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From discrete biofilm model formation to queuing transient analysis

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Dedicatory

To Manuel Rodrigues de Carvalho

In memoriam

Acknowledgments

One can´t dissociate the acknowledgments that the candidate makes at the beginning of his doctoral thesis from all the personal relationships he has experienced over the time, prior to the day he writes them.
The theme of this thesis has proved to be inter- and multidisciplinary, growing rapidly in difficulty but also in intellectual interest.
And it had a beginning.
An idea about the shape of the space occupied by the biofilm(s).
It is a question of considering a biofilm as a territorial plot, also adopting the contour lines to characterize its micro-orography adding still the demanding theoretical condition that between two level curves the difference of dimension is equal to the average thickness of a monolayer of bacteria.
This was the idea of Professor Manuel Carrondo, inspired by a geometric concept similar to that used in the BET isothermal theory (Brunauer, Emmett, Teller).
This was the beginning of this whole thesis.
Five more were added to this initial idea
Three found in scattered literature and two of the candidate, one of them inspired in the literature and another of the own
And the six provided a theoretical challenge, first in the area initially chosen (biofilm), then in the area of Queue Theory, then in Teletraffic, ... and, ... later ... and, ...
As consequence of this task, and of my aptitude for Mathematics, I have consolidated the habit of participating in international Mathematics conferences, specifically Special Functions and Orthogonal Polynomials, Equations of Differences and Argument Delayed, Integral Systems, and alike subjects.
This is how I met Professor Amílcar Branquinho of the Department of Mathematics of the University of Coimbra at a congress held in Munich, who in turn introduced me to Professor Francisco Marcellán (Paco) of the Department of Mathematics of the Superior Polytechnic School of Carlos III University of Madrid.
Even before the congress in Munich I had the objective of embracing the new and fascinating, for me, profession of Mathematician, and this objective began to materialize with the initiative both of them had to introduce me to the PhD program in Mathematical Engineering, which I am currently doing very enthusiastically and according to the competence and the requirement, but also the sympathy of the faculty I know in Madrid.
Many thanks Amilcar and Paco.
I am especially grateful to my teachers for the PhD in Mathematical Engineering, Professors José Manuel Rodríguez, Jorge Arvesú, Eva Touris, António García, Elena Romera, Julio Moro, Miguel Moscoso, Domingo Pestana, Hector Pijeira and Jorge Sánchez Ruiz of the Carlos III University of Madrid and also to Visiting Professors, Yamilet Quintana, from Simon Bolivar University (Venezuela), Antonio Marquina, from University of Valencia, Ricardo Soto, from Universidad Católica del Norte (Chile) and Ana Portilla, from Saint Louis University (Campus of Madrid).
To my Thesis Director at the Carlos III University of Madrid, Professor Luis Lopez Bonilla, who is also at the faculty of the doctoral program in Mathematical Engineering, and impeller of the University Institute on Modeling and Simulation in Fluid Dynamics, Nanoscience and Industrial Mathematics "Gregorio Millán Barbany" I appreciate and gratefully acknowledge his availability, guidance, and the interest and recognition he has always shown for my work.
Also to Professor Manuel Carretero, current Director of the Gregorio Millán Institute, a word of sympathy and thanks is due.
At the Faculty of Science and Technology of the of Lisbon New University I am grateful for the exchange of ideas with Professors Manuel Duarte Ortigueira from the Department of Electrical Engineering and with Professors Carlos Agra Coelho and Manuel Esquivel from the Department of Mathematics.

In addition to Professor Manuel Carrondo, from the Department of Chemistry, I also always found words of encouragement from Professor Manuel Nunes da Ponte, João Paulo Crespo, Luis Sousa Lobo, Fernando Pina and José Paulo Mota.
After initial times of skepticism, I found that the results I achieved started to be interesting.
I wish to acknowledge my intellectual debt to the book Kleinrock, Leonard: Queueing Systems, vol.1: Theory, Wiley, New York (1975) where I´ve found the indication that I was in a disheartening mathematical context, in which one only can expect to find ugly expressions, and also greater and greater complexity and obscurity, as one attempts to deduce transient exact analytic solutions of more and more general queuing systems.
This indication, was found after already having reached some nice results in the subject of biofilm model. It gives me confidence to prosecute, not only in this last mentioned subject, but also in queuing systems transient regimes, profiting the acquired skills and accepting this new additional challenge with enthusiasm.
I conclude by renewing my thanks to all, and apologizing to those whom I will involuntarily forget to mention.

Abstract

The initial motivation of this work was to establish a theoretical model that could mathematically translate the evolution over time of growth and accumulation of biofilm bacterial mass.
The model, although based only on six very simple qualitative ideas, once put in equation, implies the resolution of a system of differential-difference equations similar to those one can find in queues transient phases, but far more complex.
Besides that version of the model, based on Monod kinetic law, four more versions, less complicated are considered.
Namely, two upper bond approaches,
- *Zero order kinetics (the less tight one)*
- *"Blackman inspired" kinetics (the more tight one)*

... and two lower bonds approaches,
- *"Averaged" first order kinetics (the more tight one)*
- *"Mono-layered concentrated growth" kinetics (the less tight one).*

The " Mono-layered concentrated growth" kinetics version allows to reach an exact analytical solution for those early stages of biofilm formation and growth.
But it is even more difficult to handle analytically than that of M/M/1 queue.
However application of an algorithm like Miller´s tree term recurrence, already known for Bessel functions of first kind (and also the modified ones) allows an exact calculation of such kind of solutions, for a wide range of parameters values and time.
For biofilm model a five term recurrence was deduced and applied in backwards calculation. A suitable normalization condition completes the reach of the solution.
After the deduction in this simplest version ("Mono-layered concentrated growth" kinetics) the mathematical skills reached directed us to construct a guess solution with the purpose, accomplished, of solve the model in his most general version (Monod kinetic law).
Besides the utilization of bond concept, already known, namely in electro-technical literature (theoretical radar context) the concept of "mathematical meeting point" with the purpose to "erode" near intractable models is introduced.
The skills gained in biofilm model context were applied in queuing systems transient regimes opening a new subject of interest.
The beginning of an handbook of exactly analytic solutions in queuing systems transient regime is aborded.

Kei words: biofilm, Monod kinetics, Miller recurrent algorithm, queuing theory, transient regime, recurrences.

Resumo

A motivação inicial deste trabalho foi a de conseguir obter um modelo teórico que pudesse traduzir matematicamente a evolução, ao longo do tempo, da acumulação e do crescimento de biomassa em biofilmes bacterianos.

Embora o modelo se baseie em apenas seis ideias qualitativas e muito simples, implica a resolução de um sistema de equações diferenciais de diferença, semelhante aos que governam os regimes transitórios das Filas de Espera sendo, a pesar dessa semelhança, bastante mais complexo.

Além dessa versão, mais complicada, baseada na cinética de Monod, consideraram-se também outras quatro versões, mais simples, a saber,

Duas aproximações representando cotas superiores á cinética de Monod,

- *Cinética de ordem zero (a cota menos apertada, ou acima mais distante)*
- *Cinética inspirada na de Blackman (a cota mais apertada, ou acima mais próxima)*

E duas aproximações representando cotas inferiores á cinética de Monod,,

- *Cinética de 1ª ordem média (a cota mais apertada, ou abaixo mais próxima)*
- *Cinética de crescimento concentrado numa só monocamada (a cota menos apertada ou abaixo mais distante)*

A versão da cinética de crescimento concentrado numa só monocamada permite obter uma solução analítica exacta para os estágios iniciais de formação e crescimento do biofilme. Mas é ainda mais difícil de manejar analiticamente do que a da fase transitória da fila M/M/1.

Se, contudo, aplicarmos um algoritmo aparentado com o de Miller, para recorrências homogéneas de três termos, já conhecido no cálculo das funções de Bessel de 1ª espécie, podemos calcular com exactidão esse género de soluções, para uma escolha de parâmetros sem quaisquer restrições e para um vasto intervalo de tempo.

No caso da solução encontrada para o modelo do biofilme foi necessário deduzir uma recorrência homogénea com cinco termos e aplicá-la de forma regressiva.Com uma adequada e óbvia condição de normalização fica completo o cálculo exacto da solução.

Uma vez deduzidas a solução e a recorrência para a versão mais simples, da cinética de crescimento concentrado numa só monocamada, tentou-se uma solução pré definida com a finalidade, conseguida, de resolver o modelo da formação e crescimento iniciais do biofilme, na sua versão mais geral, baseada na cinética de Monod.

Além da utilização do conceito de versões aproximadas como cotas, superiores e inferiores, já conhecido, nomeadamente da literatura da Engenharia Electrotécnica (no contexto da teoria do radar, por exemplo) também se introduz o novo conceito de "ponto de encontro matemático" como metodologia para resolver modelos quase intractáveis.

As capacidades ganhas no contexto do modelo do biofilme passaram a ser aplicados no das Fases Transitórias das Filas de Espera e o começo da elaboração de um pequeno "handbook" com as correspondentes soluções exactas começou a ser abordado.

Palavras - chave: biofilme, cinética de Monod, Algoritmo recorrente de Miller, Teoria das Filas de Espera, regime transitório, recorrencias.

x

Preface

Discrete mathematical models in space, and continuous in time, arise in many different contexts of Physics, Chemistry and Engineering.

Ever since, consideration of more sophisticated exact mathematical solutions, for transient regimes, invariantly leads, in known bibliography, to numerical and approximated methods application.

Such approach neglects all the theoretical information contained in an exact solution, which usually consists of special functions or, more frequently, complex combinations between their (in particular an infinite sum of terms with Bessel functions of first kind, modified or not).

As a matter of fact the accurate calculation of special functions seems far from being an inviting task for the vast majority of leading authors in several fields of exact sciences, namely Operational Research [1,2], Teletraffic Theory [3], Manufacturing Systems [4], etcetera.

Particularly notorious is the commentary in [2] and subsequent ones, with the same inspiration, many years after [5], and even nowadays [6].

Functions like that one of transient M/M/1 queue also came into play in Physical-Chemistry (nucleation, polymerization, crystallization, ... , aggregation processes in general), in Radar Theory [7], in Computer Networks Theory [8], and Congestion phenomenon [9].

From those exact solutions little information is brought to light, with the exception of long time behaviour (steady state) which is a limit easy to deduce. Normally transient scenery is avoided and quickly authors came on analysing steady state.

If transient regime is considered, having already an exact solution, a formalism of type Fokker-Planck is usually chosen to approximate such regime. The purpose is to mimic a process discrete in space and continuous in time by a process continuous both in space and time.

Doing things like that the initial infinite system of differential-difference equations is transformed in an equation with partial derivatives (Fokker-Planck equation).

On the other hand special functions, that are part of the solution or, even better saying, that are the "backbone" of the solution, are simply eliminated from all the analysis, and so we lose all their formal beauty.

Formalism conducting to Fokker-Planck equation started when Frenkel brought Becker-Doring theoretical work on nucleation kinetics to “its modern form”, as Goodrich sais [10].

Seems that such “modern form”, or others similar approximation approaches, has been generally accepted by the vast majority of authors in many domains of scientific knowledge already cited above.

In past years a theoretical model to account quantitatively evolution over time of growth and accumulation of biofilm biomass lead to a few similar exact solutions [11,12,13].

In particular, the simplest version of all variants analysed them, lead to an analytical solution more complex than that of the M/M/1 queue [11].

To compute the solution obtained with accuracy one must before deduce adequate homogeneous recurrence relations of a part of it containing, at least, an infinite sum of that solution already refereed . Even better if it is possible to deduce the homogeneous recurrence for the complete solution.

This has been in fact the case for the aforesaid simplest version of the biofilm model.

A suitable adaptation of the already known three terms recurrence (TTR) Miller algorithm [14] for the functions $J_n(x)$ and $I_n(x)$ to this new context in which recurrences have now more than tree terms has been done.

Miller applied backward recurrence “for evaluating sequences of functions $\{ f_k \}$ when the recurrence connecting successive members was unstable for increasing k” as published Thacher [15].

Applying an algorithm like Miller´s (TTR) [14], allowed reach an exact calculation and graphic representation of such kind of solutions, with a five term recurrence, also applied in backwards calculation.

Consequently we are applying a generalization of that algorithm including the whole infinite sum in functions { f_k } and not only one Bessel function. So the recurrences obtained have more than tree terms, as expected according to the additional complexity of the functions.

The early stages of growth and accumulation of biofilm included in that model, now revised and enlarged, will be a subject of this thesis, jointly with the search of how to deal when facing theoretical models near the border of mathematic intractability.

Accordingly, and going deeper in that hard mathematical context, those early stages are considered more complicated than the referred simplest solution, which is only a first approach.

Besides that first approach, using what we call "Mono-layered concentrated growth" kinetics, we will, also for sake of completeness, describe and deduce solutions for three more approaches named, Zero order kinetics, "Blackman inspired" kinetics and "Averaged" first order kinetics, all known in biotechnological literature.

Those approaches are, however, only particular cases of the main task, which is the mathematical deduction of the exact solution for Monod kinetic law. This will be the principal purpose of the revisitation of our old model.

We reach such general version of the model profiting the skills acquired solving the simplest version [12].

As consequence a theoretical Bessel functions framework seems to be an appropriate mathematical ambient for modeling biofilm formation and growth dynamics.

The ideas and the model described here are only the beginning of the modeling possibilities offered by that mathematic framework.

In Queuing Theory utilization of State-Transition-Rate Diagrams is a customary tool for description of a particular queue dynamics under analysis.

We will construct the equivalent diagram for biofilm model, lightning this way an "isomorphism" between the two contexts.

Accordingly with this circumstance, a careful analysis of Clarke classical solution [16] and our old work [12] allows to put in equation and solve many more queues, with finite or infinite storage room, in a unifying way, obtaining, with a unique deduction, an exactly solved general solution.

As we can conclude, by the aforesaid considerations, transient phases in Queuing Theory is a subject, although not the only one, of most interest for our purposes.

But we will leave that broader task for immediate future work.

List of Figures

List of Tables

List of Boxes

Contents

Mathematical tables Transient regime at M/M/1 queuing system.

{Parameters: $\rho = 0,8$, $i = 100$}

Pages: From [0-41/25-29,5] to [0-41/340-358]

CHAPTER I

Introduction

(A Multilayer Model for Early Stages Formation and Growth of Biofilms)

What is called biofilm, in the corresponding technical and (or) so called scientific literature, simply does not exist as a sufficiently well defined entity, and therefore liable to be mathematically modelled.

In fact, a biofilm can be a large panoply of entities.

A biofilm can be bacterial mass formed:

… on the hull of a ship,

… within the various pipes (in industrial facilities, in domestic and hotel facilities, drainage infrastructures, collection and transport of urban sewage, …),

… as part of the overall operating process of a chemical / biochemical reactor,

… as in the oral cavity adhering to the tooth, forming plaque,

… as in many clinical prostheses (e.g. in heart by-pass) thereby disturbing its correct functioning,

… covering the inert support of a percolator filter,

… and so on …

Besides a biofilm can be formed not only for just one species of bacteria but several, and, adding this, not only bacteria can inhabit and (or) interact with it.

Also protozoa, fungi (including yeast), algae, …, … and more?

We can conclude that biofilms have environmental, medical, and industrial implications. Most of biofilm formation consequences are harmful, but beneficial exceptions exist [17].

They grow on almost all surfaces immersed in natural aquatic environments, either on biological surfaces (aquatic plants and animals) or on inorganic surfaces (stones, particles, metals).

Such biofilms degrade polluting organic compounds thus constituting a mean of purifying the environment. This fact is used in the treatment of waste water, percolating filters and fluidized bed reactors, where purposely exist a large surface area for biofilm growth

Biofilms are also used in the industry to purify wastewater before reject them into the sewer. In human and animal intestinal tracts specific "natural" biofilms are formed which protect against pathogens.

However, they also have detrimental effects: they are the seats for the biocorrosion of metals (ship hulls, pipelines, …) and increase liquid flow turbulence inside pipes, and consequently resistance by friction also rises, which implies a negative economic factor.

Also their formation inside heat exchangers makes these almost inoperative decreasing conductive transfer of heat, this decrease not being compensated by the convective transfer increase associated with flow turbulence rise.

From the medical point of view they can infect human body prostheses and implants by strong adherence to their surfaces. This is the case for Staphylococcus epidermidis and other species. Internal bacteria remain protected from antibiotic action and the biofilm will persist as a source of infection, affecting other and several body parts due to desorption of outermost bacteria. In the case of artificial hearts (pacemakers) this can even cause death. In the case of cystic fibrosis, Pseudomonas aeruginosa produces a large amount of exopolymer (alginate), which limits the diffusion of antibiotics, thus also preventing the control of biofilm growth.

Other examples are dental plaque which is a biofilm that can lead to tooth loss and contact lens where bacteria attachment can cause irritation and inflammation.

Biofilms are also, in general, cause of health problems when they develop in domestic water distribution systems.

Bryers gives us his definition of biofilm:

A biofilm is a set of microorganisms, mostly bacteria, in a three-dimensional extracellular polymer matrix secreted by bacteria themselves [18].

We can reply with our definition of town:

A town is a set of living beings, mostly humans, in a three-dimensional concrete matrix, constructed by humans themselves.

The point is that model a biofilm is as meaning (or meaningless!) as trying to analyse in the same context traffic problems and sustainability of towns as disparate as Halifax, Athens, Kabul, Tokyo, Delhi and Timbuktu. Saying only a few!

"Community" is, in fact, a frequently used word in biofilm literature to characterize the aforesaid panoply of situations and (or) systems.

"Community" is also what a town represents. And the degree of complexity is similar for both systems of systems. Similar in the sense that both are too complex for description in only one set of model starting assumptions.

Biofilm literature is replete of qualitative concepts and also replete of mathematical models that are not much more than a mixture of known equations, all of them joined together with little criteria, in a computer tool which provides, as output, nice graphic visualizations.

One can always conclude that those graphic visualizations are in good harmony with our general, intuitive and preconceived, ideas about this subject (or subjects).

Lets now refer a few of those aforesaid qualitative concepts.

Microbial cells adhere tightly to almost all surfaces. These immobilized cells grow, reproduce, and produce extracellular polymer which packs them in a matrix (Characklis W.G. and Marshall K.C., [19]).

This set (bacteria and polymer) forms a three-dimensional space that is not necessarily continuous because not all the solid surface will be covered. Add to this that it also exhibits a local variation in its thickness and, if it has not reached the steady state , the whole structure varies dynamically over time. So, it is not necessarily uniform in time and space.

Biofilm initial formation is generally described as a consequence of two phenomena: adsorption and growth [20a - 20e].

Bacteria detachment at these biofilm formation early stages is not normally considered for modelling purpose.

Polymers produced by the bacteria are polysaccharides and this may facilitate and strengthen the initial adhesion to polar substrates via hydrogen bonds or dipole-dipole interactions [21].

However a physiological response by bacteria is not always necessary to stick to it, because some adhere quickly and strongly to solid non polar surfaces. In this case the adsorption to the solid will be more probable as the number of collisions increases, and these will depend on two factors: the mobility of the bacterium itself (if it is mobile) and its brownian motion or water currents leading to the solid.

The fact that desorption is not considered in these early stages of colonization can be explained observing that desorption can only be done if the extracellular polymer between the bacterium and the solid itself desorbs, which is unlikely since it comprises many binding sites to the solid.

Bacteria desorption is generally taken in consideration only after the thickness of the biofilm or, if preferred, the height of the colonies is large enough for frictional forces between the biofilm and the liquid flowing above becomes an important factor [22].

It is not, however, only a question of hydrodynamic friction, since the biofilm itself changes in its internal forces of cohesion, diminishing these as it grows, as well as diminish the forces that bind it to the support itself.

The decrease in the intensity of those internal forces of cohesion causes such desorption taking place preferentially at the biofilm-liquid interface by the release of small layers of bacteria, or even of individual bacteria (erosion), whereas the reduction of the forces that bind it to the support leads to a loss of biomass through the rush out of large, in relative terms, biomass volumes (sloughing) [23].

On the other hand, the phenomenon of desorption influences the structure of the biofilm by eliminating its outermost, and generally less dense and more porous zones. Here we understood as porous biofilm not that one with many pores inside the solid phase, but that one which has such a phase consisting mainly of filaments (non-porous) that will, sooner or later, be pulled out by frictional forces.

The initial motivation of this work was to establish a theoretical model that could mathematically translate the evolution, over time, of growth and accumulation of biofilm bacterial mass, not only in global terms but also in a micro scale context, reaching as well a quantitative (statistical) description of locally variable biofilm thickness.

This is possible but one must be aware that facing such variable and complex system(s) a mathematical model, deserving such accurate qualification on proper way, will be only a starting point of view and, consequently, begins by a rude approximation to the simplest situations.

The inclusion of the desorption phenomenon in the very initial assumptions of a model is nevertheless essential, either by erosion and (or) discontinuous sloughing, because these are the only partial phenomenon that implies a limit on the growth of biofilm thickness.

Besides, the positive parameter defined for such purpose can always be arbitrarily small, or even equal to zero, when desorption must be irrelevantly accounted for. Doing things like this, one never loses mathematical generality.

The final structure of a biofilm is influenced by two factors: the consumption of substrate with the consequent growth of biomass and factors that lead to desorption [24].

However this is not enough. One may or, even better, must also include the initial and (or) continuous adhesion in time, also referred to as flocculation. As consequence a bacteria adhesion flux must be defined, similar, but not equal, to Calldwel´s constant attachment rate A [20a - 20e].

It is not a straight expectation that, under conditions of low substrate concentration, and therefore of high gradients not only in the liquid-biofilm interface, but also in the layers adjacent to it (in both directions: liquid and solid) biofilms reach a very filamentous structure or, if preferred, "porous" structure..

With regard to gradients in substrate concentration, it is particularly important, for mathematical modelling purposes, to take into account the internal gradient.

In fact it is the substrate concentration internal gradient that locally determines different values for bacterial growth specific rate. Ultimately, in the innermost volume spaces, such concentration will be zero, thus determining a population of non-breeding bacteria (inactive bacteria).

Usually biofilm development comprises six phases: latent, dynamic, linear, decreasing, stabilization and sloughing phases.

More description details can be obtained in [25].

This Capdeville and Nguyen work will play an important role in the establishment of our model assumptions, namely to take into account the division between active (more external) and inactive (innermost) bacteria.

It is also a very interesting macro semi quantitative description of biofilm kinetics evolution from latent to sloughing phases going through all the known "S-shaped" line.

Coming down to the detailed aspects, biofilm growth kinetics can be described, at least in a qualitative first approach, according to the following six steps, already referred [25].

1º) Latent phase, comprising two sub-phases:

Passive sub-phase, during which organic compounds are adsorbed on solid support surface. It is therefore a strictly physic-chemical phenomenon.

Active sub-phase, during which cell adhesion already starts, being this adhesion reversible or irreversible, all depending on several factors: surface support wettability, nature of the cell membranes and others related to the biological activity of the same (for example, excretion of exopolymers).

2º) Dynamic or "accelerated growth" phase, during which a clear biomass accumulation over the surface is observed, either as a result of already adhering micro-colonies growth or as a result of the arrival and adherence of new bacteria and (or) micro-colonies. When this dynamic phase ends, the surface is completely covered by a continuous film of cells, with a thickness varying between 50 and 100 µm. This film is what we mean by biofilm and is physically constituted by a juxtaposition of numerous colonies of variable dimensions. Ad consequence of such juxtaposition, a variable topography results, from point to point, explaining the variable thickness.

According to B. Capdeville and K. M. Nguyen, at the end of this second phase, the film "average thickness" continues to increase, although, and at the same time, the substrate consumption has already reached a maximum and constant limit value.

They then refer that the "biological potential" of the adherent bacteria population tended to his maximum peak.

These authors reconcile the simultaneous existence of a stationary regime in the liquid phase (with respect to the consumption of substrate) and a transient regime in the biofilm (which continues to grow) through the consideration of two different populations of bacteria in the biofilm:

1) The population of active bacteria, which remove the substrate, and which have a specific growth rate μ_0. These active cells are located either at growing sites of new colonies (sites from which they will originate those new colonies) or at the periphery of existing colonies, and placed to a limited local depth, defined by the substrate gradient.

2) … and the population of the inactive, inert or deactivated bacteria, which do not remove anymore the substrate, although they may still have some enzymatic activity. These inert cells are located within the colonies, and do not metabolize the substrate due to inhibitory phenomena, such as:

Accumulation of fermentation products or inhibitory metabolites within the film, or from bacteria themselves.

Confinement effect related to the appearance of new cells around the previous ones, which will hinder the substrate diffusion to the inner cells and also the fermentation metabolites diffusion, in reverse direction, to the liquid phase (and many of these metabolites are growth inhibitors).

3º) Linear growth phase, in which biofilm accumulates at a (now) constant velocity and, in the liquid phase, the substrate concentration remains constant and at the minimum value already mentioned.

Then the following phases are deceleration (4th stage), stabilization (5th stage) and possibly a final decreasing phase (6th stage). This last one is essentially mechanical, when substantial biofilm parts break and go away. Such parts are thus no longer connected to the remaining biofilm and (or) the support, and become planktonic colonies

This phenomenon is what in literature is designed by sloughing.

It is in the 4th, 5th and 6th phases that the combined desorption, erosion and sloughing phenomena become more important, thus leading to a constant (or even decreasing) average thickness (biofilm does not grow anymore).

Before the end of the dynamic phase the surface is totally covered and the purpose of the prototype model, for this thesis, is to account biomass accumulation since the first bacteria is attached to the surface until all the surface is covered.

Further similar considerations will allow to extend the model, in future work, to the remaining dynamic phase and, after that, till an all linear phase.

So, as we said, in our model we will start and limit ourselves by considering only the first two phases described by Capdeville [25], thus deducing a growth kinetics that begins at the moment when the first bacterium adheres by flocculation to the solid support ($t = 0$), and that ends as soon as all the solid support surface is completely covered. These limits span the entire active sub-phase of the latent phase as well as all the dynamic phase. Pertinent and similar adaptations, will allow to apply and extend the model, in future work, to the remaining dynamic phase and, after that, to the phases of linear growth, deceleration, stabilization and stabilization with sloughing at unexpected times each one followed by a raise to the "steady" plateau and (or) even a decreasing phase due to starvation. .

Situations in which there is no continued adherence flux of new cells, but only the growth of a pre-existing small patchy biofilm (we may call it an " inoculum") can as well be contemplated establishing initial conditions with non bare support area fractions, in general even an initial statistical distribution. This will be briefly referred, but leaf to future work.

Additionally to the division of bacteria into two populations, active and inactive, it is necessary to establish a biofilm geometric idealization that adequately divides the three-dimensional space it occupies, and on which appropriate biomass balance equations will allow to describe biofilm growth kinetics quantitatively.

Consideration of substrate concentration internal gradient is also compulsory and will be in close relation with the aforesaid division of bacteria into two populations. Later we will return to, and remind, this point, establishing pertinent assumptions.

The purpose of the model to be constructed is to establish not only a formal global kinetic account, in mathematical terms, of the biomass accumulation process on a flat solid support, but also a transient statistical description of biofilm "orography".

The starting assumptions refer to the early phenomena, which are generally designed as "surface colonization" (Caldwell group [20a - 20e]).

However, this description of the phenomenon, based on the rather chaotic geometric image of "colony", is not necessarily the only one adequate to the idealization of the progressively accumulated biomass on the solid flat surface. An analogy has already been made between bacterial adhesion and molecular adsorption, favouring the application of the Langmuir isotherm, with suitable adaptations and modifications, to the adhesion of bacteria (Fletcher, [21]). However this isotherm is only valid for a single-layer coverage and biofilm is a three-dimensional object.

In the same way as the BET isotherm is a more realistic geometric idealization than Langmuir's isotherm, once it predicts stacking in overlapping layers, the proposed model will also consist of admitting accumulation of cells by adhesion not only on the support but also on other cells already adhered, thus giving rise to an analogous spatial overlap. The model therefore predicts, for a start, biofilm formation due to an initial, and continued over time, adsorption process. Flocculation is the term used by Wanner, O. and Guger, W. [26].

Besides this process, it is also necessary to simultaneously conjugate the proper adhered bacterial mass growth dynamics through the adequate definition of local specific growth rates (μ) that will be different for different locations within the biofilm, as well as a process of desorption ("dettachment"). This last process will be required, as we said before, so that the biofilm does not increase indefinitely in volume.

Based on the simultaneity and mutual competition of all these processes the model will describe very different "topographies" for the progressively accumulated biofilm.

To achieve this goal, principles and simplifications are established, constituting the starting "axioms" and, based on them, we deduce the biofilm geometric shape progression over time.

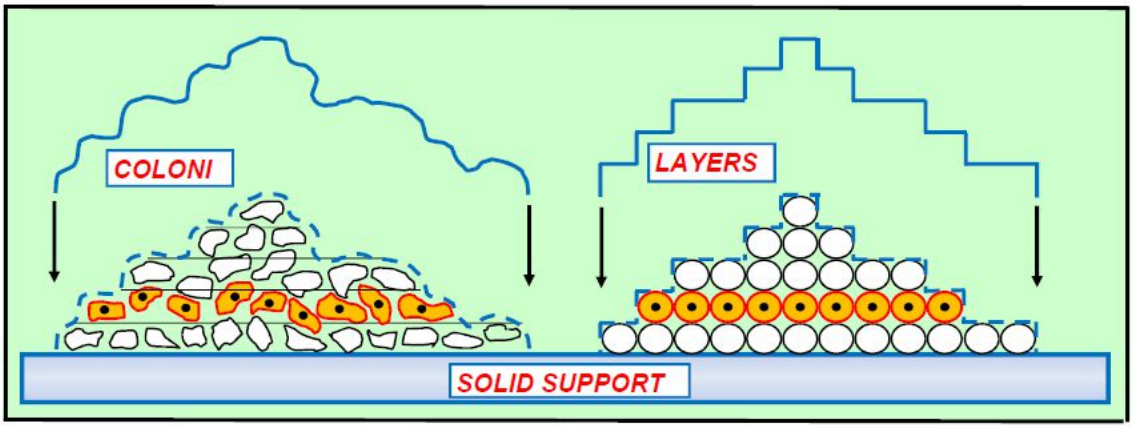


Figure 1 The model geometric image is defined dividing the biofilm occupied space by layers each of them with height equal to the mean linear bacteria dimension (including the surrounding extracellular polymer). Each bacterium belongs to the layer in which its centre of gravity is located. For example, in colony layer nº 2 there are 9 bacteria and consequently, the same number is represented in the model geometric image idealization, on the right side of the figure.

Figure 1 shows the correspondence between the concepts of accumulation of cells by micro-colonies and superposed layers. This last image is less realistic because it is not as random as the colony configuration is, but it is more "manageable", from the mathematical point of view.

Also invoking mathematical tractability, we now criticise ourselves pointing out explicitly some a priori limitations in our theoretical approach:

- mono species and not multi species biofilms,
- hydrodynamic conditions surrounding biofilm almost neglected, and their effect on erosion substituted by an empirical parameter for that purpose, which also can account detachment,
- sloughing, considered as unexpected and complicate phenomena, consequently leaved for future theoretical refinements, if feasible at all,
- and so on ...

The reward of such great simplifications is, as we shall see, to reach new mathematical formulae for quantitative account of biofilm biomass growth kinetics.

That mathematical framework results from a starting exact solution, obtained after the definition of a, first version, governing system of equations. Later this first version system can be refined, introducing more complex initial assumptions.

From the perspective of Chemical Engineers, interested in their profession theoretical foundations, it is desirable to propose a mathematical model starting by taking just a limited number of initial hypotheses, corresponding to an greatly simplified object study.

But, on the other hand, that simplification must not be more extreme than those already known in models of other contexts. However, in this respect, there is room to build a theoretical biofilm model that is not only a grossly outrageous caricature of reality. As a matter of fact abundant model examples already exist, which assume, in their theoretical first postulates, even more radical simplifications than ours.

In Chemical Engineering, sometimes one associates, more or less consciously, certain entities and (or) phenomena to a certain mathematical machinery. For example, in heterogeneous kinetics inside solid porous catalyst pellets , taking in consideration internal diffusion limitations, to everyone mind occurs the shape of Effectiveness Factor graphic versus Thiele Modulus as well, almost without discerning , that all this theoretical context involves hyperbolic functions mathematics.

In models for the formation and growth of biofilms this association of ideas does not yet exist but, as will be shown, the content of this work points a way towards a mathematical framework dominated by Bessel functions of the first kind, $J(x)$, and also the modified ones $I(x)$.

Add to this, another happy remark allows, as we shall see, to connect functions $J(x)$ with continuous biofilm and functions $I(x)$ with patchy biofilm. In dynamic system concepts with permanent growth state and steady growth state, respectively for continuous and patchy biofilm

The entire model will be based on six very simple and quite obvious ideas:

-1st idea)

A "geometric- geographic" idea: It is strange that no one has remembered yet to apply a "geographic" representation of locally variable biofilm thickness employing the concept of level contours as a describing method of biofilm structure. Like in maps describing the orography of mountains.

This "geographic" concept of level contours is side by side with the concept of parallel layers to the inert support plan. The level of each contour coincides with an entire number of layers. Such concept is also implicit in isotherm BET theory for molecular multi layer adsorption over a solid inert surface. In Figure 1 we found a schematic illustration of this concept.

-2nd idea)

The idea of the flocculation (or adhesion / attachment) arrival flow of bacteria to bare inert support or to the local areas where there is already one or more layers of adsorbed bacteria.

And it is very simple: lacking a better option, we will consider a constant and uniform arrival flow. This simplified approach can be found also in Douglas Calldwel, and his group work [20a - 20e]. We will apply it choosing adequate dimensions.

-3rd idea)

A view of the biofilm internal bacterial population structure (Bernard Capdeville contribution [25]): It is necessary to divide this population into two sub-populations, one active, with access to nutrient(s) (s) and another inactive, located in the innermost bacterial mass without such access and, consequently, which can´t reproduce.

Adding to this, active population has a decreasing specific growth rate as the distance from local interface biofilm/liquid into biomass volume increases, according to substrate concentration gradient. The model includes, as parameter, the number L of layers, counted in each location from interface biofilm/liquid into biofilm volume, where substrate concentration is not null. In locals with biofilm thickness less than L layers, all bacteria contact with substrate concentration not null, so in that locals all bacteria are active.

-4th idea)

Substrate concentration gradients can be always considered "instantly established" because nutrient(s) spread into biofilm volume is much faster than the phenomena that we are going to model: attachment, growth, detachment, erosion, sloughing, starvation, etcetera.

See, for reference purpose, Figure 2 in [27], where various time-scales for biofilm processes are put together and their relative rates (characteristic times) are quantitatively compared.

-5th idea)

We also add to our model an assumption that bacteria can reversibly belong to active and inactive populations, depending, respectively, whether substrate concentration is not null or null. Explaining in more detail: if, as a consequence of the entire dynamic processes, layers increase in number, the before innermost active bacteria afterwards will no longer be in contact with substrate as the distance to biofilm/liquid interface is greater than L layers; in such scenery bacteria switch instantly from active to non active because they no longer have access to substrate . On the other hand, if the entire dynamic process implies a local decrease in biofilm thickness, the new enumeration of L layers, from biofilm/liquid interface into deeper biofilm volume will reach bacteria that before where inactive and now have switched instantly from inactive to active (now can access to substrate).

-6th idea)

Since the very beginning of surface colonization our model foresaw a parameter bigger than zero to account for partial processes leading to diminishing biofilm biomass, like erosion and desorption. Proceeding like this one prevents starting the model without limiting global growth factors which, sooner or later would be of compulsory inclusion. At an initial stage, to avoid greater mathematic complexity, we don´t distinguish between bacterial desorption from solid surface and bacterial erosion from collective colonies. However the model conception leaves in this, like in other many aspects, room for further refinements and generalizations.

With these six simple ideas, this matter (biofilm) may well come to "navigate" in the context of Bessel functions [$J\ (x)$ and $I\ (x)$], in the same way as heterogeneous catalysis has already "sailed" for many years in the context of Hyperbolic Functions.

These six assumptions lead to deductions that will let one apply a mathematical formalism similar to those found in many other topics of Applied Mathematics, namely in Transient Queue Phases, where exact solutions are akin to those found later, along all this thesis.

They are, however, in all cases difficult to analyze solutions, brushing the edge of intractability.

In the sequel we will resume how current most important models are classified, trying to insert ours in that general scenery. Theoretical models can fall in two types:

a) Individual based models (IbM):

In these models bacterial individual properties are inventoried first, and them one must reach, as objective, a description of biofilm quantitative characteristics as a whole Kreft , J.-U. et al., [28a, 28b]. The most important variables in such models are, the cell mass, its volume, possibly its geometric shape, and so on.

b) Biomass based models (BbM):

In these models the biofilm is considered as a multiphase system.

What is called a "representative volume element" is their starting point, associated with some average values. As an example of an average value: cell mass per unit volume. It seems that such "representative volume elements" must be bigger than bacterial size (typically in the order of ~1μm) and also bigger than the average distance between then.

But, on the other hand, they must be smaller than the scale where biomass changes significantly its physical and (or) (bio)chemical properties.

With "representative volume elements" smaller than this scale the model reaches a "good resolution".

Biomass-based models can also be divided into two sub-classes, depending on how they describe the increase in biofilm volume due to their global biomass spreading:

1) Discrete models (cellular automata) in which biomass expands only through a finite number of directions, those directions corresponding to the neighbouring volume elements, closest to the one firstly, considered as departure point. Biomass expansion takes place according to rules previously established in model initial assumptions.(Picioreanu, C. et al., [29a, 29b]).

2) Continuous biomass models in which the biofilm is described as a continuous phase. Bacterial growth at a certain point imply an increase in local pressure at that point.

This pressure, exerted by the growing biomass, originates a liquid-biofilm interface "movement" towards the liquid, and thus the volume increase is described as a consequence of such "movement" (Eberl, HJ et al., [30] and Dockery, J. and Klapper, I ., [31]).

So we can summarize the previous existing models classification this way (Picioreanu C. and van Loosdrecht M.C.M, [32]).

--- a) IbM (Individual based Model)
--- b) BbM (Biomass based Model) :
    1) DbM (Discrete based Model)
    2) CbM (Continuos based Model)

According to our chosen idealization we can conclude that in the direction perpendicular to the solid flat support the space is divided according to the maximum resolution criterion: the average size of a bacterium, as in an DbM model.

In the other two directions, that is, in the planes parallel to the surface of the solid support, by the introduced simplifications, we can consider that the model is classified as continuous, that is to say: a CbM model.

Several recent reviews about biofilm partial processes and growth kinetics can be found in General Bibliography.

We reserved this Item for references not cited in the text.

However, at a quick glance, one can conclude that nothing like our approach has been proposed till now.

We indicate those references as a source for more, mainly qualitative, information.

In this general bibliography also references about quantitative information (mathematical models) can be found, but they are all much more numerical experiences than deductive new formulae.

All the details of the model, the initial "axioms" as well as the governing initial partial processes equations, will be described rigorously in the next Chapter II.

In Chapter III the get together of all those partial processes will lead to the governing equations and, in the sequel, model resolution will be developed in Chapter IV.

CHAPTER II

Mathematical model development

(A Multilayer Model for Early Stages Formation and Growth of Biofilms)

The model is based on the following simplifying assumptions:

Assumption A) We define a constant irreversible attachment flux (f) of planktonic bacteria perpendicularly over the surface (which is considered to be flat and uniform in all physical and chemical aspects) and also an average projected area per attached bacterium (a_P) including the glycocalyx matrix or extracellular polymeric substances surrounding each bacteria.
The dimensions of f are:

$$\frac{Number\ of\ bacteria}{Arrival\ area\ (S_T) \cdot Time\ (t)} \equiv L^{-2}T^{-1} \qquad [\mathbf{2.1}]$$

However, for deduction and calculation purposes, it is more convenient to define the flux F:

$$F = a_P \cdot f \qquad [\mathbf{2.2}]$$

where a_P is the flat carrier area covered by a single attached bacteria. Such area includes not only bacteria own dimension but also the surrounding extracellular space in biofilm. In Figure 2 we distinguish this area a_P from the area a_P^* of a bacterium not yet adhered to either the bare support or the other bacteria already in the biofilm. In the sequel we will designate those not adhered bacteria as "planktonic".
For our modelling purposes it is a_P the area that must be accounted for, rather than a_P^* .
The dimensions of F are:

$$\frac{Number\ of\ bacteria \cdot a_P}{Arrival\ area\ (S_T)\ \cdot\ Time\ (t)} \equiv T^{-1} \qquad [\mathbf{2.3}]$$

(Such as those of a specific growth rate.)

Assumption B) Instead of being physically constituted by randomly juxtaposed bacterial colonies, a biofilm structure consisting of bacterial layers which are locally overlapped one to each other is assumed. Thus, the non-uniformity in film thickness is determined by the number of overlapping layers that will exist, at a given instant, at each point of the flat support surface (such layers number will vary from point to point and from time to time).

Assumption C) The size of each bacteria, both in terms of volume and particularly in terms of projected area, already defined as a_P, is considered uniform or, alternatively, constant and equal to a weighted average value between the dimension at birth time (or subdivision of cells time) and the dimension in later instants, till the eventual next subdivision. This dimension includes the adjacent exopolymer.

Assumption D) The model do not distinguish between different bacterial orientations relative to the support plane, because don´t establish any particular bacteria shape geometry. We can therefore consider, for modelling purposes, each local layer of bacteria as a geometric continuous entity, changing in time.

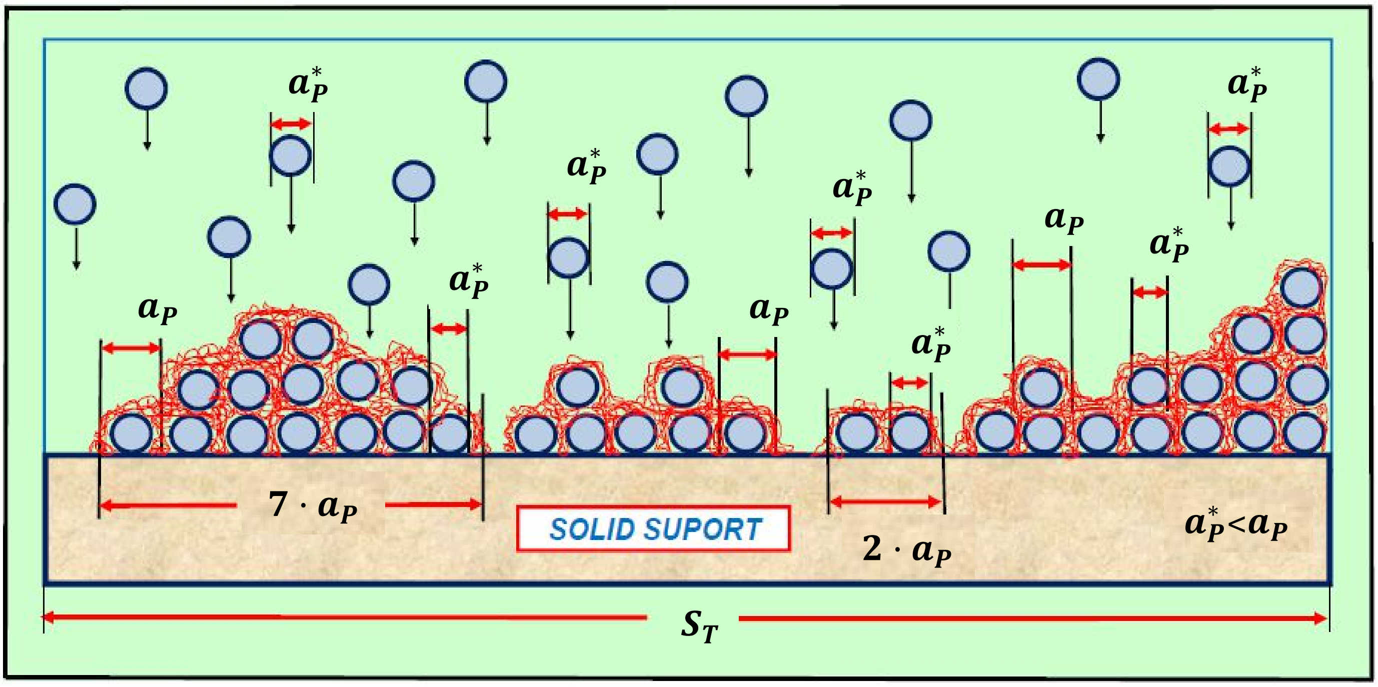


Figure 2 - In this figure we represent the bacteria by circles, as in Figure 1, but we distinguish between the projected area of the planktonic bacteria (a_P^*) and that of the bacteria in the biofilm (a_P). The latter is larger because it includes the adjacent exopolymer.

Assumption E) The total surface of the support, S_T, is subdivided into surfaces S_0, S_1, S_2, ..., S_n, ... corresponding to the area values respectively, uncovered, covered with one layer, covered with two layers, ... , covered with n layers, ..., .
Physically, each area value S_n, $(n = 0, 1, 2, \ldots, n, \ldots)$ does not evidently represents a single continuous surface. Each S_n, is segmented and distributed across multiple locations over the support surface. In Fig. 3, as well as in the respective legend, this segmentation is illustrated and explained.

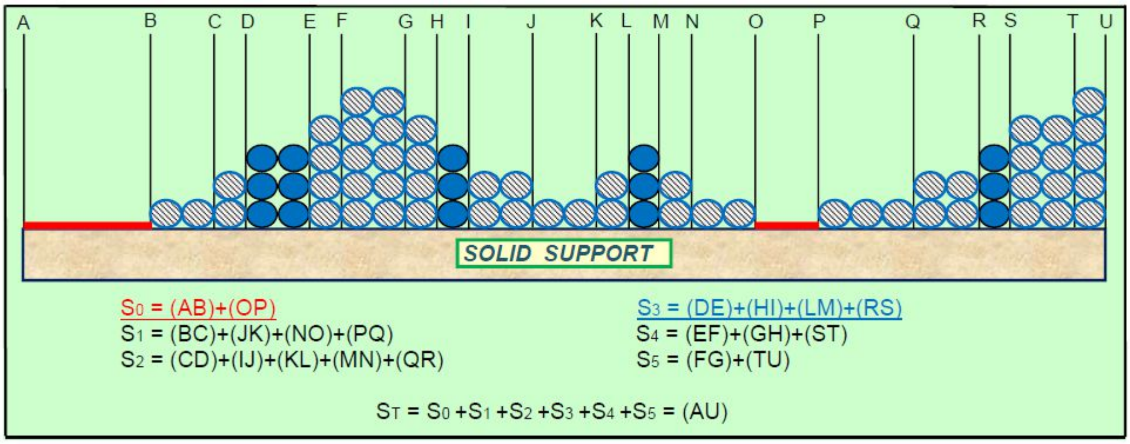


Figure 3 - This figure illustrate and explains the S_T quantitative segmentation referred in the text.

Assumption F) We define an integer value L, as a model parameter, that represents the maximum number of active layers in each area S_n, numbered from top to down. Thus, if $0 < n < L$, all the layers of the corresponding S_n will be build up by active bacteria and, if $n > L$, then the $(n - L)$ innermost layers in S_n are made of inactive bacteria. Those bacteria do not reproduce. Figure 4 shows this division in active and inactive populations for the simplest case: $L = 1$.

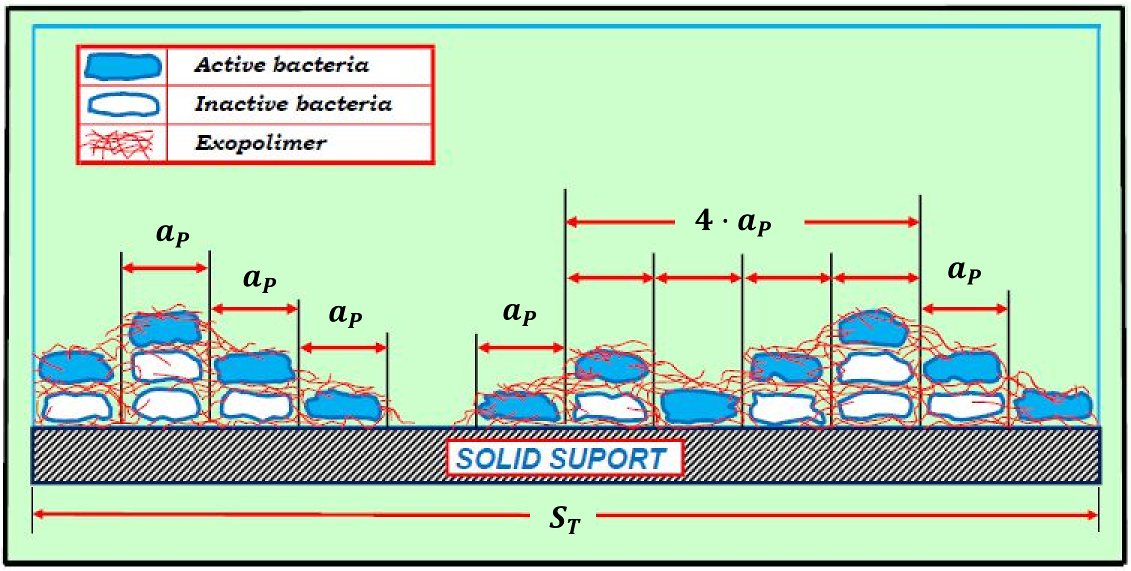


Figure 4 - Bacteria in the biofilm can be active (blue), if they are located in the outermost layers, where the concentration of substrate is still not zero, or inactive (white) if located at greater depths inside the biofilm, where there is no substrate (depletion zones). Only the active bacteria will grow and reproduce. The number $L \geq 1$ of outer layers considered active is a model parameter. In this figure $L = 1$. For the more realistic situation, with $L > 1$, a substrate concentration gradient must be included in this active zone as another model assumption.

Assumption G) We account for two specific growth rates: one ($\mu^{\rightarrow}$) in the direction parallel to the support if, when one cell divides in two, they remain in the same layer and another one ($\mu^{\uparrow}$), perpendicular to the support if one of the two daughter cells is conditioned in the immediately above adjacent layer. Both specific growth rates depend on the kinetic equation to be locally applied and, consequently, they also depend on substrate concentration in the bacterial layer in consideration, that is, the local inner biofilm point in question. Schügerl and Bellgardt [33] present a list of known kinetic equations: Monod, Blackman, Haldane, ... , either of which can be used as input of the model.
Exhaustive description of such multiplicity of choices is an intricate task and would confer to the model an enormous amount of work, deductive and computational. However we will return to this subject in assumption I, below.

