chi_{\Omega\setminus\widetilde{B}(x_0,t)}\|_{L^{p(\cdot)}(\Omega)},\tag{7.2}$$

we estimate $\|Tf\chi_{\Omega\setminus\widetilde{B}(x_0,t)}\|_{L^{p(\cdot)}(\Omega)}$ by means of Theorem 7.1 and obtain

$$\|Tf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)}\le C\sup_{t>0}\frac{1}{\omega_2(t)}\int_0^t r^{\frac{n}{p(x_0)}-1}\|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}dr$$
$$\le C\|f\|_{{}^{\complement}\mathcal{M}^{\{x_0\}}_{p(\cdot),\omega_1}(\Omega)}\sup_{t>0}\frac{1}{\omega_2(t)}\int_0^t\omega_1(r)\frac{dr}{r}\le C\|f\|_{{}^{\complement}\mathcal{M}^{\{x_0\}}_{p(\cdot),\omega_1}(\Omega)}.\quad\square$$

Corollary 7.3. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded set, $p\in WL(\Omega)$ satisfy the assumption (3.1), $x_0\in\Omega$ and $0\le\lambda<n$, $\lambda\le\mu\le n$. Then the singular integral operator T is bounded from ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda}_{\{x_0\}}(\Omega)$ to ${}^{\complement}\mathcal{L}^{p(\cdot),\mu}_{\{x_0\}}(\Omega)$.*

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