Assumption H) For the sake of simplicity, we postulate that, in the above-mentioned active layers, the substrate concentration decreases linearly along the depth from the liquid-biofilm interface to inner layers, and that the value in the outermost top layer (which is the layer number n in S_n) is the same for all S_n. We can designate this common value of the substrate concentration in the layer of order n in S_n, as C_1.
This means that liquid phase substrate concentration gradient accommodates to the biofilm external topography in a "parallel" way, like shown in figure 5.
Due to the linearity of the substrate concentration gradient, inside the biofilm, the value we must assign to a still active layer of order i, in which $(n - L + 1) \leq i \leq n$, must be:

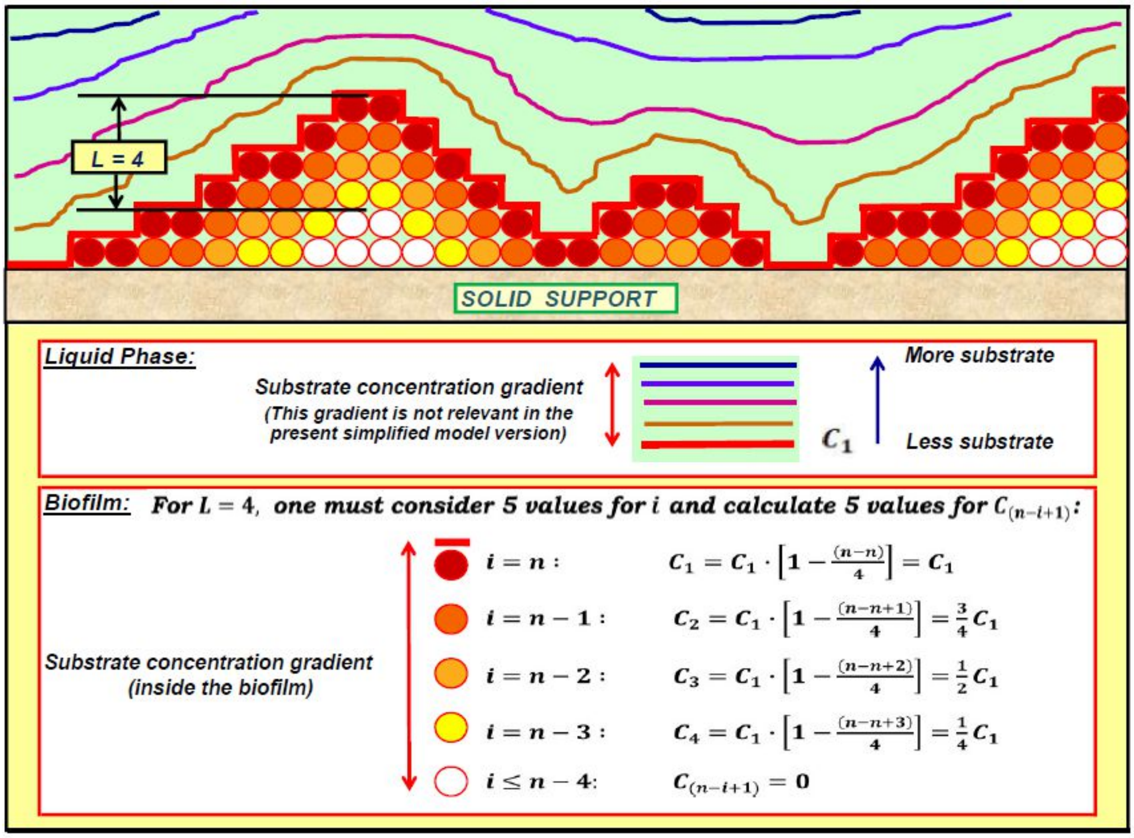
$$C_{(n-i+1)} = C_1 - \frac{(n-i)}{L} \cdot C_1 = C_1 \cdot \left[1 - \frac{(n-i)}{L}\right] \qquad [2.4]$$

Note that when $i = (n - L)$ this formula is also valid because gives $C_{(L+1)} = 0$, as expected for a layer already inactive.
In general, and for all the layers of order i located on the surface S_n, we can attribute a substrate concentration given by $C_{(n-i+1)}$:

$$\boldsymbol{C_{(n-i+1)}} = \begin{cases} \boldsymbol{C_1 \cdot \left[1 - \frac{(n-i)}{L}\right] \ldots if \ \ (n-L) \leq i \leq n} \\ \\ \boldsymbol{0 \ldots if \ \ i \leq (n-L)} \end{cases} \qquad [2.5]$$

(Observe that, for the borderline case, $\boldsymbol{i = (n - L)}$, both hypothesis coincide)

A clear diagrammatic explanation of these calculus can also be found in Figure 5.



<u>Figure 5</u> - Active bacteria in the biofilm access to different not null substrate concentrations depending of their respective layers location. This means that active population is not homogeneous, being the innermost ones those with access to less substrate. In this figure such active population heterogeneity is illustrated for L = 4 *assuming a linear substrate concentration gradient, decreasing from top to down till to attain null substrate concentration, where bacterial becomes inactive (depletion zones). Those bacterial (in white) will not grow and neither reproduce.*

<u>Assumption I</u>) As a consequence, and according to assumption G, we must assign to every single layer of order $\boldsymbol{i}$, being a building part of every $\boldsymbol{S_n}$ surfaces, two specific growth velocity values: one for accounting parallel growth to the support ($\mu^{\rightarrow}$) and another for accounting perpendicular growth ($\mu^{\uparrow}$). Such assign requires a kinetic law, defining how local substrate concentration determines, in a formal mathematical relationship, the aforesaid specific growth rates values.

Monod kinetics equation is the most popular and generalized one, among all listed by Schügerl and Bellgardt [33], and we will also chose it here as the most general model input.
By choosing Monod equation, as we shall see, one faces to great mathematical complexity leading to an inherent and almost inevitable need for definition of simpler formal relations. We will define below four of such possible simpler relations.
Before proceed to adapt known bulk kinetic processes to confined space ones, inside the biofilm, let review Monod equation:

$$\mu = \frac{\mu_{máx.} \cdot C_S}{K_S + C_S} \qquad [2.6]$$

... and, put it in dimensionless form,

$$y = \frac{x}{1+x} \qquad [2.7]$$

... where,

$$x = \frac{C_S}{K_S}, \text{ and } y = \frac{\mu}{\mu_{máx.}} \qquad [2.8]$$

In figure 6 a plot of [2.7] can be found and also the corresponding plots for the aforesaid four simpler formal relations that we now enumerate.
- "Zero order inspired" kinetics, in the sense that y (and so also μ) is insensible to substrate concentration changes. Consequently if we start by defining, in each upper layer, a specific growth rate value corresponding to point B, we maintain this value in all the others innermost still active layers.
- Other simplification is "Blackman inspired" kinetics in which the specific growth rate value of point B is also adopted in some next layers located more inside the biofilme but only in a enough number for substrate concentration to achieve a value as low as that of point F (see Figure 6). This point is calculated considering that straight line AF is tangent to Monod curve at point A.
- The "averaged" first order kinetics means to follow straight line AB, from B to A, as deep inside the biofilm increases. In this case, substrate concentration and bacterial specific growth rate both decrease linearly when profundity inside biofilm increases.
- Lastly the "mono layered concentrated growth" kinetics ($L = 1$), in which all the biological growth potential of bacterial inside the biofilm is concentred and accounted only in the outermost layer of every area S_n. Such a radical simplification allows us to reach a close form exact solution after a quite evolving deduction. From the details of that deduction is possible extract a few insights, namely to establish guess solutions, for solve the other more complicated cases. Another particularly interesting fact is that, if $L = 1$, all the five described hypothesis coincide in this choice of L.
An immediate observation of Figure 6 concludes that is possible to obtain an ordered classification according to the biological growth potential of each kinetic. In fact, in any substrate concentration, "zero inspired order" always exhibits the biggest specific growth rate, and from point B to point F equals the "Blackman inspired" case. This one has the second biggest biological growth potential (from point F to C is less than the "zero order inspired" case).
Monod kinetics is, in all substrate concentration values, except in points A and B, under those two. Then the "averaged" first order kinetics is under Monod red curve. And lastly the "mono layered concentrated growth" kinetics exhibits the smaller biological growth potential of all the five described cases.

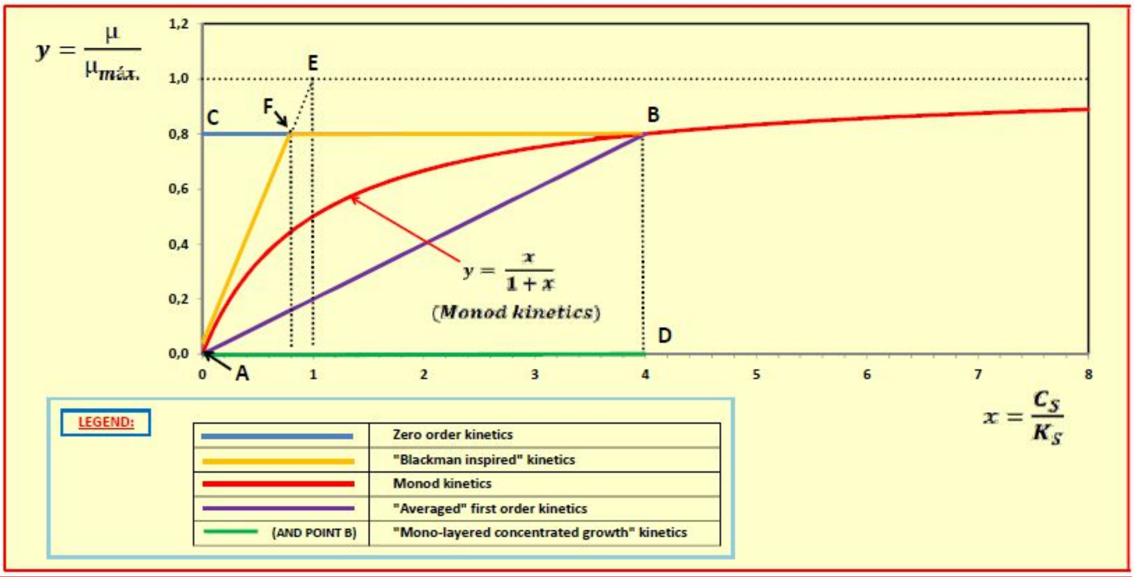


Figure 6 - *Point A to point B, in the red curve, represents the Monod kinetics plot in dimensionless form. Substrate dimensionless concentration at point B (x_B) represents the concentration at the highest layer of each partial area S_n (layer of order n). So x_B is the dimensionless concentration $x_B = C_1/K_S$, and, for coherence in notation, must be written $x_B = x_1$ ($x_1 = 4$, in this case).*
Point B substrate concentration (x_1) and the corresponding specific growth rate (y_1) refers both to conditions in the uppermost layer of all partial areas S_n (n>0) in the total plane support area S_T (layer of order $n = 1, 2, \ldots$.
Monod kinetics is the more realistic and generalized choice for account quantitative specific growth rates of biofilm active bacterial population. However, direct insertion of Monod kinetics in the model framework is a very difficult task, near mathematic intractability.
Point A to point F and point F to point B , straight orange lines, represents an hypothetic "Blackman inspired" kinetics plot, also in dimensionless form, as all the others. It constitutes an upper bound to the Monod kinetics plot, in terms of bacterial growth potential.
Point C to point B , straight blue line, represents a zero order kinetics plot. It constitutes also an upper bound to the Monod kinetics but, from point C to point F, a less tight bound than the "Blackman inspired" kinetics plot.
Point A to point B, straight violet line, represents the "averaged" first order kinetics plot. It is a lower bound to the Monod kinetic plot.
Point A to point D, straight green line replacing, at the end, point D by point B, represents the "mono-layered concentrated growth" kinetics plot. It is also a lower bound to the Monod kinetics plot, but less tight than the "averaged" first order kinetics in all the range of x, from $x = 0$ to $x = x_B$. In this plot $x_B = x_1 = 4$ (only for exemplifying purposes).

Since Monod kinetics is the essential and more realistic one, being all the others considered as simplified approximations, we can classify these ones, in mathematical terms, according of their biological growth potential, as upper and lower bounds.
It is a known strategy, in many hard mathematical models, the approximation consisting to solve it for an upper and also for a lower bound, thus beaconing the exact, tough to obtain, solution. The greater will be the tightness of both bounds better will be the approximation achieved.
In the aforesaid description "Blackman inspired" kinetics represents a tighter upper bound than "Zero order inspired", which is also an upper bound.

We can call "Blackman inspired" kinetics the near upper bound, and "Zero order inspired" kinetics the far upper bound, in a relative criterion.
In the same way, "averaged" first order kinetics can be designed the near (so the tighter one) lower bound and "mono layered concentrated growth" kinetics the far lower bound.
We can always insert the substrate concentrations linear gradient in any other assumed kinetic equation: Haldane, Blakman, Teissier, Aiba and Edwards, Webb, and many others (see Schügerl and Bellgardt [33], for their mathematical definitions).
More detailed diagrammatical illustrations for the five kinetics described in Figure 6 can be found in Figures 7, 8, 9, 10 and 11, at the end of this Chapter. In all those diagrams we provide a clear and rigorous definition of the layers, their nature and role in biofilm architecture, as well as their location related identifying labels.
Returning now to Monod equation [2.6], the more suitable way to insert growth anisotropy along the tow directions, parallel and perpendicular to the flat solid support, is to decompose $\mu_{máx.}$ in two parcels: $\mu^{\rightarrow}_{máx.}$ and $\mu^{\uparrow}_{máx.}$, accounting respectively the mentioned parallel and perpendicular specific growth rates.
Consequently, from formulas [2.6], [2.7] and [2.8] we can get:

$$\mu_{máx.} = \mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.} \qquad [\mathbf{2.9}]$$

$$\mu = \frac{(\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.})\cdot C_S}{K_S+C_S} = \frac{\mu^{\rightarrow}_{máx.}\cdot C_S}{K_S+C_S} + \frac{\mu^{\uparrow}_{máx.}\cdot C_S}{K_S+C_S} \qquad [\mathbf{2.10}]$$

To the constant $\mu_{máx.}$ in the Monod equation we attribute two different values for $\mu^{\rightarrow}_{máx.}$ and $\mu^{\uparrow}_{máx.}$, as consequence of this decomposition, and we maintain K_S equal in equations [2.11],

$$\mu^{\rightarrow} = \frac{\mu^{\rightarrow}_{máx.}\cdot C_S}{K_S+C_S}, \quad \text{and} \quad \mu^{\uparrow} = \frac{\mu^{\uparrow}_{máx.}\cdot C_S}{K_S+C_S} \qquad [\mathbf{2.11}]$$

$$\mu = \mu^{\rightarrow} + \mu^{\uparrow} \qquad [\mathbf{2.12}]$$

For the limit cases of orders zero and one, only equations [2.10] and [2.11] need to be rewritten.
First order (K_S>>C_S),

$$\mu = \frac{(\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.})\cdot C_S}{K_S} = \frac{\mu^{\rightarrow}_{máx.}\cdot C_S}{K_S} + \frac{\mu^{\uparrow}_{máx.}\cdot C_S}{K_S} \qquad [\mathbf{2.13}]$$

$$\mu^{\rightarrow} = \frac{\mu^{\rightarrow}_{máx.}\cdot C_S}{K_S}, \quad \text{and} \quad \mu^{\uparrow} = \frac{\mu^{\uparrow}_{máx.}\cdot C_S}{K_S} \qquad [\mathbf{2.14}]$$

… and, zero order (K_S<<C_S),

$$\mu = \mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.} \qquad [\mathbf{2.15}]$$

$$\mu^{\rightarrow} = \mu^{\rightarrow}_{máx.}, \quad \text{and} \quad \mu^{\uparrow} = \mu^{\uparrow}_{máx.} \qquad [\mathbf{2.16}]$$

In the case of zero-order kinetics, the substrate concentration internal gradient will not influence the model, except in what concerns its active bacterial depth (L).

<u>Assumption J</u>) We postulate that the relative relation between the values of $\mu^{\rightarrow}_{máx.}$ and $\mu^{\uparrow}_{máx.}$, and consequently also between $\mu^{\rightarrow}$ *and* $\mu^{\uparrow}$, is equal in all layers of order i of all surfaces S_n. That is, the ratio is the same across all the biofilm active bacterial population. We only need to define one parameter and we chose the fraction of the total growth that is effected in the support perpendicular direction.

That is,

$$\Phi = \frac{\mu^{\uparrow}_{máx.}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}} = \frac{\mu^{\uparrow}}{\mu^{\rightarrow}+\mu^{\uparrow}} \qquad [2.17]$$

Defined Φ, the fraction of the total growth that is effected in the support parallel direction is also immediately defined,

$$1-\Phi = \frac{\mu^{\rightarrow}_{máx.}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}} = \frac{\mu^{\rightarrow}}{\mu^{\rightarrow}+\mu^{\uparrow}} \qquad [2.18]$$

Now it is straightforward to put this anisotropic scenario in dimensionless terms, building homologue equations corresponding to [2.7] and [2.8].
From [2.11], dividing by $\mu = \mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}$,

$$\frac{\mu^{\rightarrow}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}} = \frac{\mu^{\rightarrow}_{máx.}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}}\cdot\frac{C_S}{K_S+C_S}, \quad \text{and} \quad \frac{\mu^{\uparrow}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}} = \frac{\mu^{\uparrow}_{máx.}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}}\cdot\frac{C_S}{K_S+C_S} \qquad [2.19]$$

… and, defining the dimensionless parameters,

$$y^{\rightarrow} = \frac{\mu^{\rightarrow}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}}, \quad \text{and} \quad y^{\uparrow} = \frac{\mu^{\uparrow}}{\mu^{\rightarrow}_{máx.}+\mu^{\uparrow}_{máx.}} \qquad [2.20]$$

… one gets,

$$y^{\rightarrow} = (1-\Phi)\cdot\frac{x}{1+x}, \quad \text{and} \quad y^{\uparrow} = \Phi\cdot\frac{x}{1+x} \qquad [2.21]$$

Of course,

$$y^{\rightarrow} + y^{\uparrow} = \frac{x}{1+x} = y \qquad [2.22]$$

… as expected.

<u>Assumption K</u>) A process of biomass desorption (or detachment) by erosion is included in the model. In this process biofilm bacterial loss is rendered concrete by the outermost layer partial desorption.
For each layer of order n in all partial areas S_n ($n > 0$) a kinetic constant k_n is defined by,

$$k_n = \frac{(Bacterial\ desorbed\ number\ from\ layer\ of\ order\ n\ in\ S_n)\cdot a_P}{S_n\cdot Time} \qquad [2.23]$$

The dimension of k_n is $[Time]^{-1}$ like those of a specific growth rate.

Detachment processes generally only have quantitative relevance at more advanced time in biofilm formation. An elegant method for modelling this erosion, accounting this increasing intensity over time, can be to establish an explicit empirical expression for k_n $(n = 1, 2, \dots, n, \dots)$, in such a way that the erosion intensity increases as n increases.
An example to explore could be a power law $k_n = k_1 \cdot n^{\alpha}$
In any case, we do not establish, in this model simplified version, a definitive choice for an explicit k_n formula.
Nevertheless we observe that, simply taking into account an indexation to n one can always introduce this variation with the local biofilm height n.
Being such indexation an empirical relation we have some freedom in choosing this variation, in mathematical terms.
In the present model version, however, we will not include any detachment increase intensity variation with the local n height of the biofilm. The power law can be included in the Excell spreadsheet, or any other computation tool, but we have always used in our old model version, $\alpha = 0$. So in ours analyzed cases all the constants k_n will have the

same value k_1 in all surfaces S_n. It will be analyzed the effect, in biofilm structure, of the variation in this k_n $(= k_1)$ parameter common value.
As previously indicated, and in the logic sequel of Figure 6, each one of the five kinetic choices illustrated there is now represented in diagrams in the next Figures 7, 8, 9, 10 and 11, respectively for Monod, "Blackman inspired", zero order, "averaged" first order and "mono-layered concentrated growth" kinetics.

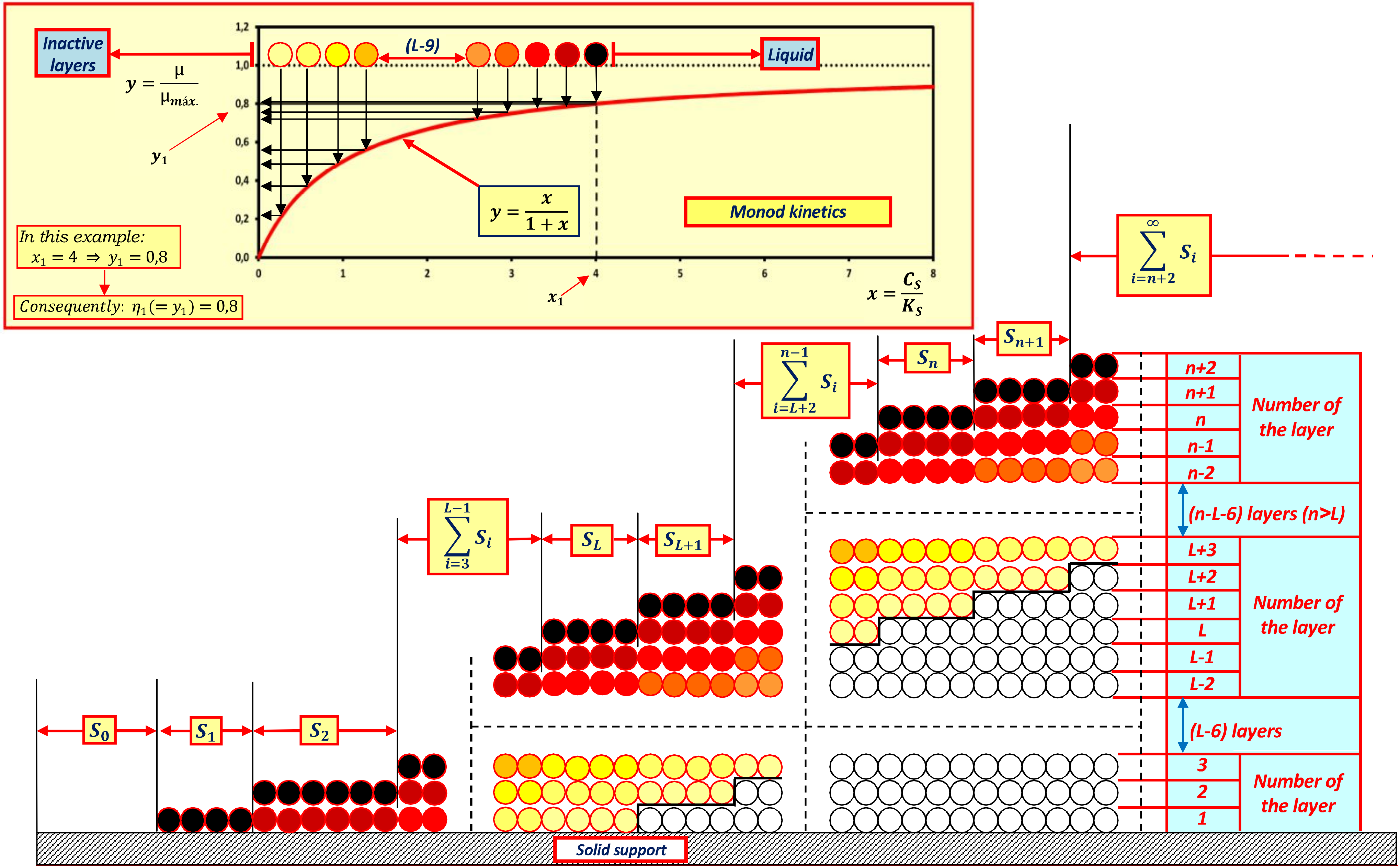


Figure 7 - Spatial distribution of biofilm layers according to Monod kinetics. Active bacterial population is located in the L outermost layers or, in places where biofilm don´t have such great thickness, in all layers deep enough to reach support. However active population is not, for model purposes, considered homogeneous because the substrate concentration diminishes linearly with deep, inside the biofilm. As consequence of this linear gradient also the bacterial specific growth rate diminishes with deep but, not linearly. Bacterial specific growth rate will diminish according to the kinetic mathematical equation relating with the substrate concentration. For Monod kinetics, which is the most usual adopted equation, this variation is illustrated in the dimensionless graphic in the upper left side of the figure.
Besides L, also x_1, the dimensionless substrate concentration in every outermost layer, is a model parameter. We assume that this is also the substrate concentration all over the liquid surrounding the biofilm.
On the right side of the diagram the system of numbering all the layers is explicitly represented without ambiguity. Also all the little boxes above the layers define precisely the meaning of all the partial areas S_j $(j = 0, 1, 2, \dots, j, \dots)$.

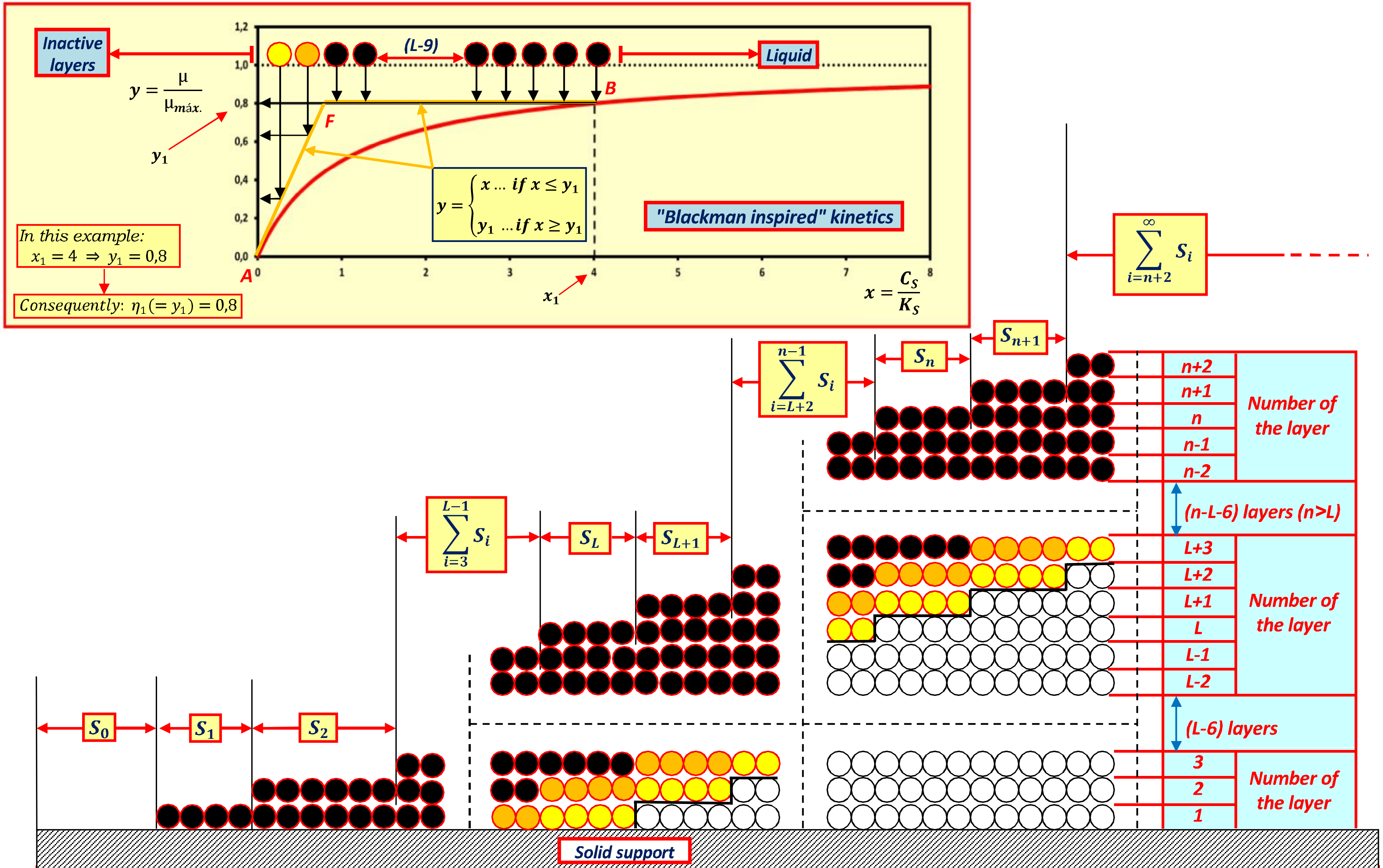


Figure 8 - Spatial distribution of biofilm layers according to "Blackman inspired" kinetics. The only difference between this figure and Figure 7 is the change in the kinetic equation. In the dimensionless graphic in the upper left side of the figure one can verify that this is a simplification of hyperbole Monod curve. The criterion is to approximate Monod kinetics by a first order one if substrate concentration is low, and by a zero order kinetics if substrate concentration is high.
The derivative of Monod equation at $x = 0$ is 1, corresponding to the straight line (in orange) from point A to point F (coherently with Figure 6). Then, at point F, that straight line intercepts the horizontal one corresponding to $y = y_1$ (and, of course also $x = y_1$). We must remind that x_1 is an initial parameter of the model. For substrate concentrations $x > y_1$ this kinetic approach becomes like a zero order one, and the orange line becomes horizontal from point F to point B.
Active population, from every outermost layers till an inside one where x diminishes (linearly) to a small value as y_1 have the same specific growth rate. This homogeneity is signalized, in the diagram, by using the same colour (black) for those layers.
The system of numbering all the layers (in the right side of the diagram) and all the little boxes above the layers, are equal to those of Figure 7, and define all the model geometry without ambiguity.

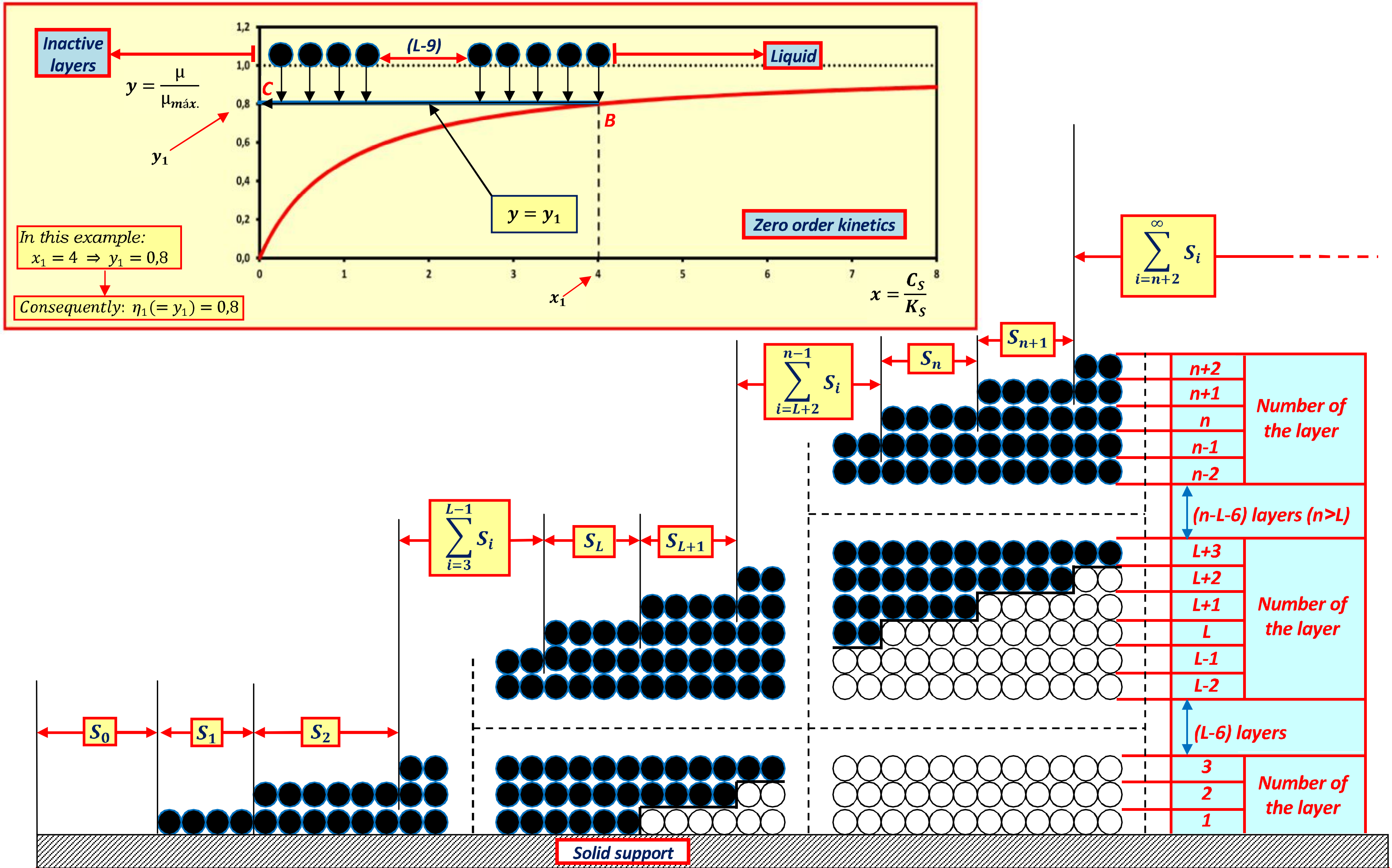


Figure 9 - Spatial distribution of biofilm layers according to zero order kinetics. This figure is like Figure 8 without consideration of the first order kinetic range, in small substrate concentrations. In the dimensionless graphic in the upper left side of the figure one can verify that a constant specific growth rate equal to y_1, in dimensionless form, is assumed all over the entire bacterial active population. Such homogeneity is illustrated by using the same collar (black) all over such active population..
In this case the linear, decreasing with deep; substrate concentration gradient don´t have any influence in the specific growth rate which maintains is value, of y_1, all over the active population.
In this figure, and also in early Figures 7 and 8, as well as in next Figures 10 and 11, the linearity of the substrate concentration gradient is illustrated by the mutual equidistance between the circles on top of the kinetic graphic (in upper left side).
Differentiation in the collar circles, starting with black and successively fading away to near white, illustrates, in that order, the decreasing in specific growth rates (also in all the figures, from 7 to11).
The system of numbering all the layers (in the right side of the diagram) and all the little boxes above the layers, are also equal in those figures (7 to 11), and define all the model geometry without ambiguity.

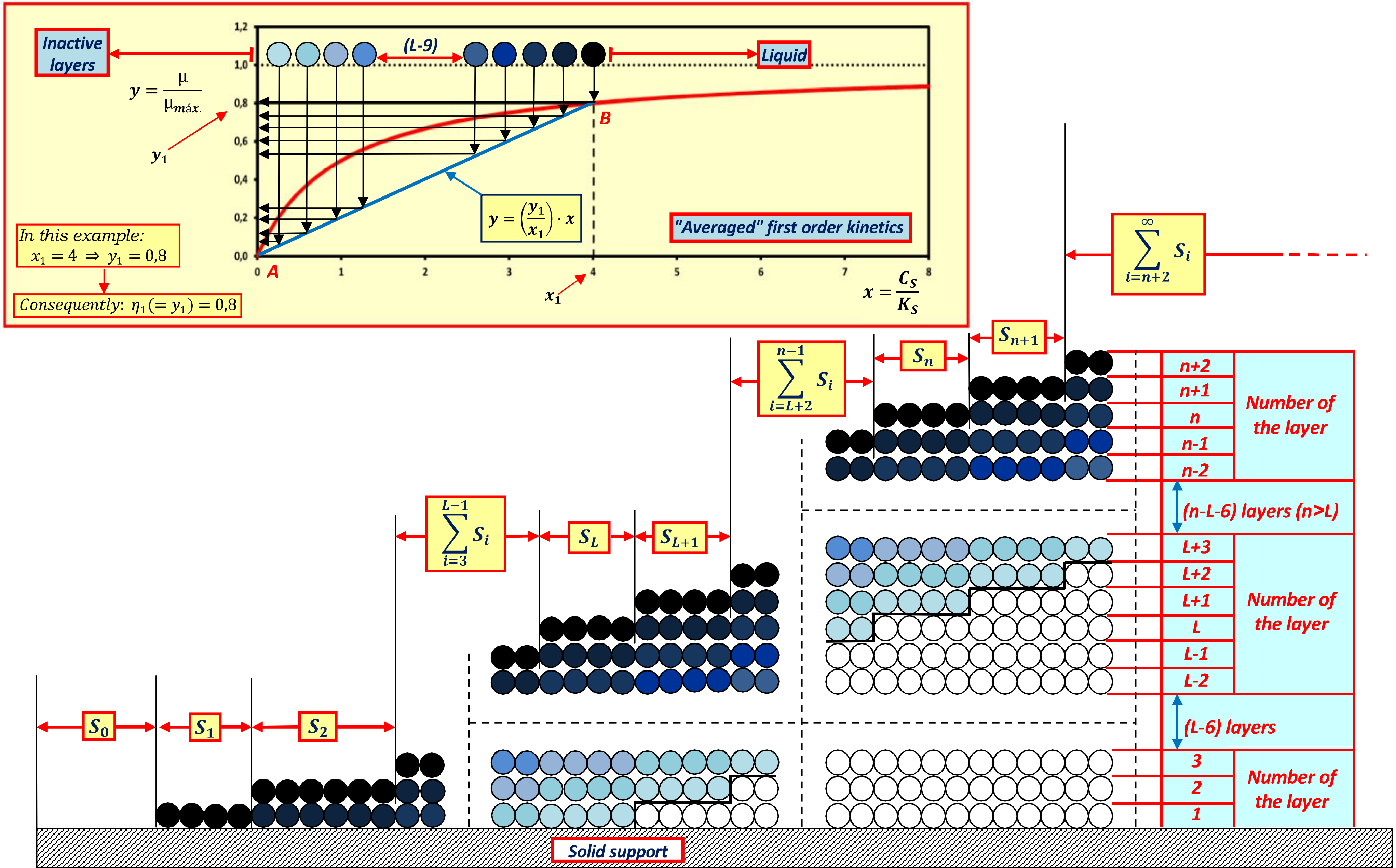


Figure 10 - Spatial distribution of biofilm layers according to "averaged first order" kinetics. This figure simply assumes that K_S is notably greater than C_S ($K_S >> C_S$) what is equivalent to ($1 >> x$) in all the range under analysis. This implies that we can to neglect x in the denominator of Monod equation. Add to this, the point $x = x_1$ and $y = y_1$ must belong to the kinetic first order straight line. So the slope of that straight line is (y_1/x_1), and that line is as drawn in the upper left dimensionless graphic in blue collar, from point A to point B. In this case not only the substrate concentration gradient decreases linearly with deep. The bacterial specific growth rate also decreases linearly with deep inside the biofilm.
Differentiation in the collar circles, starting with black and successively fading away to blue (near white) illustrates, in that order, the decreasing in specific growth rates.
The system of numbering all the layers (in the right side of the diagram) and all the little boxes above those layers define all the model geometry without ambiguity.

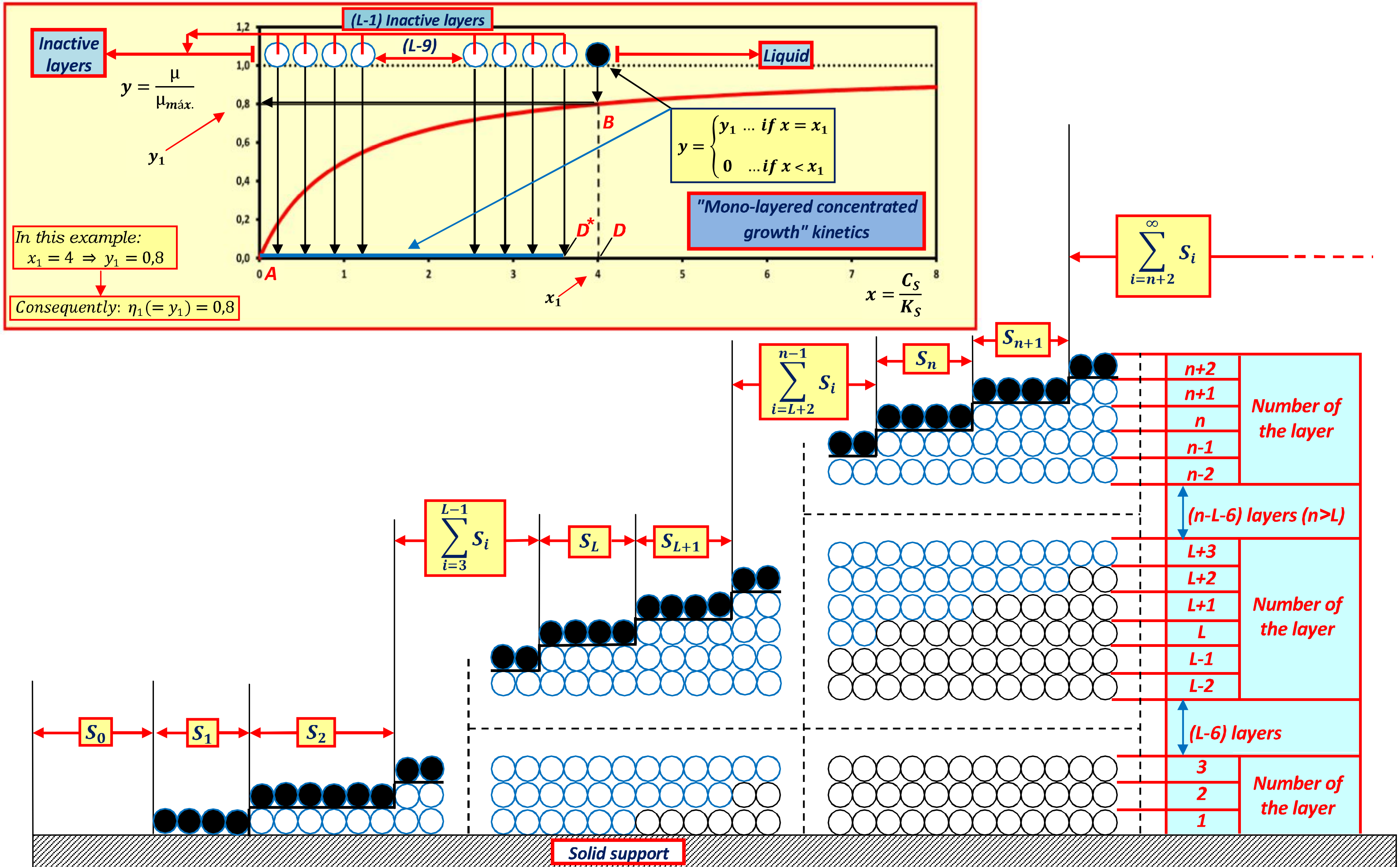


Figure 11 - Spatial distribution of biofilm layers according to "mono-layered concentrated growth" kinetics. In this figure the bacterial active population is occupying only the uppermost layer in every area S_j . We can interpret such radical simplification in two ways. In one way we observe that all cases portrayed in the previous figures 7, 8, 9 and 10 fall in this one when $L = 1$. Such circumstance confers to this case a quality of "meeting point" of all of those previous cases. In other words: if we reach to solve exactly this "mono-layered concentrated growth" kinetics model we solve all the others in their most simplified versions. In the other way we can propose this particular case as a theoretical model by its own right, concentrating all the bacterial growth in the biofilm top layers as an intrinsic model premise.
The points A, B and D, in the upper left graphic are coherent with Figure 7. The point D^* define the straight horizontal line from A to D^* where the bacterial specific growth rate is null. This point D^* corresponds to $x = [(L-1)/L] \cdot x_1$.
The system of numbering all the layers (in the right side of the diagram) and all the little boxes above those layers define all the model geometry without ambiguity.

CHAPTER III

Partial processes dynamics equations

(A Multilayer Model for Early Stages Formation and Growth of Biofilms)

1. Text

As it has been described so far, the model will inevitably lead to a very complicated system of differential equations if we use the kinetics of Monod or any other than the simpler ones, like "zero-order kinetics or "mono layered concentrated growth" kinetics. Even so with the aforesaid assumptions in the previous Chapter II we can construct the general and most elaborate differential-difference system of equations for the variation of $S_n\,(n = 0, 1, 2, \dots, n, \dots)$ with time. After, only as few as necessary simplifications will be introduced.

For sake of achieve total generality, our starting point now must be to adopt a suitable modification about the already defined substrate concentrations $C_{(n-i+1)}$ and specific growth rates $\mu^{\rightarrow}_{(n-i+1)}$ and $\mu^{\uparrow}_{(n-i+1)}$:

For enumeration convenience let put,

$$j = n - i + 1 \qquad [3.1]$$

With this j we number, from 1 to L, and from top outermost layer to down innermost still active layer all active biofilm layers for the specific purpose of distinguish among all of their, in terms of substrate concentrations and specific growth rates.

According to [3.1], $C_{(n-i+1)}$, $\mu^{\rightarrow}_{(n-i+1)}$ and $\mu^{\uparrow}_{(n-i+1)}$ must be written C_j, $\mu^{\rightarrow}_j$ and $\mu^{\uparrow}_j$.

In Monod kinetics context the new equations are,

$$\mu^{\rightarrow}_j = \frac{\mu^{\rightarrow}_{máx.} \cdot C_j}{K_S + C_j}, \quad \text{and} \quad \mu^{\uparrow}_j = \frac{\mu^{\uparrow}_{máx.} \cdot C_j}{K_S + C_j} \qquad [3.2]$$

Which are formally,

$$\mu^{\rightarrow}_j = \mu^{\rightarrow}_{máx.} \cdot \eta_j, \quad \text{and} \quad \mu^{\uparrow}_j = \mu^{\uparrow}_{máx.} \cdot \eta_j \qquad [3.3]$$

… where,

$$\eta_j = \frac{C_j}{K_S + C_j} \qquad [3.4]$$

Each study situation links to a set with L of these parameters η_j $(\eta_1, \eta_2, \dots, \eta_L)$ accounting for the scalar and biofilm deep variable factor in the specific growth rates, $\mu^{\rightarrow}_j$ and $\mu^{\uparrow}_j$.

The other two factor parameters, $\mu^{\rightarrow}_{máx.}$ and $\mu^{\uparrow}_{máx.}$ contribute with, besides their own value, the growth anisotropy, in two directions, already described.

Equations [3.3] can be applied to any a priori chosen kinetic law, not only to Monod´s case. Judicious analysis of all the other simplified kinetics described ("Zero order inspired", "Blackman inspired" , "averaged" first order and "mono layered concentrated growth" kinetics) allows to get the set of η_j parameters in all the cases.

Another advantage of [3.3] is the economy and simplicity of mathematical notation particularly suitable for master equations deduction.

Lets now start such deductive task deriving the variation of a general surface area S_n with time.

I) If $n > L$, S_n will be submitted to eight ways of changing.

S_n will increase because:

a) S_n will increase because the active layers of order n in S_n, $S_{n+1}, S_{n+2} \dots$, and S_{n+L-1} , standing at his same height, will growth horizontally, each one respectively associated to the parameter $\eta_1, \eta_2, \ldots, \eta_L$. Therefore we have a partial variation of,

$$\left(\frac{dS_n}{dt}\right)_a = + \mu^{\rightarrow}_{máx.} \cdot \sum_{j=1}^{L} \eta_j S_{n+j-1} \qquad (Process\ a) \qquad [3.5]$$

b) Over S_{n-1} falls the constant irreversible attachment flux $F = a_P \cdot f$ and therefore S_{n-1} is promoted to S_n, and this one increases. This partial variation is,

$$\left(\frac{dS_n}{dt}\right)_b = +F \cdot S_{n-1} \qquad (Process\ b) \qquad [3.6]$$

c) S_n will increase because all the active layers in S_{n-1}, will growth perpendicularly to the solid support, and consequently S_{n-1} will be promoted to S_n, and this one increases. All those active layers in S_{n-1} have the same size but, from top to down, in S_{n-1}, they are associated successively with the parameters $\eta_1, \eta_2, \ldots, \eta_L$.
The corresponding partial variation is,

$$\left(\frac{dS_n}{dt}\right)_c = + \mu^{\uparrow}_{máx.} \cdot \left(\sum_{j=1}^{L} \eta_j\right) \cdot S_{n-1} \qquad (Process\ c) \qquad [3.7]$$

d) S_{n+1} will detach by erosion and therefore will come down to S_n, and this one increases. In S_{n+1} the uppermost layer, of order $(n+1)$, suffers erosion by a first order process with kinetic constant, according to [2.23], k_{n+1}. The corresponding S_n variation will be:

$$\left(\frac{dS_n}{dt}\right)_d = +k_{n+1} \cdot S_{n+1} \qquad (Process\ d) \qquad [\mathbf{3}.8]$$

Respectively, by the same reasons, S_n will decrease because:
e) S_{n+1} will increase, growing horizontally over S_n, because the active layers of order $(n+1)$ in $S_{n+1}, S_{n+2}, \ldots,$ and S_{n+L} , standing at his same height, will growth horizontally, each one respectively associated to the parameter $\eta_1, \eta_2, \ldots, \eta_L$.
Therefore we have a partial variation of,

$$\left(\frac{dS_n}{dt}\right)_e = - \mu^{\rightarrow}_{máx.} \cdot \sum_{j=1}^{L} \eta_j S_{n+j} \qquad (Process\ e) \qquad [3.9]$$

f) Over S_n falls the constant irreversible attachment flux $F = a_P \cdot f$ and therefore S_n is promoted to S_{n+1}. Consequently S_n decreases. This partial variation is,

$$\left(\frac{dS_n}{dt}\right)_f = -F \cdot S_n \qquad (Process\ f) \qquad [3.10]$$

g) All the active layers in S_n, will growth perpendicularly to the solid support, and consequently S_n will be promoted to S_{n+1}. As result S_n decreases. All those growing active layers in S_n have the same size but, from top to down, in S_n, they are associated successively with the parameters $\eta_1, \eta_2, \ldots, \eta_L$.
The corresponding partial variation is,

$$\left(\frac{dS_n}{dt}\right)_g = - \mu^{\uparrow}_{máx.} \cdot \left(\sum_{j=1}^{L} \eta_j\right) \cdot S_n \qquad (Process\ g) \qquad [3.11]$$

h) S_n will detach by erosion and therefore will come down to S_{n-1}, and this one increases. In S_n the uppermost layer, of order n, suffers erosion by a first order process with kinetic constant, according to [2.23], k_n. The result is that S_n decreases according to the following differential balance:

$$\left(\frac{dS_n}{dt}\right)_h = -k_n \cdot S_n \qquad (Process\ h) \qquad [3.12]$$

II) For $(2 \leq n \leq L)$, the area S_n will change the same way, also by these eight processes, except that in those processes concerning support perpendicular growth, the suns limits in *Processes* c and g must be modified because each S_n has as many active layers as his totality, and this one depends on n.

The adapted partial differential balances for *Processes* c and g, when $2 \leq n \leq L$ reads,

$$\left(\frac{dS_n}{dt}\right)_c = +\,\mu^{\uparrow}_{máx.} \cdot \left(\sum_{j=1}^{n-1} \eta_j\right) \cdot S_{n-1} \qquad (Process\ c) \qquad [3.13]$$

… and,

$$\left(\frac{dS_n}{dt}\right)_g = -\,\mu^{\uparrow}_{máx.} \cdot \left(\sum_{j=1}^{n} \eta_j\right) \cdot S_n \qquad (Process\ g) \qquad [3.14]$$

III) If $n = 1$, S_1 will change like in the case of $(2 \leq n \leq L)$, except that *Processes* c does not exist because the bare surface S_0 does not growth. Consequently we must eliminate $\left(\frac{dS_n}{dt}\right)_c$ for global differential balance calculus purpose.

IV) If $n = 0$, S_0 will change only accordingly to the *Processes* d, e and f.

V) Wherever, by anyone of the *Processes* e, f and g, the surface S_n (with $n > 0$) is promoted to S_{n+1}, cells of the innermost active layer of S_n (that is, layer of order $n - L + 1$ in S_n) will change from active to inactive, because such layer will locate in S_{n+1} deep enough to become inactive.

VI) Correspondently, wherever, by anyone of the *Processes* a, b and c, S_{n-1} is promoted to S_n cells of the layer $(n - L)$ in S_{n-1} change from active to inactive, when S_{n-1} becomes S_n.

VII) Lastly, wherever S_{n+1} and S_n detach, respectively *Processes* d and h, layer cells of orders $(n + L + 1)$ in S_{n+1} and $(n + L)$ in S_n change from inactive to active

These tree last rules, V, VI and VII, are direct consequence of 4th idea and 5th idea, already expressed in Chapter I (Introduction).

2. Companion schematic diagrams

For clear description purposes in Figure 12 one can find the enumeration system of all the areas S_n and also, in each S_n , the labels j for their n component layers as well as the border between active and inactive bacteria layers.
This diagrammatical enumeration system in Figure 12, and the following four figures 13, 14, 15 and 16 facilitates the understanding of all the dynamic processes just defined in this Chapter III.
After each one of those four figures, 13, 14, 15 and 16, a companion Box respectively 1, 2, 3 and 4 complements the explanation.
Lastly, in Box 5, a synthesis of Boxes 1, 2, 3 and 4, explains how to obtain the total system of differential-difference equations for all this formation and growth of biofilm dynamic.

The ordination for the next pages is,

LEGEND

***n** - Number of layers heaped up in area S_n and also the number of a particular layer which his above (n-1) layers in any area S_m being ($n \leq m$).*
***j** - Enumeration, from top to down, of active layers in each area S_n .*
***L** - Maximum number of active layers in each S_n .*
○ - *Active layers.*
○ - *Inactive layers.*

RELATIONS

If** $(0 < n \leq L)$**: All the layers in** S_n **are active bacterial layers
***… and the range of the enumeration index j is, from top to down,** $j = 1, 2, 3, \ldots, (n-1), n$*
If** $(n > L)$**: Only the L layers counted in** S_n **, from top to down, are bacterial active layers
***… and the range of the enumeration index j is, from top to down,** $j = 1, 2, 3, \ldots, (L-1), L$*

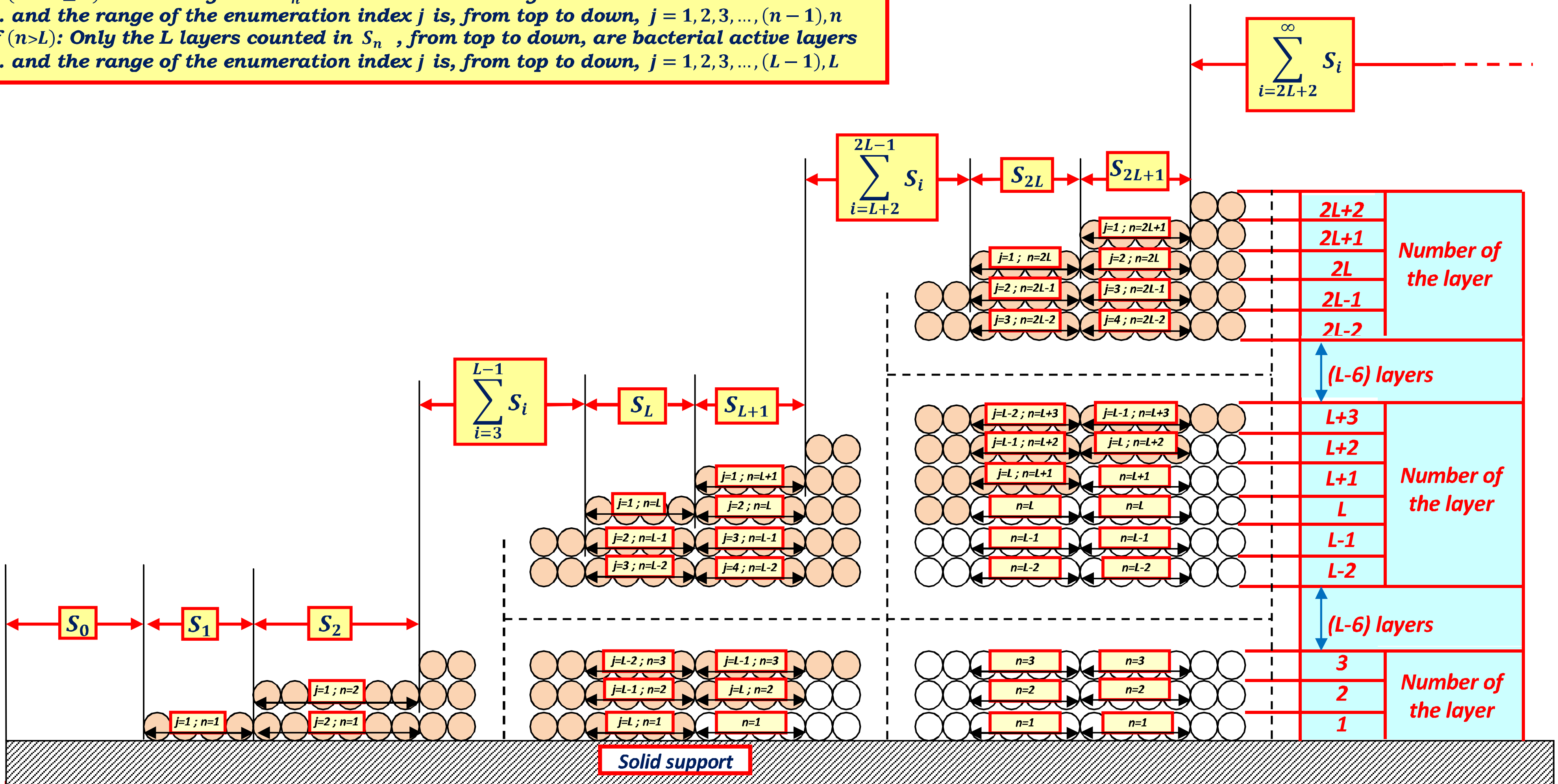


Figure 12 - Spatial system of enumeration of biofilm layers.
In this diagram we define all the layered geometry intrinsic to the model. Layers are not only numbered by n , in the right side, but also in the places where they are located. The little boxes above the layers define the enumeration of areas S_n.
Lastly, in every layer location the value of index j , already defined in the text for enumeration of bacterial active layers, is found explicitly.
For the purpose of this diagram the substrate concentration gradient is not relevant and consequently is omitted. But for the definition of parameter L and the border between active and inactive layers is necessary differentiate those two populations by circles of two collars (pink for actives and white for inactives).

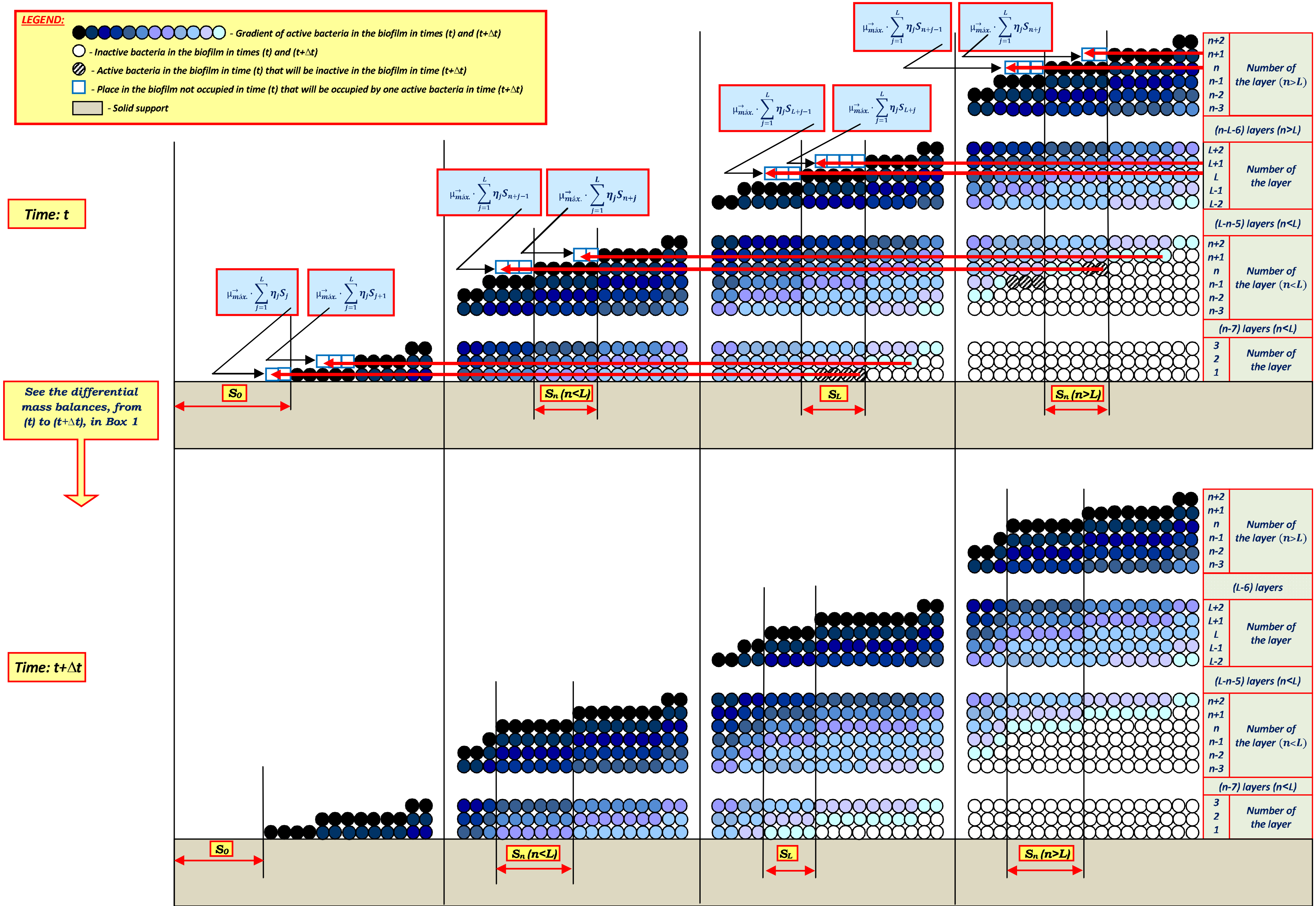


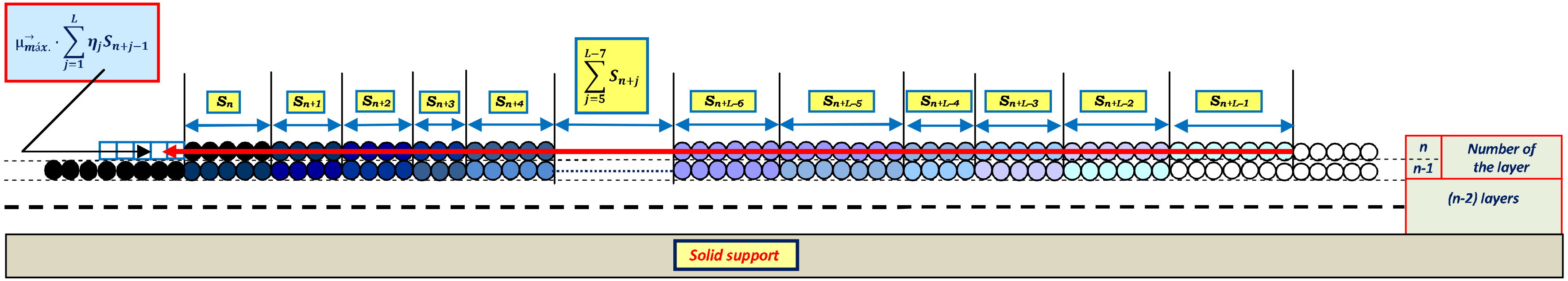
Fig. 13 : Dynamic effects of parallel growth on the morphology of the biofilm after an increment Δt in time .
See Box 1 for details of the quantitative balance that describes the corresponding variations in the values of S_n *, for* $n = 0, 1, 2, \ldots, n(n < L), \ldots, L, \ldots, n\ (n>L), \ldots$
Also in Box 1 some diagrams explains specific growth rates gradients rearrangements, from time (t) to time (t+Δt).

Box 1 - Parallel growth balances (This Box explains Figure 13) [1/3]

Elementary object of analysis (labeled generically by n): *The set of all adjacent active layers located at height n, that´s to say all the active layers of number n, numbered from down (solid support level) up to the top.*

Explicitly: first top layer of area S_n, second top layer of area S_{n+1}, third top layer of area S_{n+2}, ... , in general, (j) top layer of area S_{n+j-1}, ... , $(L-1)$ top layer of area S_{n+L-2} and L top layer of area S_{n+L-1}.

We thus define the index "j" from $(j=1)$ to $(j=L)$ and adopt S_{n+j-1} as the generic element of the defined elementary object of analysis.



Each layer of this set, generically designed in this diagram by S_{n+j-1} *, will growth in the solid support parallel direction, according to the red arrows, here and in Figure 13, and their corresponding single contribution is:*

Now the explanation follows by this block diagram:

Specific growth rate: $\vec{\mu}_j = \vec{\mu}_{máx.} \cdot \eta_j \quad (j = 1, 2, \ldots, j, \ldots, L)$

$$\vec{\mu}_{máx.} \cdot \eta_j = \left(\frac{\Delta S_{n+j-1}}{S_{n+j-1}}\right) \cdot \frac{1}{\Delta t}$$

Single contribution of area S_{n+j-1} for total growth of the elementary object of analysis:

$$\Delta S_{n+j-1} = \vec{\mu}_{máx.} \cdot \eta_j \cdot S_{n+j-1} \cdot \Delta t$$

Alternative definition of specific growth rate:

$$\vec{\mu}_j = \left(\frac{\Delta S_{n+j-1}}{S_{n+j-1}}\right) \cdot \frac{1}{\Delta t}$$

Equivalence between relative biomass growth and relative area layer growth:

$$\left(\frac{\Delta M_{n+j-1}}{M_{n+j-1}}\right) = \left(\frac{\Delta S_{n+j-1}}{S_{n+j-1}}\right)$$

Definition of specific biomass growth rate:

$$\vec{\mu}_j = \left(\frac{\Delta M_{n+j-1}}{M_{n+j-1}}\right) \cdot \frac{1}{\Delta t}$$

Total growth of the elementary object of analysis (labeled by n):

$$\sum_{j=1}^{L} \Delta S_{n+j-1} = \vec{\mu}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j S_{n+j-1}\right\} \cdot \Delta t$$

Total growth rate of the elementary object of analysis (labeled by n):

$$\frac{1}{\Delta t} \cdot \sum_{j=1}^{L} \Delta S_{n+j-1} = \vec{\mu}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j S_{n+j-1}\right\}$$

Observe that the right hand sides are the quantities explicitly represented in the blue rectangles in Figure 13.
Are positive quantities, according to their definition.

The sum in the left hand side is the increase, in biomass or volume, between (t) and $(t+\Delta t)$, of the elementary object of analysis, labeled by n :

$$\sum_{j=1}^{L} \Delta S_{n+j-1} = (number\ of\ new\ arbitrarily\ areal\ units)_n \cdot (arbitral\ unit)$$

The most logic, and coherent whith the text, choise for the arbitral areal unit is a_P, the flat carrier area covered by a single attached bacteria. Such area includes not only bacteria own dimension but also extracellular space in biofilm.

In Figure 13 we represent each a_P by a circle.

Let make:

$$(number\ of\ new\ arbitrarily\ areal\ units)_n = (N_{a.u.})_{n\rightarrow}\ and\ (arbitral\ unit) = a_P$$

Consequently:

$$\sum_{j=1}^{L} \Delta S_{n+j-1} = (N_{a.u.})_{n\rightarrow} \cdot a_P$$

Total growth rate of the elementary object of analysis (labeled by n) in terms of areal units a_P :

$$\frac{(N_{a.u.})_{n\rightarrow} \cdot a_P}{\Delta t} = \vec{\mu}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j S_{n+j-1}\right\}$$

A→ Follows in B

B→ Sequel from A

Box 1 - Parallel growth balances (This Box explains Figure 13) [2/3]

Total growth rate of the elementary object of analysis (labeled by n) in terms of areal units a_P:

$$\frac{(N_{a.u.})_{n\rightarrow}\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j S_{n+j-1}\right\}$$

Applying this formula the total growth rates of the eight elementary objects of analysis, represented in Figure 13 by the red arrows, are straightforward obtainned: *(Check the values of $(N_{a.u.})_{n\rightarrow}$ by inspection of Figure 13)*

The total growth of such an elementary object of analysis, labeled by (n), has two effects on the values of the areas S_n:
The first effect is the obvious raise of the value of S_n.
The second effect is the decrease in the value of S_{n-1}, because the refered elementary set of active layers grows over the inmediatly lower layer of order $(n-1)$ pertaining to S_{n-1}
Of course, the area S_n also diminishes as consequence of the growth of the elementary object labeled by $(n+1)$, located inmediatly above the one labeled by (n).
The effect of the parallel growth in area S_0 is, obviously, allways a permanent decrease of his value.

Applying these reasonings one can easily establish the differential balances for all the areas S_n.

Elementary object of analysis $n=1$: $\frac{2\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\}$

Elementary object of analysis $n=2$: $\frac{3\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+1}\right\}$

Elementary object of analysis n $(n<L)$: $\frac{3\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n-1}\right\}$

Elementary object of analysis $[n+1]$ $(n<L)$: $\frac{2\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n}\right\}$

Elementary object of analysis L: $\frac{3\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+L-1}\right\}$

Elementary object of analysis $[L+1]$: $\frac{4\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+L}\right\}$

Elementary object of analysis n $(n>L)$: $\frac{3\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n-1}\right\}$

Elementary object of analysis $[n+1]$ $(n>L)$: $\frac{2\cdot a_P}{\Delta t}=\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n}\right\}$

Combining the left hand sides

$$\left(\frac{\Delta S_0}{\Delta t}\right)=-\frac{2a_P}{\Delta t}$$

$$\left(\frac{\Delta S_1}{\Delta t}\right)=+\frac{2a_P}{\Delta t}-\frac{3a_P}{\Delta t}$$

$$\left(\frac{\Delta S_n}{\Delta t}\right)_{(n<L)}=+\frac{3a_P}{\Delta t}-\frac{2a_P}{\Delta t}$$

$$\left(\frac{\Delta S_L}{\Delta t}\right)=+\frac{3a_P}{\Delta t}-\frac{4a_P}{\Delta t}$$

$$\left(\frac{\Delta S_n}{\Delta t}\right)_{(n>L)}=+\frac{3a_P}{\Delta t}-\frac{2a_P}{\Delta t}$$

Inserting the right hand sides

$$\left(\frac{\Delta S_0}{\Delta t}\right)=-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{\Delta S_1}{\Delta t}\right)=+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot (S_j-S_{j+1})\right\}$$

$$\left(\frac{\Delta S_n}{\Delta t}\right)_{(n<L)}=+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot (S_{j+n-1}-S_{j+n})\right\}$$

$$\left(\frac{\Delta S_L}{\Delta t}\right)=+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot (S_{j+L-1}-S_{j+L})\right\}$$

$$\left(\frac{\Delta S_n}{\Delta t}\right)_{(n>L)}=+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot (S_{j+n-1}-S_{j+n})\right\}$$

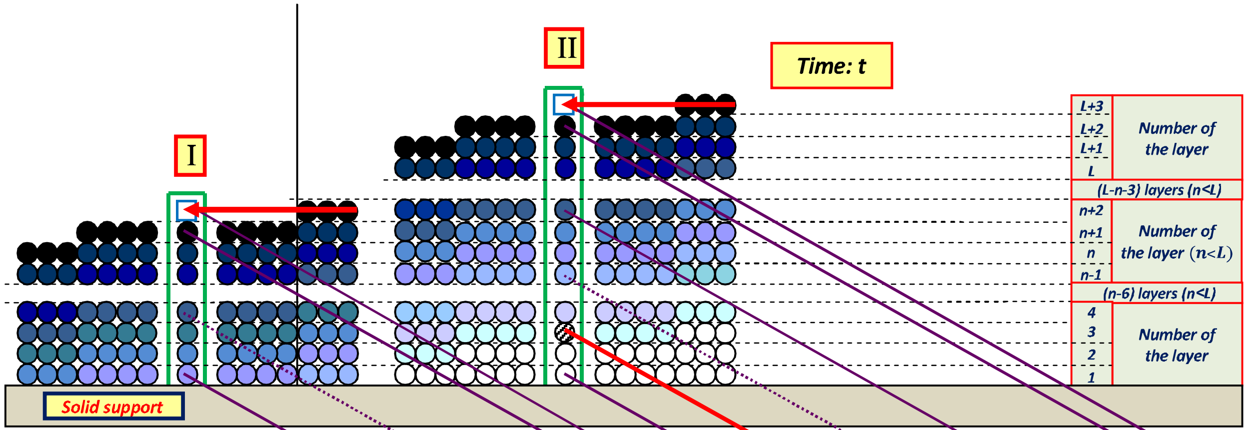
Now, applying the limit Δt→0, and avoiding to write repeated formulas, we get the system of differential difference equations concerning only the effects of parallel growth of active bacterial (processes designed before as "a" and "e").

$$\left(\frac{dS_0}{dt}\right)_{a,e}=-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{dS_n}{dt}\right)_{a,e}=+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot (S_{j+n-1}-S_{j+n})\right\}\quad (for\ n\geq 1)$$

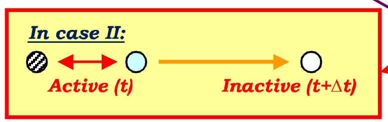
PARTIAL SYSTEM Nº1

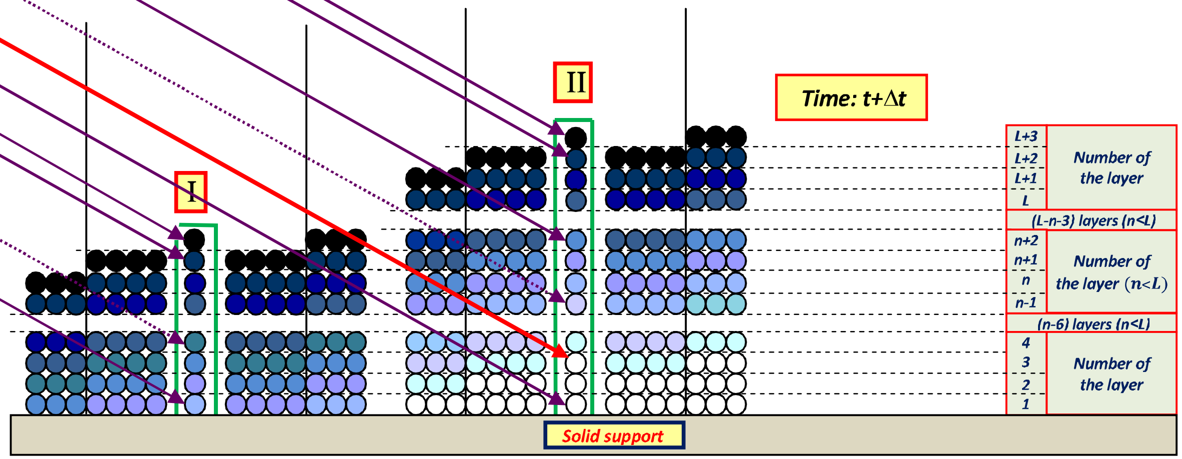
Box 1- Specific growth rates gradient rearrangements after parallel growth (Explains gradient details in Figure 13) [3/3]



The substrate concentration gradient, and the corresponding monotone change in the local specific growth rate, decreasing with deep inside the biofilm, rearranges in all locals where, as result of parallel growth, a raise in thickness (in one layer more) occurs.
In the new local top layer substrate concentration and specific growth rate have the values of the previous top layer which, after time interval Δt, is now the second top layer. Consequently, this one assumes the values of the previous second top one, so decreasing substrate concentration and specific growth rate. And so on: the layer with label (j), counted from the top, at time (t) assumes the values of the layer with label (j+1) at that same time after time interval (Δt) elapsed.

Like signalized in this diagram, two situations must be distinguished: I and II (See the block in the left down side of this page)





In case (I) the number of local layers at time (t) is less than (L). So the gradient rearrangement don´t have, as consequence, the inactivation of the active innermost initial bacteria.
In case (II) the number of local layers at time (t) is (L) or more. Consequently the gradient rearrangement implies that the active innermost bacteria, numbered (L) from top to down at time (t) , will have number (L+1) at time (t+Δt) and so will already pertain to the inactive population.

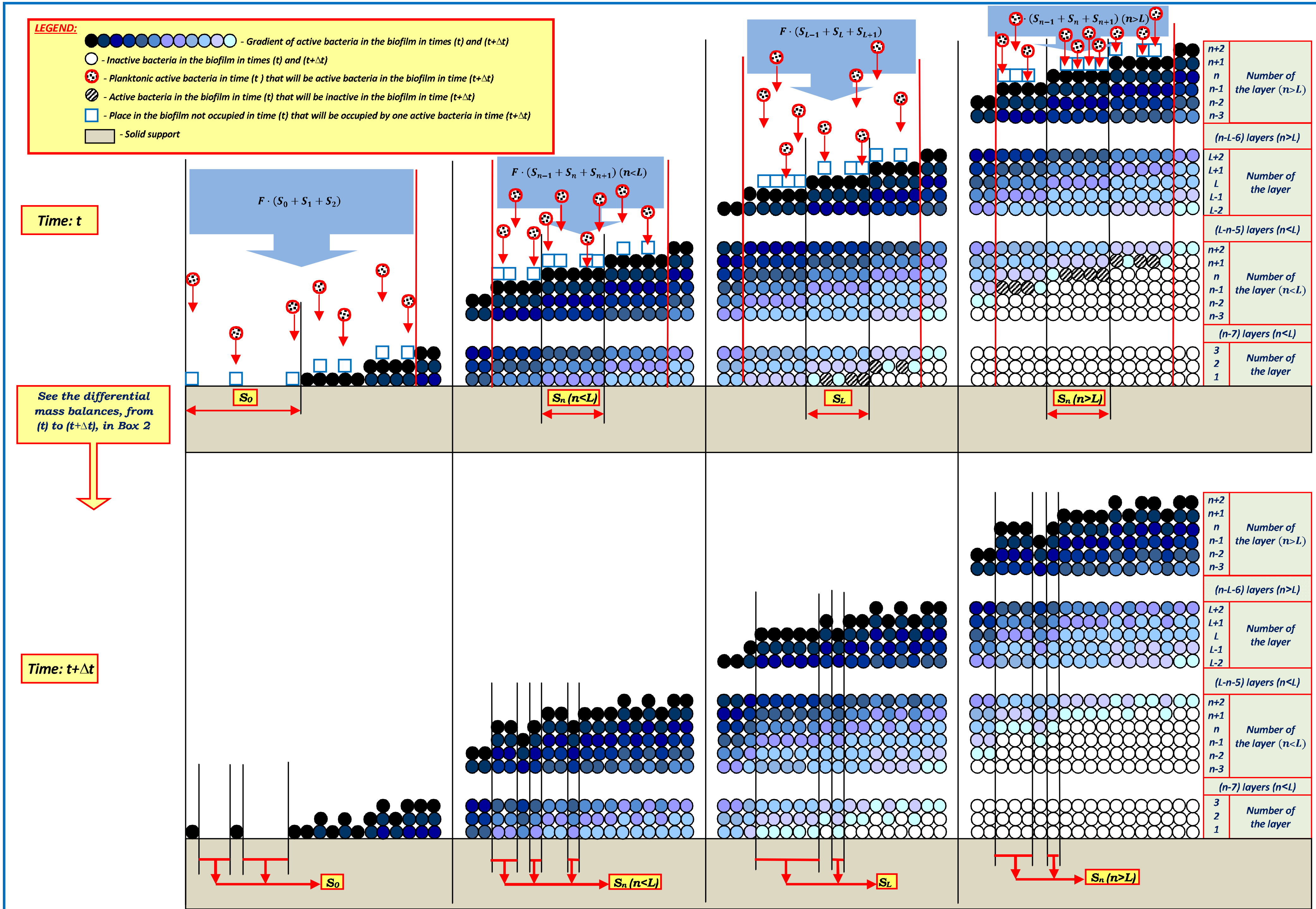
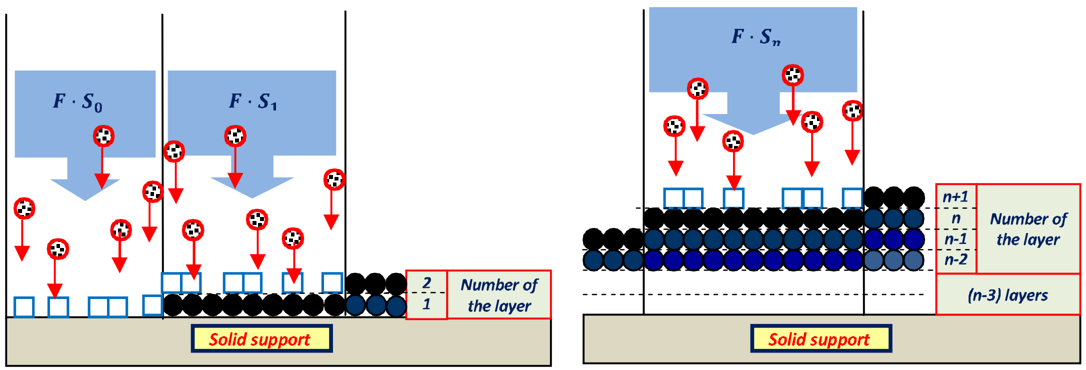


Fig. 14 : Dynamic effects of the attachment flux on the morphology of the biofilm after an increment Δt in time .
In Box 2 we define quantitatively the balances for the variations in the values of S_n , for $n = 0, 1, 2, \dots, n (n < L), \dots, L, \dots, n (n>L), \dots$
Also in Box 2 some diagrams explains specific growth rates gradients rearrangements, from time (t) to time (t+Δt).

Box 2 - Attachment flux balances (This Box explains Figure 14) [1/3]

Elementary object of analysis: *The set of all top layers of all the areas S_n, including S_0. That´s to say all the exposed area all over the solid plane support.*
Explicitly: the bare area S_0, the layer of order 1 in area S_1, the layer of order 2 in area S_2, the layer of order 3 in area S_3, ... , and, in general, the layer of order n in area S_n.



Each area of this set, generically designed by S_n, will increase locally the number of her own piled layers, by one single unity, as consequence of the arrival of an uniform and constant planktonic bacterial flux that we assume to attach irreversibly to the bare area S_0 or to the already existing biofilm areas S_n.

An uniform flux of arriving planktonic bacteria is defined on the basis to the total solid support area:

$$f_T = \frac{\text{Number of arriving planktonic bacteria}}{\text{Arrival area } (S_T) \cdot \text{Time } (t)} \equiv L^{-2}T^{-1}$$

In the same way we can define fluxes based on every area S_n $(n = 0, 1, 2, \dots, n, \dots)$:

$$f_n = \frac{\text{Number of arriving planktonic bacteria}}{\text{Arrival area } (S_n) \cdot \text{Time } (t)} \equiv L^{-2}T^{-1} \qquad (n = 0, 1, 2, \dots, n, \dots)$$

Flux uniformity allows to write: $f_0 = f_1 = f_2 = \cdots = f_n = \cdots = f_T$
... and, consequently we can define only one flux f without specify the area on which is based:

$$f = \frac{\text{Number of arriving planktonic bacteria}}{\text{Arrival area } (S_T) \cdot \text{Time } (t)} \equiv L^{-2}T^{-1}$$

For our calculations it is more suitable express the flux of planktonic bacteria in terms of covered area:

$$(\text{Covered area by arriving planktonic bacterial flux}) = (\text{Number of arriving planktonic bacteria}) \cdot (\text{Projected area of one bacteria in the biofilm})$$

The projected area of one bacteria in the biofilm is, as usual, our areal unit (a_P).
It means the flat carrier area covered by a single attached bacteria.
Such area includes not only bacteria own dimension but also extracellular space in biofilm.

Let also make:

$$(\text{Number of arriving planktonic bacteria}) = (N_{p.b.})$$

Consequently:

$$(\text{Covered area by arriving planktonic bacterial flux}) = (N_{p.b.}) \cdot a_P$$

The flux in terms of covered area is defined by:

$$F = \frac{(\text{Covered area by arriving planktonic bacterial})}{[\text{Arrival area } (S_T)] \cdot [\text{Time } (t)]} \equiv T^{-1}$$

Now we can establish a little formulary:

$$F = \frac{(N_{p.b.}) \cdot (a_P)}{(S_T) \cdot (t)} \equiv T^{-1} \qquad f = \frac{(N_{p.b.})}{(S_T) \cdot (t)} \equiv L^{-2}T^{-1}$$

$$F = (a_P) \cdot f$$

When the planktonic flux falls over the area S_{n-1}, between time (t) and time $(t + \Delta t)$, the covered area by this process, promote area S_{n-1} to area S_n.
So S_n increases by $(+\Delta S_n)$ and, correspondingly, S_{n-1} diminishes by the same quantity $(-\Delta S_{n-1})$:

$$(+\Delta S_n) = (-\Delta S_{n-1})$$

$$(\text{Covered area by arriving planktonic bacterial}) = (+\Delta S_n) = (-\Delta S_{n-1})$$

$$F = \left(\frac{+\Delta S_n}{\Delta t \cdot S_{n-1}}\right) = \left(\frac{-\Delta S_{n-1}}{\Delta t \cdot S_{n-1}}\right)$$

For interpreting Figure 14 is more adequate to account local area variations in function of the number of arriving planktonic bacterial:

$$(+\Delta S_n) = (-\Delta S_{n-1}) = +(N_{p.b.})_{(n-1)} \cdot (a_P)$$

$(N_{p.b.})_{(n-1)}$ is the number of arriving planktonic bacteria to S_{n-1} between (t) and $(t + \Delta t)$.

A→ Follows in B

B→ Sequel from A

Box 2- Attachment flux balances (This Box explains Figure 14) [2/3]

Applying this formula

$$F = \frac{(N_{p.b.})_n \cdot (a_P)}{(S_n) \cdot (\Delta t)}$$

refereed to each arrival area S_n it is possible to give significance to all the products $\left[(N_{p.b.})_n \cdot (a_P)\right]$ in Figure 14:

$$(N_{p.b.})_n \cdot (a_P) = F \cdot (S_n) \cdot (\Delta t)$$

With this formula:

$$(+\Delta S_n) = +(N_{p.b.})_{(n-1)} \cdot (a_P) - (N_{p.b.})_{(n)} \cdot (a_P)$$

we can now to account local area partial variations in function of the number of arriving planktonic bacterial flux:

- **Those arriving at S_{n-1}, are designed $(N_{p.b.})_{(n-1)}$, and promote this area to S_n. So they contribute to increase S_n.**
- **Those arriving at S_n, are designed $(N_{p.b.})_{(n)}$, and promote this area to S_{n+1}. So they contribute to decrease S_n.**
- **The effect of the planktonic bacterial attachment flux in area S_0 is, obviously, a permanent decrease of his value.**

Over S_0: $3a_P = F \cdot S_0 \cdot \Delta t$

Over S_1: $2a_P = F \cdot S_1 \cdot \Delta t$

Over S_2: $2a_P = F \cdot S_2 \cdot \Delta t$

Over S_{n-1} (n<L): $3a_P = F \cdot S_{n-1} \cdot \Delta t$

Over S_n (n<L): $4a_P = F \cdot S_n \cdot \Delta t$

Over S_{n+1} (n<L): $2a_P = F \cdot S_{n+1} \cdot \Delta t$

Over S_{L-1}: $4a_P = F \cdot S_{L-1} \cdot \Delta t$

Over S_L: $3a_P = F \cdot S_L \cdot \Delta t$

Over S_{L+1}: $2a_P = F \cdot S_{L+1} \cdot \Delta t$

Over S_{n-1} (n>L): $3a_P = F \cdot S_{n-1} \cdot \Delta t$

Over S_n (n>L): $4a_P = F \cdot S_n \cdot \Delta t$

Over S_{n+1} (n>L): $3a_P = F \cdot S_{n+1} \cdot \Delta t$

Substituting the left hand sides	**Substituting the right hand sides**
$\Delta S_0 = -3a_P$	$\Delta S_0 = -F \cdot S_0 \cdot \Delta t$
$\Delta S_1 = +3a_P - 2a_P$	$\Delta S_1 = +F \cdot S_0 \cdot \Delta t - F \cdot S_1 \cdot \Delta t$
$\Delta S_2 = +2a_P - 2a_P$	$\Delta S_2 = +F \cdot S_1 \cdot \Delta t - F \cdot S_2 \cdot \Delta t$
- - - - - -	- - - - - -
$\Delta S_n(n<L) = +3a_P - 4a_P$	$\Delta S_n(n<L) = +F \cdot S_{n-1} \cdot \Delta t - F \cdot S_n \cdot \Delta t$
$\Delta S_{n+1}(n<L) = +4a_P - 2a_P$	$\Delta S_{n+1}(n<L) = +F \cdot S_n \cdot \Delta t - F \cdot S_{n+1} \cdot \Delta t$
- - - - - -	- - - - - -
$\Delta S_L = +4a_P - 3a_P$	$\Delta S_L = +F \cdot S_{L-1} \cdot \Delta t - F \cdot S_L \cdot \Delta t$
$\Delta S_{L+1} = +3a_P - 2a_P$	$\Delta S_{L+1} = +F \cdot S_L \cdot \Delta t - F \cdot S_{L+1} \cdot \Delta t$
- - - - - -	- - - - - -
$\Delta S_n(n>L) = +3a_P - 4a_P$	$\Delta S_n(n>L) = +F \cdot S_{n-1} \cdot \Delta t - F \cdot S_n \cdot \Delta t$
$\Delta S_{n+1}(n>L) = +4a_P - 3a_P$	$\Delta S_{n+1}(n>L) = +F \cdot S_n \cdot \Delta t - F \cdot S_{n+1} \cdot \Delta t$

Now, dividing by Δt, applying the limit Δt→0, and avoiding to write repeated formulas, we get the system of differential difference equations concerning only the effects of attachment flux of active bacterial (processes designed before as "b" and "f").

$$\left(\frac{dS_0}{dt}\right)_{b,f} = -F \cdot S_0$$

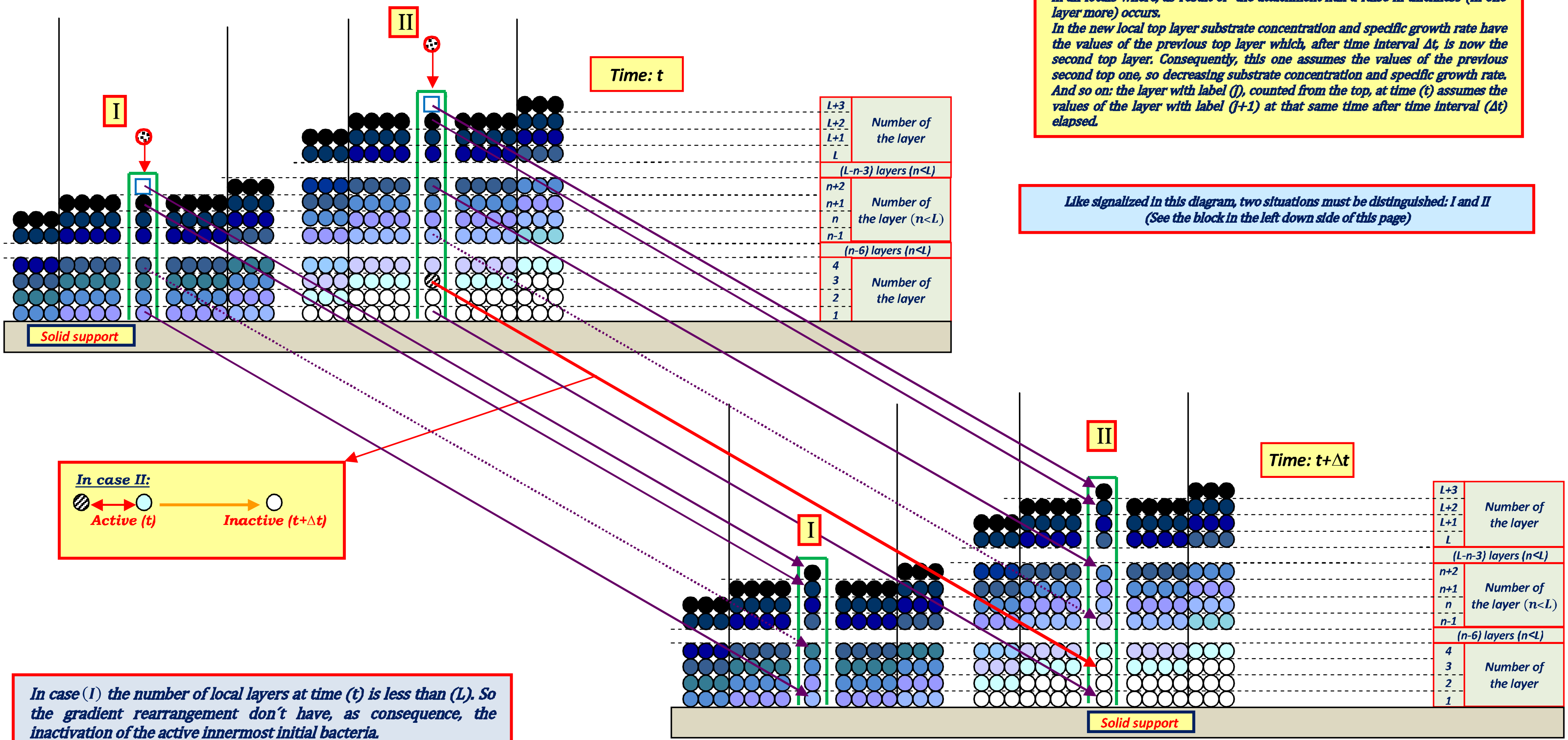
$$\left(\frac{dS_n}{dt}\right)_{b,f} = F \cdot S_{n-1} - F \cdot S_n \quad (for\ n \geq 1)$$

PARTIAL SYSTEM Nº2

Box 2- Specific growth rates gradient rearrangements after attachment (Explains gradient details in Figure 14) [3/3]

The substrate concentration gradient, and the corresponding monotone change in the local specific growth rate, decreasing with deep inside the biofilm, rearranges in all locals where, as result of the attachment flux a raise in thickness (in one layer more) occurs.
In the new local top layer substrate concentration and specific growth rate have the values of the previous top layer which, after time interval Δt, is now the second top layer. Consequently, this one assumes the values of the previous second top one, so decreasing substrate concentration and specific growth rate. And so on: the layer with label (j), counted from the top, at time (t) assumes the values of the layer with label (j+1) at that same time after time interval (Δt) elapsed.

Like signalized in this diagram, two situations must be distinguished: I and II (See the block in the left down side of this page)



In case (I) the number of local layers at time (t) is less than (L). So the gradient rearrangement don´t have, as consequence, the inactivation of the active innermost initial bacteria.
In case (II) the number of local layers at time (t) is (L) or more. Consequently the gradient rearrangement implies that the active innermost bacteria, numbered (L) from top to down at time (t), will have number (L+1) at time (t+Δt) and so will already pertain to the inactive population.

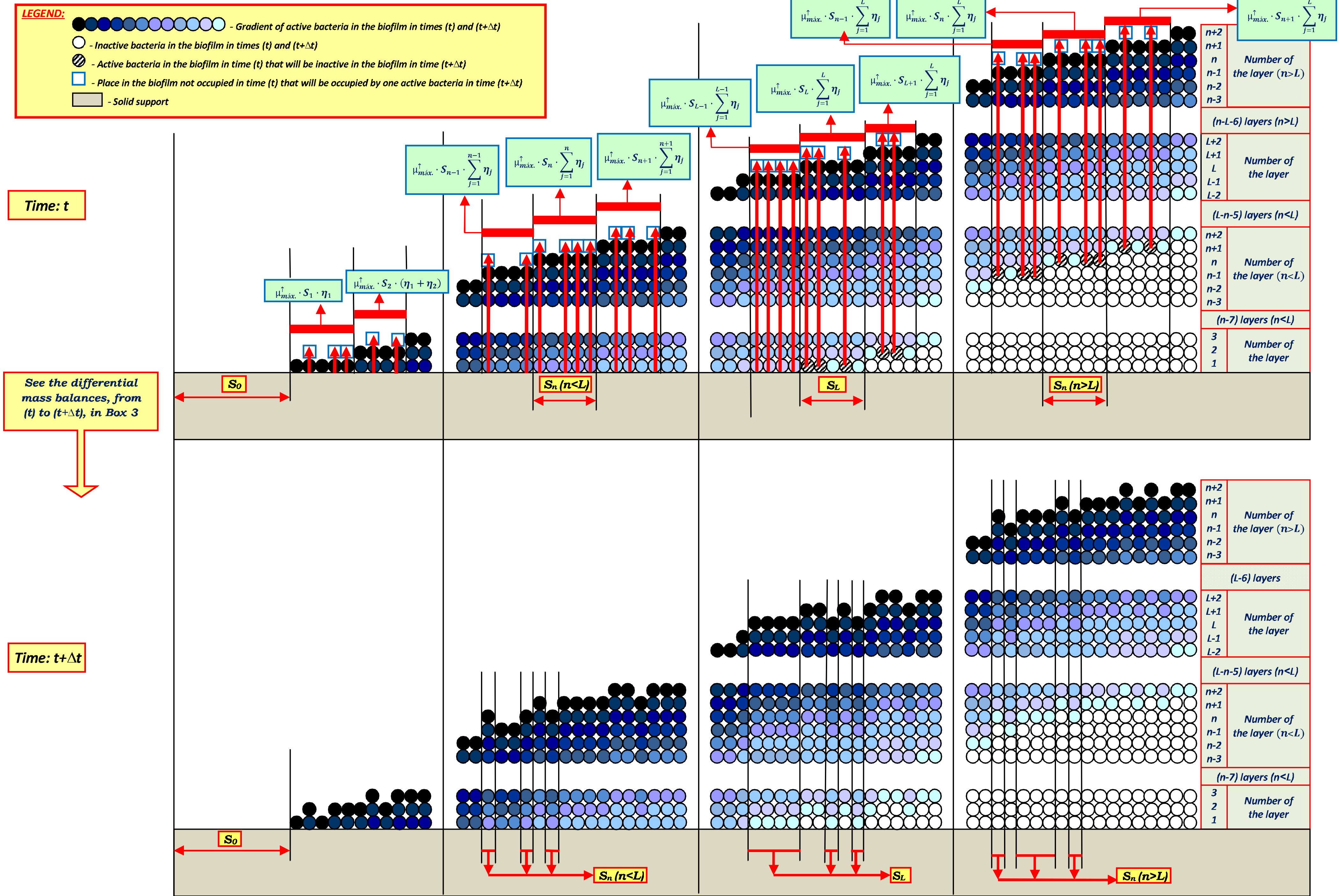


Fig. 15 : This picture illustrates the dynamical consequences of perpendicular growth of active bacteria , in the morphology of the biofilm , after an increment Δt in time. Box 3 expose how we can deduce balances for S_n with $n = 0, 1, 2, \ldots, n(n < L), \ldots, L, \ldots, n\,(n>L), \ldots$ Also in Box 3 some diagrams explains specific growth rates gradients rearrangements, from time (t) to time (t+Δt).

Box 3 - Perpendicular growth balances (This Box explains Figure 15) [1/5]

Elementary object of analysis: ***The set of all adjacent active layers pertaining to an area S_n.***

One must distinguish two situations:

In locations [I] where the thickness of area S_n is less than (or equal to) that equivalent to the maximum possible number of active layers (entire parameter L), that´s to say if $(n \le L)$ all the layers pertaining to S_n are active.

In locations [II] where the thickness of area S_n is bigger than that equivalent to the maximum possible number of active layers (entire parameter L), that´s to say if $(n > L)$ only the first L layers, counted from top to down and pertaining to S_n are active.

In both cases all the aforesaid active layers, pertaining to a given S_n, have the same area, which is equal to S_n.

As usual we define the index "j" from $(j = 1)$ to $(j = L)$ for kinetics purposes, accounted in η_j and numbered from top exposed layer down inside the biofilm. There is not necessarily adopt more definition details because all the active layers, pertaining to a given S_n, have the same area, which is equal to S_n.

Each layer of this set, will growth in the solid support perpendicular direction, according to the red arrows, here and in Figure 15, and their corresponding single contributions are:

Specific growth rates:

$$\mu_j^{\uparrow} = \mu_{máx.}^{\uparrow} \cdot \eta_j \quad (j = 1, 2, \ldots, j, \ldots, n) \quad \ldots if\ (n \le L), like\ in\ [I]$$

$$\mu_j^{\uparrow} = \mu_{máx.}^{\uparrow} \cdot \eta_j \quad (j = 1, 2, \ldots, j, \ldots, L) \quad \ldots if\ (n \ge L), like\ in\ [II]$$

Now the explanation follows by this block diagram:

Definition of specific biomass growth rate:

$$\mu_j^{\uparrow} = \left(\frac{\Delta M_j}{M_j}\right) \cdot \frac{1}{\Delta t}$$

Equivalence between relative biomass growth and relative area layer growth:

$$\left(\frac{\Delta M_j}{M_j}\right) = \left(\frac{\Delta S_j}{S_j}\right)$$

Remark: *in this case the initial area is the same for all the elements of the object under analysis, and is equal to S_n.*

So we can write:

$M_j = M_n$ *and* $S_j = S_n$ *for all labels j (but not* $\Delta M_j = \Delta M_n$ *and* $\Delta S_j = \Delta S_n$*)*

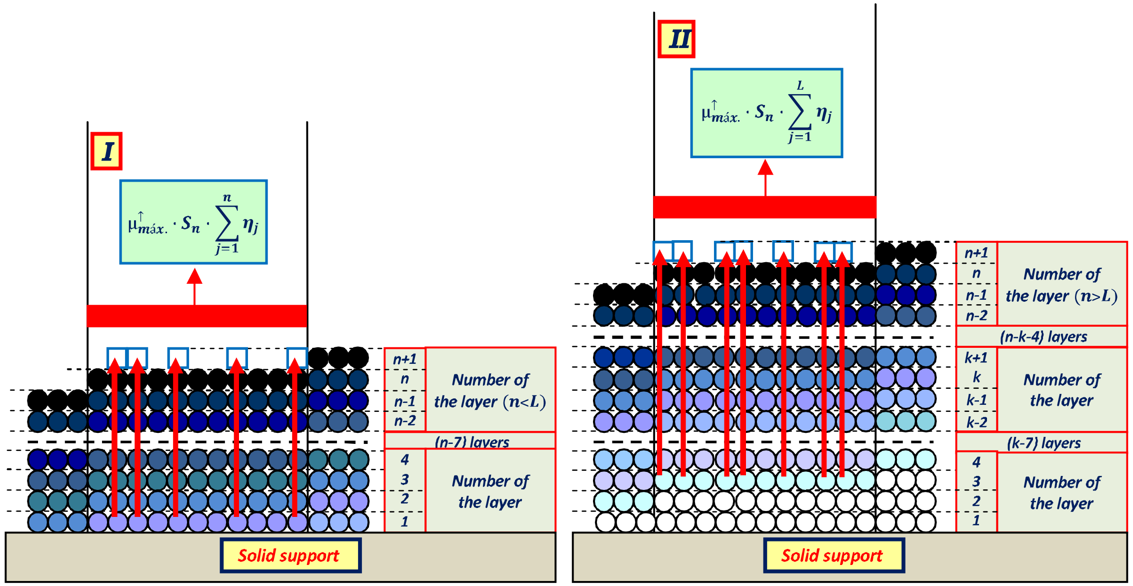
$$\ldots consequently: \quad \left(\frac{\Delta M_j}{M_j}\right) = \left(\frac{\Delta M_j}{M_n}\right) \quad and \quad \left(\frac{\Delta S_j}{S_j}\right) = \left(\frac{\Delta S_j}{S_n}\right)$$

Alternative definition of specific growth rate:

$$\mu_j^{\uparrow} = \left(\frac{\Delta S_j}{S_j}\right) \cdot \frac{1}{\Delta t}$$

$$\mu_{máx.}^{\uparrow} \cdot \eta_j = \left(\frac{\Delta S_j}{S_n}\right) \cdot \frac{1}{\Delta t} \quad (j = 1, 2, \ldots, j, \ldots, n) \quad \ldots if\ (n \le L), like\ in\ [I]$$

$$\mu_{máx.}^{\uparrow} \cdot \eta_j = \left(\frac{\Delta S_j}{S_n}\right) \cdot \frac{1}{\Delta t} \quad (j = 1, 2, \ldots, j, \ldots, L) \quad \ldots if\ (n \ge L), like\ in\ [II]$$



Single contribution of layer area S_j for total growth of the elementary object of analysis:

$$\Delta S_j = \mu_{máx.}^{\uparrow} \cdot \eta_j \cdot S_n \cdot \Delta t \qquad (j = 1, 2, \ldots, j, \ldots, n) \quad \ldots if\ (n \le L), like\ in\ [I]$$

$$\Delta S_j = \mu_{máx.}^{\uparrow} \cdot \eta_j \cdot S_n \cdot \Delta t \qquad (j = 1, 2, \ldots, j, \ldots, L) \quad \ldots if\ (n \ge L), like\ in\ [II]$$

A→ Follows in B

Total growth of the elementary object of analysis (labeled by n):

$$\sum_{j=1}^{n}\Delta S_j = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n}\eta_j\right\}\cdot S_n\cdot\Delta t \qquad (j=1,2,\dots,j,\dots,n)\quad \dots if\ (n\le L), like\ in\ [I]$$

$$\sum_{j=1}^{L}\Delta S_j = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_n\cdot\Delta t \qquad (j=1,2,\dots,j,\dots,L)\quad \dots if\ (n\ge L), like\ in\ [II]$$

Total growth rate of the elementary object of analysis (labeled by n):

$$\frac{1}{\Delta t}\cdot\sum_{j=1}^{n}\Delta S_j = \mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{n}\eta_j\right\}\cdot S_n \qquad (j=1,2,\dots,j,\dots,n)\quad \dots if\ (n\le L), like\ in\ [I]$$

$$\frac{1}{\Delta t}\cdot\sum_{j=1}^{L}\Delta S_j = \mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_n \qquad (j=1,2,\dots,j,\dots,L)\quad \dots if\ (n\ge L), like\ in\ [II]$$

Total growth rate of the elementary object of analysis (labeled by n) in terms of areal units a_P:

$$\frac{(N_{a.u.})_{n\uparrow}\cdot a_P}{\Delta t} = \mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{n}\eta_j\right\}\cdot S_n \qquad (j=1,2,\dots,j,\dots,n)\quad \dots if\ (n\le L), like\ in\ [I]$$

$$\frac{(N_{a.u.})_{n\uparrow}\cdot a_P}{\Delta t} = \mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_n \qquad (j=1,2,\dots,j,\dots,L)\quad \dots if\ (n\ge L), like\ in\ [II]$$

Applying these two formulas the total growth rates of the eleven elementary objects of analysis, represented in Figure 15 by the red arrows, are straightforward obtainned:
(Check the values of $(N_{a.u.})_{n\uparrow}$ by inspection of Figure 15)

Observe that the right hand sides are the quantities explicitly represented in the blue rectangles in Figure 15.
Are positive quantities, according to their definition.

Top layer $(j=1)$ *over* S_1: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{1}\eta_j\right\}\cdot S_1$

Top layer $(j=1)$ *over* S_2: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{2}\eta_j\right\}\cdot S_2$

Top layer $(j=1)$ *over* S_{n-1} $(n<L)$: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n-1}\eta_j\right\}\cdot S_{n-1}$

Top layer $(j=1)$ *over* S_n $(n<L)$: $\frac{4a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n}\eta_j\right\}\cdot S_n$

Top layer $(j=1)$ *over* S_{n+1} $(n<L)$: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n+1}\eta_j\right\}\cdot S_{n+1}$

Top layer $(j=1)$ *over* S_{L-1}: $\frac{4a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{L-1}\eta_j\right\}\cdot S_{L-1}$

Top layer $(j=1)$ *over* S_L: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_L$

Top layer $(j=1)$ *over* S_{L+1}: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_{L+1}$

Top layer $(j=1)$ *over* S_{n-1} $(n>L)$: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_{n-1}$

Top layer $(j=1)$ *over* S_n $(n>L)$: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_n$

Top layer $(j=1)$ *over* S_{n+1} $(n>L)$: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_{n+1}$

The sums in the left hand side is the increase, in biomass or volume, between (t) and $(t+\Delta t)$, of the elementary object of analysis, labeled by n:

$$\sum_{j=1}^{n}\Delta S_j = (number\ of\ new\ arbitrarily\ areal\ units)_n\cdot(arbitral\ unit) \qquad (j=1,2,\dots,j,\dots,n)\quad \dots if\ (n\le L), like\ in\ [I]$$

$$\sum_{j=1}^{L}\Delta S_j = (number\ of\ new\ arbitrarily\ areal\ units)_n\cdot(arbitral\ unit) \qquad (j=1,2,\dots,j,\dots,L)\quad \dots if\ (n\ge L), like\ in\ [II]$$

The most logic, and coherent whith the text, choise for the arbitral areal unit is, as usual, a_P, the flat carrier area covered by a single attached bacteria. Such area includes not only bacteria own dimension but also extracellular space in biofilm.

In Figure 15 we represent each a_P by a circle.

Let make:

$$(number\ of\ new\ arbitrarily\ areal\ units)_n = (N_{a.u.})_{n\uparrow}\ and\ (arbitral\ unit) = a_P$$

Consequently:

$$\sum_{j=1}^{n}\Delta S_j = (N_{a.u.})_{n\uparrow}\cdot a_P \qquad (j=1,2,\dots,j,\dots,n)\quad \dots if\ (n\le L), like\ in\ [I]$$

$$\sum_{j=1}^{L}\Delta S_j = (N_{a.u.})_{n\uparrow}\cdot a_P \qquad (j=1,2,\dots,j,\dots,L)\quad \dots if\ (n\ge L), like\ in\ [II]$$

C→ Follows in D

D→ Sequel from C

Box 3 - Perpendicular growth balances (This Box explains Figure 15) [3/5]

Here we write again, for convenience, the eleven last equations of the previous page:

The total growth of such an elementary object of analysis, designed S_n and labeled by (n), decreases his own value because that area S_n is promoted to S_{n+1}. Correspondly, the total growth of the elementary object of analysis, designed S_{n-1} and labeled by $(n-1)$, increases the value of the area S_n because that area S_{n-1} is promoted to S_n. Of course, the area S_1 only can diminish as consequence of its own growth and consequent promotion to S_2. And is also obvious that the process of perpendicular biomass growth has absolutely none effect on the value of the bare area S_0.

Top layer $(j=1)$ over S_1: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{1} \eta_j\right\} \cdot S_1$

Top layer $(j=1)$ over S_2: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{2} \eta_j\right\} \cdot S_2$

Top layer $(j=1)$ over S_{n-1} $(n<L)$: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n-1} \eta_j\right\} \cdot S_{n-1}$

Top layer $(j=1)$ over S_n $(n<L)$: $\frac{4a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n} \eta_j\right\} \cdot S_n$

Top layer $(j=1)$ over S_{n+1} $(n<L)$: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n+1} \eta_j\right\} \cdot S_{n+1}$

Top layer $(j=1)$ over S_{L-1}: $\frac{4a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L-1} \eta_j\right\} \cdot S_{L-1}$

Top layer $(j=1)$ over S_L: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_L$

Top layer $(j=1)$ over S_{L+1}: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_{L+1}$

Top layer $(j=1)$ over S_{n-1} $(n>L)$: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_{n-1}$

Top layer $(j=1)$ over S_n $(n>L)$: $\frac{3a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_n$

Top layer $(j=1)$ over S_{n+1} $(n>L)$: $\frac{2a_P}{\Delta t} = \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_{n+1}$

Substituting the left hand sides

Additional equation

$$\left(\frac{\Delta S_0}{\Delta t}\right)_{Total\uparrow} = 0$$

$$\left(\frac{\Delta S_1}{\Delta t}\right)_{Total\uparrow} = -\frac{3a_P}{\Delta t}$$

$$\left(\frac{\Delta S_2}{\Delta t}\right)_{Total\uparrow} = +\frac{3a_P}{\Delta t} - \frac{2a_P}{\Delta t}$$

$$\left(\frac{\Delta S_n}{\Delta t}\right)_{Total\uparrow} = +\frac{2a_P}{\Delta t} - \frac{4a_P}{\Delta t} \quad (n<L)$$

$$\left(\frac{\Delta S_{n+1}}{\Delta t}\right)_{Total\uparrow} = +\frac{4a_P}{\Delta t} - \frac{3a_P}{\Delta t} \quad (n<L)$$

$$\left(\frac{\Delta S_L}{\Delta t}\right)_{Total\uparrow} = +\frac{4a_P}{\Delta t} - \frac{3a_P}{\Delta t}$$

$$\left(\frac{\Delta S_{L+1}}{\Delta t}\right)_{Total\uparrow} = +\frac{3a_P}{\Delta t} - \frac{2a_P}{\Delta t}$$

$$\left(\frac{\Delta S_n}{\Delta t}\right)_{Total\uparrow} = +\frac{3a_P}{\Delta t} - \frac{3a_P}{\Delta t} \quad (n>L)$$

$$\left(\frac{\Delta S_{n+1}}{\Delta t}\right)_{Total\uparrow} = +\frac{3a_P}{\Delta t} - \frac{2a_P}{\Delta t} \quad (n>L)$$

Applying these reasonings one can easily establish the differential balances for all the areas S_n. The following obvious formulas can be used for that purpose:

$$\left(\frac{\Delta S_0}{\Delta t}\right)_{Total\uparrow} = 0$$

$$\left(\frac{\Delta S_1}{\Delta t}\right)_{Total\uparrow} = -\frac{(N_{a.u.})_{1\uparrow} \cdot a_P}{\Delta t}$$

$$\left(\frac{\Delta S_n}{\Delta t}\right)_{Total\uparrow} = +\frac{(N_{a.u.})_{(n-1)\uparrow} \cdot a_P}{\Delta t} - \frac{(N_{a.u.})_{n\uparrow} \cdot a_P}{\Delta t}$$

Substituting the right hand sides

	Equation kind
$\left(\frac{\Delta S_0}{\Delta t}\right) = 0$	$n = 0$
$\left(\frac{\Delta S_1}{\Delta t}\right) = -\mu^{\uparrow}_{máx.} \cdot \eta_1 \cdot S_1$	$n = 1$
$\left(\frac{\Delta S_2}{\Delta t}\right) = +\mu^{\uparrow}_{máx.} \cdot \eta_1 \cdot S_1 - \mu^{\uparrow}_{máx.} \cdot (\eta_1 + \eta_2) \cdot S_2$	$2 \le n \le L$
$\left(\frac{\Delta S_n}{\Delta t}\right)_{(n<L)} = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n-1} \eta_j\right\} \cdot S_{n-1} - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n} \eta_j\right\} \cdot S_n$	$2 \le n \le L$
$\left(\frac{\Delta S_{n+1}}{\Delta t}\right)_{(n<L)} = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n} \eta_j\right\} \cdot S_n - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n+1} \eta_j\right\} \cdot S_{n+1}$	$2 \le n \le L$
$\left(\frac{\Delta S_L}{\Delta t}\right) = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L-1} \eta_j\right\} \cdot S_{L-1} - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_L$	$2 \le n \le L$
$\left(\frac{\Delta S_{L+1}}{\Delta t}\right) = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_L - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_{L+1}$	$n>L$
$\left(\frac{\Delta S_n}{\Delta t}\right)_{(n>L)} = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_{n-1} - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_n$	$n>L$
$\left(\frac{\Delta S_{n+1}}{\Delta t}\right)_{(n>L)} = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_n - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_{n+1}$	$n>L$

E→ Follows in F

Box 3 - Perpendicular growth balances (This Box explains Figure 15) [4/5]

F→ Sequel from E

Now, applying the limit $\Delta t \to 0$, and avoiding to write repeated formulas, we get the system of differential difference equations concerning only the effects of perpendicular growth of active bacterial (processes designed before as "c" and "g").

$$\left(\frac{dS_0}{dt}\right)_{c,g} = 0$$

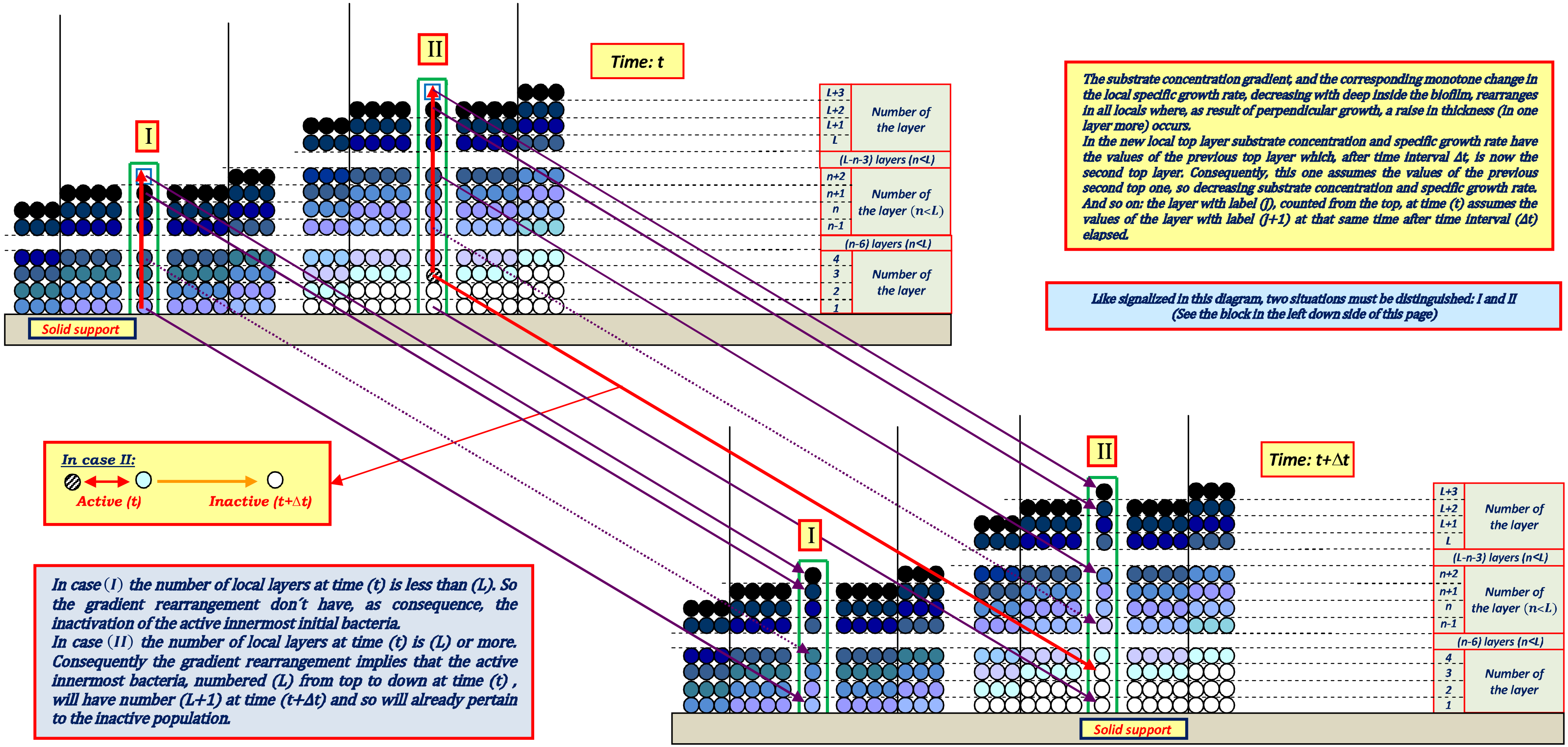
$$\left(\frac{dS_1}{dt}\right)_{c,g} = -\mu^{\uparrow}_{máx.} \cdot \eta_1 \cdot S_1$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n-1} \eta_j\right\} \cdot S_{n-1} - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{n} \eta_j\right\} \cdot S_n \qquad (for\ 2 \le n \le L)$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_{n-1} - \mu^{\uparrow}_{máx.} \cdot \left\{\sum_{j=1}^{L} \eta_j\right\} \cdot S_n \qquad (for\ n > L)$$

PARTIAL SYSTEM Nº3

Box 3 - Specific growth rates gradient rearrangements after perpendicular growth (Explains gradient details in Figure 15) [5/5]



The substrate concentration gradient, and the corresponding monotone change in the local specific growth rate, decreasing with deep inside the biofilm, rearranges in all locals where, as result of perpendicular growth, a raise in thickness (in one layer more) occurs.
In the new local top layer substrate concentration and specific growth rate have the values of the previous top layer which, after time interval Δt, is now the second top layer. Consequently, this one assumes the values of the previous second top one, so decreasing substrate concentration and specific growth rate. And so on: the layer with label (j), counted from the top, at time (t) assumes the values of the layer with label (j+1) at that same time after time interval (Δt) elapsed.

Like signalized in this diagram, two situations must be distinguished: I and II (See the block in the left down side of this page)

In case (I) the number of local layers at time (t) is less than (L). So the gradient rearrangement don´t have, as consequence, the inactivation of the active innermost initial bacteria.
In case (II) the number of local layers at time (t) is (L) or more. Consequently the gradient rearrangement implies that the active innermost bacteria, numbered (L) from top to down at time (t), will have number (L+1) at time (t+Δt) and so will already pertain to the inactive population.

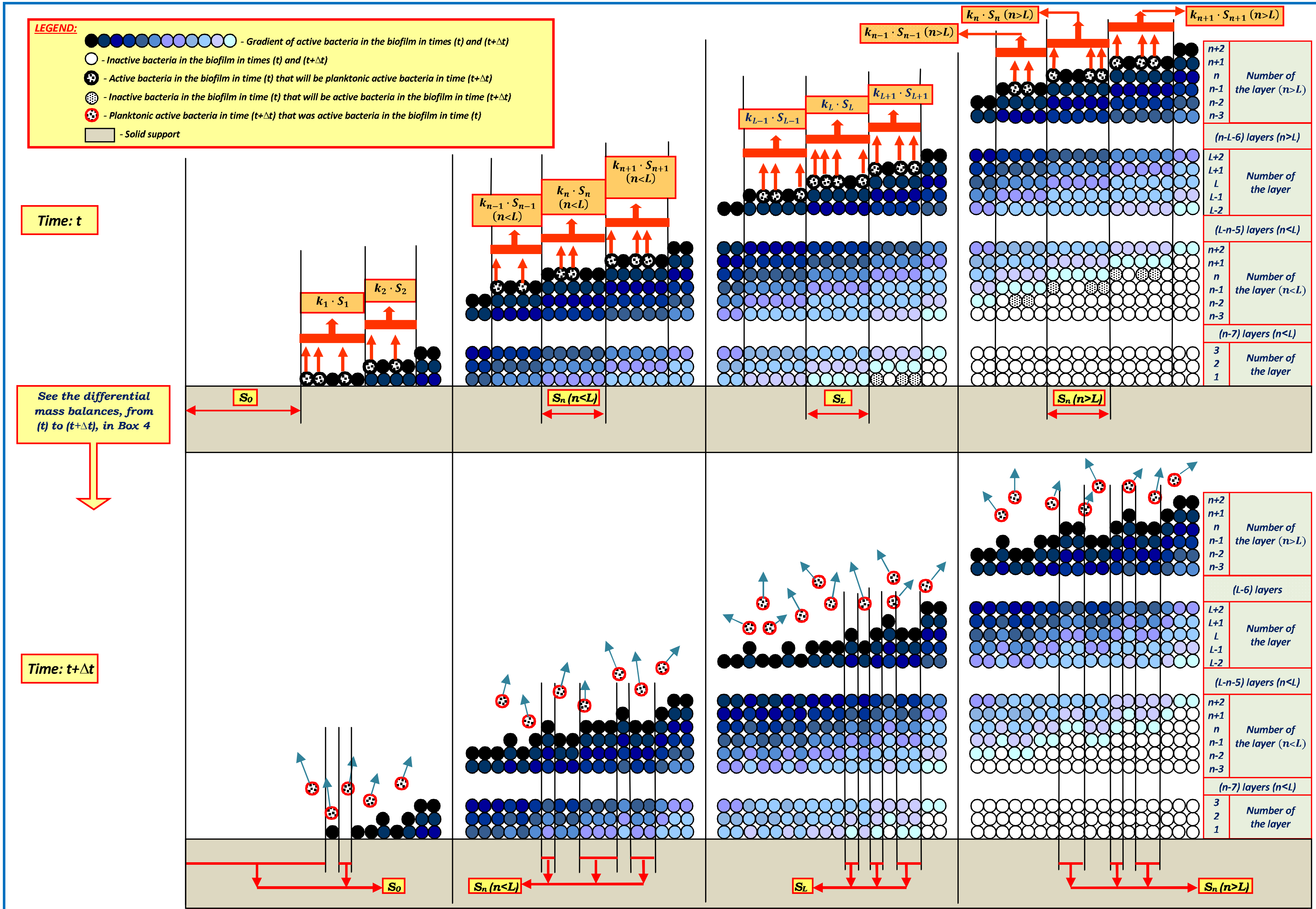
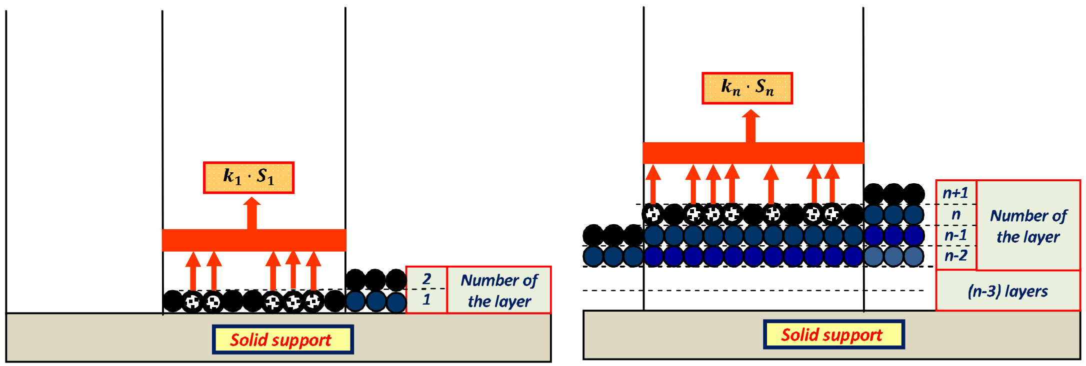


Fig. 16 : Dynamical effects of detachment (erosion , in this case) after an increment Δt , in time.
The details defining the corresponding quantitative balances , for n = 0 , 1 , 2 , … , n(n < L) , … , L , … , n (n>L) , … , are given in Box 4
Also in Box 4 some diagrams explains specific growth rates gradients rearrangements, from time (t) to time (t+Δt).

Box 4 - Detachment (erosion) balances (This Box explains Figure 16) [1/3]

Elementary object of analysis: *The set of all top layers of all the areas S_n, excluding $S0$ because this area can´t suffer detachment.*
Explicitly: the layer of order 1 in area S_1, the layer of order 2 in area S_2, the layer of order 3 in area S_3, … , and, in general, the layer of order n in area S_n.



Each aforesaid exposed top layer of this set, generically designed in this diagram by S_n, will suffer detachment (erosion), according to the orange arrows, here and in Figure 16, and their corresponding single contribution to this global biofilm erosion is:

$$\frac{[Number\ of\ sessile\ bacteria\ leaving\ area\ (S_n)]}{[Departing\ area\ (S_n)]\ \cdot\ [Time\ (t)]} \equiv L^{-2}T^{-1}$$

For our calculations it is more suitable express the number of sessile bacteria leaving area S_n in terms of (dis)covered area:

$$[(Dis)covered\ area\ by\ sessile\ bacteria\ leaving\ area\ S_n] = [Number\ of\ sessile\ bacteria\ leaving\ area\ (S_n)] \cdot (Projected\ area\ of\ one\ bacteria\ in\ the\ biofilm)$$

The projected area of one bacteria in the biofilm is, as usual, our areal unit (a_P).
It means the flat carrier area covered by a single attached bacteria. Such area includes not only bacteria own dimension but also extracellular space in biofilm.

Let also make:

$$[Number\ of\ sessile\ bacteria\ leaving\ area\ (S_n)] = (N_{s.b.})$$

Consequently:

$$[(Dis)covered\ area\ by\ sessile\ bacteria\ leaving\ area\ S_n] = (N_{s.b.}) \cdot a_P$$

The detachment kinetic constant in terms of (dis)covered area is defined by:

$$k_n = \frac{[(Dis)covered\ area\ by\ sessile\ bacteria\ leaving\ area\ S_n]}{[Departing\ area\ (S_n)]\ \cdot\ [Time\ (t)]} \equiv T^{-1}$$

Now we can write the formula:

$$k_n = \frac{(N_{s.b.}) \cdot (a_P)}{(S_n) \cdot (t)} \equiv T^{-1}$$

When the sessile bacterial detach from the area S_n, between time (t) and time $(t + \Delta t)$, that eroded area lowers from S_n to S_{n-1}.
So S_{n-1} increases by $(-\Delta S_n)$ and, correspondingly, S_n diminishes by the same quantity $(+\Delta S_{n-1})$:

$$(+\Delta S_{n-1}) = (-\Delta S_n)$$

$$[(Dis)covered\ area\ by\ sessile\ bacteria\ leaving\ area\ S_n] = (+\Delta S_{n-1}) = (-\Delta S_{n-})$$

$$k_n = \left(\frac{+\Delta S_{n-1}}{\Delta t \cdot S_n}\right) = \left(\frac{-\Delta S_n}{\Delta t \cdot S_n}\right)$$

For interpreting Figure 16 is more adequate to account local area variations in function of the number of detached sessile bacterial:

$$(+\Delta S_{n-1}) = (-\Delta S_n) = +(N_{s.b.})_{(n)} \cdot (a_P)$$

$(N_{s.b.})_{(n)}$ is the number of detached sessile bacterial from S_n between (t) and $(t + \Delta t)$.

A→ Follows in B

B→ Sequel from A

Box 4 - Detachment (erosion) balances (This Box explains Figure 16) [2/3]

Applying this formula

$$k_n = \frac{(N_{s.b.})_n \cdot (a_P)}{(S_n) \cdot (\Delta t)}$$

refereed to each eroded area S_n it is possible to give significance to all the products $[(N_{s.b.})_n \cdot (a_P)]$ in Figure 16:

$$(N_{s.b.})_n \cdot (a_P) = k_n \cdot (S_n) \cdot (\Delta t)$$

With this formula:

$$(+\Delta S_n) = +(N_{s.b.})_{(n+1)} \cdot (a_P) - (N_{s.b.})_{(n)} \cdot (a_P)$$

we can now to account local area partial variations in function of the number of detached sessile bacterial:

- ***Those detaching at S_{n+1}, are designed $(N_{s.b.})_{(n+1)}$, and decline this area to S_n. So they contribute to increase S_n.***
- ***Those detaching at S_n, are designed $(N_{s.b.})_{(n)}$, and decline this area to S_{n-1}. So they contribute to decrease S_n.***
- ***The effect of the sessile biofilm bacterial detachment in area S_0 is, obviously, a permanent increase of his value as consequence of detachment of sessile bacterial in S_1.***

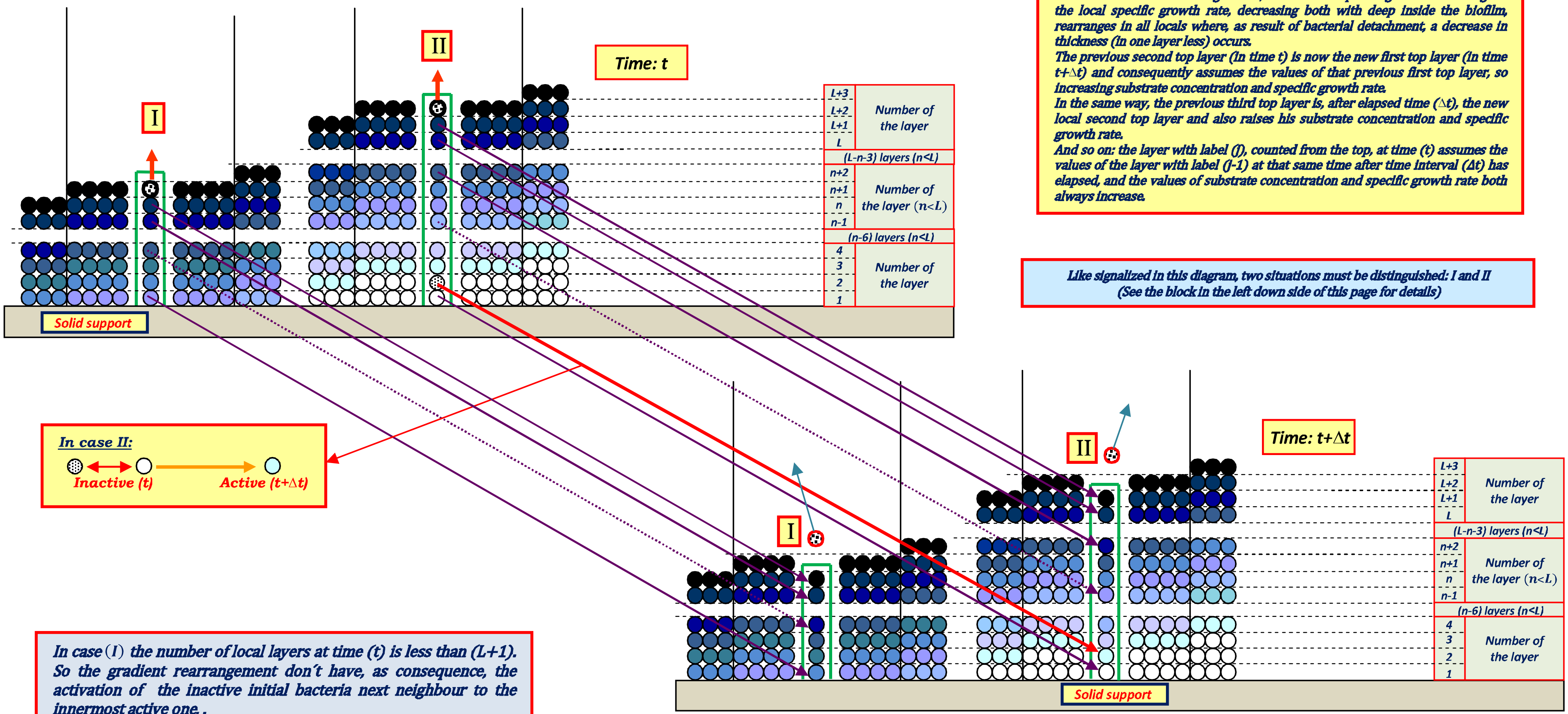
Detachment		Substituting the left hand sides	Substituting the right hand sides
		$\Delta S_0 = +3a_P$	$\Delta S_0 = +k_1 \cdot S_1 \cdot \Delta t$
Detachment from S_1:	$3a_P = k_1 \cdot S_1 \cdot \Delta t$		
Detachment from S_2:	$2a_P = k_2 \cdot S_2 \cdot \Delta t$	$\Delta S_1 = +2a_P - 3a_P$	$\Delta S_1 = +k_2 \cdot S_2 \cdot \Delta t - k_1 \cdot S_1 \cdot \Delta t$
		- - - - - -	- - - - - -
Detachment from S_{n-1} (n<L):	$2a_P = k_{n-1} \cdot S_{n-1} \cdot \Delta t$	$\Delta S_{n-1}(n<L) = +2a_P - 2a_P$	$\Delta S_{n-1}(n<L) = +k_n \cdot S_n \cdot \Delta t - k_{n-1} \cdot S_{n-1} \cdot \Delta t$
Detachment from S_n (n<L):	$2a_P = k_n \cdot S_n \cdot \Delta t$	$\Delta S_n(n<L) = +3a_P - 2a_P$	$\Delta S_n(n<L) = +k_{n+1} \cdot S_{n+1} \cdot \Delta t - k_n \cdot S_n \cdot \Delta t$
Detachment from S_{n+1} (n<L):	$3a_P = k_{n+1} \cdot S_{n+1} \cdot \Delta t$		
		- - - - - -	- - - - - -
Detachment from S_{L-1}:	$3a_P = k_{L-1} \cdot S_{L-1} \cdot \Delta t$	$\Delta S_{L-1} = +4a_P - 3a_P$	$\Delta S_{L-1} = +k_L \cdot S_L \cdot \Delta t - k_{L-1} \cdot S_{L-1} \cdot \Delta t$
Detachment from S_L:	$4a_P = k_L \cdot S_L \cdot \Delta t$	$\Delta S_L = +3a_P - 4a_P$	$\Delta S_L = +k_{L+1} \cdot S_{L+1} \cdot \Delta t - k_L \cdot S_L \cdot \Delta t$
Detachment from S_{L+1}:	$3a_P = k_{L+1} \cdot S_{L+1} \cdot \Delta t$		
		- - - - - -	- - - - - -
Detachment from S_{n-1} (n>L):	$2a_P = k_{n-1} \cdot S_{n-1} \cdot \Delta t$	$\Delta S_{n-1}(n>L) = +3a_P - 2a_P$	$\Delta S_{n-1}(n>L) = +k_n \cdot S_n \cdot \Delta t - k_{n-1} \cdot S_{n-1} \cdot \Delta t$
Detachment from S_n (n>L):	$3a_P = k_n \cdot S_n \cdot \Delta t$	$\Delta S_n(n>L) = +3a_P - 3a_P$	$\Delta S_n(n>L) = +k_{n+1} \cdot S_{n+1} \cdot \Delta t - k_n \cdot S_n \cdot \Delta t$
Detachment from S_{n+1} (n>L):	$3a_P = k_{n+1} \cdot S_{n+1} \cdot \Delta t$		
		- - - - - -	- - - - - -

Now, dividing by Δt, applying the limit Δt→0, and avoiding to write repeated formulas, we get the system of differential difference equations concerning only the effects of detachment of active sessile bacterial from the biofilm out to liquid phase (processes designed before as "d" and "h").

$$\left(\frac{dS_0}{dt}\right)_{d,h} = +k_1 \cdot S_1$$

$$\left(\frac{dS_n}{dt}\right)_{d,h} = +k_{n+1} \cdot S_{n+1} - k_n \cdot S_n \quad (for\ n \geq 1)$$

PARTIAL SYSTEM Nº4



The substrate concentration gradient, and the corresponding monotone change in the local specific growth rate, decreasing both with deep inside the biofilm, rearranges in all locals where, as result of bacterial detachment, a decrease in thickness (in one layer less) occurs.
The previous second top layer (in time t) is now the new first top layer (in time t+Δt) and consequently assumes the values of that previous first top layer, so increasing substrate concentration and specific growth rate.
In the same way, the previous third top layer is, after elapsed time (Δt), the new local second top layer and also raises his substrate concentration and specific growth rate.
And so on: the layer with label (j), counted from the top, at time (t) assumes the values of the layer with label (j-1) at that same time after time interval (Δt) has elapsed, and the values of substrate concentration and specific growth rate both always increase.

Like signalized in this diagram, two situations must be distinguished: I and II (See the block in the left down side of this page for details)

In case (I) the number of local layers at time (t) is less than (L+1). So the gradient rearrangement don´t have, as consequence, the activation of the inactive initial bacteria next neighbour to the innermost active one. .
In case (II) the number of local layers at time (t) is (L+1) or more. Consequently the gradient rearrangement implies that the inactive initial bacteria next neighbour to the innermost active one, numbered (L+1) from top to down at time (t) , will have number (L) at time (t+Δt) and so will already pertain to the active population.

BOX 5: TOTAL SYSTEM OF DIFFERENTIAL-DIFFERENCE EQUATIONS (SYNTHESIS OF BOXES: 1, 2, 3 AND 4) [1/3]

From the four Figures 13, 14, 15 and 16, and their corresponding four "companion boxes" we collect the four partial systems of differential difference equations concerning all the dynamic processes considered in this model:

1) - effects of parallel growth of active bacterial (processes designed before as "a" and "e").
2) - effects of attachment flux of active bacterial (processes designed before as "b" and "f").
3) - effects of perpendicular growth of active bacterial (processes designed before as "c" and "g").
4) - effects of detachment of active sessile bacterial from the biofilm out to liquid phase (processes designed before as "d" and "h").

The demonstration of this "Total system of differential-difference equations", where the terms in the right hand sides are ordered by increasing (n) in the fractions S_n can be understood following the blocks in this and the next two pages.

TOTAL SYSTEM OF DIFFERENTIAL-DIFFERENCE EQUATIONS:

$$\left(\frac{dS_0}{dt}\right) = -F\cdot S_0 + (k_1 - \mu^{\rightarrow}_{\text{máx.}}\cdot\eta_1)\cdot S_1 - \mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=2}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{dS_1}{dt}\right) = +F\cdot S_0 - (F + \mu^{\uparrow}_{\text{máx.}}\cdot\eta_1 + k_1 - \mu^{\rightarrow}_{\text{máx.}}\cdot\eta_1)\cdot S_1 + [\mu^{\rightarrow}_{\text{máx.}}(\eta_2-\eta_1)+k_2]\cdot S_2 + \mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=3}^{L}(\eta_j-\eta_{j-1})\cdot S_j\right\} - \mu^{\rightarrow}_{\text{máx.}}\cdot\eta_L\cdot S_{L+1}$$

$$\left(\frac{dS_n}{dt}\right) = +\left\{F + \mu^{\uparrow}_{\text{máx.}}\cdot\left[\sum_{j=1}^{n-1}\eta_j\right]\right\}\cdot S_{n-1} - \left\{F + \mu^{\uparrow}_{\text{máx.}}\cdot\left[\sum_{j=1}^{n}\eta_j\right] - \mu^{\rightarrow}_{\text{máx.}}\cdot\eta_1 + k_n\right\}\cdot S_n + \left\{\mu^{\rightarrow}_{\text{máx.}}\cdot(\eta_2-\eta_1)+k_{n+1}\right\}\cdot S_{n+1} + \mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{j+n}\right\} - \mu^{\rightarrow}_{\text{máx.}}\cdot\eta_L\cdot S_{n+L}$$

$$(2 \le n \le L)$$

$$\left(\frac{dS_n}{dt}\right) = +\left\{F + \mu^{\uparrow}_{\text{máx.}}\cdot\left[\sum_{j=1}^{L}\eta_j\right]\right\}\cdot S_{n-1} - \left\{F - \mu^{\rightarrow}_{\text{máx.}}\cdot\eta_1 + \mu^{\uparrow}_{\text{máx.}}\cdot\left[\sum_{j=1}^{L}\eta_j\right] + k_n\right\}\cdot S_n + \left\{\mu^{\rightarrow}_{\text{máx.}}\cdot(\eta_2-\eta_1)+k_{n+1}\right\}\cdot S_{n+1} + \mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{n+j}\right\} - \mu^{\rightarrow}_{\text{máx.}}\cdot\eta_L\cdot S_{n+L}$$

$$(n > L)$$

The first step is to collect the equations (obtained in "Boxes" 1, 2, 3 and 4) according to the classification in "Box" 3 because that is the case where more values of index are distinguished one to each others.

So we must contemplate four cases:
- S_0
- S_1
- S_n ***, with*** $(2 \le n \le L)$
- S_n ***, with*** $(n > L)$

Rearrangement of differential-difference equations from the four partial systems deduced in "Boxes 1, 2, 3 and 4. Here classified by fractions S_n, concerning their derivatives.

$$\left(\frac{dS_0}{dt}\right)_{a,e} = -\mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{dS_0}{dt}\right)_{b,f} = -F\cdot S_0$$

$$\left(\frac{dS_0}{dt}\right)_{c,g} = 0$$

$$\left(\frac{dS_0}{dt}\right)_{d,h} = +k_1\cdot S_1$$

$$\left(\frac{dS_1}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_j - S_{j+1})\right\} = +\mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\} - \mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=2}^{L+1}\eta_{j-1}\cdot S_j\right\}$$

$$\left(\frac{dS_1}{dt}\right)_{b,f} = F\cdot S_0 - F\cdot S_1$$

$$\left(\frac{dS_1}{dt}\right)_{c,g} = -\mu^{\uparrow}_{\text{máx.}}\cdot\eta_1\cdot S_1$$

$$\left(\frac{dS_1}{dt}\right)_{d,h} = +k_2\cdot S_2 - k_1\cdot S_1$$

$$\left(\frac{dS_n}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_{j+n-1} - S_{j+n})\right\} = +\mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n-1}\right\} - \mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=2}^{L+1}\eta_{j-1}\cdot S_{j+n-1}\right\}$$

$$\left(\frac{dS_n}{dt}\right)_{b,f} = F\cdot S_{n-1} - F\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu^{\uparrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{n-1}\eta_j\right\}\cdot S_{n-1} - \mu^{\uparrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{n}\eta_j\right\}\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{d,h} = +k_{n+1}\cdot S_{n+1} - k_n\cdot S_n$$

$$(2 \le n \le L)$$

$$\left(\frac{dS_n}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_{j+n-1} - S_{j+n})\right\}$$

$$\left(\frac{dS_n}{dt}\right)_{b,f} = F\cdot S_{n-1} - F\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu^{\uparrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_{n-1} - \mu^{\uparrow}_{\text{máx.}}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{d,h} = +k_{n+1}\cdot S_{n+1} - k_n\cdot S_n$$

$$(n > L)$$

BOX 5: *Next step consists in transforming the four equations relatives to parallel growth (processes "a" and "e") obtaining a suitable discrimination of terms , as well as their disposition in increasing order, for the purpose of make an easy summation of the four partial equations in each set. The transformations of three of those equations are in this page. The remaining fourth is in next page.* **[2/3]**

$$\left(\frac{dS_0}{dt}\right)_{a,e} = -\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{dS_0}{dt}\right)_{b,f} = -F\cdot S_0$$

$$\left(\frac{dS_0}{dt}\right)_{c,g} = 0$$

$$\left(\frac{dS_0}{dt}\right)_{d,h} = +k_1\cdot S_1$$

Transformation of $\left(\frac{dS_0}{dt}\right)_{a,e}$

$$\left(\frac{dS_0}{dt}\right)_{a,e} = -\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\} =$$

$$= -\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_1-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{dS_0}{dt}\right)_{a,e} = -\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_1-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{dS_0}{dt}\right)_{b,f} = -F\cdot S_0$$

$$\left(\frac{dS_0}{dt}\right)_{c,g} = 0$$

$$\left(\frac{dS_0}{dt}\right)_{d,h} = +k_1\cdot S_1$$

Sum

$$\left(\frac{dS_0}{dt}\right)_{TOTAL} = -F\cdot S_0+(k_1-\mu^{\rightarrow}_{máx.}\cdot\eta_1)\cdot S_1-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}\eta_j\cdot S_j\right\}$$

$$\left(\frac{dS_1}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_j-S_{j+1})\right\} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L+1}\eta_{j-1}\cdot S_j\right\}$$

$$\left(\frac{dS_1}{dt}\right)_{b,f} = F\cdot S_0-F\cdot S_1$$

$$\left(\frac{dS_1}{dt}\right)_{c,g} = -\mu^{\uparrow}_{máx.}\cdot\eta_1\cdot S_1$$

$$\left(\frac{dS_1}{dt}\right)_{d,h} = +k_2\cdot S_2-k_1\cdot S_1$$

Transformation of $\left(\frac{dS_1}{dt}\right)_{a,e}$

$$\left(\frac{dS_1}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_j-S_{j+1})\right\} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L+1}\eta_{j-1}\cdot S_j\right\} =$$

$$= +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_1+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}\eta_j\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}\eta_{j-1}\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{L+1} =$$

$$= +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_1+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}(\eta_j-\eta_{j-1})\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{L+1} =$$

$$= +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_1+\mu^{\rightarrow}_{máx.}\cdot(\eta_2-\eta_1)\cdot S_2+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=3}^{L}(\eta_j-\eta_{j-1})\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{L+1}$$

$$\left(\frac{dS_1}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_1+\mu^{\rightarrow}_{máx.}\cdot(\eta_2-\eta_1)\cdot S_2+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=3}^{L}(\eta_j-\eta_{j-1})\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{L+1}$$

$$\left(\frac{dS_1}{dt}\right)_{b,f} = F\cdot S_0-F\cdot S_1$$

$$\left(\frac{dS_1}{dt}\right)_{c,g} = -\mu^{\uparrow}_{máx.}\cdot\eta_1\cdot S_1$$

$$\left(\frac{dS_1}{dt}\right)_{d,h} = +k_2\cdot S_2-k_1\cdot S_1$$

Sum

$$\left(\frac{dS_1}{dt}\right)_{TOTAL} = +F\cdot S_0-(F+\mu^{\uparrow}_{máx.}\cdot\eta_1+k_1-\mu^{\rightarrow}_{máx.}\cdot\eta_1)\cdot S_1+[\mu^{\rightarrow}_{máx.}(\eta_2-\eta_1)+k_2]\cdot S_2+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=3}^{L}(\eta_j-\eta_{j-1})\cdot S_j\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{L+1}$$

$$\left(\frac{dS_n}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_{j+n-1}-S_{j+n})\right\} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n-1}\right\}-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L+1}\eta_{j-1}\cdot S_{j+n-1}\right\}$$

$$\left(\frac{dS_n}{dt}\right)_{b,f} = F\cdot S_{n-1}-F\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n-1}\eta_j\right\}\cdot S_{n-1}-\mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n}\eta_j\right\}\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{d,h} = +k_{n+1}\cdot S_{n+1}-k_n\cdot S_n$$

$(2\le n\le L)$

Transformation of $\left(\frac{dS_n}{dt}\right)_{a,e}$ ***(2 ≤ n ≤ L)***

$$\left(\frac{dS_n}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_{j+n-1}-S_{j+n})\right\} = +\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n-1}\right\}-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L+1}\eta_{j-1}\cdot S_{j+n-1}\right\} =$$

$$= +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_n+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}\eta_j\cdot S_{j+n-1}\right\}-\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}\eta_{j-1}\cdot S_{j+n-1}\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{n+L} =$$

$$= +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_n+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L}(\eta_j-\eta_{j-1})\cdot S_{j+n-1}\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{n+L} =$$

$$= +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_n+\mu^{\rightarrow}_{máx.}\cdot(\eta_2-\eta_1)\cdot S_{n+1}+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=3}^{L}(\eta_j-\eta_{j-1})\cdot S_{j+n-1}\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{n+L} =$$

$$= +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_n+\mu^{\rightarrow}_{máx.}\cdot(\eta_2-\eta_1)\cdot S_{n+1}+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{j+n}\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{n+L} =$$

$$\left(\frac{dS_n}{dt}\right)_{a,e} = +\mu^{\rightarrow}_{máx.}\cdot\eta_1\cdot S_n+\mu^{\rightarrow}_{máx.}\cdot(\eta_2-\eta_1)\cdot S_{n+1}+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{j+n}\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{n+L}$$

$$\left(\frac{dS_n}{dt}\right)_{b,f} = F\cdot S_{n-1}-F\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n-1}\eta_j\right\}\cdot S_{n-1}-\mu^{\uparrow}_{máx.}\cdot\left\{\sum_{j=1}^{n}\eta_j\right\}\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{d,h} = +k_{n+1}\cdot S_{n+1}-k_n\cdot S_n$$

$(2\le n\le L)$

Sum

$$\left(\frac{dS_n}{dt}\right)_{TOTAL} = +\left\{F+\mu^{\uparrow}_{máx.}\cdot\left[\sum_{j=1}^{n-1}\eta_j\right]\right\}\cdot S_{n-1}-\left\{F+\mu^{\uparrow}_{máx.}\cdot\left[\sum_{j=1}^{n}\eta_j\right]-\mu^{\rightarrow}_{máx.}\cdot\eta_1+k_n\right\}\cdot S_n+\left\{\mu^{\rightarrow}_{máx.}\cdot(\eta_2-\eta_1)+k_{n+1}\right\}\cdot S_{n+1}+\mu^{\rightarrow}_{máx.}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{j+n}\right\}-\mu^{\rightarrow}_{máx.}\cdot\eta_L\cdot S_{n+L}$$

$(2\le n\le L)$

$$\left(\frac{dS_n}{dt}\right)_{a,e} = +\mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_{j+n-1}-S_{j+n})\right\}$$

$$\left(\frac{dS_n}{dt}\right)_{b,f} = F\cdot S_{n-1} - F\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu_{\text{máx.}}^{\uparrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_{n-1} - \mu_{\text{máx.}}^{\uparrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{d,h} = +k_{n+1}\cdot S_{n+1} - k_n\cdot S_n$$

$(n > L)$

Transformation of $\left(\frac{dS_n}{dt}\right)_{a,e}$ ***(n > L)***

$$\begin{aligned}
\left(\frac{dS_n}{dt}\right)_{a,e} &= +\mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot(S_{j+n-1}-S_{j+n})\right\}\\
&= +\mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n-1}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\cdot S_{j+n}\right\}\\
&= +\mu_{\text{máx.}}^{\rightarrow}\cdot\eta_1\cdot S_n + \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_2\cdot S_{n+1} + \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=3}^{L}\eta_j\cdot S_{j+n-1}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_1\cdot S_{n+1} - \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=2}^{L-1}\eta_j\cdot S_{j+n}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_L\cdot S_{n+L}\\
&= +\mu_{\text{máx.}}^{\rightarrow}\cdot\eta_1\cdot S_n + \mu_{\text{máx.}}^{\rightarrow}\cdot(\eta_2-\eta_1)\cdot S_{n+1} + \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=3}^{L}\eta_j\cdot S_{j+n-1}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=2}^{L-1}\eta_j\cdot S_{j+n}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_L\cdot S_{n+L}\\
&= +\mu_{\text{máx.}}^{\rightarrow}\cdot\eta_1\cdot S_n + \mu_{\text{máx.}}^{\rightarrow}\cdot(\eta_2-\eta_1)\cdot S_{n+1} + \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=2}^{L-1}\eta_{j+1}\cdot S_{j+n}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=2}^{L-1}\eta_j\cdot S_{j+n}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_L\cdot S_{n+L}\\
&= +\mu_{\text{máx.}}^{\rightarrow}\cdot\eta_1\cdot S_n + \mu_{\text{máx.}}^{\rightarrow}\cdot(\eta_2-\eta_1)\cdot S_{n+1} + \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{n+j}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_L\cdot S_{n+L}
\end{aligned}$$

$$\left(\frac{dS_n}{dt}\right)_{a,e} = +\mu_{\text{máx.}}^{\rightarrow}\cdot\eta_1\cdot S_n + \mu_{\text{máx.}}^{\rightarrow}\cdot(\eta_2-\eta_1)\cdot S_{n+1} + \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{n+j}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_L\cdot S_{n+L}$$

$$\left(\frac{dS_n}{dt}\right)_{b,f} = F\cdot S_{n-1} - F\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{c,g} = +\mu_{\text{máx.}}^{\uparrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_{n-1} - \mu_{\text{máx.}}^{\uparrow}\cdot\left\{\sum_{j=1}^{L}\eta_j\right\}\cdot S_n$$

$$\left(\frac{dS_n}{dt}\right)_{d,h} = +k_{n+1}\cdot S_{n+1} - k_n\cdot S_n$$

$(n > L)$

Sum

$$\left(\frac{dS_n}{dt}\right)_{TOTAL} = +\left\{F + \mu_{\text{máx.}}^{\uparrow}\cdot\left[\sum_{j=1}^{L}\eta_j\right]\right\}\cdot S_{n-1} - \left\{F - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_1 + \mu_{\text{máx.}}^{\uparrow}\cdot\left[\sum_{j=1}^{L}\eta_j\right] + k_n\right\}\cdot S_n + \left\{\mu_{\text{máx.}}^{\rightarrow}\cdot(\eta_2-\eta_1) + k_{n+1}\right\}\cdot S_{n+1} + \mu_{\text{máx.}}^{\rightarrow}\cdot\left\{\sum_{j=2}^{L-1}(\eta_{j+1}-\eta_j)\cdot S_{n+j}\right\} - \mu_{\text{máx.}}^{\rightarrow}\cdot\eta_L\cdot S_{n+L}$$

$(n > L)$

CHAPTER IV

General model resolution: firsts steps (near mathematical intractability borderline)

1. Model governing dimensionless master equations

Now we start the resolution of our model rewriting the total system of differential-difference equations obtained in the previous Chapter III, in "Box 5".
That system accounts the variation with time of all S_n $(n = 0, 1, 2, \ldots, n, \ldots)$. All terms with common S_n are colected together and are disposed in increased order (n) in the right hand sides. The system reads:

$$\left(\frac{dS_0}{dt}\right) = -\boldsymbol{F} \cdot S_0 + \left(k_1 - \mu^{\rightarrow}_{m\acute{a}x.} \cdot \eta_1\right) \cdot S_1 - \mu^{\rightarrow}_{m\acute{a}x.} \cdot \left(\sum_{j=2}^{L} \eta_j \cdot S_j\right) \qquad [4.1]$$

$$\left(\frac{dS_1}{dt}\right) = \boldsymbol{F} \cdot S_0 + \left[\left(\mu^{\rightarrow}_{m\acute{a}x.} - \mu^{\uparrow}_{m\acute{a}x.}\right) \cdot \eta_1 - \boldsymbol{F} - k_1\right] \cdot S_1 + \left[k_2 + \mu^{\rightarrow}_{m\acute{a}x.} \cdot \left(\eta_2 - \eta_1\right)\right] \cdot S_2 +$$
$$+ \mu^{\rightarrow}_{m\acute{a}x.} \cdot \left[\sum_{j=3}^{L} (\eta_j - \eta_{j-1}) \cdot S_j\right] - \mu^{\rightarrow}_{m\acute{a}x.} \cdot \eta_L \cdot S_{L+1} \qquad [4.2]$$

$$\left(\frac{dS_n}{dt}\right) = \left[\boldsymbol{F} + \mu^{\uparrow}_{m\acute{a}x.} \cdot \left(\sum_{j=1}^{n-1} \eta_j\right)\right] \cdot S_{n-1} + \left[-\boldsymbol{F} + \mu^{\rightarrow}_{m\acute{a}x.} \cdot \eta_1 - \mu^{\uparrow}_{m\acute{a}x.} \cdot \left(\sum_{j=1}^{n} \eta_j\right) - k_n\right] \cdot S_n +$$
$$+ \left[\mu^{\rightarrow}_{m\acute{a}x.} \cdot (\eta_2 - \eta_1) + k_{n+1}\right] \cdot S_{n+1} + \mu^{\rightarrow}_{m\acute{a}x.} \cdot \left[\sum_{j=2}^{L-1} (\eta_{j+1} - \eta_j) \cdot S_{n+j}\right] - \mu^{\rightarrow}_{m\acute{a}x.} \cdot \eta_L \cdot S_{n+L}$$
$$\ldots \text{ if } (2 \leq n \leq L) \qquad [4.3]$$

$$\left(\frac{dS_n}{dt}\right) = \left[\boldsymbol{F} + \mu^{\uparrow}_{m\acute{a}x.} \cdot \left(\sum_{j=1}^{L} \eta_j\right)\right] \cdot S_{n-1} + \left[-\boldsymbol{F} + \mu^{\rightarrow}_{m\acute{a}x.} \cdot \eta_1 - \mu^{\uparrow}_{m\acute{a}x.} \cdot \left(\sum_{j=1}^{L} \eta_j\right) - k_n\right] \cdot S_n +$$
$$+ \left[\mu^{\rightarrow}_{m\acute{a}x.} \cdot (\eta_2 - \eta_1) + k_{n+1}\right] \cdot S_{n+1} + \mu^{\rightarrow}_{m\acute{a}x.} \cdot \left[\sum_{j=2}^{L-1} (\eta_{j+1} - \eta_j) \cdot S_{n+j}\right] - \mu^{\rightarrow}_{m\acute{a}x.} \cdot \eta_L \cdot S_{n+L}$$
$$\ldots \text{ if } (n > L) \qquad [4.4]$$

We observe that the system structure consists in,

$\left(\frac{dS_0}{dt}\right)$: one differential equation

$\left(\frac{dS_1}{dt}\right)$: one differential equation

$\left(\frac{dS_n}{dt}\right)$ if $(2 \leq n \leq L)$: $(L-1)$ differential equations

$\left(\frac{dS_n}{dt}\right)$ if $(n>L)$: infinite number of differential equations

The dimensionless variables and parameters that we choose are defined from expression [4.5] to expression [4.11]:

Dimensionless variables and parameters definition

$$\tau = \left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right) \cdot \eta_1 \cdot t \qquad [4.5]$$

$$\theta_n = \frac{S_n}{S_T} \qquad [4.6]$$

$$\rho = \frac{F}{\left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right) \cdot \eta_1} \qquad [4.7]$$

$$\Phi = \frac{\mu^{\uparrow}_{máx.}}{\left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right)} \qquad [4.8]$$

$$(1-\Phi) = \frac{\mu^{\rightarrow}_{máx.}}{\left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right)} \qquad [4.9]$$

$$\delta_n = \frac{k_n}{\left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right) \cdot \eta_1} \qquad [4.10]$$

$$\psi_j = \frac{\eta_j}{\eta_1} \qquad [4.11]$$

Before insert the dimensionless parameters it is convenient put in evidence the kinetic factor η_1 whenever it occurs to made easy the substitution of ψ_j .
This is done from expression [4.12] to expression [4.15].

$$\left(\frac{dS_0}{dt}\right) = -\boldsymbol{F} \cdot S_0 + \left(k_1 - \mu^{\rightarrow}_{máx.} \cdot \eta_1\right) \cdot S_1 - \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=2}^{L} \left(\frac{\eta_j}{\eta_1}\right) \cdot S_j\right] \qquad [4.12]$$

$$\left(\frac{dS_1}{dt}\right) = \boldsymbol{F} \cdot S_0 + \left[\left(\mu^{\rightarrow}_{máx.} - \mu^{\uparrow}_{máx.}\right) \cdot \eta_1 - \boldsymbol{F} - k_1\right] \cdot S_1 + \left[k_2 + \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left(\frac{\eta_2}{\eta_1} - 1\right)\right] \cdot S_2 +$$
$$+ \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=3}^{L} \left(\frac{\eta_j}{\eta_1} - \frac{\eta_{j-1}}{\eta_1}\right) \cdot S_j\right] - \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left(\frac{\eta_L}{\eta_1}\right) \cdot S_{L+1} \qquad [4.13]$$

$$\left(\frac{dS_n}{dt}\right) = \left\{\boldsymbol{F} + \mu^{\uparrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=1}^{n-1}\left(\frac{\eta_j}{\eta_1}\right)\right]\right\} \cdot S_{n-1} - \left\{\boldsymbol{F} - \mu^{\rightarrow}_{máx.} \cdot \eta_1 + \mu^{\uparrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=1}^{n}\left(\frac{\eta_j}{\eta_1}\right)\right] + k_n\right\} \cdot S_n +$$

$$+\left[\mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left(\frac{\eta_2}{\eta_1} - 1\right) + k_{n+1}\right] \cdot S_{n+1} + \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=2}^{L-1}\left(\frac{\eta_{j+1}}{\eta_1} - \frac{\eta_j}{\eta_1}\right) \cdot S_{n+j}\right] - \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left(\frac{\eta_L}{\eta_1}\right) \cdot S_{n+L}$$

$$\ldots \text{ if } (2 \le n \le L) \qquad [4.14]$$

$$\left(\frac{dS_n}{dt}\right) = \left\{\boldsymbol{F} + \mu^{\uparrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=1}^{L}\left(\frac{\eta_j}{\eta_1}\right)\right]\right\} \cdot S_{n-1} - \left\{\boldsymbol{F} - \mu^{\rightarrow}_{máx.} \cdot \eta_1 + \mu^{\uparrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=1}^{L}\left(\frac{\eta_j}{\eta_1}\right)\right] + k_n\right\} \cdot S_n +$$

$$+\left[\mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left(\frac{\eta_2}{\eta_1} - 1\right) + k_{n+1}\right] \cdot S_{n+1} + \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left[\sum_{j=2}^{L-1}\left(\frac{\eta_{j+1}}{\eta_1} - \frac{\eta_j}{\eta_1}\right) \cdot S_{n+j}\right] - \mu^{\rightarrow}_{máx.} \cdot \eta_1 \cdot \left(\frac{\eta_L}{\eta_1}\right) \cdot S_{n+L}$$

$$\ldots \text{ if } (n > L) \qquad [4.15]$$

As consequence of [4.5] and [4.6] the derivatives $\left(\frac{dS_n}{dt}\right)$ on the left hand sides assume the form:

$$\left(\frac{dS_n}{dt}\right) = \frac{d(\theta_n \cdot S_T)}{d\left[\dfrac{\tau}{\eta_1\left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right)}\right]} = \left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right) \cdot \eta_1 \cdot S_T \cdot \left(\frac{d\theta_n}{d\tau}\right) \qquad [4.16]$$

The next step is to divide all the equations by $\left[\left(\mu^{\rightarrow}_{máx.} + \mu^{\uparrow}_{máx.}\right) \cdot \eta_1 \cdot S_T\right]$ and we get immediately the dimensionless system, equations [4.17] to [4.20].
Besides initial condition [4.21] is part of the system, consequently also must be included.

$$\left(\frac{d\theta_0}{d\tau}\right) = -\rho \cdot \theta_0 + [\delta_1 - (1 - \Phi)] \cdot \theta_1 - (1 - \Phi) \cdot \left(\sum_{j=2}^{L} \psi_j \cdot \theta_j\right) \qquad [4.17]$$

$$\left(\frac{d\theta_1}{d\tau}\right) = \rho \cdot \theta_0 + [(1 - 2 \cdot \Phi) - \rho - \delta_1] \cdot \theta_1 + [\delta_2 + (1 - \Phi) \cdot (\psi_2 - 1)] \cdot \theta_2 +$$

$$+(1 - \Phi) \cdot \left[\sum_{j=3}^{L}(\psi_j - \psi_{j-1}) \cdot \theta_j\right] - (1 - \Phi) \cdot \psi_L \cdot \theta_{L+1} \qquad [4.18]$$

$$\left(\frac{d\theta_n}{d\tau}\right) = \left[\rho + \Phi \cdot \left(\sum_{j=1}^{n-1} \psi_j\right)\right] \cdot \theta_{n-1} + \left[(1-\Phi) - \rho - \Phi \cdot \left(\sum_{j=1}^{n} \psi_j\right) - \delta_n\right] \cdot \theta_n +$$

$$+\left[(1-\Phi) \cdot (\psi_2 - 1) + \delta_{n+1}\right] \cdot \theta_{n+1} + (1-\Phi) \cdot \left[\sum_{j=2}^{L-1} (\psi_{j+1} - \psi_j) \cdot \theta_{n+j}\right] - (1-\Phi) \cdot \psi_L \cdot \theta_{n+L}$$

$$\ldots \; if \; (2 \leq n \leq L) \qquad [4.19]$$

$$\left(\frac{d\theta_n}{d\tau}\right) = \left[\rho + \Phi \cdot \left(\sum_{j=1}^{L} \psi_j\right)\right] \cdot \theta_{n-1} + \left[(1-\Phi) - \rho - \Phi \cdot \left(\sum_{j=1}^{L} \psi_j\right) - \delta_n\right] \cdot \theta_n +$$

$$+\left[(1-\Phi) \cdot (\psi_2 - 1) + \delta_{n+1}\right] \cdot \theta_{n+1} + (1-\Phi) \cdot \left[\sum_{j=2}^{L-1} (\psi_{j+1} - \psi_j) \cdot \theta_{n+j}\right] - (1-\Phi) \cdot \psi_L \cdot \theta_{n+L}$$

$$\ldots \; if \; (n>L) \qquad [4.20]$$

The initial condition must, of course, be:

$$\theta_0(0) = 1, \; and \;\; \theta_n(0) = 0 \;\; for \; n>0 \qquad [4.21]$$

(at the beginning (τ=0) al the solid support is bare)

2. General model resolution strategy: concept of "Mathematical meeting point"

This system of equations is composed by the so-called differential equations of difference. It is a traditionally difficult domain and suitable theory for this subject can be found in Pinney [34], Bellman and Cooke [35] and El´sgol´ts and Norkin [36] and certainly other more recent works.

Also formally similar problems arise in Operational Research literature, in particular when transient states are analysed in Queuing Theory, both in book chapters and in articles.

In the present case the system has an analytical solution. However that solution can´t be represented, for the most general version, by a closed form, like a final expression of type $\theta_n = \theta_n(\tau)$, and we will only solve the model for a simplified case and extract observations from that resolution.

Such observations will allow to establish the method for more general cases.

A formalism similar to that of Clarke ([37], [38]) will be our working tool. Suitable modifications will be introduced to adapt such solving way to the particular details of our case. Clarke formalism can also be found described in Saaty [39] and Srivastava and Kashyap [40].

We will use the following strategy, in tow steps:
First step) to solve the system for $L = 1$ and $\delta_1 = \delta_2 = \cdots = \delta_n = \cdots = \delta$. That is, for the case where there is only one active layer and erosion is of equal intensity in all the surface fractions θ_n $(n \geq 1)$. This one will be named the simplified version of the model. More specifically we can name this version the "Mono layered concentrated" growth kinetics variant, already defined in the previous Chapter III.
Second step) To profit from the solution obtained in the first step, inspecting their mathematical structure, and constructing a suitable Guess Solution that, as we will see when arriving at that point, will reduce all the solution of the complete system to an algebraic problem.
The algebraic problem defined in this second step falls in the context of algebraic difference equations and will be somewhat cumbersome. However it will be demonstrated that the complete resolution is possible.
The aforesaid Guess Solution contains an infinite series of Bessel functions, each one multiplied by a respective labelled factor. All those factors must then be analytically determined on a recurring suitable scheme..
Several mathematical and technical items, used in the first step, will be analyzed and selectively profited, with appropriate modifications, in the general resolution of the second step.
The system of the simplified "Mono layered concentrated" growth kinetics version, corresponding to the aforesaid first step, is easily obtained from [4.17] to [4.21] inserting the conditions:

$$\delta_1 = \delta_2 = \cdots = \delta_n = \cdots = \delta \qquad [4.22]$$

... and,

$$\psi_j = 0\,, \dots\dots\dots\ for\ \ j > 1\ \ \dots\dots\dots leaving\ L \geq 2 \qquad [4.23]$$

Condition [4.23] is equivalent to impose $L = 1$ because all kinetic parameters ψ_j are annulled for $j \geq 2$.
We also observe that [4.23] implicates the sameness between [4.18] and [4.19].
Consequently, the system for the simplified version is:

$$\left(\frac{d\theta_0}{d\tau}\right) = -\rho \cdot \theta_0 + [\delta - (1 - \Phi)] \cdot \theta_1 \qquad [4.24]$$

$$\left(\frac{d\theta_1}{d\tau}\right) = \rho \cdot \theta_0 + [(1 - 2 \cdot \Phi) - \rho - \delta] \cdot \theta_1 + [\delta - (1 - \Phi)] \cdot \theta_2 \qquad [4.25]$$

$$\left(\frac{d\theta_n}{d\tau}\right) = (\rho + \Phi) \cdot \theta_{n-1} + [(1 - \Phi) - \rho - \Phi - \delta] \cdot \theta_n + [\delta - (1 - \Phi)] \cdot \theta_{n+1}$$

$$\dots\ if\ (n \geq 2) \qquad [4.26]$$

With initial condition:

$$\theta_0(0) = 1, \quad and \quad \theta_n(0) = 0\ \ for\ n>0 \qquad [4.27]$$

We have already defined, as work context, the five specific growth rate kinetics:
- Monod kinetics
- Zero order kinetics
- "Blackman inspired" kinetics
- "Averaged" first order kinetics
- "Mono-layered concentrated growth" kinetics

When $L = 1$ the first four (Monod, Zero order, "Blackman inspired" and "Averaged" first order) reduce in complexity and becomes equal to "Mono-layered concentrated growth" kinetics model. So we can conclude that to solve this last model is equivalent to solve the most simplified version of all the other four. We can consider "Mono-layered concentrated growth" kinetics as a sort of "mathematical meeting point" of all the kinetics under our analysis.

3. "Mono layered concentrated" growth kinetics: exact solution

3.1 Common initial steps: generating function formalism

The first step to start the resolution is to make the substitution,

$$\theta_n(\tau) = e^{-(\rho+2\cdot\Phi-\delta-1)\cdot\tau} \cdot Q_n(\tau) \quad [4.28]$$

This substitution aims to simplify the system by eliminating the term relative to index (n) on the right hand side, in all the equations for ($n \geq 1$).
After simplification we get,

$$\left(\frac{dQ_0}{d\tau}\right) = (2\cdot\Phi+\delta-1)\cdot Q_0 - (1-\Phi-\delta)\cdot Q_1 \quad [4.29]$$

$$\left(\frac{dQ_1}{d\tau}\right) = \rho\cdot Q_0 - (1-\Phi-\delta)\cdot Q_2 \quad [4.30]$$

$$\left(\frac{dQ_n}{d\tau}\right) = (\rho+\Phi)\cdot Q_{n-1} - (1-\Phi-\delta)\cdot Q_{n+1} \quad \ldots\ if\ (n \geq 2) \quad [4.31]$$

With initial condition:

$$Q_0(0) = 1, \quad and \quad Q_n(0) = 0 \quad for\ n>0 \quad [4.32]$$

Let define the following generating function,

$$G(z,\tau) = \sum_{n=0}^{\infty} a_n \cdot Q_n(\tau) \cdot \frac{(\tau - z)^n}{n!} \quad [4.33]$$

… similar to that one of Clarke ([37], [38]),

$$G(z,\tau) = \sum_{n=0}^{\infty} Q_n(\tau) \cdot \frac{(z-\tau)^n}{n!} \quad [4.34]$$

… but with additional coefficients a_n which remain to be determined and, for convenience of calculation, with changed signals at τ and z .
These modifications are justified because the system solved by Clarke is,

$$\left(\frac{dQ_0}{d\tau}\right) = Q_0 + Q_1 \quad [4.35]$$

$$\left(\frac{dQ_n}{d\tau}\right) = \rho\cdot Q_{n-1} + Q_n \quad \ldots\ if\ (n \geq 1) \quad [4.36]$$

Therefore this is a simpler system because Φ and δ are null which is not the case of our system.
This explains the need to include the coefficients a_n .

Another difference lies in the exchanged signals in the functions $Q_n(\tau)$ which explains the need to change the signals in τ and z in the generating function.
Like in Clarke ([37], [38]) here we also seek for a generating function, $G(z,\tau)$ such that obeys a differential equation like,

$$\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z} = Constant \cdot\, G(z,\tau) \qquad [4.37]$$

Which is the so called Telegraph equation.
The suitable deduction of coefficients a_n will accomplish that goal.
First, $\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z}$ must be obtained from [4.33]. We get,

$$\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z} = -a_1 \cdot \frac{dQ_1}{d\tau} - \sum_{n=2}^{\infty} a_n \cdot \frac{dQ_n}{d\tau} \cdot \frac{(\tau-z)^{n-1}}{(n-1)!} - \sum_{n=2}^{\infty} a_n \cdot Q_n \cdot \frac{(\tau-z)^{n-2}}{(n-2)!} \qquad [4.38]$$

The order of partial derivations is indifferent,

$$\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z} = \frac{\partial^2 G(z,\tau)}{\partial z\partial\tau} \qquad [4.39]$$

Now, taking the expressions for $\frac{dQ_n}{d\tau}$ with $n \geq 1$ given by [4.30] and [4.31] we get,

$$\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z} = -a_1 \cdot \rho \cdot Q_0 - (\rho+\Phi) \cdot \sum_{n=2}^{\infty} a_n \cdot Q_{n-1} \cdot \frac{(\tau-z)^{n-1}}{(n-1)!} -$$

$$-\sum_{n=2}^{\infty} [a_{n-1} \cdot (1-\Phi-\delta) - a_n] \cdot Q_n \cdot \frac{(\tau-z)^{n-2}}{(n-2)!} \qquad [4.40]$$

The usefulness of coefficients a_n is now proved inspecting the second summation because we want to cancel it. The reason is that the exponents in $(\tau - z)$ is two units lower than the corresponding order of functions Q_n and this summation cancelation would not be possible if all coefficients a_n where equal to 1, like in Clarke´s generating function.
Determination of coefficients a_n is easy and follows straightaway.
Must be,

$$a_{n-1} \cdot (1-\Phi-\delta) - a_n = 0 \quad \dots for\ (n \geq 2) \qquad [4.41]$$

We get immediately the solution,

$$a_n = a_1 \cdot (1-\Phi-\delta)^{n-1} \quad \dots for\ (n \geq 2) \qquad [4.42]$$

Substituting [4.42] in [4.40] the Telegraph equation now reads,

$$\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z} = -(\rho+\Phi) \cdot (1-\Phi-\delta) \cdot \left[a_1 \cdot \frac{\rho}{(\rho+\Phi)\cdot(1-\Phi-\delta)} \cdot Q_0 + a_1 \cdot \sum_{n=1}^{\infty} (1-\Phi-\delta)^{n-1} \cdot Q_n \cdot \frac{(\tau-z)^n}{(n)!}\right]$$

[4.43]

Seting,

$$a_0 = a_1 \cdot \frac{\rho}{(\rho+\Phi)\cdot(1-\Phi-\delta)} \qquad [4.44]$$

… equation [4.43] becomes,

$$\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z} = -(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\sum_{n=0}^{\infty} a_n\cdot Q_n\cdot\frac{(\tau-z)^n}{(n)!} \qquad [4.45]$$

This right hand side is equal to $G(z,\tau)$ multiplied by $[-(\rho+\Phi)\cdot(1-\Phi-\delta)]$, which is constant. As proposed, a Telegraph equation is now reached.
The coefficient a_0 can be equal to 1 for the sake of simplification. Consequently all the a_n are now defined,

$$a_0 = 1 \qquad [4.46]$$

… and $$a_n = \frac{(\rho+\Phi)}{\rho}\cdot(1-\Phi-\delta)^n \quad \text{… for } (n\geq 1) \qquad [4.47]$$

Till now holds,

$$\frac{\partial^2 G(z,\tau)}{\partial\tau\,\partial z} = -\,(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot G(z,\tau) \quad \text{(Telegraph equation)} \qquad [4.48]$$

… where,

$$G(z,\tau) = Q_0(\tau) + \frac{(\rho+\Phi)}{\rho}\cdot\sum_{n=1}^{\infty}(1-\Phi-\delta)^n\cdot Q_n\,(\tau)\cdot\frac{(z-\tau)^n}{n!} \qquad [4.49]$$

Now equation [4.29] and condition [4.32] will come into play.
According to [4.32], at time $(\tau = 0)$, the generating function must be $G(z,\tau) = Q_0(0) = 1$.
So, condition [4.32] now reads. in generating function formalism,

$$G(z,0) = 1 \qquad [4.50]$$

With [4.49] the partial derivative $\left(\frac{\partial G}{\partial\tau}\right)$ can be easily obtained,

$$\left(\frac{\partial G}{\partial\tau}\right) = \frac{dQ_0}{d\tau} + \frac{(\rho+\Phi)}{\rho}\cdot\sum_{n=1}^{\infty}(1-\Phi-\delta)^n\cdot\frac{dQ_n}{d\tau}\cdot\frac{(\tau-z)^n}{n!} +$$

$$+\frac{(\rho+\Phi)}{\rho}\cdot\sum_{n=1}^{\infty}(1-\Phi-\delta)^n\cdot Q_n\cdot\frac{(\tau-z)^{n-1}}{(n-1)!} \qquad [4.51]$$

When $(z=\tau)$,

$$\left(\frac{\partial G}{\partial\tau}\right)_{z=\tau} = \frac{dQ_0}{d\tau} + \frac{(\rho+\Phi)}{\rho}\cdot(1-\Phi-\delta)\cdot Q_1 \qquad [4.52]$$

Inserting here equation [4.29] one gets,

$$\left(\frac{\partial G}{\partial\tau}\right)_{z=\tau} = (2\cdot\Phi+\delta-1)\cdot Q_0 + \frac{(1-\Phi-\delta)\cdot\Phi}{\rho}\cdot Q_1 \qquad [4.53]$$

But,

$$Q_0(\tau) = G(\tau,\tau) \qquad [4.54]$$

… and,

$$\frac{\partial G}{\partial z} = -\frac{(\rho+\Phi)}{\rho}\cdot\sum_{n=1}^{\infty}(1-\Phi-\delta)^n\cdot Q_n(\tau)\cdot\frac{(\tau-z)^{n-1}}{(n-1)!} \qquad [4.55]$$

What means,

$$\left(\frac{\partial G}{\partial z}\right)_{z=\tau} = -\frac{(\rho+\Phi)}{\rho}\cdot(1-\Phi-\delta)\cdot Q_1(\tau) \qquad [4.56]$$

.. and, consequently,

$$Q_1(\tau) = -\frac{\rho}{(\rho+\Phi)\cdot(1-\Phi-\delta)}\cdot\left(\frac{\partial G}{\partial z}\right)_{z=\tau} \qquad [4.57]$$

Equations [4.54] and [4.57] are now included in [4.53] and, in this way, the last condition for complete definition of the problem, in generating function context, is,

$$\left(\frac{\partial G}{\partial \tau}\right)_{z=\tau} = (2\cdot\Phi+\delta-1)\cdot G(\tau,\tau) - \frac{\Phi}{(\rho+\Phi)}\cdot\left(\frac{\partial G}{\partial z}\right)_{z=\tau} \qquad [4.58]$$

Once deduced, the generating function $G(z,\tau)$ will conduct easily to all the functions $Q_n(\tau)$.
In fact from [4.49] one can conclude that,

$$Q_0(\tau) = G(\tau,\tau) \qquad (already\ obtained\ before) \qquad [4.59]$$

... and,

$$Q_n(\tau) = -\frac{(-1)^n\cdot\rho}{(\rho+\Phi)\cdot(1-\Phi-\delta)^n}\cdot\left(\frac{\partial^n G}{\partial z^n}\right)_{z=\tau} \quad \ldots\ldots for\ (n\geq 1) \qquad [4.60]$$

This last formula is easy to demonstrate, so we omit the details.
Lastly, with [4.28] fractions $\theta_n(\tau)$ are calculated from the functions $Q_n(\tau)$.
Then we will reach the solution of the simplified version of the model which is also the "mathematical meeting point" for all the kinetic variants under analysis.
A summary of what has been obtained so far and will be used in the sequel is,

... from (I) to (VII),

<u>*Generating function formalism:*</u>

$I)$ $\left(\frac{\partial^2 G}{\partial z\,\partial \tau}\right) = -(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot G(z,\tau)$ $\quad (Telegraph\ equation)$ [4.48]

$II)$ $G(z,0) = 1$ $\quad (Initial\ time\ condition)$ [4.50]

$III)$ $\left(\frac{\partial G}{\partial \tau}\right)_{z=\tau} = (2\cdot\Phi+\delta-1)\cdot G(\tau,\tau) - \frac{\Phi}{(\rho+\Phi)}\cdot\left(\frac{\partial G}{\partial z}\right)_{z=\tau}$ $\quad (First\ \ equation\ border\ condition)$ [4.58]

$IV)$ $G(z,\tau) = Q_0(\tau) + \frac{(\rho+\Phi)}{\rho}\cdot\sum_{n=1}^{\infty}(1-\Phi-\delta)^n\cdot Q_n(\tau)\cdot\frac{(z-\tau)^n}{n!}$ $\quad (Generating\ function)$ [4.49]

$V)$ $Q_0(\tau) = G(\tau,\tau)$ $\quad (Obtaining\ Q_0)$ [4.59]

$VI)$ $Q_n(\tau) = -\frac{(-1)^n\cdot\rho}{(\rho+\Phi)\cdot(1-\Phi-\delta)^n}\cdot\left(\frac{\partial^n G}{\partial z^n}\right)_{z=\tau}$ $\ldots\ldots for\ (n\geq 1)$ $\quad (Obtaining\ Q_n\ for\ n\geq 1)$ [4.60]

$VII)$ $\theta_n(\tau) = e^{-(\rho+2\cdot\Phi-\delta-1)\cdot\tau}\cdot Q_n(\tau)$ $\quad (Obtaining\ \theta_n\ from\ Q_n)$ [4.28]

Telegraph equation [4.48] is solved applying Laplace transform over variable z.
Applying first to the left hand side, one gets,

$$\mathcal{L}\left\{\frac{\partial}{\partial z}\cdot\left[\frac{\partial G}{\partial \tau}\right]\right\} = \int_0^\infty e^{-s\cdot z}\cdot\frac{\partial}{\partial z}\cdot\left[\frac{\partial G}{\partial \tau}\right]\cdot dz \qquad (definition\ of\ Laplace\ transform) \qquad [4.61]$$

$$\int_0^\infty e^{-s\cdot z}\cdot\frac{\partial}{\partial z}\cdot\left[\frac{\partial G}{\partial \tau}\right]\cdot dz = s\cdot\mathcal{L}\left[\frac{\partial G}{\partial \tau}\right] - \left(\frac{\partial G}{\partial \tau}\right)_{z=0} = s\cdot\frac{\partial g(s,\tau)}{\partial \tau} - f(\tau) \qquad [4.62]$$

The function $g(s,\tau)$ is the Laplace transform of $G(z,\tau)$ and,

$$f(\tau) = \left[\frac{\partial G(z,\tau)}{\partial \tau}\right]_{z=0} \qquad [4.63]$$

Similarly, applying the transform to the right hand side,

$$\mathcal{L}[-(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot G(z,\tau)] = -(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot g(s,\tau) \qquad [4.64]$$

The right had sides in [4.62] and [4.64] must be equal,

$$s\cdot\frac{\partial g(s,\tau)}{\partial \tau} - f(\tau) = -(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot g(s,\tau) \qquad [4.65]$$

The initial time condition [4.50] after Laplace transform reads,

$$\mathcal{L}[G(z,0)] = \mathcal{L}(1) \qquad [4.66]$$

… which is equivalent to,

$$g(s,0) = \frac{1}{s} \qquad [4.67]$$

The general solution of [4.65] is known from mathematical tables,

$$g(s,\tau) = e^{-(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\frac{\tau}{s}}\cdot\left[\frac{1}{s}\cdot\int f(\tau)\cdot e^{-(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\frac{\tau}{s}}\cdot d\tau + C(s)\right] \qquad [4.68]$$

The integration "constant" $C(s)$ is only constant relatively to time τ and only depends on s.
Inserting condition [4.67] and making $\tau = 0$ one can get $C(s)$,
From [4.68],

$$g(s,0) = \frac{1}{s}\cdot\left[\int f(\tau)\cdot e^{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\frac{\tau}{s}}\cdot d\tau\right]_{\tau=0} + C(s) \qquad [4.69]$$

Substitution of [4.67] here leads to,

$$C(s) = \frac{1}{s}\cdot\left\{1 - \left[\int f(\tau)\cdot e^{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\frac{\tau}{s}}\cdot d\tau\right]_{\tau=0}\right\} \qquad [4.70]$$

Now is easy to particularize the general solution [4.68] for our case, inserting there this expression of $C(s)$,

$$g(s,\tau) = \frac{e^{-(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\frac{\tau}{s}}}{s} + \int_0^\tau f(\sigma)\cdot\frac{e^{-(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\frac{(\tau-\sigma)}{s}}}{s}\cdot d\sigma \qquad [4.71]$$

An observation must be made about the change of τ by σ as integration variable. This has been done with the purpose of avoid confusion between that variable and the superior limit in the integral.

3.2 Trifurcation stage.
Continuous biofilm, patchy biofilm and border case

In the sequel, application of inverse Laplace transform leads back again to the generating function $G(z, \tau)$.
But before apply inverse transform tree hypotheses came into play depending on the relative value of parameter δ when compared to the value of $(1-\Phi)$. As bigger is δ, or as littler is $(1-\Phi)$,
more the process of erosion is important in their intensity, comparatively to the process of bacterial growth inside the biofilm.
In mathematical context this is compulsory because if the signal in the exponentials is positive the inverse transform is one and if it is negative is another.
Explicitly,

Case 1) If $\delta<(1-\Phi)$, the inverse transform reads,

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\cdot\tau/s}}{s}\right\} = J_0\big(2\cdot\sqrt{a\cdot\tau\cdot z}\big) \ldots\ldots where\ a = (\rho+\Phi)\cdot(1-\Phi-\delta)>0 \qquad [4.72]$$

Case 2) If $\delta>(1-\Phi)$, the inverse transform reads,

$$\mathcal{L}^{-1}\left\{\frac{e^{+a\cdot\tau/s}}{s}\right\} = I_0\big(2\cdot\sqrt{a\cdot\tau\cdot z}\big) \ldots\ldots where\ a = (\rho+\Phi)\cdot(\Phi+\delta-1)>0 \qquad [4.73]$$

Case 3) If $\delta=(1-\Phi)$, the inverse transform reads,

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \ldots\ldots in\ this\ borderline\ case,\ a=0 \qquad [4.74]$$

The functions in cases 1 and 2 are respectively Bessel functions and Modified Bessel functions of first kind of order zero.
Bessel function of first kind of order n is defined by,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\,\Gamma(n+k+1)} \qquad [4.75]$$

… and Modified Bessel function of first kind of order n by,

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\,\Gamma(n+k+1)} \qquad [4.76]$$

Here $\Gamma(x)$ is the gamma function. When x =1, 2, 3, … then,

$$\Gamma(x) = (x-1)! \qquad [4.77]$$

In our case $n=0$, so $\Gamma(n+k+1) = \Gamma(k+1) = k!$ and we can write,

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2} \qquad [4.78]$$

… and,

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2} \qquad [4.79]$$

As indicated before, in the first stages of biofilm formation erosion is seldom the most intense process. In most of situations growth as consequence of active bacterial reproduction is nearly the only relevant process. This corresponds, in mathematical terms to the aforesaid case 1.And we can classify the obtained biofilm as a "continuous biofilm".
However, patchy biofilms, also exists elsewhere if erosion becomes strong enough in those initial times of development, leading to a steady state where bare areas of solid support till exist in finite extension, after a long time, and biomass remains only accumulated in colonies or clusters without a complete coverage of solid support area.
The designation "patchy biofilm" is opposed, or complementary to "continuous biofilm" in this physical classification.
Consequently we observe that this mathematical formalism can translate this duality in biofilm structure, relating different biofilm morphologies with different mathematical functions.
Such translation gets more visibility, principally when achieved formulas are graphically represented.
The border case, as will be demonstrated, corresponds to a steady state where the limit value of the bare fraction, $\theta_0(\tau)$, is zero at infinite time τ, as expected.
Continuous biofilm, after some time, reach a stadium when coverage of initial support bare area is rendered complete. After that time τ, the fraction $\theta_0(\tau)$ assume negative values, and those values are not physically possible. Consequently one must initiate a new calculation time interval where $\theta_0(\tau)$ no longer came into play. and successively so on later with $\theta_1(\tau), \theta_2(\tau)$ … by the same reasons.

3.3 Continuous biofilm solution

Now follows the solution for continuous biofilm, case 1.
Inverse Laplace transform applied to [4.71] leads to,

$$G(z,\tau) = J_0\left(2 \cdot \sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\tau\cdot z}\right) + \int_0^{\tau} J_0\left(2\cdot\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot(\tau-\sigma)\cdot z}\right)\cdot f(\sigma)\cdot d\sigma\cdot$$

[4.80]

At this point, it is convenient to define other functions, related to Bessel functions, but easier for calculation purposes.
We will need to derive partially relatively to τ and z when the conditions in generating function formalism, before established, must be satisfied. However such derivations applied directly in Bessel functions are somewhat cumbersome.
Such more adequate functions are,

$$V_n(\sigma,\tau,z) = \left[\frac{(\tau-\sigma)}{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot z}\right]^{n/2} \cdot J_n\left(2\cdot\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot(\tau-\sigma)\cdot z}\right)$$

[4.81]

Their derivatives are very simple to use, according to the following rules,

$$\frac{\partial V_n(\sigma,\tau,z)}{\partial z} = -(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot V_{n+1}(\sigma,\tau,z)$$

[4.82]

$$\frac{\partial V_n(\sigma,\tau,z)}{\partial \tau} = V_{n-1}(\sigma,\tau,z)$$

[4.83]

Demonstration of these rules are long and tedious. So we will pursue simply accepting these rules.
Inserting [4.81] in [4.80],

$$G(z,\tau) = V_0(0,\tau,z) + \int_0^{\tau} V_0(\sigma,\tau,z) \cdot f(\sigma) \cdot d\sigma \qquad [4.84]\cdot$$

Now this solution must satisfy condition [4.58], which has not yet been taken in consideration.
For achieve that goal determination of function $f(\sigma)$ is the way.
Let define $f(\sigma)$ by an infinite power series,

$$f(\sigma) = \sum_{k=0}^{\infty} c_k \cdot \sigma^k \qquad [4.85]$$

... and include the power series in [4.84],

$$G(z,\tau) = V_0(0,\tau,z) + \sum_{k=0}^{\infty} c_k \cdot \left[\int_0^{\tau} V_0(\sigma,\tau,z) \cdot \sigma^k \ \cdot d\sigma\right] \qquad [4.86]$$

The integrals,

$$\int_0^{\tau} V_0(\sigma,\tau,z) \cdot \sigma^k \ \cdot d\sigma$$

... can be calculated introducing the definition of the function $V_0(\sigma,\tau,z)$ and applying the identity,

$$\int_0^{\tau} (\tau-\sigma)^i \cdot \sigma^k \ \cdot d\sigma = \frac{k!\,i!}{(k+i+1)!} \cdot \tau^{k+i+1} \qquad [4.87]$$

The result, which we limit ourselves to only indicate, is,

$$\int_0^{\tau} V_0(\sigma,\tau,z) \cdot \sigma^k \ \cdot d\sigma = k! \cdot V_{k+1}(0,\tau,z) \qquad [4.88]$$

The consequent sequel is that [4.86] becomes,

$$G(z,\tau) = V_0(0,\tau,z) + \sum_{k=0}^{\infty} c_k \cdot k! \cdot V_{k+1}(0,\tau,z) \qquad [4.89]$$

Rules [4.82] and [4.83] came now into play to accomplish condition [4.58] in this last equation.
Calculating the partial derivatives of the generating function in [4.89] they reads,

$$\frac{\partial G(z,\tau)}{\partial \tau} = V_{-1}(0,\tau,z) + \sum_{k=0}^{\infty} c_k \cdot k! \cdot V_k(0,\tau,z) \qquad [4.90]$$

$$\frac{\partial G(z,\tau)}{\partial z} = -(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\left[V_1(0,\tau,z) + \sum_{k=0}^{\infty} c_k \cdot k! \cdot V_{k+2}(0,\tau,z)\right] \qquad [4.91]$$

Easily one can prove,

$$V_{-n}(\sigma,\tau,z)=\left[\frac{(-1)(\rho+\Phi)(1-\Phi-\delta)}{(\tau-\sigma)}\right]^{n}\cdot V_n(\sigma,\tau,z) \qquad [4.92]$$

Formula [4.92] avoids the utilization of functions of negative order.

It can be used straightaway in [4.90],

$$\frac{\partial G(z,\tau)}{\partial \tau}=-\frac{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot z}{\tau}\cdot V_1(0,\tau,z)+\sum_{k=0}^{\infty}c_k\cdot k!\cdot V_k(0,\tau,z) \qquad [4.93]$$

At last the two partial derivatives of the generating function at $z=\tau$ can be written,

$$\left[\frac{\partial G(z,\tau)}{\partial \tau}\right]_{z=\tau}=-(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot V_1(0,\tau,\tau)+\sum_{k=0}^{\infty}c_k\cdot k!\cdot V_k(0,\tau,\tau) \qquad [4.94]$$

$$\left[\frac{\partial G(z,\tau)}{\partial z}\right]_{z=\tau}=-(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot\left[V_1(0,\tau,\tau)+\sum_{k=0}^{\infty}c_k\cdot k!\cdot V_{k+2}(0,\tau,\tau)\right] \qquad [4.95]$$

The left hand side of [4.58] is [4.94], and is now possible to write it putting the functions $V_k(0,\tau,\tau)$ in increasing order n,

$$\left[\frac{\partial G(z,\tau)}{\partial \tau}\right]_{z=\tau}=c_0\cdot V_0(0,\tau,\tau)+[c_1-(\rho+\Phi)\cdot(1-\Phi-\delta)]\cdot V_1(0,\tau,\tau)+\sum_{k=2}^{\infty}c_k\cdot k!\cdot V_k(0,\tau,\tau)$$

[4.96]

In the same way, the right hand side of [4.58] can be rewritten substituting there [4.89] and [4.95]. And also with the functions $V_k(0,\tau,\tau)$ in increasing order n,

$$(2\cdot\Phi+\delta-1)\cdot G(\tau,\tau)-\frac{\Phi}{(\rho+\Phi)}\cdot\left(\frac{\partial G}{\partial z}\right)_{z=\tau}=$$

$$=(2\cdot\Phi+\delta-1)\cdot V_0(0,\tau,\tau)+[(2\cdot\Phi+\delta-1)\cdot c_0-\Phi\cdot(1-\Phi-\delta)]\cdot V_1(0,\tau,\tau)+$$

$$+\sum_{k=0}^{\infty}\left[\Phi\cdot(1-\Phi-\delta)\cdot c_k\cdot k!+(2\cdot\Phi+\delta-1)\cdot c_{k+1}\cdot(k+1)!\right]\cdot V_{k+2}(0,\tau,\tau)$$

[4.97]

Equation [4.58] implies to equalize the right hand sides of [4.96] and [4.97]. Such equalization only will be valid if the factors affecting each function $V_k(0,\tau,\tau)$ are equal on those both right hand sides. In this way a system of algebraic equations is established and, from that system, all the coefficients c_k defined without ambiguity. And, at last, the generating function is completely defined.

The aforesaid system, extracted from [4.96] and [4.97], is,

$$c_0=(2\cdot\Phi+\delta-1) \qquad [4.98\ (0)]$$

$$c_1-(\rho+\Phi)\cdot(1-\Phi-\delta)=\Phi\cdot(1-\Phi-\delta)+(2\cdot\Phi+\delta-1)\cdot c_0 \qquad [4.98\ (1)]$$

$$c_2\cdot 2!=\Phi\cdot(1-\Phi-\delta)\cdot c_0+(2\cdot\Phi+\delta-1)\cdot c_1 \qquad [4.98\ (2)]$$

$$c_3\cdot 3!=\Phi\cdot(1-\Phi-\delta)\cdot c_1+(2\cdot\Phi+\delta-1)\cdot c_2\cdot 2! \qquad [4.98\ (3)]$$

$$c_4 \cdot 4! = \Phi \cdot (1 - \Phi - \delta) \cdot c_2 \cdot 2! + (2 \cdot \Phi + \delta - 1) \cdot c_3 \cdot 3! \qquad [4.98\ (4)]$$

− −

$$c_{k+2} \cdot (k+2)! = \Phi \cdot (1 - \Phi - \delta) \cdot c_k \cdot k! + (2 \cdot \Phi + \delta - 1) \cdot c_{k+1} \cdot (k+1)! \qquad [4.98\ (k+2)]$$

− −

The two first [4.98 (0)] and [4.98 (1)] are preliminary conditions and all the others have the general form given by [4.98 (k+2)].
This last one is a difference equation which form can be also put like,

$$b_{k+2} + A \cdot b_{k+1} + B \cdot b_k = 0 \qquad [4.99]$$

… where,

$$b_k = c_k \cdot k! \qquad [4.100]$$

$$A = (1 - 2 \cdot \Phi - \delta) \qquad [4.101]$$

$$B = \Phi \cdot (\Phi + \delta - 1) \qquad [4.102]$$

The general solution of [4.99] is known elsewhere,

$$b_k = \frac{[b_1 + b_0(A + r_1)]}{(r_1 - r_2)} \cdot r_1^k - \frac{[b_1 + b_0(A + r_2)]}{(r_1 - r_2)} \cdot r_2^k \qquad [4.103]$$

… where the roots are,

$$r_1 = \frac{-A + \sqrt{A^2 - 4 \cdot B}}{2} \qquad r_2 = \frac{-A - \sqrt{A^2 - 4 \cdot B}}{2} \qquad [4.104]$$

In the case under analysis, inserting [4.101] and [4.102] in [4.104],

$$r_1 = \Phi \qquad r_2 = (\Phi + \delta - 1) \qquad [4.105]$$

Substitution in [4.103] and consideration of [4.100] leads to the coefficients c_k,

$$c_k \cdot k! = \frac{1}{(1 - \delta)} \cdot \{[c_1 + c_0 \cdot (1 - \Phi - \delta)] \cdot \Phi^k - (c_1 - c_0 \cdot \Phi) \cdot (\Phi + \delta - 1)^k\} \qquad [4.106]$$

The coefficient c_0 is given by [4.98 (0)] and c_1 can be obtained by [4.98 (1)],

$$c_1 = (1 - \Phi - \delta) \cdot (\rho + 2 \cdot \Phi) + (2 \cdot \Phi + \delta - 1)^2 \qquad [4.107]$$

Now c_0 and c_1 are eliminated from [4.106] by substitution,

$$c_k = \frac{1}{(1 - \delta) \cdot k!} \cdot \{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^k + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^{k+1}\} \qquad [4.108]$$

Formula [4.108] is valid for $k \geq 1$.
For $k = 0$, the formula [4.98 (0)] must be used for calculate c_0.
Considering [4.100] we can use, from now on, the coefficients b_k or the c_k.
For sake of economy in notation is better to chose, from now on, only the coefficients b_k.

What is achieved till now about these coefficients can be appropriately summarized,

$$b_k = \begin{cases} (2 \cdot \Phi + \delta - 1) \ldots for\ k = 0 & [4.109\ (0)] \\ \\ \dfrac{\{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^k + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^{k+1}\}}{(1 - \delta)} \cdot \ldots for\ k \geq 1 & [4.109\ (k)] \end{cases}$$

Taking in consideration [4.100] and substituting these two expressions in [4.89] one gets the final wanted formula for the generating function,

$$G(z,\tau) = V_0(0,\tau,z) + (2\cdot\Phi+\delta-1)\cdot V_1(0,\tau,z) +$$

$$+\frac{1}{(1-\delta)}\cdot\sum_{k=1}^{\infty}\{[\rho\cdot(1-\Phi-\delta)+\Phi\cdot(1-\delta)]\cdot\Phi^k+(\rho+1-\delta)\cdot(\Phi+\delta-1)^{k+1}\}\cdot V_{k+1}(0,\tau,z)$$

[4.110]

Functions $Q_n(\tau)$ are defined by [4.60]. But let first generalize [4.82] for n derivations (is convenient change n by k in [4.82] and then to derive n times),

$$\frac{\partial^n V_k(\sigma,\tau,z)}{\partial z^n} = (-1)^n\cdot[(\rho+\Phi)\cdot(1-\Phi-\delta)]^n\cdot V_{k+n}(\sigma,\tau,z)$$

[4.111]

Applying in [4.110],

$$\frac{\partial^n G(z,\tau)}{\partial z^n} = (-1)^n\cdot[(\rho+\Phi)\cdot(1-\Phi-\delta)]^n\cdot\left\{V_n(0,\tau,z)+b_0\cdot V_{n+1}(0,\tau,z)+\sum_{k=1}^{\infty}b_k\cdot V_{n+k+1}(0,\tau,z)\right\}$$

[4.112]

Where b_0 and b_k are defined by [4.109 (0)] and [4.109 (k)].

Lastly, functions $Q_n(\tau)$, for $n \geq 1$,

$$Q_n(\tau) = -\frac{(-1)^n\cdot\rho}{(\rho+\Phi)\cdot(1-\Phi-\delta)^n}\cdot\left(\frac{\partial^n G}{\partial z^n}\right)_{z=\tau}\quad \ldots\ldots for\ (n\geq 1) \qquad [4.113]$$

Resulting, for $n \geq 1$,

$$Q_n(\tau) = \rho\cdot(\rho+\Phi)^{n-1}\cdot\left\{V_n(0,\tau,\tau)+b_0\cdot V_{n+1}(0,\tau,\tau)+\sum_{k=1}^{\infty}b_k\cdot V_{n+k+1}(0,\tau,\tau)\right\}$$

[4.114]

The fractions $\theta_n(\tau)$, for $n \geq 1$ are reached with [4.28],

$$\theta_n(\tau) = e^{-(\rho+2\cdot\Phi-\delta-1)\cdot\tau}\cdot Q_n(\tau) \qquad [4.115]$$

The fractions $\theta_0(\tau)$ is obtained from [4.109], putting $z=\tau$, because $Q_0(\tau) = G(\tau,\tau)$, and then also applying [4.28].

Inside the box in next page the solution for case 1 can be found in terms of Bessel functions of first kind, $J_n(x)$. For such achievement one only need to apply formula [4.81], but inverting for made explicit the functions $J_n\left(2\cdot\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot(\tau-\sigma)\cdot z}\right)$,

$$J_n\left(2\cdot\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot(\tau-\sigma)\cdot z}\right) = \left[\frac{(\rho+\Phi)\cdot(1-\Phi-\delta)\cdot z}{(\tau-\sigma)}\right]^{n/2}\cdot V_n(\sigma,\tau,z)$$

[4.116]

SOLUTION FOR CASE 1: CONTINUOUS BIOFILM $[\boldsymbol{\delta}<(\mathbf{1}-\boldsymbol{\Phi})]$

$$\theta_n(\tau) = \alpha_n \cdot e^y \cdot \sum_{k=0}^{\infty} \beta_k \cdot J_{n+k}(x) \qquad [4.117]$$

Where:

$$x = 2 \cdot \tau \cdot \sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta)}$$

$$y = (1 - \rho - 2 \cdot \Phi - \delta) \cdot \tau$$

$$\alpha_0 = 1$$

$$\alpha_n = \frac{\rho \cdot (\rho + \Phi)^{n-1}}{\sqrt{(\rho + \Phi)^n \cdot (1 - \Phi - \delta)^n}} \dots only\ for\ n \geq 1$$

$$\beta_0 = 1$$

$$\beta_1 = \frac{(2 \cdot \Phi + \delta - 1)}{\sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta)}}$$

$$\beta_k = \frac{\{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^{k-1} + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^k\}}{(1 - \delta) \cdot \sqrt{(\rho + \Phi)^k \cdot (1 - \Phi - \delta)^k}}$$

$$\dots only\ for\ k \geq 2$$

3.4 Patchy biofilm solution

Let recall that the solution [4.117] is only the solution for the before named case 1, corresponding to continuous biofilm.
As signalled in the preceding box, [4.117] corresponds to the condition $\delta<(1-\Phi)$.
The "complementary" condition, $\delta>(1-\Phi)$, designated before as case 2, relates the evolution towards a steady state where the bare fraction θ_0 don´t never get null. In physical terms this means the formation, at times large enough, of a patchy biofilm.
The solution for case 2 can be obtained simply profiting solution [4.117], being aware that whenever root $\sqrt{1-\Phi-\delta}$ occurs substitution by $i\cdot\sqrt{\Phi+\delta-1}$ must be done.
Consequently, in [4.117] the variable x and the coefficients α_n and β_k are modified as follows,

Variable x changes this way,

$$x = 2\cdot\tau\cdot\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)}$$

$$2\cdot\tau\cdot\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)} = i\cdot 2\cdot\tau\cdot\sqrt{(\rho+\Phi)\cdot(\Phi+\delta-1)}$$

$$i\cdot 2\cdot\tau\cdot\sqrt{(\rho+\Phi)\cdot(\Phi+\delta-1)} = i\cdot x^*$$

So the change in variable x reads,

$$x \to (i\cdot x^*) \qquad [4.118]$$

Coefficient α_n changes this way,

$$\frac{\rho\cdot(\rho+\Phi)^{n-1}}{\sqrt{(\rho+\Phi)^n\cdot(1-\Phi-\delta)^n}} = \frac{\rho\cdot(\rho+\Phi)^{n-1}}{i^n\cdot\sqrt{(\rho+\Phi)^n\cdot(\Phi+\delta-1)^n}}$$

$$\frac{\rho\cdot(\rho+\Phi)^{n-1}}{i^n\cdot\sqrt{(\rho+\Phi)^n\cdot(\Phi+\delta-1)^n}} = i^{-n}\cdot\alpha_n^*$$

So the change in coefficient α_n reads,

$$\alpha_n \to (i^{-n}\cdot\alpha_n^*) \qquad [4.119]$$

And the new coefficient α_n^* is now, for all values of n,

$$\alpha_n^* = \begin{cases} 1 \;\; \ldots for\ n=0 & [4.120\ (0)\ ] \\ \dfrac{\rho\cdot(\rho+\Phi)^{n-1}}{\sqrt{(\rho+\Phi)^n\cdot(\Phi+\delta-1)^n}} \ldots only\ for\ n\geq 1 & [4.120\ (k)\ ] \end{cases}$$

Coefficient β_1 changes this way,

$$\beta_1 = \frac{(2\cdot\Phi+\delta-1)}{\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)}}$$

$$\frac{(2\cdot\Phi+\delta-1)}{\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)}} = \frac{(2\cdot\Phi+\delta-1)}{i\cdot\sqrt{(\rho+\Phi)\cdot(\Phi+\delta-1)}}$$

$$\frac{(2\cdot\Phi+\delta-1)}{i\cdot\sqrt{(\rho+\Phi)\cdot(\Phi+\delta-1)}} = i^{-1}\cdot\beta_1^*$$

So the change in coefficient β_1 reads,

$$\beta_1 \rightarrow (i^{-1} \cdot \beta_1^*) \qquad [4.121]$$

Coefficient β_k, for $(k \geq 2)$, changes this way,

$$\beta_k = \frac{\{[\rho \cdot (1-\Phi-\delta) + \Phi \cdot (1-\delta)] \cdot \Phi^{k-1} + (\rho+1-\delta) \cdot (\Phi+\delta-1)^k\}}{(1-\delta) \cdot \sqrt{(\rho+\Phi)^k \cdot (1-\Phi-\delta)^k}}$$

... but the numerator can be written using the b_k definition,

$$b_{k-1} = \{[\rho \cdot (1-\Phi-\delta) + \Phi \cdot (1-\delta)] \cdot \Phi^{k-1} + (\rho+1-\delta) \cdot (\Phi+\delta-1)^k\}$$

(observe that in the numerator b_{k-1} doesn´t exist any square root, so remains the same)

... thus lightning the notation,

$$\beta_k = \frac{b_{k-1}}{\sqrt{(\rho+\Phi)^k \cdot (1-\Phi-\delta)^k}}$$

$$\frac{b_{k-1}}{\sqrt{(\rho+\Phi)^k \cdot (1-\Phi-\delta)^k}} = \frac{b_{k-1}}{i^k \cdot \sqrt{(\rho+\Phi)^k \cdot (\Phi+\delta-1)^k}}$$

$$\frac{b_{k-1}}{i^k \cdot \sqrt{(\rho+\Phi)^k \cdot (\Phi+\delta-1)^k}} = i^{-k} \cdot \beta_k^*$$

So the change in coefficient β_k, wherever occurs, reads,

$$\beta_k \rightarrow \left(i^{-k} \cdot \beta_k^*\right) \qquad [4.122]$$

And the new coefficient β_k^* is now, for all values of n,

$$\beta_0^* = 1 \qquad [4.123\ (0)\ ]$$

$$\beta_1^* = \frac{(2 \cdot \Phi + \delta - 1)}{\sqrt{(\rho+\Phi) \cdot (\Phi+\delta-1)}} \qquad [4.123\ (1)\ ]$$

$$\beta_k^* = \frac{\{[\rho \cdot (1-\Phi-\delta) + \Phi \cdot (1-\delta)] \cdot \Phi^{k-1} + (\rho+1-\delta) \cdot (\Phi+\delta-1)^k\}}{(1-\delta) \cdot \sqrt{(\rho+\Phi)^k \cdot (\Phi+\delta-1)^k}} \quad \ldots only\ for\ k \geq 2 \qquad [4.123\ (k)\ ]$$

Lastly and easily the solution for case 2 (patchy biofilm) can now be written taking [4.117] as starting point, and applying the preceding obtained changes in coefficients.
Step by step:

$$\theta_n(\tau) = \alpha_n \cdot e^y \cdot \sum_{k=0}^{\infty} \beta_k \cdot J_{n+k}(x) \qquad [4.117]$$

Using [4.118], [4.119] and [4.122],

$$\alpha_n \cdot e^y \cdot \sum_{k=0}^{\infty} \beta_k \cdot J_{n+k}(x) = i^{-n} \cdot a_n^* \cdot e^y \cdot \sum_{k=0}^{\infty} i^{-k} \cdot \beta_k^* \cdot J_{n+k}(i \cdot x^*) \qquad [4.124]$$

Recalling that $J_n(i \cdot x) = i^n \cdot I_n(x)$,

$$i^{-n} \cdot a_n^* \cdot e^y \cdot \sum_{k=0}^{\infty} i^{-k} \cdot \beta_k^* \cdot J_{n+k}(i \cdot x^*) = i^{-n} \cdot a_n^* \cdot e^y \cdot \sum_{k=0}^{\infty} i^{-k} \cdot \beta_k^* \cdot i^{n+k} \cdot I_{n+k}(x^*) \qquad [4.125]$$

Associating all the factors i^{-n}, i^{-k} and i^{n+k},

$$i^{-n} \cdot a_n^* \cdot e^y \cdot \sum_{k=0}^{\infty} i^{-k} \cdot \beta_k^* \cdot i^{n+k} \cdot I_{n+k}(x^*) = a_n^* \cdot e^y \cdot \sum_{k=0}^{\infty} \beta_k^* \cdot I_{n+k}(x^*) \qquad [4.126]$$

Finally, concatenating [4.117] with [4.124], then with [4.125] and lastly with [4.126] the searched solution reads,

$$\theta_n(\tau) = a_n^* \cdot e^y \cdot \sum_{k=0}^{\infty} \beta_k^* \cdot I_{n+k}(x^*) \qquad [4.127]$$

And the solution for case 2 (Patchy biofilm) can be completely written down like in this box,

<u>SOLUTION FOR CASE 2: PATCHY BIOFILM</u> $[\boldsymbol{\delta} > (\mathbf{1} - \boldsymbol{\Phi})]$

$$\theta_n(\tau) = \alpha_n^* \cdot e^y \cdot \sum_{k=0}^{\infty} \beta_k^* \cdot I_{n+k}(x^*) \qquad [4.127]$$

Where:

$$x^* = 2 \cdot \tau \cdot \sqrt{(\rho + \Phi) \cdot (\Phi + \delta - 1)}$$

$$y = (1 - \rho - 2 \cdot \Phi - \delta) \cdot \tau$$

$$\alpha_0^* = 1$$

$$\alpha_n^* = \frac{\rho \cdot (\rho + \Phi)^{n-1}}{\sqrt{(\rho + \Phi)^n \cdot (\Phi + \delta - 1)^n}} \ldots only\ for\ n \geq 1$$

$$\beta_0^* = 1$$

$$\beta_1^* = \frac{(2 \cdot \Phi + \delta - 1)}{\sqrt{(\rho + \Phi) \cdot (\Phi + \delta - 1)}}$$

$$\beta_k^* = \frac{\{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^{k-1} + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^k\}}{(1 - \delta) \cdot \sqrt{(\rho + \Phi)^k \cdot (\Phi + \delta - 1)^k}}$$

$$\ldots only\ for\ k \geq 2$$

3.5 Frontier between continuous and patchy biofilm solution

The solution for case 3 can be easily deduced by induction after doing the corresponding simplifications in the initial system of differential-difference equations, just before any other step.
It is so simple that we indicate the solution,

$$\theta_0(\tau) = e^{-\rho\tau} \qquad [4.128(0)]$$

$$\theta_n(\tau) = e^{-(\rho+\Phi)\cdot\tau} \cdot \frac{\rho\cdot(\rho+\Phi)^{n-1}}{\Phi^n}\left\{e^{\Phi\tau} - \sum_{k=0}^{n-1}\frac{(\tau\cdot\Phi)^k}{k!}\right\} \ldots for\ (n>0) \qquad [4.128(n)]$$

This solution is coherent with solution [4.117] for case 1and it is important and interesting to check that coherence.
We can achieve that goal written solution [4.117] in this alternative form,

SOLUTION FOR CASE 1: CONTINUOUS BIOFILM $[\boldsymbol{\delta}<(\mathbf{1}-\boldsymbol{\Phi})]$
(ALTERNATIVE REPRESENTATION)

$$\theta_n(\tau) = A_n \cdot e^y \cdot \sum_{k=0}^{\infty} B_k \cdot \frac{J_{n+k}(x)}{\gamma^{n+k}} \qquad [4.129]$$

Where:

$$x = 2\cdot\tau\cdot\gamma$$
$$y = (1-\rho-2\cdot\Phi-\delta)\cdot\tau$$
$$\gamma = \sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)}$$
$$A_0 = 1$$
$$A_n = \rho\cdot(\rho+\Phi)^{n-1} \ldots only\ for\ n \geq 1$$
$$B_0 = 1$$
$$B_1 = (2\cdot\Phi+\delta-1)$$
$$B_k = \frac{\{[\rho\cdot(1-\Phi-\delta)+\Phi\cdot(1-\delta)]\cdot\Phi^{k-1} + (\rho+1-\delta)\cdot(\Phi+\delta-1)^k\}}{(1-\delta)} \ldots only\ for\ k \geq 2$$

When $\delta = 1-\Phi$, $\gamma = 0$ and, in the summation in [4.129], each Bessel function term assumes the following value,

$$\frac{J_{n+k}(x)}{\gamma^{n+k}} = \frac{1}{\gamma^{n+k}} \cdot \sum_{i=0}^{\infty} \frac{(-1)^i\cdot(\tau\cdot\gamma)^{n+k+2\cdot i}}{i!\cdot(n+k+i)!}$$

$$\frac{1}{\gamma^{n+k}} \cdot \sum_{i=0}^{\infty} \frac{(-1)^i\cdot(\tau\cdot\gamma)^{n+k+2\cdot i}}{i!\cdot(n+k+i)!} = \sum_{i=0}^{\infty} \frac{(-1)^i\cdot\tau^{n+k+2\cdot i}\cdot\gamma^{2\cdot i}}{i!\cdot(n+k+i)!}$$

Consequently, if $\gamma = 0$,

$$\frac{J_{n+k}(x)}{\gamma^{n+k}} = \frac{\tau^{n+k}}{(n+k)!}$$

And inserting in [4.129],

$$\theta_n(\tau) = A_n \cdot e^y \cdot \sum_{k=0}^{\infty} B_k \cdot \frac{\tau^{n+k}}{(n+k)!}$$

Coefficients B_k simplify,

$$B_k = \Phi^k \ldots for\ (k \geq 0)$$

... as well as y,

$$y = -(\rho + \Phi) \cdot \tau$$

Now we conclude that,

$$\theta_0(\tau) = e^{-(\rho+\Phi)\cdot\tau} \cdot \sum_{k=0}^{\infty} \Phi^k \cdot \frac{\tau^k}{k!}$$

... and,

$$e^{-(\rho+\Phi)\cdot\tau} \cdot \sum_{k=0}^{\infty} \Phi^k \cdot \frac{\tau^k}{k!} = e^{-(\rho+\Phi)\cdot\tau} \cdot e^{\Phi\cdot\tau}$$

... and,

$$e^{-(\rho+\Phi)\cdot\tau} \cdot e^{\Phi\cdot\tau} = e^{-\rho\cdot\tau}$$

Concatenating those last three equalities one obtains [4.128(0)],

And, when $(n \geq 1)$,

$$\theta_n(\tau) = \rho \cdot (\rho + \Phi)^{n-1} \cdot e^{-(\rho+\Phi)\cdot\tau} \cdot \sum_{k=0}^{\infty} \Phi^k \cdot \frac{\tau^{n+k}}{(n+k)!}$$

... and also,

$$\rho \cdot (\rho + \Phi)^{n-1} \cdot e^{-(\rho+\Phi)\cdot\tau} \cdot \sum_{k=0}^{\infty} \Phi^k \cdot \frac{\tau^{n+k}}{(n+k)!} = e^{-(\rho+\Phi)\cdot\tau} \cdot \frac{\rho \cdot (\rho + \Phi)^{n-1}}{\Phi^n} \cdot \sum_{k=0}^{\infty} \frac{(\Phi \cdot \tau)^{n+k}}{(n+k)!}$$

But, once the following identity, holds,

$$\sum_{k=0}^{\infty} \frac{(\Phi \cdot \tau)^{n+k}}{(n+k)!} = e^{-\rho\cdot\tau} - \sum_{k=0}^{n-1} \frac{(\Phi \cdot \tau)^k}{k!}$$

... we get,

$$\theta_n(\tau) = e^{-(\rho+\Phi)\cdot\tau} \cdot \frac{\rho \cdot (\rho + \Phi)^{n-1}}{\Phi^n} \left\{ e^{\Phi\tau} - \sum_{k=0}^{n-1} \frac{(\tau \cdot \Phi)^k}{k!} \right\} \ldots for\ (n>0)$$

... which is equal to [4.128(n)].

3.6 Concluding remark.

All the consistency and coherence demonstrated about the changes between the solutions of the three analysed cases, considering in each transformation the appropriate mathematics modifications, leaves to conclude that a unique recurrent algorithm is enough for applications.

In fact, the solution of case 1 conglobates the other two, after suitable modifications.

CHAPTER V

Miller Generalized Algorithm, and M/M/1 Queue Recurrence Machinery. (eroding mathematical intractability)

1. Preliminary introduction.

The solution reached in chapter IV for the "Mono-layered concentrated growth" kinetics, defined by formulas [4.117] for continuous biofilm (case 1) and by [4.127] for patchy biofilm (case 2) are much more difficult to handle than the frontier between those two (case 3) given by [4.128(0)] and [4.128(n)]. Besides the case 3 is the less important, representing only a very specific choice of parameters with the purpose of corroborate the mathematical coherence in all the deductions made.
From these proof we concluded that to handle case 1 solution [4.117], leads to a mathematical difficulty similar to handle case 2 solution [4.127]. And also leads to the same recurrent result, as we shall see.
Consequently we focus now our attention exclusively in continuous biofilm solution [4.117].
This solution is not easy to handle for numeric calculation and therefore also not easy for graphic and intelligible representation of the model.
The solution obtained is complex because it consists of an infinite series of terms and the main difficulty lies on the fact that each term, in the infinite sum, contains a Bessel function of first kind, $J_{n+k}(x)$.
It is not because the summation is infinite in the number of terms why it is difficult reach a rigorous calculation. The truth is that terms of bigger order will always diminish in absolute value, and this circumstance allows, sooner or later, to despise those upper order terms becoming possible to accomplish all the computation without error.
The main problem to solve refers to the rigorous calculation of each Bessel function not only because extensive ranges of orders $(n + k)$ are required but also because their argument values, proportional with dimensionless time (τ), must vary over a large interval. As a matter of fact, we do not always find values for arguments that, in our case, are (or can be) relatively high. There is out of question have time scale limitations when analysing a transient exact solution.
A priori do not exist tables providing such exigent demands on Bessel function orders and argument values.
The infinite series of terms with Bessel functions , which we have obtained in our case, can´t (or should not) be calculated using published tables not only because it is necessary to use very high order functions, not always tabulated, but also because, even if the tables have these high order Bessel functions, the accuracy is not the same as the lower-order functions $(J_0(x), J_1(x), J_2(x), \dots)$.
This is an additional inconvenient question related to want of accuracy uniformity in data provided by known tables. Bessel functions of lower order (0, 1, 2, ...) usually are tabulated with much more decimal significant places than the others (Abramowitz and Stegun, [41]). Thus, when one intends to calculate some more terms in the series, in the purpose of not to neglect meaningful terms, coherence is lost because their accuracy is almost always inferior to that of the initial terms.
Consequently we must work only with computer tools paying attention to achieve the better exactitude disposable nowadays.
We choose to compute in Excel spreadsheet not only because it stores numerical values as "Double Precision Floating Point", representing numbers accurate to something like 15 decimal places, but also because it allows to design the layout of the algorithms, over the spreadsheet with great personalized liberty.
Another important advantage lies on the fact that one can "see" the contents of each cell and easily detect and correct errors and (or) improve the global efficiency of the entire algorithm.
Once discharged the numeric values published in those tables our attention goes to the theoretic method used, in old past epochs, for construct such Bessel functions tables.

The central and fundamental tool for such purpose was the Three Terms Recurrence (TTR) Miller´s algorithm (Miller, [42]).
The immediate obvious way to solve the problem, of handle solution [4.117] with adequate accuracy, is to apply TTR Miller´s algorithm, term by term, till reach a Bessel function order high enough for neglect the absolute value of such term and also the absolute values of all following terms.
Another, more efficient and elegant, alternative concerns with considering all the infinite series summation, and so all the function $\theta_n(\tau)$, as a special function passible for application of an algorithm related to, but more general than, TTR Miller´s algorithm.
In more explicit words: functions $\theta_n(\tau)$ will came into play relatively to this new algorithm in a similar way Bessel functions of first kind play their role relatively to the already known TTR Miller´s algorithm.
Application of this new approach implies, as first step, the deduction of a corresponding homogeneous recurrence for the special functions $\theta_n(\tau)$.
In the following section we will deduce that homogeneous recurrence and explain how can it be used for accurate computation of functions $\theta_n(\tau)$, translating to this new method a similar reasoning to that one of the old TTR Miller´s algorithm method.
These two methods can be jointly, and independently, applied and so we will be able to validate mutually both.

2. Generalized Miller´s like algorithm construction.

The calculation of the model equation [4.117], will be done, in this section, applying a method alike that of Miller [42] but more general.
This method applies to the so-called special functions when one can construct recurrence formulas between them. Knowing the functions of lower orders we can calculate those of higher orders with a formula of progressive recurrence and vice versa with a regressive formula.
However, if the special functions are decreasing when their order increases the application of the progressive recurrence will imply the use of the subtraction operation and the loss of significant numbers (S. Zhang and J. Jin, [43]) each time the recurrence computation applies , making the method numerically unstable.
This is the case for Bessel functions of first kind and therefore also the case for the functions $\theta_n(\tau)$ at the initial values of (τ). But not only in the initial values of (τ) since there will always exist surface fractions $\theta_n(\tau)$, in the largest orders (n) that are in stage of increasing its value, having therefore a low value, since the kinetic process is an overall continuous accumulation of biomass.
Therefore, it is necessary to use a regressive formula in all values of dimensionless time (τ). That is the same to say that we have to compute the functions $\theta_n(\tau)$ with the smallest orders (n) depending of those ones with the biggest orders (n).
Miller [42] solved the problem by assigning an arbitrary value to the function from where the recurrence computation begins, which is the one of the highest order(n).
By applying the homogeneous recurrence formula one gets the relative correct magnitudes between all the functions under calculation. After a suitable common normalization factor must be applied to all the functions of the sequence. The result of this multiplication is the exact value of all and each one of the functions.
The aforesaid suitable common normalization factor is defined by the known value, in the particular case under analysis, of an infinite series with form,

$$\sum_{n=0}^{\infty} c_n \cdot f_n(\tau) \tag{5.1}$$

The homogeneous recurrence formula to be constructed must, for the functions $\theta_n(\tau)$, read like,

$$\theta_n(\tau) = F[\,\theta_{n+1}(\tau), \theta_{n+2}(\tau), \ldots, \theta_{n+k}(\tau)]$$

… where,

$$k \geq 1 \qquad [5.2]$$

… and being $\theta_n(\tau)$ obtained depending of $\theta_{n+1}(\tau), \theta_{n+2}(\tau), \dots, \dots$ and $\theta_{n+k}(\tau)$.
We call [5.2] a $(k+1)$-terms homogeneous recurrence.
Going now to the homogeneous recurrence deduction, and for start, let define the auxiliary functions $W_n(x)$, for $(n \geq 0)$,

$$W_n(x) = \sum_{k=0}^{\infty} \beta_k \cdot J_{n+k}(x) \qquad [5.3-A]$$

$$W_n(x) = \beta_0 \cdot J_n(x) + \beta_1 \cdot J_{n+1}(x) + \sum_{k=2}^{\infty} \beta_k \cdot J_{n+k}(x) \qquad [5.3-B]$$

… where coefficients β_k are defined like they have been defined in the model equation [4.117],

$$\beta_k = \begin{cases} 1 \dots if\ k=0 \\ \\ \dfrac{b_{k-1}}{\gamma^k} \dots if\ k \geq 1 \end{cases} \quad \dots being,\ \gamma = \sqrt{(1-\Phi-\delta)\cdot(\rho+\Phi)}\ \dots\ and,\ b_k = c_k \cdot k! \qquad [5.4]$$

Following a brief revision from Chapter IV, coefficients c_k are, in their turn, defined by equations [4.98(0)] and [4.108],

$$c_k = \begin{cases} (2\cdot\Phi+\delta-1)\ \dots\ if\ k=0 & [4.98(0)] \\ \\ \dfrac{\{[\rho\cdot(1-\Phi-\delta)+\Phi\cdot(1-\delta)]\cdot\Phi^k+(\rho+1-\delta)\cdot(\Phi+\delta-1)^{k+1}\}}{(1-\delta)\cdot k!}\ \dots\ if\ k\geq 1 & [4.108] \end{cases}$$

And, also from Chapter IV, consequently the coefficients b_k obey equation [4.99],

$$b_{k+2} + A\cdot b_{k+1} + B\cdot b_k = 0 \quad \dots with\ (k \geq 0) \qquad [4.99]$$

… where,

$$A = (1-2\cdot\Phi-\delta) \qquad [4.101]$$

$$B = \Phi\cdot(\Phi+\delta-1) \qquad [4.102]$$

Let use functions $W_n(x)$ in the form,

$$W_n(x) = J_n(x) + \sum_{k=1}^{\infty} \frac{b_{k-1}}{\gamma^k}\cdot J_{n+k}(x) \qquad [5.5]$$

Our purpose is now to obtain a recurrence relation between the functions $W_n(x)$ in which the functions $J_n(x)$ are not present. And, in the sequel, applying the relationship between functions $W_n(x)$ and functions $\theta_n(\tau)$, that recurrence will be easily converted into a recurrence between the functions $\theta_n(\tau)$, as desired.
Recurrences concerning Bessel functions of first kind imply $J_n(x)$, $J_{n+1}(x)$ and $J_{n+2}(x)$. For this reason let now write explicitly $W_n(x)$, $W_{n+1}(x)$ and $W_{n+2}(x)$ in the following way,

$$W_n(x) = J_n(x) + \frac{b_0}{\gamma}\cdot J_{n+1}(x) + \frac{b_1}{\gamma^2}\cdot J_{n+2}(x) + \sum_{k=3}^{\infty}\frac{b_{k-1}}{\gamma^k}\cdot J_{n+k}(x)$$

[5.6(0)]

$$W_{n+1}(x) = J_{n+1}(x) + \frac{b_0}{\gamma} \cdot J_{n+2}(x) + \sum_{k=2}^{\infty} \frac{b_{k-1}}{\gamma^k} \cdot J_{n+k+1}(x)$$

[5.6(1)]

$$W_{n+2}(x) = J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{b_{k-1}}{\gamma^k} \cdot J_{n+k+2}(x)$$

[5.6(2)]

Remodelling the summations so that everyone starts with $(k = 1)$,

$$W_n(x) = J_n(x) + \frac{b_0}{\gamma} \cdot J_{n+1}(x) + \frac{b_1}{\gamma^2} \cdot J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{b_{k+1}}{\gamma^{k+2}} \cdot J_{n+k+2}(x)$$

[5.7(0)]

$$W_{n+1}(x) = J_{n+1}(x) + \frac{b_0}{\gamma} \cdot J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{b_k}{\gamma^{k+1}} \cdot J_{n+k+2}(x)$$

[5.7(1)]

$$W_{n+2}(x) = J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{b_{k-1}}{\gamma^k} \cdot J_{n+k+2}(x)$$

[5.7(2)]

Now we properly multiply [5.7(1)] and [5.7(2)] in such a way that, by associating the same Bessel function in the three equations [5.7(0)], [5.7(1)] and [5.7(2)], equality [4.99] can be used thus eliminating the infinite terms summation:

$$W_n(x) = J_n(x) + \frac{b_0}{\gamma} \cdot J_{n+1}(x) + \frac{b_1}{\gamma^2} \cdot J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{b_{k+1}}{\gamma^{k+2}} \cdot J_{n+k+2}(x)$$

[5.8(0)]

$$\frac{A}{\gamma} \cdot W_{n+1}(x) = \frac{A}{\gamma} \cdot J_{n+1}(x) + \frac{A \cdot b_0}{\gamma^2} \cdot J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{A \cdot b_k}{\gamma^{k+2}} \cdot J_{n+k+2}(x)$$

[5.8(1)]

$$\frac{B}{\gamma^2} \cdot W_{n+2}(x) = \frac{B}{\gamma^2} \cdot J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{B \cdot b_{k-1}}{\gamma^{k+2}} \cdot J_{n+k+2}(x)$$

[5.8(2)]

Adding the three equations [5.8(0)], [5.8(1)] and [5.8(2)] we obtain,

$$W_n(x) + \frac{A}{\gamma} \cdot W_{n+1}(x) + \frac{B}{\gamma^2} \cdot W_{n+2}(x) = J_n(x) + \frac{(b_0 + A)}{\gamma} \cdot J_{n+1}(x) +$$

$$+ \frac{(b_1 + A \cdot b_0 + B)}{\gamma^2} \cdot J_{n+2}(x) + \sum_{k=1}^{\infty} \frac{(b_{k+1} + A \cdot b_k + B \cdot b_{k-1})}{\gamma^{k+2}} \cdot J_{n+k+2}(x)$$

[5.9]

Observing that, from $b_k = c_k \cdot k!$, and also [4.98 (0)], [4.98 (1)], [4.101] and [4.102], the following three equalities are valid,

$$b_0 + A = 0$$ [5.10]

$$b_1 + A \cdot b_0 + B = \gamma^2 \qquad [5.11]$$

$$b_{k+1} + A \cdot b_k + B \cdot b_{k-1} = 0 \quad \dots \; for \; (k \geq 1) \qquad [5.12]$$

… we are now able to insert they in [5.9] , and therefore getting in it a great simplification,

$$W_n(x) + \frac{A}{\gamma} \cdot W_{n+1}(x) + \frac{B}{\gamma^2} \cdot W_{n+2}(x) = J_n(x) + J_{n+2}(x) \qquad [5.13]$$

Let now add 2 in the discrete order (n) at [5.13],

$$W_{n+2}(x) + \frac{A}{\gamma} \cdot W_{n+3}(x) + \frac{B}{\gamma^2} \cdot W_{n+4}(x) = J_{n+2}(x) + J_{n+4}(x) \qquad [5.14]$$

… and call into play the identities,

$$J_n(x) + J_{n+2}(x) = \frac{2 \cdot (n+1)}{x} \cdot J_{n+1}(x) \qquad [5.15(0)]$$

$$J_{n+2}(x) + J_{n+4}(x) = \frac{2 \cdot (n+3)}{x} \cdot J_{n+3}(x) \qquad [5.15(2)]$$

… which must be substituted in [5.13] and [5.14] ,

$$W_n(x) + \frac{A}{\gamma} \cdot W_{n+1}(x) + \frac{B}{\gamma^2} \cdot W_{n+2}(x) = \frac{2 \cdot (n+1)}{x} \cdot J_{n+1}(x) \qquad [5.16]$$

$$W_{n+2}(x) + \frac{A}{\gamma} \cdot W_{n+3}(x) + \frac{B}{\gamma^2} \cdot W_{n+4}(x) = \frac{2 \cdot (n+3)}{x} \cdot J_{n+3}(x) \qquad [5.17]$$

… reducing this way to only one the Bessel functions of first kind figured in the right hand sides.
Now we explicit those Bessel functions in the left hand side,

$$J_{n+1}(x) = \frac{x}{2 \cdot (n+1)} \cdot \left\{ W_n(x) + \frac{A}{\gamma} \cdot W_{n+1}(x) + \frac{B}{\gamma^2} \cdot W_{n+2}(x) \right\} \qquad [5.18]$$

$$J_{n+3}(x) = \frac{x}{2 \cdot (n+3)} \cdot \left\{ W_{n+2}(x) + \frac{A}{\gamma} \cdot W_{n+3}(x) + \frac{B}{\gamma^2} \cdot W_{n+4}(x) \right\} \qquad [5.19]$$

Back to equation [5.13] let write it again but adding this time 1 in the discrete order (n),

$$W_{n+1}(x) + \frac{A}{\gamma} \cdot W_{n+2}(x) + \frac{B}{\gamma^2} \cdot W_{n+3}(x) = J_{n+1}(x) + J_{n+3}(x) \qquad [5.20]$$

The right hand side at [5.20] only depends on $J_{n+1}(x)$ and $J_{n+3}(x)$ which in turn are given by [5.18] and [5.19] depending only on $W_{n+k}(x)$ with $k = 0, 1, 2, 3$ or 4.
Inserting the equations [5.18] and [5.19] in the right hand side at [5.20] we obtain an equation where all Bessel functions of first kind have been eliminated by suitable substitutions, as desired.
That result reads,

$$W_{n+1}(x) + \frac{A}{\gamma} \cdot W_{n+2}(x) + \frac{B}{\gamma^2} \cdot W_{n+3}(x) =$$

$$= \frac{x}{2 \cdot (n+1)} \cdot \left\{ W_n(x) + \frac{A}{\gamma} \cdot W_{n+1}(x) + \frac{B}{\gamma^2} \cdot W_{n+2}(x) \right\} +$$

$$+ \frac{x}{2 \cdot (n+3)} \cdot \left\{ W_{n+2}(x) + \frac{A}{\gamma} \cdot W_{n+3}(x) + \frac{B}{\gamma^2} \cdot W_{n+4}(x) \right\}$$

[5.21]

The expression we have reached, [5.21], can now be transformed into a progressive recurrence if we explicit $W_{n+4}(x)$ as depending of the functions of lesser orders, or in a regressive recurrence if we explicit $W_n(x)$ as depending of all the others which have higher orders.
For the sake to of attain numerical stability we are interested in the regressive recurrence for a proper application of the Generalized Miller´s like algorithm we seek for construction.
Before manipulation work at [5.21] is convenient establish an alternative expression for $\left(\frac{2}{x}\right)$.
From the definitions $x = 2 \cdot \tau \cdot \sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta)}$ and $\gamma = \sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta)}$ the result follows immediately,

$$\frac{2}{x} = \frac{1}{\gamma \cdot \tau} \qquad [5.22]$$

The wanted regressive recurrence is now reached easily from [5.21], taking [5.22] in consideration,

$$W_n(x) = \left[\frac{(n+1)}{\gamma \cdot \tau} - \frac{A}{\gamma}\right] \cdot W_{n+1}(x) + \left[\frac{(n+1) \cdot A}{\gamma^2 \cdot \tau} - \frac{B}{\gamma^2} - \frac{(n+1)}{(n+3)}\right] \cdot W_{n+2}(x) +$$

$$+\left[\frac{(n+1) \cdot B}{\gamma^3 \cdot \tau} - \frac{(n+1) \cdot A}{(n+3) \cdot \gamma}\right] \cdot W_{n+3}(x) - \frac{(n+1) \cdot B}{(n+3) \cdot \gamma^2} \cdot W_{n+4}(x) \qquad [5.23]$$

What logically follows must be to relate functions $W_n(x)$ with functions $\theta_n(\tau)$ and then insert that relation in [5.23] and finally obtain a regressive recurrence for these last one functions.

From the model solution, [64] and the definition of $W_n(x)$, $[5.3 - A]$,

$$\theta_0(\tau) = e^y \cdot W_0(x) \qquad [5.24(0)]$$

$$\theta_n(\tau) = \frac{\rho \cdot (\rho + \Phi)^{n-1}}{\gamma^n} \cdot e^y \cdot W_n(x) \quad \ldots \; for \; (n \geq 1) \qquad [5.24(n)]$$

Next step is to obtain explicit formulas for $W_n(x)$ depending on $\theta_n(\tau)$,

$$W_0(x) = e^{-y} \cdot \theta_0(\tau) \qquad [5.25(0)]$$

$$W_n(x) = e^{-y} \cdot \frac{\gamma^n}{\rho \cdot (\rho + \Phi)^{n-1}} \cdot \theta_n(\tau) \ldots for \; (n \geq 1) \qquad [5.25(n)]$$

Regarding the fact that the function $W_0(x)$ has a dependence, different from all the others, on the corresponding function $\theta_0(\tau)$ one is compelled to conclude that also a different and unique regressive recurrence relation must be established for $\theta_0(\tau)$.
Accordingly, inserting [5.25(0)] and [5.25(n)] in [5.23], and making $(n = 0)$, the recurrence for $\theta_0(\tau)$ is achieved and reads,

$$\theta_0(\tau) = \left[\frac{1}{\tau} - A\right] \cdot \frac{\theta_1(\tau)}{\rho} + \left[\frac{A}{\tau} - B - \frac{\gamma^2}{3}\right] \cdot \frac{\theta_2(\tau)}{\rho \cdot (\rho + \Phi)} +$$

$$+\left[\frac{B}{\tau} - \frac{A \cdot \gamma^2}{3}\right] \cdot \frac{\theta_3(\tau)}{\rho \cdot (\rho + \Phi)^2} - \frac{B \cdot \gamma^2}{3} \cdot \frac{\theta_4(\tau)}{\rho \cdot (\rho + \Phi)^3} \qquad [5.26(0)]$$

... and inserting only [5.25(n)] in [5.23], and making $(n \geq 1)$, we reach the formula that allows the regressive computation of all the others $\theta_n(\tau)$,

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$$\theta_n(\tau) = \left[\frac{(n+1)}{\tau} - A\right] \cdot \frac{\theta_{n+1}(\tau)}{(\rho+\Phi)} + \left[\frac{(n+1)\cdot A}{\tau} - B - \frac{(n+1)\cdot\gamma^2}{(n+3)}\right] \cdot \frac{\theta_{n+2}(\tau)}{(\rho+\Phi)^2} +$$

$$+\left[\frac{(n+1)\cdot B}{\tau} - \frac{(n+1)\cdot A\cdot\gamma^2}{(n+3)}\right] \cdot \frac{\theta_{n+3}(\tau)}{(\rho+\Phi)^3} - \frac{(n+1)\cdot B\cdot\gamma^2}{(n+3)} \cdot \frac{\theta_{n+4}(\tau)}{(\rho+\Phi)^4} \qquad [5.26(n)]$$

The last two regressive recurrences, [5.26(0)] and [5.26(n)] can be clustered in only one, which reads like,

GENERAL REGRESSIVE RECURRENCE FOR FUNCTIONS $\theta_n(\tau)$
[BIOFILM EARLY STAGES FORMATION AND GROWTH MODEL]

$$\theta_n(\tau) = \left[\frac{(n+1)}{\tau} - A\right] \cdot \frac{\theta_{n+1}(\tau)}{D_1} + \left[\frac{(n+1)\cdot A}{\tau} - B - \frac{(n+1)\cdot C}{(n+3)}\right] \cdot \frac{\theta_{n+2}(\tau)}{D_2} +$$

$$+\left[\frac{(n+1)\cdot B}{\tau} - \frac{(n+1)\cdot A\cdot C}{(n+3)}\right] \cdot \frac{\theta_{n+3}(\tau)}{D_3} - \frac{(n+1)\cdot B\cdot C}{(n+3)} \cdot \frac{\theta_{n+4}(\tau)}{D_4} \qquad [5.27]$$

Where:

$A = (1 - 2\cdot\Phi - \delta)$

$B = \Phi\cdot(\Phi + \delta - 1)$

$C = (\rho + \Phi)\cdot(1 - \Phi - \delta)$

$$D_k = \left\{ \begin{array}{ll} \rho\cdot(\rho+\Phi)^{k-1} & \ldots for\ (n=0) \\ (\rho+\Phi)^k & \ldots for\ (n\geq 1) \end{array} \right\} \quad k\ in\ the\ range: \quad (1 \leq k \leq 4)$$

REMARK: The designation "General" for this recurrence means that it is valid in the three cases analysed in Chapter IV:
CASE 1: Continuous Biofilm [$\delta<(1-\Phi)$]
CASE 2: Patchy Biofilm [$\delta>(1-\Phi)$]
CASE 3: Frontier between Continuous and Patchy Biofilm [$\delta=(1-\Phi)$]

Our goal is to apply correctly alike ideas to those applied by Miller [42] but now with this five terms recurrence (FTR) instead of the already known three term recurrence (TTR) classical algorithm.
For start, formula [5.26(n)] is applied at an order $(n+4)$ high enough to allow us to set $\theta_{n+4}(\tau)=0$, $\theta_{n+3}(\tau)=0$ and $\theta_{n+2}(\tau)=0$. That´s to say: these orders, $(n+4)$, $(n+3)$ and $(n+2)$, must be so high that the corresponding fractions of solid support area $\theta_{n+4}(\tau)$, $\theta_{n+3}(\tau)$ and $\theta_{n+2}(\tau)$ represent, in their turn, an accumulation of pilled layers so big that was not yet achieved at any point in the biofilm, at the value of dimensionless time (τ) under consideration.
In the same formula, [5.26(n)], a value of 1 must be assigned to $\theta_{n+1}(\tau)$ if we follow strictly Miller´s criterion (Miller [42]).
However we introduce in this stage an improvement consisting in assign to $\theta_{n+1}(\tau)$ a small positive value, ε_{n+1}, but not necessarily equal to 1, and preferably less than 1.
This allows a more conservative increase in absolute values of the successive $\theta_{n-k}(\tau)$ under computation avoiding to reach the upper absolute value limit of Excel, which is 10^{+308}, or, in inverse reasoning, allows star the recurrence in higher orders $(n+4)$, $(n+3)$ and $(n+2)$

accomplishing this way computations much more extensive in their number of recurrent steps.
So, the first recurrent computation consists in doing $\theta_{n+4}(\tau)=0$, $\theta_{n+3}(\tau)=0$, $\theta_{n+2}(\tau)=0$ and $\theta_{n+1}(\tau)=\varepsilon_{n+1}$ where ($0<\varepsilon_{n+1}\leq 1$),insert these values in [5.26(n)], and so get the non normalized value of $\theta_n(\tau)$, which we designate by ε_n.
The second recurrent computation consists in apply once again [5.26(n)], but diminishing 1 to all orders $(n\pm k)$. We designate by [5.26($n-1$)] such "new" equation,

$$\theta_{n-1}(\tau)=\left[\frac{(n)}{\tau}-A\right]\cdot\frac{\theta_n(\tau)}{(\rho+\Phi)}+\left[\frac{(n)\cdot A}{\tau}-B-\frac{(n)\cdot\gamma^2}{(n+2)}\right]\cdot\frac{\theta_{n+1}(\tau)}{(\rho+\Phi)^2}+$$

$$+\left[\frac{(n)\cdot B}{\tau}-\frac{(n)\cdot A\cdot\gamma^2}{(n+2)}\right]\cdot\frac{\theta_{n+2}(\tau)}{(\rho+\Phi)^3}-\frac{(n)\cdot B\cdot\gamma^2}{(n+2)}\cdot\frac{\theta_{n+3}(\tau)}{(\rho+\Phi)^4}\qquad[5.26(n-1)]$$

Now we insert the former values of $\theta_{n+3}(\tau)=0$, $\theta_{n+2}(\tau)=0$, $\theta_{n+1}(\tau)=\varepsilon_{n+1}$ and $\theta_n(\tau)=\varepsilon_n$ and obtain the non normalized value of $\theta_{n-1}(\tau)$, coherently designated by ε_{n-1}.
This procedure is done repeatedly $(n-2)$ more times till the non normalized value of $\theta_1(\tau)$, designed ε_1, is calculated.

In general, the k-th recurrent computation, being $(1\leq k\leq n)$, is realized with equation [5.26($n-k+1$)], which reads,

$$\theta_{n-k+1}(\tau)=\left[\frac{(n-k+2)}{\tau}-A\right]\cdot\frac{\theta_{n-k+2}(\tau)}{(\rho+\Phi)}+\left[\frac{(n-k+2)\cdot A}{\tau}-B-\frac{(n-k+2)\cdot\gamma^2}{(n-k+4)}\right]\cdot\frac{\theta_{n-k+3}(\tau)}{(\rho+\Phi)^2}+$$

$$+\left[\frac{(n-k+2)\cdot B}{\tau}-\frac{(n-k+2)\cdot A\cdot\gamma^2}{(n-k+4)}\right]\cdot\frac{\theta_{n-k+4}(\tau)}{(\rho+\Phi)^3}-\frac{(n-k+2)\cdot B\cdot\gamma^2}{(n-k+4)}\cdot\frac{\theta_{n-k+5}(\tau)}{(\rho+\Phi)^4}$$

$$[5.26(n-k+1)]$$

The four initial postulated values, $\theta_{n+4}(\tau)=0$, $\theta_{n+3}(\tau)=0$, $\theta_{n+2}(\tau)=0$ and $\theta_{n+1}(\tau)=\varepsilon_{n+1}$ are used in the first recurrent computation. Then, from these, the three values $\theta_{n+3}(\tau)=0$, $\theta_{n+2}(\tau)=0$ and $\theta_{n+1}(\tau)=\varepsilon_{n+1}$, are still necessary in the second recurrent computation. In the same way, the two values $\theta_{n+2}(\tau)=0$ and $\theta_{n+1}(\tau)=\varepsilon_{n+1}$ are needed for the third computation, and also the value $\theta_{n+1}(\tau)=\varepsilon_{n+1}$ still for the fourth one.
In the fifth and subsequent recurrent computations only calculated values in former computations are needed. None of the initial fourth postulated values figures anymore.
In all this regressive and recurrent computations, equations [5.26(n)], [5.26($n-1$)], [5.26($n-2$)], … , [5.26($n-k+1$)], … , [5.26(3)], [5.26(2)] and [5.26(1)] are applied and the non normalized values of $\theta_n(\tau)$, $\theta_{n-1}(\tau)$, $\theta_{n-2}(\tau)$, … , $\theta_{n-k+1}(\tau)$, … , $\theta_3(\tau)$, $\theta_2(\tau)$ and $\theta_1(\tau)$ are respectively obtained.
Lastly, the non normalized value of $\theta_0(\tau)$, is computed applying equation [5.26(0)] inserting in it the non normalized values of $\theta_1(\tau)$, $\theta_1(\tau)$, $\theta_1(\tau)$ and $\theta_1(\tau)$ obtained, respectively, in the former n-th, $(n-1)$-th, $(n-2)$-th and $(n-3)$-th recurrent computations.
In the following table (next page) all the sequence of these recurrent computations for the non normalized values of all the functions $\theta_k(\tau)$, being $(0\leq k\leq n+4)$, is succinctly and clearly explained.

TABLE 1

RECURRENT COMPUTATION N°/ / APPLIED EQUATION	NOT NORMALIZED INPUT VALUES:				RESULT
1 $(*)$	$\boldsymbol{\theta_{n+4}(\tau)}$	$\boldsymbol{\theta_{n+3}(\tau)}$	$\boldsymbol{\theta_{n+2}(\tau)}$	$\boldsymbol{\theta_{n+1}(\tau)}$	$\boldsymbol{\theta_{n}(\tau)}$
$[5.26(\boldsymbol{n})]$	0	0	0	$0 < \varepsilon_{n+1} \leq 1$	ε_n
2 $(*)$	$\boldsymbol{\theta_{n+3}(\tau)}$	$\boldsymbol{\theta_{n+2}(\tau)}$	$\boldsymbol{\theta_{n+1}(\tau)}$	$\boldsymbol{\theta_{n}(\tau)}$	$\boldsymbol{\theta_{n-1}(\tau)}$
$[5.26(\boldsymbol{n-1})]$	0	0	$0 < \varepsilon_{n+1} \leq 1$	ε_n	ε_{n-1}
3 $(*)$	$\boldsymbol{\theta_{n+2}(\tau)}$	$\boldsymbol{\theta_{n+1}(\tau)}$	$\boldsymbol{\theta_{n}(\tau)}$	$\boldsymbol{\theta_{n-1}(\tau)}$	$\boldsymbol{\theta_{n-2}(\tau)}$
$[5.26(\boldsymbol{n-2})]$	0	$0 < \varepsilon_{n+1} \leq 1$	ε_n	ε_{n-1}	ε_{n-2}
4 $(*)$	$\boldsymbol{\theta_{n+1}(\tau)}$	$\boldsymbol{\theta_{n}(\tau)}$	$\boldsymbol{\theta_{n-1}(\tau)}$	$\boldsymbol{\theta_{n-2}(\tau)}$	$\boldsymbol{\theta_{n-3}(\tau)}$
$[5.26(\boldsymbol{n-3})]$	$0 < \varepsilon_{n+1} \leq 1$	ε_n	ε_{n-1}	ε_{n-2}	ε_{n-3}
5 $(*)$	$\boldsymbol{\theta_{n}(\tau)}$	$\boldsymbol{\theta_{n-1}(\tau)}$	$\boldsymbol{\theta_{n-2}(\tau)}$	$\boldsymbol{\theta_{n-3}(\tau)}$	$\boldsymbol{\theta_{n-4}(\tau)}$
$[5.26(\boldsymbol{n-4})]$	ε_n	ε_{n-1}	ε_{n-2}	ε_{n-3}	ε_{n-4}
6 $(*)$	$\boldsymbol{\theta_{n-1}(\tau)}$	$\boldsymbol{\theta_{n-2}(\tau)}$	$\boldsymbol{\theta_{n-3}(\tau)}$	$\boldsymbol{\theta_{n-4}(\tau)}$	$\boldsymbol{\theta_{n-5}(\tau)}$
$[5.26(\boldsymbol{n-5})]$	ε_{n-1}	ε_{n-2}	ε_{n-3}	ε_{n-4}	ε_{n-5}
…………	…………	…………	…………	…………	…………
	…………	…………	…………	…………	…………
$\boldsymbol{k}$ $(*)$	$\boldsymbol{\theta_{n-k+5}(\tau)}$	$\boldsymbol{\theta_{n-k+4}(\tau)}$	$\boldsymbol{\theta_{n-k+3}(\tau)}$	$\boldsymbol{\theta_{n-k+2}(\tau)}$	$\boldsymbol{\theta_{n-k+1}(\tau)}$
$[5.26(\boldsymbol{n-k+1})]$	ε_{n-k+5}	ε_{n-k+4}	ε_{n-k+3}	ε_{n-k+2}	ε_{n-k+1}
…………	…………	…………	…………	…………	…………
	…………	…………	…………	…………	…………
$\boldsymbol{n-4}$ $(*)$	$\boldsymbol{\theta_{9}(\tau)}$	$\boldsymbol{\theta_{8}(\tau)}$	$\boldsymbol{\theta_{7}(\tau)}$	$\boldsymbol{\theta_{6}(\tau)}$	$\boldsymbol{\theta_{5}(\tau)}$
$[5.26(5)]$	ε_9	ε_8	ε_7	ε_6	ε_5
$\boldsymbol{n-3}$ $(*)$	$\boldsymbol{\theta_{8}(\tau)}$	$\boldsymbol{\theta_{7}(\tau)}$	$\boldsymbol{\theta_{6}(\tau)}$	$\boldsymbol{\theta_{5}(\tau)}$	$\boldsymbol{\theta_{4}(\tau)}$
$[5.26(4)]$	ε_8	ε_7	ε_6	ε_5	ε_4
$\boldsymbol{n-2}$ $(*)$	$\boldsymbol{\theta_{7}(\tau)}$	$\boldsymbol{\theta_{6}(\tau)}$	$\boldsymbol{\theta_{5}(\tau)}$	$\boldsymbol{\theta_{4}(\tau)}$	$\boldsymbol{\theta_{3}(\tau)}$
$[5.26(3)]$	ε_7	ε_6	ε_5	ε_4	ε_3
$\boldsymbol{n-1}$ $(*)$	$\boldsymbol{\theta_{6}(\tau)}$	$\boldsymbol{\theta_{5}(\tau)}$	$\boldsymbol{\theta_{4}(\tau)}$	$\boldsymbol{\theta_{3}(\tau)}$	$\boldsymbol{\theta_{2}(\tau)}$
$[5.26(2)]$	ε_6	ε_5	ε_4	ε_3	ε_2
$\boldsymbol{n}$ $(*)$	$\boldsymbol{\theta_{5}(\tau)}$	$\boldsymbol{\theta_{4}(\tau)}$	$\boldsymbol{\theta_{3}(\tau)}$	$\boldsymbol{\theta_{2}(\tau)}$	$\boldsymbol{\theta_{1}(\tau)}$
$[5.26(1)]$	ε_5	ε_4	ε_3	ε_2	ε_1
$\boldsymbol{n+1}$ $(**)$	$\boldsymbol{\theta_{4}(\tau)}$	$\boldsymbol{\theta_{3}(\tau)}$	$\boldsymbol{\theta_{2}(\tau)}$	$\boldsymbol{\theta_{1}(\tau)}$	$\boldsymbol{\theta_{0}(\tau)}$
$[5.26(0)]$	ε_4	ε_3	ε_2	ε_1	ε_0

SHORT LEGEND

$(*)$ −***Regressive Recurrences used in the computations numbers*** (k), ***with*** $(1 \leq k \leq n)$ **:** $[5.26(k)]$

$(**)$ −***Regressive Recurrence used in the computation number*** $(n+1)$**:** $[5.26(0)]$

Till this stage we only have the non normalized values of all functions $\theta_k(\tau)$. Opportunely we now recall that they relative proportions are already correct.
In the sequel we must add up all those non normalized values and divide all of them by the value of that summation, which, in this way, works like a normalization factor, because the summation of all the fractions $\theta_k(\tau)$ must be equal to 1.
The initial postulated order, from where the recurrent computations must begin to be applied, ought to be large enough for the positive, non normalized, value assigned to $\theta_{n+1}(\tau)$ be much lower or, at least, alike to the reached, non normalized value of $\theta_0(\tau)$, at the other extreme of the entire computation. This last scenario is only valid if $\theta_0(\tau)$ attained already very low values as consequence of having elapsed enough time since the beginning at $(\tau = 0)$ and, therefore, almost support area is just at hand to be completely covered.
In other words: the highest order $(n + 1)$ among all the $\theta_k(\tau)$ computed and, so, considered not null must be high enough to realty represent a $\theta_k(\tau)$ with a positive value, but very low, very near to zero. Only this way the postulated null values of $\theta_{n+2}(\tau)$, $\theta_{n+3}(\tau)$, $\theta_{n+4}(\tau)$, and beyond that can be considered correctly valued.
The aforesaid need for start the recurrent computations in an order $(n + 4)$ that can be very high directs us to the new problem of avoid that the maximum value of the non normalized functions $\theta_k(\tau)$ don´t have an absolute value so high that exceeds the positive numerical limit of Excel spreadsheet. This limit has already been mentioned hand is around 10^{+308}. From our numerical experiments we concluded that this big upper positive limit conjugated with the possibility of start with a postulated value for $\theta_{n+1}(\tau)$ very low, albeit still positive, $(\varepsilon_{n+1} \gtrsim 0)$, leads always, meaning this for all values of dimensionless time (τ), to a possible complete and accurate computation of all the functions $\theta_k(\tau)$.
Logically, as the dimensionless time (τ) increases, more and more fractions $\theta_k(\tau)$, where order (k) is higher, must be calculated because they represent already absolute positive values not anymore negligible.
Consequently once fixed the model parameters values, in a numerical experiment, , as time (τ) increases a successive need for start the algorithm in an order $(n + 4)$ higher, and higher again, come into play. Facing this circumstance the accurate recurrent computation of the kinetic evolution, over a broad range of time $(\tau \geq 0)$, will always imply the construction of a series of similar Generalized Miller´s like algorithms in the same spreadsheet, but where the starting computation order $(n + 4)$ increases from one to the next algorithm. Bigger values of time (τ) demand bigger initial $(n + 4)$ orders as the natural reflex of the physical growth of the biofilm with the continuous accumulation of biomass and concomitant attaining of fractions at solid support area covered by more and more pilled layers.

3. Unavoidable inviting incursion in M/M/1 Queue Recurrence Machinery.
3.1. Brief remarks about M/M/1 Queue Classical Exact Solution

The logical sequel, after having deduced and applied the homogeneous recurrences, reaching an exact graphical translation of biofilm early stages formation and growth model, is now to yield to temptation of made an incursion in the context of Queuing Transient Regime analysis, pursuing a new contribution relatively to their accurate computation.
As we can conclude, by the aforesaid considerations, transient phases in Queuing Theory is a subject, although not the only one, of most interest for our purposes.

The goal is to avoid approximated methods that neglect all the theoretical information contained in an exact solution, and also this way avoid the simple elimination of Bessel Functions from all the analysis, thus preserving to lose their formal beauty.
The immediate choice is M/M/1 queue, not only because one must start by the simpler, or less complicated, case but also because this solution has an infinite series of Modified Bessel Functions of First Kind. Consequently the acquired experience with the work just described in the former section can yield us a nice return.
On the other hand, it is also our goal to conserve Special Functions into play all through the analysis because they are part of the exact solution or, even better saying, because they are the "backbone" of the exact solution.

The performance of an operation involving queuing models will be completely understood only if the transient state probabilities can be computed with high accuracy, and for a large range of parameters and time, at least till steady state is reached. The problem of the M/M/1 queue, considering the aforesaid goals, is a long lived, not already solved question.
The transient solution under analysis was first deduced by A.B. Clarke in 1952 [44] but published and also obtained later, by different methods, by Ledermann and Reuter [45], N. Bailey [46], P. M. Morse [47] and Champernowne [48].
A. B. Clarke [49] deduced again the same solution but generalized to arrival and service rates time depending.
Since those early years M/M/1 transient behaviour has been object of an huge amount of publications describing a great variety of new methodologies, most of them approximated, trying to minimize the inherent computation intractability. Nevertheless, the classical Bessel form, though representing the most antique of all, still constitute a not ended subject.
For the classical exact solution under consideration, we adopt the rescaled version deduced by Clarke [49],

$$P_{i,j}(\tau) = e^{-(1+\rho)\tau} \cdot \{ \rho^{(j-i)/2}\, I_{j-i}\big(2\tau\sqrt{\rho}\big) + \rho^{(j-i-1)/2}\, I_{j+i+1}\big(2\tau\sqrt{\rho}\big) + +(1-\rho)\,\rho^{j} \sum_{k=j+i+2}^{\infty} \rho^{-k/2}\, I_k\big(2\tau\sqrt{\rho}\big)\} \qquad [5.27]$$

... with rescaled time $\tau = \mu\, t$, but constant relative traffic intensity $\rho = \lambda / \mu$.
An important remark concerning the significance of the two entire indexes, i and j, must be, right now, done. The index i refers to the queue state where the transient evolution begins. That´s to say, at initial time $(\tau = 0)$ there are i costumers waiting in the queue. In turn the index j refers to all the possible queue states, including the state corresponding to index i. Therefore the transient regime initial condition is defined by the index i which is this way a constant for each evolution. After the transient "collective" evolution begins, all the states will change as time elapses and such evolution depends on the "starting point" of all the dynamic under analysis, so depends on i. Each value of i defines a particular evolution to steady state. In turn the steady state, that´s to say the "arrival point", is unique.
The more frequently signalled difficulty refers to handle the infinite series summation of terms containing the Modified Bessel Functions of First Kind. However, seems to us that the main difficulty lies on the existing first term in which the corresponding Bessel Function has an order, $(j-i)$, and therefore displaced by $[(j+i)-(j-i)] = 2 \cdot i$ entire units from the value that ought have if, as lowest order term, the continuous discrete ordination $(j+i), (j+i+1), (j+i+$ *2, ...* would be attained.
Consequently, and putting in realization this criterion, we will pay, for start, attention to the whole solution but without the first term $(F.T.)$ which, free of the exponential, reads,

$$F.T. = \rho^{(j-i)/2}\, I_{j-i}\big(2\tau\sqrt{\rho}\big) \qquad [5.28]$$

3.2. M/M/1 Queue Classical Exact Solution: Some preliminary definitions

For systematization purpose let, first of all, define the functions,

$$W_n^*(\tau) = \rho^{-n/2} I_n\big(2\tau\sqrt{\rho}\big) \qquad [5.29]$$

<u>Remark:</u> The asterisk in $W_n^*(\tau)$ has the objective of prevent confusion with the functions $W_n(x)$ already defined in $[5.3-A]$ and $[5.3-B]$

The M/M/1 queue exact solution multiplied by $e^{+(1+\rho)\tau}$ is defined by,

$$Q_{i,j}(\tau) = \rho^{(j-i)/2} I_{j-i}\left(2\tau\sqrt{\rho}\right) + \rho^{(j-i-1)/2} I_{j+i+1}\left(2\tau\sqrt{\rho}\right) + (1-\rho)\,\rho^{j} \sum_{k=j+i+2}^{\infty} \rho^{-k/2} I_k\left(2\tau\sqrt{\rho}\right) \quad [5.30]$$

Now, applying [5.29] in [5.30] we can read $Q_{i,j}(\tau)$ like,

$$Q_{i,j} = W^*_{i-j} + \rho^j W^*_{i+j+1} + (1-\rho)\,\rho^j \sum_{k=i+1}^{\infty} W^*_{k+j+1} \quad [5.31]$$

From this definition let to separate as well $R_{i,j}(\tau)$ which is all the part of $Q_{i,j}(\tau)$ where the orders of Modified Bessel functions or, alternatively, of functions $W^*_n(\tau)$ follow an uninterrupted ordination, from $(i+j+1)$ to infinite.
Such function reads,

$$R_{i,j} = \rho^j W^*_{i+j+1} + (1-\rho)\,\rho^j \sum_{k=i+1}^{\infty} W^*_{k+j+1} \quad [5.32]$$

At [5.31] and [5.32], for the sake of avoid heavy notation, dimensionless time dependence of functions $Q_{i,j}(\tau)$, $R_{i,j}(\tau)$ and $W^*_k(\tau)$ is not explicit. Also, in all the sequel, we excuse from include time dependence of functions $P_{i,j}(\tau)$, $I_n\left(2\tau\sqrt{\rho}\right)$, $W^*_n(\tau)$, $Q_{i,j}(\tau)$ and $R_{i,j}(\tau)$.
Of course, the first term at [5.28] is,

$$W^*_{i-j} = \rho^{(j-i)/2} I_{j-i}\left(2\tau\sqrt{\rho}\right) \quad [5.33]$$

.. and functions $Q_{i,j}$ can shortly be written,

$$Q_{i,j} = W^*_{i-j} + R_{i,j} \quad [5.34]$$

In the same way also functions $P_{i,j}$ can be simply represented like,

3.3. M/M/1 Queue Classical Exact Solution: Brief description of the proposed Miller´s adapted method

We are going to adapt Three Term Recurrence (T. T. R.) Miller algorithm, which was used to construct tables of Modified Bessel Functions of the First Kind (see Abrammovitch and Stegun [41]) to calculate accurately the M/M/1 queue exact solution.
For start we deduce straight away a four term recurrence formula for the already defined functions $R_{i,j}$. So the infinite series of Bessel functions is included in such four term recurrence, but the first term W^*_{i-j} is not. That formula will be used to compute all the transient state probabilities with a backwards recurrence algorithm similar to that described by Miller [42] to compute Bessel functions. In this new method we will use the aforementioned four terms recurrence (F. T. R.) and, for account and locate into play the first, not included, term a suitable normalization condition will be established. Recall that functions $R_{i,j}$, like functions W^*_{i-j}, $Q_{i,j}$ and $P_{i,j}$, have all of them the same two indexes, i and j, with the significance already explained. We are interested into analyse a general evolution in transient regime once fixed the initial condition, that´s to say, once fixed index i. Consequently the homogeneous recurrences, we are interested for, are those with index i fixed, and index j changing, by discrete orders placed each to each other by contiguous variations of one unity: $R_{i,j}$, $R_{i,j+1}, R_{i,j+2}$, ... and so on. Others kinds of recurrences will be deduced latter on for the sake of completeness.
The numerical computation will also be done by a direct computation of the entire solution, term by term, with values of the Bessel modified function calculated by the already known tree term recurrence backwards Miller algorithm. This method is not innovative but is seldom applied in queuing theory literature, if it is even applied at all.
The two mentioned independent methods will validate each other.

In the new and adapted method, where the four term recurrence plays an essential role, the functions $R_{i,j}$ are considered as a Special Function by its own, like happened in the biofilm model, regarding then the $W_n(x)$ and the $\theta_n(\tau)$ functions, albeit all they contain an infinite series of terms each one of them having a Bessel Function of First Kind (modified or not).
Finally we will deduce the homogeneous recurrence for the whole solution $P_{i,j}(\tau)$.
It will be observed that for this complete solution one obtains a six terms recurrence.
This last formula will be only used as another independent validation of the two mentioned methods. All the computations have been done in Excel.

3.4. M/M/1 Queue Classical Exact Solution: Function $R_{i,j}(\tau)$ four terms homogeneous recurrence deduction.

The first step in this derivation is multiply [5.32] by ρ and rearrange the result in an obvious way.
We get,

$$\rho R_{i,j} = \rho^{j+1} W^*_{i+j+1} + (1-\rho)\,\rho^{j+1} W^*_{i+j+2} + (1-\rho)\,\rho^{j+1} \sum_{k=i+1}^{\infty} W^*_{k+j+2} \qquad [5.36]$$

Next add 1 to j in [5.32],

$$R_{i,j+1} = \rho^{j+1} W^*_{i+j+2} + (1-\rho)\,\rho^{j+1} \sum_{k=i+1}^{\infty} W^*_{k+j+2} \qquad [5.37]$$

Now we subtract [5.37] from [5.36], and add again 1 to j, getting,

$$\rho R_{i,j} - R_{i,j+1} = \rho^{j+1} W^*_{i+j+1} - \rho^{j+2} W^*_{i+j+2} \qquad [5.38]$$

… and,

$$\rho R_{i,j+1} - R_{i,j+2} = \rho^{j+2} W^*_{i+j+2} - \rho^{j+3} W^*_{i+j+3} \qquad [5.39]$$

$$W^*_n(\tau) = \rho^{-n/2} I_n\left(2\tau\sqrt{\rho}\right) \qquad [5.29]$$

Inserting [5.29] in the known recursion, for modified Bessel functions of first kind,

$$I_{n+1} = I_{n-1} - \frac{n}{\tau\sqrt{\rho}} I_n \qquad [5.40]$$

… the equivalent recursion for W^*_n reads,

$$\rho W^*_{n+1} = W^*_{n-1} - \frac{n}{\tau} W^*_n \qquad [5.41]$$

In this recursion we put $n = i + j + 2$, and easily we obtain,

$$[W^*_{i+j+1} - \rho W^*_{i+j+3}] = \frac{(i+j+2)}{\tau} W^*_{i+j+2} \qquad [5.42]$$

Now divide [5.38] by ρ^{j+1} , [5.39] by ρ^{j+2} and add those two results getting,

$$\rho^{-j} R_{i,j} - \rho^{-j-2} R_{i,j+2} = (1-\rho) W^*_{i+j+2} + [W^*_{i+j+1} - \rho W^*_{i+j+3}] \qquad [5.43]$$

Equation [10] inserted in [11] allows us to get an expression without W^*_{i+j+1} nor W^*_{i+j+3}, and obtain a formula for W^*_{i+j+2} ,

$$W^*_{i+j+2} = \frac{\tau}{\rho^{j+2}[(1-\rho)\tau + (i+j+2)]} [\rho^2 R_{i,j} - R_{i,j+2}] \qquad [5.44]$$

And adding 1 to j,

$$W^*_{i+j+3} = \frac{\tau}{\rho^{j+3}[(1-\rho)\tau + (i+j+3)]} [\rho^2 R_{i,j+1} - R_{i,j+3}] \qquad [5.45]$$

As final step we only need substitute W^*_{i+j+2} and W^*_{i+j+3} , given by [5.44], and [5.45], in [5.39], and in such a way get the desired four terms recurrence formula:

$$\left\{\frac{\rho^2\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i,j}+\left\{-\rho-\frac{\rho^2\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i,j+1}+$$
$$+\left\{1-\frac{\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i,j+2}+\left\{\frac{\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i,j+3}=0 \quad [5.46]$$

3.5. M/M/1 Queue Classical Exact Solution: Normalization condition and mathematical outline for transient state probabilities $P_{i,j}(\tau)$ computation.

In this subsection the recurrence [5.46] will be analysed seeking for a suitable application in computing the transient regime of M/M/1 queue.
Let, for the sake of simplifying the notation, define,

$$\eta_k=\frac{\tau}{[(1-\rho)\tau+(i+\mathrm{k})]} \quad [5.47]$$

$$A_j=\rho^2\eta_{j+2} \quad [5.48(\mathrm{A})]$$

$$B_j=-\rho(1+\rho\eta_{j+3}) \quad [5.48(\mathrm{B})]$$

$$C_j=1-\eta_{j+2} \quad [5.48(\mathrm{C})]$$

$$D_j=\eta_{j+3} \quad [5.48(\mathrm{D})]$$

So [5.46] now can be written,

$$A_j\cdot R_{i,j}+B_j\cdot R_{i,j+1}+C_j\cdot R_{i,j+2}+D_j\cdot R_{i,j+3}=0 \quad [5.49]$$

… and we obtain straightaway the corresponding appropriate regressive form,

$$R_{i,j}=-\frac{B_j}{A_j}\cdot R_{i,j+1}-\frac{C_j}{A_j}\cdot R_{i,j+2}-\frac{D_j}{A_j}\cdot R_{i,j+3} \quad [5.50(\mathrm{j})-\mathrm{A}]$$

… or in an alternative equivalent form,

$$R_{i,j}=\left[\frac{1+\eta_{j+3}}{\eta_{j+2}}\right]\cdot R_{i,j+1}+\left[\frac{1}{\rho^2}\cdot\left(1-\frac{\eta_j}{\eta_{j+2}}\right)\right]\cdot R_{i,j+2}-\left(\frac{\eta_{j+3}}{\rho^2\cdot\eta_{j+2}}\right)\cdot R_{i,j+3} \quad [5.50(\mathrm{j})-\mathrm{B}]$$

This equation, $[5.50(k)-A]$ or, equivalently $[5.50(k)-B]$, where $(0\le k\le j)$, is that one which must be applied in a regressive recurrent fashion, exactly like in Section V.2. equations $[5.26(0)]$ and $[5.26(k)]$, where $(0\le k\le n)$, have been applied in the computation of the biofilm early stages formation and growth model.
Now the homogeneous recurrence relation under analysis is a four terms recurrence. For avoid confusion with the five terms recurrence at Section V.2. the system of abbreviated designations (T. T. R.), (F. T. R.) and (F. T. R.) concerning respectively Three, Four and Five Terms Recurrence, must be slightly modified.
We propose, in the same ordination, the shortenings $(T_3.T.R.)$, $(F_4.T.R.)$ and $(F_5.T.R.)$.

This way not only the confusion between four and five terms recurrences is eliminated but also nomenclature uniformity is maintained by using, in the case of Three Terms Recurrence, $(T_3.T.R.)$ instead of (T. T. R.).
Following the same reasoning, with adaptations, done at Section V.2. the first step is to apply formula [5.50(j) − A] (or [5.50(j) − B]) at an order $(j+3)$ high enough to made $R_{i,j+3}(\tau)=0$ and $R_{i,j+2}(\tau)=0$. Also in [5.50(j) − A] (or [5.50(j) − B]) a small postulated positive value, $\varepsilon_{i,j+1}$, but not necessarily equal to 1, and preferably less than 1, must be assigned to $R_{i,j+1}(\tau)$.
This way we can accomplish extensive computations in their number of recurrent steps, like before, in Section V.2.
Consequently, the first recurrent computation consists in doing $R_{i,j+3}(\tau)=0$, $R_{i,j+2}(\tau)=0$ and $R_{i,j+1}(\tau)=\varepsilon_{i,j+1}$ where $(0<\varepsilon_{i,j+1}\leq 1)$, insert these values in [5.50(j) − A] (or [5.50(j) − B]), and so get the non normalized value of $R_{i,j}(\tau)$, which we designate by $\varepsilon_{i,j}$.
The second recurrent computation consists in apply once again [5.50(j) − A] (or [5.50(j) − B]), but diminishing 1 to all orders $(j\pm k)$. We designate by [5.50(j) − A] (or [5.50(j) − B]) such "new" equation,

$$R_{i,j-1}=-\frac{B_{j-1}}{A_{j-1}}\cdot R_{i,j}-\frac{C_{j-1}}{A_{j-1}}\cdot R_{i,j+1}-\frac{D_{j-1}}{A_{j-1}}\cdot R_{i,j+2} \qquad [5.50(j-1)-A]$$

… or,

$$R_{i,j-1}=\left[\frac{1+\eta_{j+2}}{\eta_{j+1}}\right]\cdot R_{i,j}+\left[\frac{1}{\rho^2}\cdot\left(1-\frac{\eta_{j-1}}{\eta_{j+1}}\right)\right]\cdot R_{i,j+1}-\left(\frac{\eta_{j+2}}{\rho^2\cdot\eta_{j+1}}\right)\cdot R_{i,j+2} \qquad [5.50(j-1)-B]$$

Now we insert the former values of $R_{i,j+2}(\tau)=0$, $R_{i,j+1}(\tau)=\varepsilon_{i,j+1}$ and $R_{i,j}(\tau)=\varepsilon_{i,j}$ and obtain the non normalized value of $R_{i,j-1}(\tau)$, coherently designated by $\varepsilon_{i,j-1}$.
This procedure is done repeatedly $(j-1)$ more times till the non normalized value of $R_{i,0}(\tau)$, designed $\varepsilon_{i,0}$, is calculated.
In general, the k-th recurrent computation, being $(1\leq k\leq j)$, is realized with equation $[5.50(j-k+1)]$, which reads,

$$R_{i,j-k+1}=-\frac{B_{j-k+1}}{A_{j-k+1}}\cdot R_{i,j-k+2}-\frac{C_{j-k+1}}{A_{j-k+1}}\cdot R_{i,j-k+3}-\frac{D_{j-k+1}}{A_{j-k+1}}\cdot R_{i,j-k+4} \qquad [5.50(j-k+1)-A]$$

… or equivalently,

$$R_{i,j-k+1}=\left[\frac{1+\eta_{j-k+4}}{\eta_{j-k+3}}\right]\cdot R_{i,j-k+2}+$$
$$+\left[\frac{1}{\rho^2}\cdot\left(1-\frac{\eta_{j-k+1}}{\eta_{j-k+3}}\right)\right]\cdot R_{i,j-k+3}-\left(\frac{\eta_{j-k+4}}{\rho^2\cdot\eta_{j-k+3}}\right)\cdot R_{i,j-k+4} \qquad [5.50(j-k+1)-B]$$

The three initial postulated values, $R_{i,j+3}(\tau)=0$, $R_{i,j+2}(\tau)=0$ and $R_{i,j+1}(\tau)=\varepsilon_{i,j+1}$ are used in the first recurrent computation. Then, from these, the two values $R_{i,j+2}(\tau)=0$ and $R_{i,j+1}(\tau)=\varepsilon_{i,j+1}$, are still needed in the second recurrent computation. In the same way, the value $R_{i,j+1}(\tau)=\varepsilon_{i,j+1}$ is still needed for the third computation.
In the fourth and subsequent recurrent computations only calculated values in former computations are needed. None of the initial three postulated values figures anymore.
In all this regressive and recurrent computations, equations $[5.50(j)]$, $[5.50(j-1)]$, $[5.50(j-2)]$, … , $[5.50(j-k+1)]$, … , [5.50(3)], [5.50(2)], [5.50(1)], and [5.50(0)] are applied and the non normalized values of $R_{i,j}(\tau)$, $R_{i,j-1}(\tau)$, $R_{i,j-2}(\tau)$, … , $R_{i,j-k+1}(\tau)$, … , $R_{i,3}(\tau)$, $R_{i,2}(\tau)$, $R_{i,1}(\tau)$, and $R_{i,0}(\tau)$, are respectively obtained.
This time, in the actual algorithm adaptation, the last regressive recurrent computation, leading to the non normalized value of $R_{i,0}(\tau)$, is rendered concrete by the same formula as

that at all the others former computations, not coming into play a different one recurrent relation, as happened in the case of the algorithm described in Section V.2. This circumstance is merely a consequence of the different analytic structures between these two cases.
In the following table (next page) all the sequence of the described recurrent computations for the non normalized values of all the functions $R_{i,k}(\tau)$, being $(0 \leq k \leq j+3)$, is succinctly and clearly explained.

TABLE 2

RECURRENT COMPUTATION N°/ / APPLIED EQUATION	NOT NORMALIZED INPUT VALUES:			RESULT
1	$\boldsymbol{R_{i,j+3}(\tau)}$	$\boldsymbol{R_{i,j+2}(\tau)}$	$\boldsymbol{R_{i,j+1}(\tau)}$	$\boldsymbol{R_{i,j}(\tau)}$
$[\mathbf{5.50(j) - (A\ or\ B)}]$	0	0	$0 < \varepsilon_{i,j+1} \leq 1$	$\varepsilon_{i,j}$
2	$\boldsymbol{R_{i,j+2}(\tau)}$	$\boldsymbol{R_{i,j+1}(\tau)}$	$\boldsymbol{R_{i,j}(\tau)}$	$\boldsymbol{R_{i,j-1}(\tau)}$
$[\mathbf{5.50(j-1) - (A\ or\ B)}]$	0	$0 < \varepsilon_{i,j+1} \leq 1$	$\varepsilon_{i,j}$	$\varepsilon_{i,j-1}$
3	$\boldsymbol{R_{i,j+1}(\tau)}$	$\boldsymbol{R_{i,j}(\tau)}$	$\boldsymbol{R_{i,j-1}(\tau)}$	$\boldsymbol{R_{i,j-2}(\tau)}$
$[\mathbf{5.50(j-2) - (A\ or\ B)}]$	$0 < \varepsilon_{i,j+1} \leq 1$	$\varepsilon_{i,j}$	$\varepsilon_{i,j-1}$	$\varepsilon_{j,j-2}$
4	$\boldsymbol{R_{i,j}(\tau)}$	$\boldsymbol{R_{i,j-1}(\tau)}$	$\boldsymbol{R_{i,j-2}(\tau)}$	$\boldsymbol{R_{i,j-3}(\tau)}$
$[\mathbf{5.50(j-3) - (A\ or\ B)}]$	$\varepsilon_{i,j}$	$\varepsilon_{i,j-1}$	$\varepsilon_{j,j-2}$	$\varepsilon_{i,j-3}$
5	$\boldsymbol{R_{i,j-1}(\tau)}$	$\boldsymbol{R_{i,j-2}(\tau)}$	$\boldsymbol{R_{i,j-3}(\tau)}$	$\boldsymbol{R_{i,j-4}(\tau)}$
$[\mathbf{5.50(j-4) - (A\ or\ B)}]$	$\varepsilon_{i,j-1}$	$\varepsilon_{j,j-2}$	$\varepsilon_{i,j-3}$	$\varepsilon_{i,j-4}$
6	$\boldsymbol{R_{i,j-2}(\tau)}$	$\boldsymbol{R_{i,j-3}(\tau)}$	$\boldsymbol{R_{i,j-4}(\tau)}$	$\boldsymbol{R_{i,j-5}(\tau)}$
$[\mathbf{5.50(j-5) - (A\ or\ B)}]$	$\varepsilon_{j,j-2}$	$\varepsilon_{i,j-3}$	$\varepsilon_{i,j-4}$	$\varepsilon_{i,j-5}$
	…………	…………	…………	…………
…………	…………	…………	…………	…………
$\boldsymbol{k}$	$\boldsymbol{R_{i,j-k+4}(\tau)}$	$\boldsymbol{R_{i,j-k+3}(\tau)}$	$\boldsymbol{R_{i,j-k+2}(\tau)}$	$\boldsymbol{R_{i,j-k+1}(\tau)}$
$[\mathbf{5.50(j-k+1) - (A\ or\ B)}]$	$\varepsilon_{i,j-k+4}$	$\varepsilon_{i,j-k+3}$	$\varepsilon_{i,j-k+2}$	$\varepsilon_{i,j-k+1}$
	…………	…………	…………	…………
…………	…………	…………	…………	…………
$\boldsymbol{j-4}$	$\boldsymbol{R_{i,8}(\tau)}$	$\boldsymbol{R_{i,7}(\tau)}$	$\boldsymbol{R_{i,6}(\tau)}$	$\boldsymbol{R_{i,5}(\tau)}$
$[\mathbf{5.50(5) - (A\ or\ B)}]$	$\varepsilon_{i,8}$	$\varepsilon_{i,7}$	$\varepsilon_{i,6}$	$\varepsilon_{i,5}$
$\boldsymbol{j-3}$	$\boldsymbol{R_{i,7}(\tau)}$	$\boldsymbol{R_{i,6}(\tau)}$	$\boldsymbol{R_{i,5}(\tau)}$	$\boldsymbol{R_{i,4}(\tau)}$
$[\mathbf{5.50(4) - (A\ or\ B)}]$	$\varepsilon_{i,7}$	$\varepsilon_{i,6}$	$\varepsilon_{i,5}$	$\varepsilon_{i,4}$
$\boldsymbol{j-2}$	$\boldsymbol{R_{i,6}(\tau)}$	$\boldsymbol{R_{i,5}(\tau)}$	$\boldsymbol{R_{i,4}(\tau)}$	$\boldsymbol{R_{i,3}(\tau)}$
$[\mathbf{5.50(3) - (A\ or\ B)}]$	$\varepsilon_{i,6}$	$\varepsilon_{i,5}$	$\varepsilon_{i,4}$	$\varepsilon_{i,3}$
$\boldsymbol{j-1}$	$\boldsymbol{R_{i,5}(\tau)}$	$\boldsymbol{R_{i,4}(\tau)}$	$\boldsymbol{R_{i,3}(\tau)}$	$\boldsymbol{R_{i,2}(\tau)}$
$[\mathbf{5.50(2) - (A\ or\ B)}]$	$\varepsilon_{i,5}$	$\varepsilon_{i,4}$	$\varepsilon_{i,3}$	$\varepsilon_{i,2}$
$\boldsymbol{j}$	$\boldsymbol{R_{i,4}(\tau)}$	$\boldsymbol{R_{i,3}(\tau)}$	$\boldsymbol{R_{i,2}(\tau)}$	$\boldsymbol{R_{i,1}(\tau)}$
$[\mathbf{5.50(1) - (A\ or\ B)}]$	$\varepsilon_{i,4}$	$\varepsilon_{i,3}$	$\varepsilon_{i,2}$	$\varepsilon_{i,1}$
$\boldsymbol{j+1}$	$\boldsymbol{R_{i,3}(\tau)}$	$\boldsymbol{R_{i,2}(\tau)}$	$\boldsymbol{R_{i,1}(\tau)}$	$\boldsymbol{R_{i,0}(\tau)}$
$[\mathbf{5.50(0) - (A\ or\ B)}]$	$\varepsilon_{i,3}$	$\varepsilon_{i,2}$	$\varepsilon_{i,1}$	$\varepsilon_{i,0}$

Till this stage we only have the non normalized values of all functions $R_{i,k}(\tau)$, being $(0 \leq k \leq j + 3$, with their relative proportions correct, including the null values of $Ri,j+3\tau$ and $Ri,j+2\tau$ if the starting order $(j+3)$ of all the algorithm computations is high enough.

In the sequel we must search for a suitable normalization procedure. For such purpose we recall that,

$$P_{i,j} = e^{-(1+\rho)\tau} \cdot \{ W^{*}_{i-j} + R_{i,j} \} \qquad [5.35]$$

... and also that,

$$\sum_{j=0}^{\infty} P_{i,j} = 1 \qquad [5.51]$$

... therefore,

$$\sum_{j=0}^{\infty} \{W_{i-j}^{*} + R_{i,j}\} = e^{(1+\rho)\tau} \qquad [5.52]$$

... and lastly,

$$\sum_{j=0}^{\infty} R_{i,j} = e^{(1+\rho)\tau} - \sum_{j=0}^{\infty} W_{i-j}^{*} \qquad [5.53(W)]$$

Recalling [5.33],

$$W_{i-j}^{*} = \rho^{(j-i)/2}\, I_{j-i}\left(2\tau\sqrt{\rho}\right) \qquad [5.33]$$

... we can, alternatively, write [5.53(W)] like,

$$\sum_{j=0}^{\infty} R_{i,j} = e^{(1+\rho)\tau} - \sum_{j=0}^{\infty} \rho^{(j-i)/2}\, I_{j-i}\left(2\tau\sqrt{\rho}\right) \qquad [5.53(I)]$$

The right hand side $(R.H.S.)$ of [5.53(I)] is easy to compute accurately, and therefore we will adopt this identity as starting point for deduce the suitable normalization factor of functions $R_{i,k}(\tau)$.
Let the normalization factor be noted by $\{N.F.\}$ and let also recall that the non normalized values obtained after the recurrent computations are already designed by $\varepsilon_{i,j-k+1}$, being,

- ... $(1 \le k \le j+1)$ for all non null computed values
- ... $(k = 0)$ for the non null postulated value of $\varepsilon_{i,j+1}$
- ... $(k = -1, -2, -3, \dots)$ for the null postulated value of $\varepsilon_{i,j+2}$, $\varepsilon_{i,j+3}$, $\varepsilon_{i,j+4}$, … , and beyond.

Therefore the summation of all the non null neither normalized values reads,

$$\sum_{k=0}^{j+1} \varepsilon_{i,j-k+1} \qquad [5.54]$$

The expression for $\{N.F.\}$ can now be straightaway defined,

$$\{N.F.\} = \left[e^{(1+\rho)\tau} - \sum_{j=0}^{\infty} \rho^{(j-i)/2}\, I_{j-i}\left(2\tau\sqrt{\rho}\right)\right] \div \left[\sum_{k=0}^{j+1} \varepsilon_{i,j-k+1}\right] \qquad [5.55]$$

In the $(R.H.S.)$ of [5.53(I)] the infinite summation of terms with the Modified Bessel functions $I_{j-i}\left(2\tau\sqrt{\rho}\right)$ is computed applying the classical and known $(T_3.T.R.)$ Miller recurrent method.
For such purpose one of the following two normalization conditions, among others, can be easily applied,

$$e^{x} = I_0(\mathrm{x}) + 2 \cdot \sum_{n=1}^{\infty} I_n(\mathrm{x}) \qquad [5.56(e)]$$

$$1 = I_0(\mathrm{x}) + 2 \cdot \sum_{n=1}^{\infty} (-1)^n I_{2n}(\mathrm{x}) \qquad [5.56(1)]$$

Application of [5.56(e)] or [5.56(1)] for the normalization of the functions $I_{j-i}(2\tau\sqrt{\rho})$ in the infinite summation at the $(R.H.S.)$ of [5.53(I)] must be accomplished setting $x = 2\tau\sqrt{\rho}$.
Alternatively, the mentioned summation, in terms of functions $W^*_{i-j}(\tau)$, according to [5.33], is,

$$\sum_{j=0}^{\infty} \rho^{(j-i)/2}\, I_{j-i}(2\tau\sqrt{\rho}) = \sum_{j=0}^{\infty} W^*_{i-j}(\tau) \qquad [5.57]$$

And, correspondently, the two normalization conditions [5.56(e)] and [5.56(1)], in equivalent way, for functions $W^*_{i-j}(\tau)$, are written like,

$$e^{2\tau\sqrt{\rho}} = W^*_0(\tau) + 2 \cdot \sum_{n=1}^{\infty} \rho^{n/2} W^*_n(\tau) \qquad [5.58(e)]$$

$$1 = W^*_0(\tau) + 2 \cdot \sum_{n=1}^{\infty} (-\rho)^n W^*_{2n}(\tau) \qquad [5.58(1)]$$

So, having defined at [5.55] an easily accurately computable normalization factor, we only need to add up all the non normalized values $[\varepsilon_{i,j-k+1}, \ldots\ (0 \le k \le j+1)]$ of functions $R_{i,k}(\tau)$, obtaining $\sum_{k=0}^{j+1} \varepsilon_{i,j-k+1}$, divide each one of the $(j+1)$ computed, non normalized, values $[\varepsilon_{i,j-k+1}, \ldots\ (0 \le k \le$ *j+1*, by such summation and, lastly, multiply, also each one, by the *(R. H. S.)* of *[5.53(I)]*. In other words and shortly, once computed accurately $\{N.F.\}$, given by [5.55], the exact values of functions $R_{i,k}(\tau)$ are achieved multiplying each one and all of the $[\varepsilon_{i,j-k+1}, \ldots\ (0 \le k \le j+1)]$, by $\{N.F.\}$.
Like in Section V.2., the highest order $(j+1)$ among all the $R_{i,k}(\tau)$ computed and, so, considered not null must be high enough to realty represent a $R_{i,k}(\tau)$ with a positive value, but very low, very near to zero. Only this way the postulated null values of $R_{i,j+2}(\tau)$, $R_{i,j+3}(\tau)$, and beyond that, can be considered correctly valued.
The aforesaid need for start the recurrent computations in an order $(j+3)$ that can be very high directs us, also in this algorithm, to the problem of avoid that the maximum value of the non normalized functions $[\varepsilon_{i,j-k+1}, \ldots\ (0 \le k \le j+1)]$ don´t have an absolute value so high that exceeds the positive numerical limit of Excel spreadsheet. Even with this eventuality in our numerical experiments we verify that this upper positive numerical limit of Excel conjugated with the possibility of start with a postulated value for $R_{i,j+1}(\tau)$ very low, albeit still positive, $(\varepsilon_{i,j+1} \gtrsim 0)$, leads always to a possible complete and accurate computation of all the functions $R_{i,k}(\tau)$.
Logically, like in biofilm model, as the dimensionless time (τ) increases, more and more functions $R_{i,k}(\tau)$ where order (k) is higher, must be calculated because they represent already absolute positive values not anymore negligible.
In M/M/1 queue an evident criterion for starting computations, relative to low initial time values (τ), must be taken in consideration.
According to initial condition, in low time (τ) initial range, the probability functions $P_{i,k}(\tau)$ with bigger values are $P_{i,i}(\tau)$ and also the nearest ones to $P_{i,i}(\tau)$, like $P_{i,i\pm k}(\tau)$ where (k) is low: $(k = 1, 2, 3, \ldots)$.
An additional remark concerns to the obvious prevision that in all pairs of functions $P_{i,i-k}(\tau)$ and $P_{i,i+k}(\tau)$, in order places symmetrical to $P_{i,i}(\tau)$, the function $P_{i,i-k}(\tau)$, will growth more and faster than $P_{i,i+k}(\tau)$ as consequence of the evolution to steady state where the corresponding limit distribution, being $P_j = (1-\rho) \cdot \rho^j$, has his maximum value in function $P_{i,0}(\tau)$ and decrease regularly as order j increase.
Therefore starting computations in order $(j+3 = 2i)$, including this way all the function between $P_{i,0}(\tau)$ and $P_{i,i}(\tau)$ and also equal number of functions with higher order than $P_{i,i}(\tau)$ seems to be the most wise initial choice.

Consequently once fixed the model parameter value (ρ), and the initial condition (i) for the M/M/1 queue, in a numerical experiment, , as time (τ) increases the successive need for start the algorithm in an order $(j+3)$ higher, and higher again, come into play only if one wants to conduct the rigor and the extent to probability functions $P_{i,j}(\tau)$ with very high order (j) where those positive values are already very low. Nevertheless it is worthwhile to try achieve always great accuracy and wide ranges of orders (j) because queuing transient regime analysis only have useful interest if the initial congestion is significantly important $[if\ initial\ order\ (i)\ is\ very\ high]$. As a matter of fact usually, in the literature, the value chosen for (i) is too low to mimic an authentic congestion scenario. Rarely surpasses values like $(i = 3\ or\ 4)$ or, at most, $(i = 10)$.
Facing this circumstances the accurate recurrent computation of the M/M/1 evolution, over a broad range of time $(\tau \geq 0)$, and for high values of initial order (i), will always imply the construction of a series of similar Generalized Miller´s like algorithms in the same spreadsheet, but where the starting computation order $(j+3)$ increases from one to the next algorithm, just like happens in the kinetic evolution described in biofilm early stages formation and growth model. In the case of M/M/1 queue, and in the cases of all others queuing transient regimes, in general, the adoption of initial conditions very far away from steady state have great interest because those are the truly conditions of severe congestion contexts.
Bigger values of "initial congestion" (i) require bigger ranges of time (τ) where computations must be accurately accomplished and also demand bigger initial $(j+3)$ orders as the natural reflex of the complexity of this subject. In our numerical experiences we reached, with complete accuracy, the M/M/1 queue transient evolution for $(i = 100)$ with ρ in the range between $(\rho = 0{,}2)$ and $(\rho = 0{,}8)$.

3.6. M/M/1 Queue Classical Exact Solution: Complete solution $P_{i,j}(\tau)$ six terms homogeneous recurrence deduction.

In this deduction the starting point is the four terms recurrence formula for functions $R_{i,j+k}(\tau)$, already obtained,

$$A_j \cdot R_{i,j} + B_j \cdot R_{i,j+1} + C_j \cdot R_{i,j+2} + D_j \cdot R_{i,j+3} = 0 \qquad [5.49]$$

... where A_j, B_j, C_j and D_j are respectively defined by [5.48(A)], [5.48(B)], [5.48(C)] and [5.48(D)] and, for η_k, also for [5.47].

From [5.34],

$$R_{i,j} = Q_{i,j} - W^*_{i-j} \qquad [5.59]$$

Now, defining,

$$S_{i,j} = A_j \cdot Q_{i,j} + B_j \cdot Q_{i,j+1} + C_j \cdot Q_{i,j+2} + D_j \cdot Q_{i,j+3} \qquad [5.60]$$

… we can transform [5.49] obtaining,

$$S_{i,j} = A_j \cdot W^*_{i-j} + B_j \cdot W^*_{i-j-1} + C_j \cdot W^*_{i-j-2} + D_j \cdot W^*_{i-j-3} \qquad [5.61]$$

Following on, the functions W^*_{i-j} and W^*_{i-j-3} can be eliminated applying the expressions,

$$W^*_{i-j} = \frac{1}{\rho} W^*_{i-j-2} - \frac{(i-j-1)}{\rho\tau} W^*_{i-j-1} \qquad [5.62(0)]$$

$$W^*_{i-j-3} = \frac{(i-j-2)}{\tau} W^*_{i-j-2} + \rho W^*_{i-j-1} \qquad [5.62(1)]$$

… and defining,

$$E_j = [-\frac{(i-j-1)}{\rho\tau} A_j + B_j + \rho D_j\,] \qquad [5.63(E)]$$

… and also,

$$F_j = [\,\frac{1}{\rho} A_{j+1} + C_{j+1} + \frac{(i-j-3)}{\tau} D_{j+1}] \qquad [5.63(F)]$$

… one gets, from [5.61],

$$S_{i,j} = E_j \cdot W^*_{i-j-1} + F_{j-1} \cdot W^*_{i-j-2} \qquad [5.64(0)]$$

… and also, adding 1 to j,

$$S_{i,j+1} = E_{j+1} \cdot W^*_{i-j-2} + F_j \cdot W^*_{i-j-3} \qquad [5.64(1)]$$

With a little algebraic manipulation in [5.64(0)] and [5.64(1)] we can get,

$$\frac{S_{i,j+1}}{F_j} - \frac{\rho S_{i,j}}{E_j} = \left[\frac{E_{j+1}}{F_j} - \rho\,\frac{F_{j-1}}{E_j} + \frac{(i-j-2)}{\tau}\right] \cdot W^*_{i-j-2} \qquad [5.65]$$

… and defining, once more, another auxiliary parameter,

$$G_j = \left[\frac{E_{j+1}}{F_j} - \rho\,\frac{F_{j-1}}{E_j} + \frac{(i-j-2)}{\tau}\right] \qquad [5.66(G)]$$

… we can write [5.65] this way,

$$\frac{S_{i,j+1}}{F_j} - \frac{\rho S_{i,j}}{E_j} = G_j \cdot W^*_{i-j-2} \qquad [5.67]$$

… and achieve this expression for W^*_{i-j-2} ,

$$W^*_{i-j-2} = \frac{S_{i,j+1}}{F_j G_j} - \frac{\rho S_{i,j}}{E_j G_j} \qquad [5.68(0)]$$

And adding 1 to j we also have the formula for W^*_{i-j-3},

$$W^*_{i-j-3} = \frac{S_{i,j+2}}{F_{j+1} G_{j+1}} - \frac{\rho S_{i,j+1}}{E_{j+1} G_{j+1}} \qquad [5.68(1)]$$

With [5.68(0)] and [5.68(1)] W^*_{i-j-2} and W^*_{i-j-3} can disappear from [5.64(1)], and after, collecting the terms in $S_{i,j+1}$ reach,

$$\frac{\rho E_{j+1}}{E_j G_j} S_{i,j} + \left[1 - \frac{E_{j+1}}{F_j G_j} + \frac{\rho F_j}{E_{j+1} G_{j+1}}\right] S_{i,j+1} - \frac{F_j}{F_{j+1} G_{j+1}} S_{i,j+2} = 0 \qquad [5.69]$$

Now we can use [5.60] and also their corresponding variants adding 1 and 2 to j, and get the desired recurrence for the functions $Q_{i,j}(\tau)$ what is the same to say the desired recurrence for the functions $P_{i,j}(\tau)$.

The final result reads,

$$\left[A_j \frac{\rho E_{j+1}}{E_j G_j}\right] \cdot P_{i,j} +$$

$$+\left[A_{j+1}\left(1-\frac{E_{j+1}}{F_j G_j}+\frac{\rho F_j}{E_{j+1}G_{j+1}}\right)+B_j \frac{\rho E_{j+1}}{E_j G_j}\right] \cdot P_{i,j+1} +$$

$$+\left[-A_{j+2}\frac{F_j}{F_{j+1}G_{j+1}}+ B_{j+1}\left(1-\frac{E_{j+1}}{F_j G_j}+\frac{\rho F_j}{E_{j+1}G_{j+1}}\right)+C_j \frac{\rho E_{j+1}}{E_j G_j}\right] \cdot P_{i,j+2} +$$

$$+\left[-B_{j+2}\frac{F_j}{F_{j+1}G_{j+1}}+ C_{j+1}\left(1-\frac{E_{j+1}}{F_j G_j}+\frac{\rho F_j}{E_{j+1}G_{j+1}}\right)+D_j \frac{\rho E_{j+1}}{E_j G_j}\right] \cdot P_{i,j+3} +$$

$$+\left[- C_{j+2}\frac{F_j}{F_{j+1}G_{j+1}}+ D_{j+1}\left(1-\frac{E_{j+1}}{F_j G_j}+\frac{\rho F_j}{E_{j+1}G_{j+1}}\right)\right] \cdot P_{i,j+4} +$$

$$+\left[-D_{j+2}\frac{F_j}{F_{j+1}G_{j+1}}\right] \cdot P_{i,j+5} = 0 \qquad [5.70]$$

If we substitute G_j and G_{j+1} applying [5.66(G)] and also [5.66(G)] adding 1 to j, we obtain the following alternative expression,

$$\left[\frac{\tau E_{j+1}\left(\rho A_j F_j \qquad\right)}{\tau E_j E_{j+1}+(i-j-2)E_j F_j-\rho\tau F_{j-1}F_j}\right] \cdot P_{i,j} +$$

$$+\left[A_{j+1}+\frac{\tau E_{j+1}\left(\rho B_j F_j - A_{j+1}E_j\right)}{\tau E_j E_{j+1}+(i-j-2)E_j F_j-\rho\tau F_{j-1}F_j}+\frac{\tau F_j\left(\rho A_{j+1}F_{j+1} \qquad\right)}{\tau E_{j+1}E_{j+2}+(i-j-3)E_{j+1}F_{j+1}-\rho\tau F_j F_{j+1}}\right] \cdot P_{i,j+1} +$$

$$+\left[B_{j+1}+\frac{\tau E_{j+1}\left(\rho C_j F_j - B_{j+1}E_j\right)}{\tau E_j E_{j+1}+(i-j-2)E_j F_j-\rho\tau F_{j-1}F_j}+\frac{\tau F_j\left(\rho B_{j+1}F_{j+1}-A_{j+2}E_{j+1}\right)}{\tau E_{j+1}E_{j+2}+(i-j-3)E_{j+1}F_{j+1}-\rho\tau F_j F_{j+1}}\right] \cdot P_{i,j+2} +$$

$$+\left[C_{j+1}+\frac{\tau E_{j+1}\left(\rho D_j F_j - C_{j+1}E_j\right)}{\tau E_j E_{j+1}+(i-j-2)E_j F_j-\rho\tau F_{j-1}F_j}+\frac{\tau F_j\left(\rho C_{j+1}F_{j+1}-B_{j+2}E_{j+1}\right)}{\tau E_{j+1}E_{j+2}+(i-j-3)E_{j+1}F_{j+1}-\rho\tau F_j F_{j+1}}\right] \cdot P_{i,j+3} +$$

$$+\left[D_{j+1}+\frac{\tau E_{j+1}\left(\qquad - D_{j+1}E_j\right)}{\tau E_j E_{j+1}+(i-j-2)E_j F_j-\rho\tau F_{j-1}F_j}+\frac{\tau F_j\left(\rho D_{j+1}F_{j+1}-C_{j+2}E_{j+1}\right)}{\tau E_{j+1}E_{j+2}+(i-j-3)E_{j+1}F_{j+1}-\rho\tau F_j F_{j+1}}\right] \cdot P_{i,j+4} +$$

$$+\left[\frac{\tau F_j\left(\qquad - D_{j+2}E_{j+1}\right)}{\tau E_{j+1}E_{j+2}+(i-j-3)E_{j+1}F_{j+1}-\rho\tau F_j F_{j+1}}\right] \cdot P_{i,j+5} = 0$$

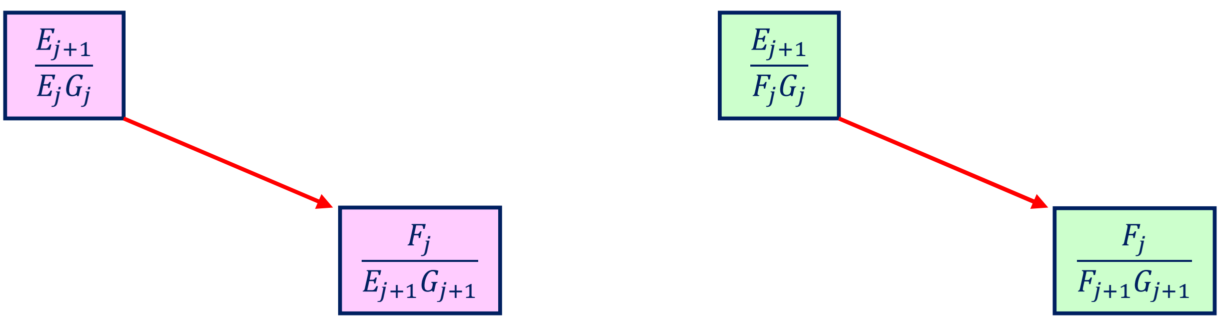
[5.71]

… looks formidable!!!

In the next page we have constructed a table were some symmetric details of coefficients in formula [5.70] are explicitly shown. We excuse ourselves from more considerations because such symmetries are quite obvious.

TABLE 3

[........ LEFT: (+) ρ					CENTRAL: (+)			RIGHT: (-) 1 ....]						.	$P_{i,j+k}$
A_j	$\frac{\rho E_{j+1}}{E_j G_j}$													.	$P_{i,j}$
B_j	$\frac{\rho E_{j+1}}{E_j G_j}$	+	A_{j+1}	$\frac{\rho F_j}{E_{j+1} G_{j+1}}$	+	A_{j+1}	1	+	$-A_{j+1}$	$\frac{E_{j+1}}{F_j G_j}$				.	$P_{i,j+1}$
C_j	$\frac{\rho E_{j+1}}{E_j G_j}$	+	B_{j+1}	$\frac{\rho F_j}{E_{j+1} G_{j+1}}$	+	B_{j+1}	1	+	$-B_{j+1}$	$\frac{E_{j+1}}{F_j G_j}$	+	$-A_{j+2}$	$\frac{F_j}{F_{j+1} G_{j+1}}$	.	$P_{i,j+2}$
D_j	$\frac{\rho E_{j+1}}{E_j G_j}$	+	C_{j+1}	$\frac{\rho F_j}{E_{j+1} G_{j+1}}$	+	C_{j+1}	1	+	$-C_{j+1}$	$\frac{E_{j+1}}{F_j G_j}$	+	$-B_{j+2}$	$\frac{F_j}{F_{j+1} G_{j+1}}$	.	$P_{i,j+3}$
			D_{j+1}	$\frac{\rho F_j}{E_{j+1} G_{j+1}}$	+	D_{j+1}	1	+	$-D_{j+1}$	$\frac{E_{j+1}}{F_j G_j}$	+	$-C_{j+2}$	$\frac{F_j}{F_{j+1} G_{j+1}}$	.	$P_{i,j+4}$
												$-D_{j+2}$	$\frac{F_j}{F_{j+1} G_{j+1}}$	.	$P_{i,j+5}$



Mnemonic remark:
In the numerators E change to F and j to $(j-1)$, and in the denominators j change to $(j+1)$.

4. A mathematical point of view: Constructing a suitable M/M/1 Queue Recurrence Formulary.

4.1. Defining (i, j) neighbourhood and solving the chart for functions $R_{i,j}$

Till now we deduced recurrence formulas for $R_{i,j}(\tau)$ and $P_{i,j}(\tau)$ functions where only index j changes, remaining i constant. This was justified by the fact that, in a particular transient regime, this is what happens. However a function depending on such two entire indexes, like are $R_{i,j}(\tau)$ and $P_{i,j}(\tau)$, is a mathematical function by its own and must be, as far as possible, analysed extensively.
Let define, for easiness and better "visualization" in the following exposition, a two dimensional discrete referential diagram where each ordered pair $(i \pm \Delta i, j \pm \Delta j)$ occupies a square being (i, j) the origin of such coordinate system, and let pay attention to the following two corresponding charts.

TABLE 4

Plane chart (i , j) for R$_{i,j}$ Functions								
	0,6							
	0,5	***1,5***						
	0,4	***1,4***	***2,4***					
	0,3	***1,3***	***2,3***	***3,3***				
↑	***0,2***	***1,2***	***2,2***	***3,2***	***4,2***			
Δj	***0,1***	***1,1***	***2,1***	***3,1***	***4,1***	***5,1***		
Δj=0	***0,0***	***1,0***	***2,0***	***3,0***	***4,0***	***5,0***	***6,0***	
(Δi , Δj)	***Δi=0***	***Δi →***						

TABLE 5

Plane chart (i , j) for Q$_{i,j}$ and P$_{i,j}$ Functions								
	0,6							
	0,5	***1,5***						
	0,4	***1,4***	***2,4***					
	0,3	***1,3***	***2,3***	***3,3***				
↑	***0,2***	***1,2***	***2,2***	***3,2***	***4,2***			
Δj	***0,1***	***1,1***	***2,1***	***3,1***	***4,1***	***5,1***		
Δj=0	***0,0***	***1,0***	***2,0***	***3,0***	***4,0***	***5,0***	***6,0***	
(Δi , Δj)	***Δi=0***	***Δi →***						

Blue columns represents the location and orientation of the recurrences deduced so far: in chart for $R_{i,j}(\tau)$ the four terms recurrence at [5.46] and in chart for $Q_{i,j}(\tau)$ $P_{i,j}(\tau)$ the six terms recurrence at [5.70] or [5.71].
Before complete the deduction of one particular recurrence one does not known by how many terms it will be composed.
The starting point are the recurrences for $R_{i,j}(\tau)$. Therefore let now explore the other three directions:

… horizontal, where recurrence is like $F\left[R_{i,j}(\tau), R_{i+1,j}(\tau), R_{i+2,j}(\tau), \dots\right] = 0$

… diagonal positive, where recurrence is like $F\left[R_{i,j}(\tau), R_{i+1,j+1}(\tau), R_{i+2,j+2}(\tau), \dots\right] = 0$

… diagonal negative, where recurrence is like $F\left[R_{i,j}(\tau), R_{i-1,j+1}(\tau), R_{i-2,j+2}(\tau), \dots\right] = 0$

Subsequently we will transform each $R_{i,j}(\tau)$ recurrence in the corresponding $Q_{i,j}(\tau)$, or $P_{i,j}(\tau)$, ones.

First nomenclature remark - The foregoing nomenclature, classifies the recurrences by their geometric position in the described charts: "vertical", "horizontal", "diagonal positive" and "diagonal negative".
This nomenclature, and also additional designations, that will be done later in appropriate turn, will be adopted in all the sequel for systematization purposes.

Let now to retake the deductions.
The $R_{i,j}(\tau)$ negative diagonal choice is the easiest. In fact, attending to [5.32],

$$R_{i,j} = \rho^j W^*_{i+j+1} + (1-\rho)\,\rho^j \sum_{k=i+1}^{\infty} W^*_{k+j+1} \qquad [5.32]$$

... subtracting 1 to i and adding 1 to j,

$$R_{i-1,j+1} = \rho^{j+1} W^*_{i+j+1} + (1-\rho)\,\rho^{j+1} \sum_{k=i}^{\infty} W^*_{k+j+2} \qquad [5.72]$$

Now multiplying [5.32] by j and comparing with [5.72] the desired result is immediate,

$$\rho \cdot R_{i,j} = R_{i-1,j+1} \qquad [5.73]$$

The simplicity of [5.73] can be profited straightaway to achieve the horizontal recurrence.
One only needs to apply [5.73],

... to $R_{i+1,j}$ one time,

$$\rho \cdot R_{i+1,j} = R_{i,j+1} \qquad [5.74(1)]$$

... to $R_{i+2,j}$ two times,

$$\rho^2 \cdot R_{i+2,j} = R_{i,j+2} \qquad [5.74(2)]$$

... and to $R_{i+3,j}$ three times,

$$\rho^3 \cdot R_{i+3,j} = R_{i,j+3} \qquad [5.74(3)]$$

Lastly recall the vertical recurrence already obtained, and given at [5.46],

$$\left\{\frac{\rho^2\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i,j} + \left\{-\rho - \frac{\rho^2\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i,j+1} +$$
$$+\left\{1 - \frac{\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i,j+2} + \left\{\frac{\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i,j+3} = 0 \qquad [5.46]$$

Inserting [5.74(1)], [5.74(2)] and [5.74(3)] in [5.46] and dividing by ρ^2 the wanted horizontal recurrence can be immediately written,

$$\left\{\frac{\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i,j}+\left\{-1-\frac{\rho\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i+1,j}+$$
$$+\left\{1-\frac{\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i+2,j}+\left\{\frac{\rho\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i+3,j}=0 \qquad [5.75]$$

At this stage only the diagonal positive recurrence is missing. In this case, a direct deduction is needed because relation [5.73] reveals useless as an auxiliary tool for short circuit the reasoning.
First multiply $R_{i,j}$ at [5.32] by ρ, and rearrange like this,

$$\rho\cdot R_{i,j}=\rho^{j+1}\cdot W^*_{i+j+1}+(1-\rho)\cdot\rho^{j+1}\cdot W^*_{i+j+2}+(1-\rho)\cdot\rho^{j+1}\cdot W^*_{i+j+3}+$$
$$+(1-\rho)\cdot\rho^{j+1}\cdot\sum_{k=i+3}^{\infty}W^*_{k+j+1} \qquad [5.76]$$

Second add 1 to i and also 1 to j at [5.32]. One obtains,

$$R_{i+1,j+1}=\rho^{j+1}\cdot W^*_{i+j+3}+(1-\rho)\cdot\rho^{j+1}\cdot\sum_{k=i+2}^{\infty}W^*_{k+j+2} \qquad [5.77]$$

It is clear that the two terms with summations at [5.76] and [5.77] are equals and, consequently, subtracting [5.77] from [5.76] conducts to,

$$\rho\cdot R_{i,j}-R_{i+1,j+1}=\rho^{j+1}\cdot W^*_{i+j+1}+(1-\rho)\cdot\rho^{j+1}\cdot W^*_{i+j+2}-\rho^{j+2}\cdot W^*_{i+j+3}=$$
$$=(1-\rho)\cdot\rho^{j+1}\cdot W^*_{i+j+2}-\left[\rho\cdot W^*_{i+j+3}-W^*_{i+j+1}\right]\cdot\rho^{j+1} \qquad [5.78]$$

Once again formula [5.41] came in to play,

$$\rho\cdot W^*_{n+1}=W^*_{n-1}-\frac{n}{\tau}\cdot W^*_n \quad \ldots\text{ or,}\quad \rho\cdot W^*_{n+1}-W^*_{n-1}=-\frac{n}{\tau}\cdot W^*_n \qquad [5.41]$$

... and making $n=i+j+2$,

$$\rho\cdot W^*_{i+j+3}-W^*_{i+j+1}=-\frac{(i+j+2)}{\tau}\cdot W^*_{i+j+2} \qquad [5.79]$$

Obviously the right hand side of [5.78] can now depend only on function W^*_{i+j+2} .
We get straightaway,

$$\rho\cdot R_{i,j}-R_{i+1,j+1}=\frac{\rho^{j+1}}{\tau}-[(1-\rho)\cdot\tau+(i+j+2)]\cdot W^*_{i+j+2} \qquad [5.80(A-2)]$$

... and with W^*_{i+j+2} explicit,

$$W^*_{i+j+2}=\frac{\tau\cdot\left[\rho\cdot R_{i,j}-R_{i+1,j+1}\right]}{\rho^{j+1}\cdot[(1-\rho)\cdot\tau+(i+j+2)]} \qquad [5.80(B-2)]$$

Follows to add 1 to i, and also 1 to j, at [5.78],

$$\rho\cdot R_{i+1,j+1}-R_{i+2,j+2}=\rho^{j+2}\cdot W^*_{i+j+3}+(1-\rho)\cdot\rho^{j+2}\cdot W^*_{i+j+4}-\rho^{j+3}\cdot W^*_{i+j+5} \qquad [5.81]$$

We need now to eliminate functions $\cdot W^*_{i+j+k}$ where k is odd. Therefore we must eliminate from [5.81] the functions W^*_{i+j+3} and W^*_{i+j+5}.
To achieve such goal let to add 1 to the sun $(i+j)$,

$$\rho \cdot W^*_{i+j+4} - W^*_{i+j+2} = -\frac{(i+j+3)}{\tau} \cdot W^*_{i+j+3} \qquad [5.82(3)]$$

… and also 3 to the same sun $(i+j)$,

$$\rho \cdot W^*_{i+j+6} - W^*_{i+j+4} = -\frac{(i+j+5)}{\tau} \cdot W^*_{i+j+5} \qquad [5.82(5)]$$

At [5.82(3)] and [5.82(5)] W^*_{i+j+3} and W^*_{i+j+5} can be, respectively, explicit and inserted at [5.81]. The result, already rearranged, reads,

$$\rho \cdot R_{i+1,j+1} - R_{i+2,j+2} = \frac{\rho^{j+2} \cdot \tau}{(i+j+3)} \cdot W^*_{i+j+2} + \\ +\rho^{j+2} \cdot \left\{(1-\rho) - \rho \cdot \tau \cdot \left[\frac{1}{(i+j+3)} + \frac{1}{(i+j+5)}\right]\right\} \cdot W^*_{i+j+4} + \\ + \frac{\rho^{j+4} \cdot \tau}{(i+j+5)} \cdot W^*_{i+j+6} \qquad [5.83]$$

Using again relation [5.80(B − 2)] suitable expressions for W^*_{i+j+4} and $\cdot W^*_{i+j+6}$ are easily set up. Adding 1 to i and also 1 to j now W^*_{i+j+4} reads,

$$W^*_{i+j+4} = \frac{\tau \cdot [\rho \cdot R_{i+1,j+1} - R_{i+2,j+2}]}{\rho^{j+2} \cdot [(1-\rho) \cdot \tau + (i+j+4)]} \qquad [5.80(\mathrm{B}-4)]$$

… and in is turn, adding 2 to i and also 2 to j, we get for W^*_{i+j+6},

$$W^*_{i+j+6} = \frac{\tau \cdot [\rho \cdot R_{i+2,j+2} - R_{i+3,j+3}]}{\rho^{j+3} \cdot [(1-\rho) \cdot \tau + (i+j+6)]} \qquad [5.80(\mathrm{B}-6)]$$

These three equations, [5.80(B − 2)], [5.80(B − 4)] and [5.80(B − 6)], are inserted in [5.83] and therefore all functions $W^*_{i+j+\mathrm{k}}$ with $(k = 2, 4\ and\ 6)$ are eliminated from this last one.
The result is,

$$\rho \cdot R_{i+1,j+1} - R_{i+2,j+2} = \frac{\rho \cdot \tau^2 \cdot (\rho \cdot R_{i,j} - R_{i+1,j+1})}{(i+j+3) \cdot [(1-\rho) \cdot \tau + (i+j+2)]} + \\ +\rho \cdot \tau^2 \cdot \left\{\frac{(1-\rho)}{\rho \cdot \tau} - \left[\frac{1}{(i+j+3)} + \frac{1}{(i+j+5)}\right]\right\} \cdot \frac{(\rho \cdot R_{i+1,j+1} - R_{i+2,j+2})}{[(1-\rho) \cdot \tau + (i+j+4)]} + \\ + \frac{\rho \cdot \tau^2 \cdot (\rho \cdot R_{i+2,j+2} - R_{i+3,j+3})}{(i+j+5) \cdot [(1-\rho) \cdot \tau + (i+j+6)]} \qquad [5.84]$$

Lastly, collecting the terms with the same common function $R_{i+k,j+k}$ with $(k = 0, 1, 2, and\ 3)$ we get the desired recurrence, observing that has four terms,

$$\left\{\frac{\rho\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}\right\}\cdot R_{i,j}+$$

$$+\left\{-\frac{\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}-\frac{\rho\cdot\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}\right\}\cdot R_{i+1,j+1}+$$

$$+\left\{\frac{\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}+\frac{\rho\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i+2,j+2}+$$

$$+\left\{-\frac{\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i+3,j+3}=0 \qquad [5.85]$$

The return to the plane chart (i,j) for $R_{i,j}$ functions allows us to make the inventory of formulary achieved till now.

TABLE 6

Plane chart (i , j) for $R_{i,j}$ Functions

	0,6							
	0,5	1,5						
	0,4	1,4	2,4					
	0,3	1,3	2,3	3,3				
	0,2	1,2	2,2	3,2	4,2			
-1,1 ↑	0,1	1,1	2,1	3,1	4,1	5,1		
↑	0,0	1,0	2,0	3,0	4,0	5,0	6,0	
Δj Δi →								

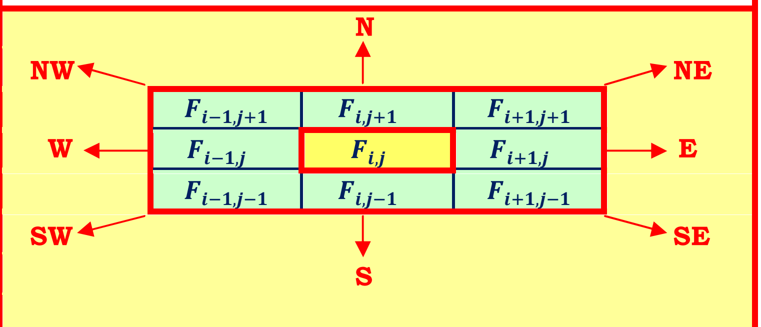
LEGEND - Contiguous straight line homogeneous recurrences for *$R_{i,j}$ Functions*

Orientation	Number of terms	Equation
Vertical	4	$F(R_{i,j},R_{i,j+1},R_{i,j+2},R_{i,j+3})=0$ [5.46]
Horizontal	4	$F(R_{i,j},R_{i+1,j},R_{i+2,j},R_{i+3,j})=0$ [5.75]
Diagonal positive	4	$F(R_{i,j},R_{i+1,j+1},R_{i+2,j+2},R_{i+3,j+3})$ [5.85]
Diagonal negative	2	$F(R_{i,j},R_{i-1,j+1})=0$ [5.73]

Second nomenclature remark - For systematization purpose the following definitions, being useful and intuitive, must be adopted.
Next neighbours functions of a general function $F_{i,j}\,(R_{i,j}, Q_{i,j}\ or\ P_{i,j})$ are the eight functions described at this table:

Definition	Designation
$F_{i,j+1}$	North next neighbour
$F_{i+1,j}$	East next neighbour
$F_{i,j-1}$	South next neighbour
$F_{i-1,j}$	West next neighbour
$F_{i+1,j+1}$	North-east next neighbour
$F_{i+1,j-1}$	South-east next neighbour
$F_{i-1,j+1}$	North-west next neighbour
$F_{i-1,j-1}$	South-west next neighbour

Geometric localization



A contiguous recurrence is composed by a set of (n) functions $\boldsymbol{F_{i,j}}\left(\boldsymbol{R_{i,j}}, \boldsymbol{Q_{i,j}}\ \boldsymbol{or}\ \boldsymbol{P_{i,j}}\right)$ in which every function is next neighbour of, at least one of the other $(n-1)$ functions $\boldsymbol{F_{i,j}}\left(\boldsymbol{R_{i,j}}, \boldsymbol{Q_{i,j}}\ \boldsymbol{or}\ \boldsymbol{P_{i,j}}\right)$.

A straight line contiguous recurrence is a contiguous recurrence located entirely in one of the four aforesaid and defined directions: vertical, horizontal, diagonal positive or diagonal negative.

So far we have solved the chart for functions $\boldsymbol{R_{i,j}}$ in all the four defined recurrences, confining ourselves to the straight line contiguous recurrence.

4.2. Two more easily obtained recurrences in the chart for functions $R_{i,j}$

For sake of our complete work advantage an observation is now due about the easiness in getting two more recurrences for functions $\boldsymbol{R_{i,j}}$.
The reach of diagonal positive four terms recurrence [5.85], and the simplicity of diagonal negative two terms recurrence [5.73] puts at hand the possibility of obtain immediately two more straight four terms recurrences (but, this time, not contiguous): $\boldsymbol{F}\left(\boldsymbol{R_{i,j}}, \boldsymbol{R_{i,j+2}}, \boldsymbol{R_{i,j+4}}, \boldsymbol{R_{i,j+6}}\right) = \boldsymbol{0}$ … and $\boldsymbol{F}\left(\boldsymbol{R_{i,j}}, \boldsymbol{R_{i+2,j}}, \boldsymbol{R_{i+4,j}}, \boldsymbol{R_{i+6,j}}\right) = \boldsymbol{0}$.
First let generalize recurrence [5.73] establishing a relation between $R_{i,j}$ and $R_{i-n,j+n}$ and also between $R_{i,j}$ and $R_{i+n,j-n}$ where n can be as big as we want: $(n = 1, 2, 3, \dots)$.
The relation between $R_{i,j}$ and $R_{i-n,j+n}$ links directly function $R_{i,j}$ to function $R_{i-n,j+n}$, located in the same negative diagonal upwards, as far as we want, and reads,

$$R_{i,j} = \rho^{-n} \cdot R_{i-n,j+n} \qquad [5.86(0)]$$

In turn the relation between $R_{i,j}$ and $R_{i+n,j-n}$ links directly function $R_{i,j}$ to function $R_{i+n,j-n}$ located in the same negative diagonal downwards, as far as we want, and reads,

$$R_{i,j} = \rho^{n} \cdot R_{i+n,j-n} \qquad [5.87(0)]$$

If at [5.86(0)] we add …

… 1 to i and also 1 to j, and make $n = 1$,
… 2 to i and also 2 to j, and make $n = 2$,
… 3 to i and also 3 to j, and make $n = 3$,

… we obtain, respectively,

$$R_{i+1,j+1} = \rho^{-1} \cdot R_{i,j+2} \qquad [5.86(1)]$$
$$R_{i+2,j+2} = \rho^{-2} \cdot R_{i,j+4} \qquad [5.86(2)]$$
$$R_{i+3,j+3} = \rho^{-3} \cdot R_{i,j+6} \qquad [5.86(3)]$$

In the same fashion if at [5.87(0)] we add …

… 1 to i and also 1 to j, and make $n = 1$,
… 2 to i and also 2 to j, and make $n = 2$,
… 3 to i and also 3 to j, and make $n = 3$,

… we obtain, this time, respectively,

$$R_{i+1,j+1} = \rho^{1} \cdot R_{i+2,j} \qquad [5.87(1)]$$
$$R_{i+2,j+2} = \rho^{2} \cdot R_{i+4,j} \qquad [5.87(2)]$$
$$R_{i+3,j+3} = \rho^{3} \cdot R_{i+6,j} \qquad [5.87(3)]$$

Inserting [5.86(1)], [5.86(2)] and [5.86(3)] in [5.85] the recurrence $\boldsymbol{F(R_{i,j},R_{i,j+2},R_{i,j+4},R_{i,j+6})=0}$ is easily achieved and reads,

$$\left\{\frac{\rho^4\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}\right\}\cdot R_{i,j}+$$

$$+\left\{-\frac{\rho^2\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}-\frac{\rho^3\cdot\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}\right\}\cdot R_{i,j+2}+$$

$$+\left\{\frac{\rho\cdot\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}+\frac{\rho^2\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i,j+4}+$$

$$+\left\{-\frac{\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i,j+6}=0 \qquad [5.88]$$

Similarly inserting [5.87(1)], [5.87(2)] and [5.87(3)] in [5.85] the recurrence $\boldsymbol{F(R_{i,j},R_{i+2,j},R_{i+4,j},R_{i+6,j})=0}$ is also easily achieved and reads,

$$\left\{\frac{\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}\right\}\cdot R_{i,j}+$$

$$+\left\{-\frac{\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}-\frac{\rho\cdot\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}\right\}\cdot R_{i+2,j}+$$

$$+\left\{\frac{\rho\cdot\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}+\frac{\rho^2\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i+4,j}+$$

$$+\left\{-\frac{\rho^2\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i+6,j}=0 \qquad [5.89]$$

What has been done in plane chart *(i , j)* for $R_{i,j}$ functions, reaching [5.88] and [5.89], can be visualized straightaway,

TABLE 7

Plane chart (i , j) for $R_{i,j}$ Functions								
	0,6							
	0,5	1,5						
	0,4	1,4	2,4					
	0,3	1,3	2,3	3,3				
	0,2	1,2	2,2	3,2	4,2			
-1,1 ↑	0,1	1,1	2,1	3,1	4,1	5,1		
↑	0,0	1,0	2,0	3,0	4,0	5,0	6,0	
Δj Δi →								

Of course, applying this method, one can also deduce so many four terms recurrences as wanted if each one of their corresponding four terms is located in one negative diagonal, being

alternate diagonals. In this table the alternate diagonal can be designed by the oriented pairs coordinate labels,
Diagonal $[\dots(0,2),(1,1),(2,0),\dots]$ then diagonal $[\dots(0,4),(1,3),(2,2),(3,1),(4,0)\dots]$ and lastly diagonal $[\dots(0,6),(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\ \dots]$.

Remark:
The complete upwards ordination of the diagonals is,
Diagonal $[\dots(-1,2),(0,1),(1.0),(2,-1)\dots]$
Diagonal $[\dots(0,2),(1,1),(2,0),\dots]\ (*)$
Diagonal $[\dots(0,3),(1,2),(2,1),(3,0)\dots]$
Diagonal $[\dots(0,4),(1,3),(2,2),(3,1),(4,0)\dots]\ (*)$
Diagonal $[\dots(0,5),(1,4),(2,3),(3,2),(4,1),(5,0),\dots]$
Diagonal $[\dots(0,6),(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\ \dots]\ (*)$

… and only at those with asterisk lies the location of the terms at [5.88] and [5.89].

Consequently, for the moment, we only could use half of all the plane chart for $R_{\,i\,,j}$ functions if we wanted deduce more recurrences.
There is no point in trying now to fulfil all the chart, "travelling recurrently" all over that plane, because we will return to this detail later, describing the mathematical method for achieve such goal.

4.3. Defining new suitable parameter notations to pursue

Let now, in logic sequel, consider the plane chart (i,j) for $Q_{i,j}$ and $P_{i,j}$ functions.
The subsequent task is to deduce the three homogeneous and contiguous straight line recurrences, for $Q_{i,j}$ and $P_{i,j}$, in the three directions not yet deduced.
Recalling that the three not yet explored directions, for recurrence deductions purpose, involving the functions $Q_{i,j}$ or, what is the same, the functions $P_{i,j}$ are,

… horizontal, where recurrence is like $F\left[Q_{i,j}(\tau),Q_{i+1,j}(\tau),Q_{i+2,j}(\tau),\dots\right]=0$ or like $F\left[P_{i,j}(\tau),P_{i+1,j}(\tau),P_{i+2,j}(\tau),\dots\right]=0$

… diagonal positive, where recurrence is like $F\left[Q_{i,j}(\tau),Q_{i+1,j+1}(\tau),Q_{i+2,j+2}(\tau),\dots\right]=0$ or like $F\left[P_{i,j}(\tau),P_{i+1,j+1}(\tau),P_{i+2,j+2}(\tau),\dots\right]=0$

… diagonal negative, where recurrence is like $F\left[Q_{i,j}(\tau),Q_{i-1,j+1}(\tau),Q_{i-2,j+2}(\tau),\dots\right]=0$ or like $F\left[P_{i,j}(\tau),P_{i-1,j+1}(\tau),P_{i-2,j+2}(\tau),\dots\right]=0$

… we will start, like before, by the recurrence at the negative diagonal direction.
And, also like before, for easiness of exposition and economy in mathematical notation, we must first reorganize the parameter definitions. The deduction of the recurrence at the negative diagonal direction will be done only after this.
Reviewing definitions already done, before the deduction of six terms recurrence at [5.70] or [5.71], we have, so far, the following ones,

$$\eta_k=\frac{\tau}{[(1-\rho)\tau+(i+k)]} \qquad [5.47]$$

$$A_j{=}\rho^2\eta_{j+2} \qquad [5.48(A)]$$

$$B_j{=}-\rho(1+\rho\eta_{j+3}) \qquad [5.48(B)]$$

$$C_j{=}1-\eta_{j+2} \qquad [5.48(C)]$$

$$D_j{=}\eta_{j+3} \qquad [5.48(D)]$$

The goal, before deduce the six terms recurrence, was to be able to write down the recurrence $F(R_{i,j}, R_{i,j+1}, R_{i,j+2}, R_{i,j+3}) = 0$, [5.46], in the more simplified form expressed at [5.49],

$$A_j \cdot R_{i,j} + B_j \cdot R_{i,j+1} + C_j \cdot R_{i,j+2} + D_j \cdot R_{i,j+3} = 0 \qquad [5.49]$$

This procedure must now be extended to the complete collection of recurrence pairs, meaning here that each such pair is composed by the recurrence in plane chart (i,j) for $R_{i,j}$ functions and the correspondent one, disposed in the same direction, in plane chart (i,j) for $Q_{i,j}$ and $P_{i,j}$ functions.

So far we have only solved the vertical pair, having obtained recurrences [5.46] and [5.70] (or [5.71]).

We have three more pairs to solve and not so many symbols or letters not still used to put into play from now on.

Consequently, is wiser to conserve the letters significance at parameters definitions [5.47] [5.48(A)], [5.48(B)], [5.48(C)] and [5.48(D)] than to use more letters, avoiding this way a probable shortage of it. Judicious choice of notation is also important to achieve, at the end, a coherent and general set of homogeneous and contiguous straight line recurrences, in all the aforesaid four directions.

Besides the new notation for A_j, B_j, C_j, and D_j, we also add now a new parameter $\sigma_k^{[+]}$ that substitutes parameter η_k which, in turn, will no longer be needed.

Therefore, in the sequel, all the set of definitions [5.47], [5.48(A)], [5.48(B)], [5.48(C)] and [5.48(D)] is substituted by the set of new definitions [5.90], [5.91(A)], [5.91(B)], [5.91(C)] and [5.91(D)], according to,

$$\sigma_k^{[+]} = (i + j + k) \qquad [5.90]$$

$$A_j^{[\uparrow]} = \frac{\rho^2 \cdot \tau}{\left[(1-\rho)\tau + \sigma_2^{[+]}\right]} \qquad [5.91(A)]$$

$$B_j^{[\uparrow]} = -\rho \cdot \left\{1 + \frac{\rho \cdot \tau}{\left[(1-\rho)\tau + \sigma_3^{[+]}\right]}\right\} \qquad [5.91(B)]$$

$$C_j^{[\uparrow]} = 1 - \frac{\tau}{\left[(1-\rho)\tau + \sigma_2^{[+]}\right]} \qquad [5.91(C)]$$

$$D_j^{[\uparrow]} = \frac{\tau}{\left[(1-\rho)\tau + \sigma_3^{[+]}\right]} \qquad [5.91(D)]$$

Remarks:

1) The arrows means that the context relates to vertical recurrences in the charts (i,j).

2) The former parameter,

$$\eta_k = \frac{\tau}{[(1-\rho)\tau + (i+k)]} \qquad [5.47]$$

… has been replaced by,

$$\sigma_k^{[+]} = (i + j + k) \qquad [5.90]$$

they relate one to each other by,

$$\eta_k = \frac{\tau}{\left[(1-\rho)\tau + \left(\sigma_k^{[+]} - j\right)\right]} \qquad [5.92]$$

... so we can conclude that $\sigma_k^{[+]}$ is "inside" η_k and, consequently, we are apparently, not economizing and simplifying the notation.
However now the goal is to construct a notation embracing coherently all the homogeneous and contiguous straight line recurrences in plane charts (i, j) concerning the functions $R_{i,j}$ and $Q_{i,j}$ (or $P_{i,j}$) along the four directions and, as we shall see, in diagonal recurrences the final result don´t allows include all parameters alike to $\sigma_k^{[+]}$ in parameters alike to η_k. In other words, in such diagonal recurrences parameters alike to $\sigma_k^{[+]}$ have their "own life".
3) The parameter $\sigma_k^{[+]}$ has, instead of an arrow, a plus signal (+) because their corresponding explicit expression, $\sigma_k^{[+]} = (i + j + k)$ will appear all over the whole recurrence machinery deductions, indistinctly at all and each of the four directions contexts.
4) The following rules must be applied,
Whenever i and (or) j, respectively, changes to $i + m$ and (or) $j + n$ the parameter $\sigma_k^{[+]}$ must, in his turn, change respectively to $\sigma_{k+m}^{[+]}$ and (or) $\sigma_{k+n}^{[+]}$. And, in the same way, whenever i and (or) j, respectively, changes to $i - m$ and (or) $j - n$ the parameter $\sigma_k^{[+]}$ must, in his turn, change respectively to $\sigma_{k-m}^{[+]}$ and (or) $\sigma_{k-n}^{[+]}$.
5) An homologue parameter of $\sigma_k^{[+]}$, but with minus signal $(-)$ must be defined right now attending to his utility for rationalization and economy of the notation purpose,

$$\sigma_k^{[-]} = (i - j + k) \qquad [5.93]$$

6) Like in remark 4, the correspondent rules for $\sigma_k^{[-]}$ must be applied,
Whenever i and (or) j, respectively, changes to $i + m$ and (or) $j + n$ the parameter $\sigma_k^{[-]}$ must, in his turn, change respectively to $\sigma_{k+m}^{[+]}$ and (or) $\sigma_{k-n}^{[+]}$. And, in the same way, whenever i and (or) j, respectively, changes to $i - m$ and (or) $j - n$ the parameter $\sigma_k^{[-]}$ must, in his turn, change respectively to $\sigma_{k-m}^{[+]}$ and (or) $\sigma_{k+n}^{[+]}$.

Along the deduction of vertical oriented six terms recurrence an auxiliary function $S_{i,j}(\tau)$ and more three parameters, E_j , F_j and G_j came into play, and defined respectively at [5.60], [5.63(E)], [5.63(F)] and [5.66(G)].
Their definitions don´t change except for the detail concerning the inclusion of the arrow, for disentangling purposes relative to similar functions and parameters we will find, later on, in following recurrences deductions.
The change of η_k by $\sigma_k^{[+]}$ don´t affect the notations of $S_{i,j}(\tau)$, E_j , F_j and G_j because has already been taken in account in the definitions of $A_j^{[\uparrow]}$, $B_j^{[\uparrow]}$, $C_j^{[\uparrow]}$ and $D_j^{[\uparrow]}$.
The new notations of $S_{i,j}(\tau)$, E_j , F_j and G_j, reads,

$$S_{i,j}^{[\uparrow]} = A_j^{[\uparrow]} \cdot Q_{i,j} + B_j^{[\uparrow]} \cdot Q_{i,j+1} + C_j^{[\uparrow]} \cdot Q_{i,j+2} + D_j^{[\uparrow]} \cdot Q_{i,j+3} \qquad [5.94]$$

$$E_j^{[\uparrow]} = \left[-\frac{\sigma_1^{[-]} A_j^{[\uparrow]}}{\rho\tau} + B_j^{[\uparrow]} + \rho D_j^{[\uparrow]} \right] \qquad [5.95(E)]$$

$$F_j^{[\uparrow]} = \left[\frac{A_{j+1}^{[\uparrow]}}{\rho} + C_{j+1}^{[\uparrow]} + \frac{\sigma_3^{[-]} D_{j+1}^{[\uparrow]}}{\tau} \right] \qquad [5.95(F)]$$

$$G_j^{[\uparrow]} = \left[\frac{E_{j+1}^{[\uparrow]}}{F_j^{[\uparrow]}} - \rho \, \frac{F_{j-1}^{[\uparrow]}}{E_j^{[\uparrow]}} + \frac{\sigma_2^{[-]}}{\tau} \right] \qquad [5.96(G)]$$

4.4. Application of the new notation to the vertical four and six terms recurrences

Applying this new notation to recurrences [5.46] and [5.70], we get,

$$\left\{\frac{\rho^2\tau}{[(1-\rho)\tau+\sigma_2^{[+]}]}\right\}\cdot R_{i,j}+\left\{-\rho-\frac{\rho^2\tau}{[(1-\rho)\tau+\sigma_3^{[+]}]}\right\}\cdot R_{i,j+1}+$$
$$+\left\{1-\frac{\tau}{[(1-\rho)\tau+\sigma_2^{[+]}]}\right\}\cdot R_{i,j+2}+\left\{\frac{\tau}{[(1-\rho)\tau+\sigma_3^{[+]}]}\right\}\cdot R_{i,j+3}=0 \qquad [5.97]$$

$$\left[A_j^{[\uparrow]}\cdot\frac{\rho\cdot E_{j+1}^{[\uparrow]}}{E_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}\right]\cdot P_{i,j}+$$
$$+\left[A_{j+1}^{[\uparrow]}\cdot\left(1-\frac{E_{j+1}^{[\uparrow]}}{F_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}+\frac{\rho\cdot F_j^{[\uparrow]}}{E_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}\right)+B_j^{[\uparrow]}\cdot\frac{\rho\cdot E_{j+1}^{[\uparrow]}}{E_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}\right]\cdot P_{i,j+1}+$$
$$+\left[-A_{j+2}^{[\uparrow]}\cdot\frac{F_j^{[\uparrow]}}{F_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}+B_{j+1}^{[\uparrow]}\cdot\left(1-\frac{E_{j+1}^{[\uparrow]}}{F_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}+\frac{\rho\cdot F_j^{[\uparrow]}}{E_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}\right)+C_j^{[\uparrow]}\cdot\frac{\rho\cdot E_{j+1}^{[\uparrow]}}{E_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}\right]\cdot P_{i,j+2}+$$
$$+\left[-B_{j+2}\cdot\frac{F_j^{[\uparrow]}}{F_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}+C_{j+1}^{[\uparrow]}\cdot\left(1-\frac{E_{j+1}^{[\uparrow]}}{F_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}+\frac{\rho\cdot F_j^{[\uparrow]}}{E_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}\right)+D_j^{[\uparrow]}\cdot\frac{\rho\cdot E_{j+1}^{[\uparrow]}}{E_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}\right]\cdot P_{i,j+3}+$$
$$+\left[-C_{j+2}\cdot\frac{F_j^{[\uparrow]}}{F_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}+D_{j+1}^{[\uparrow]}\cdot\left(1-\frac{E_{j+1}^{[\uparrow]}}{F_j^{[\uparrow]}\cdot G_j^{[\uparrow]}}+\frac{\rho\cdot F_j^{[\uparrow]}}{E_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}\right)\right]\cdot P_{i,j+4}+$$
$$+\left[-D_{j+2}^{[\uparrow]}\cdot\frac{F_j^{[\uparrow]}}{F_{j+1}^{[\uparrow]}\cdot G_{j+1}^{[\uparrow]}}\right]\cdot P_{i,j+5}=0 \qquad [5.98]$$

And applying also this new notation to recurrence [5.71], we get, in the next page,

$$\left[\frac{\tau E_{j+1}^{[\uparrow]}\left(\rho A_j^{[\uparrow]}F_j^{[\uparrow]}\qquad\qquad\right)}{\tau E_j^{[\uparrow]}E_{j+1}^{[\uparrow]}+\sigma_2^{[-]}E_j^{[\uparrow]}F_j^{[\uparrow]}-\rho\tau F_{j-1}^{[\uparrow]}F_j^{[\uparrow]}}\right]\cdot P_{i,j}+$$

$$+\left[A_{j+1}^{[\uparrow]}+\frac{\tau E_{j+1}^{[\uparrow]}\left(\rho B_j^{[\uparrow]}F_j^{[\uparrow]}-A_{j+1}^{[\uparrow]}E_j\right)}{\tau E_j^{[\uparrow]}E_{j+1}^{[\uparrow]}+\sigma_2^{[-]}E_j^{[\uparrow]}F_j^{[\uparrow]}-\rho\tau F_{j-1}^{[\uparrow]}F_j^{[\uparrow]}}+\frac{\tau F_j^{[\uparrow]}\left(\rho A_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}\qquad\qquad\right)}{\tau E_{j+1}^{[\uparrow]}E_{j+2}^{[\uparrow]}+\sigma_3^{[-]}E_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-\rho\tau F_j^{[\uparrow]}F_{j+1}^{[\uparrow]}}\right]\cdot P_{i,j+1}+$$

$$+\left[B_{j+1}^{[\uparrow]}+\frac{\tau E_{j+1}^{[\uparrow]}\left(\rho C_j^{[\uparrow]}F_j^{[\uparrow]}-B_{j+1}^{[\uparrow]}E_j\right)}{\tau E_j^{[\uparrow]}E_{j+1}^{[\uparrow]}+\sigma_2^{[-]}E_j^{[\uparrow]}F_j^{[\uparrow]}-\rho\tau F_{j-1}^{[\uparrow]}F_j^{[\uparrow]}}+\frac{\tau F_j^{[\uparrow]}\left(\rho B_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-A_{j+2}^{[\uparrow]}E_{j+1}^{[\uparrow]}\right)}{\tau E_{j+1}^{[\uparrow]}E_{j+2}^{[\uparrow]}+\sigma_3^{[-]}E_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-\rho\tau F_j^{[\uparrow]}F_{j+1}^{[\uparrow]}}\right]\cdot P_{i,j+2}+$$

$$+\left[C_{j+1}^{[\uparrow]}+\frac{\tau E_{j+1}^{[\uparrow]}\left(\rho D_j^{[\uparrow]}F_j^{[\uparrow]}-C_{j+1}^{[\uparrow]}E_j\right)}{\tau E_j^{[\uparrow]}E_{j+1}^{[\uparrow]}+\sigma_2^{[-]}E_j^{[\uparrow]}F_j^{[\uparrow]}-\rho\tau F_{j-1}^{[\uparrow]}F_j^{[\uparrow]}}+\frac{\tau F_j^{[\uparrow]}\left(\rho C_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-B_{j+2}^{[\uparrow]}E_{j+1}^{[\uparrow]}\right)}{\tau E_{j+1}^{[\uparrow]}E_{j+2}^{[\uparrow]}+\sigma_3^{[-]}E_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-\rho\tau F_j^{[\uparrow]}F_{j+1}^{[\uparrow]}}\right]\cdot P_{i,j+3}+$$

$$+\left[D_{j+1}^{[\uparrow]}+\frac{\tau E_{j+1}^{[\uparrow]}\left(\qquad-D_{j+1}^{[\uparrow]}E_j\right)}{\tau E_j^{[\uparrow]}E_{j+1}^{[\uparrow]}+\sigma_2^{[-]}E_j^{[\uparrow]}F_j^{[\uparrow]}-\rho\tau F_{j-1}^{[\uparrow]}F_j^{[\uparrow]}}+\frac{\tau F_j^{[\uparrow]}\left(\rho D_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-C_{j+2}^{[\uparrow]}E_{j+1}^{[\uparrow]}\right)}{\tau E_{j+1}^{[\uparrow]}E_{j+2}^{[\uparrow]}+\sigma_3^{[-]}E_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-\rho\tau F_j^{[\uparrow]}F_{j+1}^{[\uparrow]}}\right]\cdot P_{i,j+4}+$$

$$+\left[\frac{\tau F_j^{[\uparrow]}\left(\qquad\qquad-D_{j+2}^{[\uparrow]}E_{j+1}^{[\uparrow]}\right)}{\tau E_{j+1}^{[\uparrow]}E_{j+2}^{[\uparrow]}+\sigma_3^{[-]}E_{j+1}^{[\uparrow]}F_{j+1}^{[\uparrow]}-\rho\tau F_j^{[\uparrow]}F_{j+1}^{[\uparrow]}}\right]\cdot P_{i,j+5}=0 \qquad [5.99]$$

4.5. Deduction of the negative diagonal recurrence for functions $Q_{i,j}$ and $P_{i,j}$

As before mentioned we will start now the deduction of the homogeneous and contiguous straight line recurrence for functions $Q_{i,j}$, or, what is the same, functions $P_{i,j}$ located at the negative diagonal direction.
The starting point is recurrence [5.73],

$$\rho\cdot R_{i,j} = R_{i-1,j+1} \qquad [5.73]$$

… and also relation [5.34],

$$Q_{i,j} = W_{i-j}^{*} + R_{i,j} \qquad [5.34]$$

Subtracting 1 to i and also adding 1 to j in [5.34],

$$Q_{i-1,j+1} = W_{i-j-2}^{*} + R_{i-1,j+1} \qquad [5.100]$$

… and getting explicit $R_{i,j}$ from [5.34] we reach,

$$R_{i,j} = Q_{i,j} - W_{i-j}^{*} \qquad [5.101\ (0)]$$

Now $R_{i-1,j+1}$ is also get explicit from [5.99]

$$R_{i-1,j+1} = Q_{i-1,j+1} - W_{i-j-2}^{*} \qquad [5.101\ (1)]$$

Continuing let insert [5.101 (0)] and [5.101 (1)] in [5.73],

$$\rho\cdot\left[Q_{i,j} - W_{i-j}^{*}\right] = Q_{i-1,j+1} - W_{i-j-2}^{*} \qquad [5.102(A)]$$

… rearranging [5.102(A)],

$$Q_{i-1,j+1} - \rho \cdot Q_{i,j} = W^*_{i-j-2} - \rho \cdot W^*_{i-j} \qquad [5.102(B)]$$

The opportune following definition of $S^{[\nwarrow]}_{i,j}$,

$$S^{[\nwarrow]}_{i,j} = Q_{i-1,j+1} - \rho \cdot Q_{i,j} \qquad [5.103]$$

… allows write down [5.102(B)] this way,

$$S^{[\nwarrow]}_{i,j} = W^*_{i-j-2} - \rho \cdot W^*_{i-j} \qquad [5.104]$$

Recalling once again [5.41],

$$\rho W^*_{n+1} = W^*_{n-1} - \frac{n}{\tau} W^*_n \qquad [5.41]$$

… making $n = (i - j - 1)$,

$$\rho W^*_{i-j} = W^*_{i-j-2} - \frac{(i-j-1)}{\tau} W^*_{i-j-1} \qquad [5.105(A)]$$

… and writing this rearranged version,

$$W^*_{i-j-2} - \rho W^*_{i-j} = \frac{(i-j-1)}{\tau} W^*_{i-j-1} \qquad [5.105(B)]$$

… the relation [5.104] becomes,

$$S^{[\nwarrow]}_{i,j} = \frac{(i-j-1)}{\tau} W^*_{i-j-1} \qquad [5.106(1)]$$

Next step consists in subtract 1 to i and also add 1 to j, and, after achieve the result, repeat once more these same subtraction to i and addition to j over such result.
We reach, successively,

$$S^{[\nwarrow]}_{i-1,j+1} = \frac{(i-j-3)}{\tau} W^*_{i-j-3} \qquad [5.106(3)]$$

$$S^{[\nwarrow]}_{i-2,j+2} = \frac{(i-j-5)}{\tau} W^*_{i-j-5} \qquad [5.106(5)]$$

In these three relations [5.106(1)], [5.106(3)] and [5.106(5)], we have three functions, W^*_{i-j-1}, W^*_{i-j-3} and W^*_{i-j-5} in which the orders are distanced, one to the next other, by 2.
This circumstance requires now, before to proceed, a little algebraic work over [5.41].
Summing, at [5.41], 1 to n, subtracting 1 to n, and, lastly, putting together [5.41] and these two results we get,

$$\rho W^*_{n+2} = W^*_n - \frac{n+1}{\tau} W^*_{n+1} \qquad [5.41(+1)]$$

$$\rho W^*_{n+1} = W^*_{n-1} - \frac{n}{\tau} W^*_n \qquad [5.41]$$

$$\rho W_n^* = W_{n-2}^* - \frac{n-1}{\tau} W_{n-1}^* \qquad [5.41(-1)]$$

Next task is to eliminate W_{n+1}^* and W_{n-1}^* from this three equations system.
From [5.41(+1)] we explicit W_{n+1}^*, from [5.41(−1)] we explicit W_{n-1}^*, afterwards we insert those two results in [5.41] and, lastly, rearrange suitably.
Here is the sequence,

$$W_{n+1}^* = \frac{\tau}{n+1} \left[W_n^* - \rho W_{n+2}^*\right] \qquad [5.107(+1)]$$

$$W_{n-1}^* = \frac{\tau}{n-1} \left[W_{n-2}^* - \rho W_n^*\right] \qquad [5.107(+1)]$$

$$\frac{\rho^2\tau}{(n+1)} W_{n+2}^* - \left[\frac{\rho\tau}{(n+1)} + \frac{\rho\tau}{(n-1)} + \frac{n}{\tau}\right] W_n^* + \frac{\tau}{(n-1)} W_{n-2}^* = 0 \qquad [5.108(A)]$$

The multiplication by $[(n+1)(n-1)]$ at [5.108] directs to,

$$\rho^2\tau(n-1) W_{n+2}^* - \left[2n\rho\tau + \frac{(n+1)n(n-1)}{\tau}\right] W_n^* + (n+1)\tau W_{n-2}^* = 0 \qquad [5.108(B)]$$

Setting $n = (i-j-3)$ at [5.108(B)] we get,

$$\rho^2\tau(i-j-4) W_{i-j-1}^* - \\ - \left[2(i-j-3)\rho\tau + \frac{(i-j-2)(i-j-3)(i-j-4)}{\tau}\right] W_{i-j-3}^* + \\ + (i-j-2)\tau W_{i-j-5}^* = 0 \qquad [5.109]$$

Recalling [5.106(1)], [5.106(3)] and [5.106(5)] and setting explicit the three functions W_{i-j-1}^*, W_{i-j-3}^* and W_{i-j-5}^*

$$W_{i-j-1}^* = \frac{\tau}{(i-j-1)} S_{i,j}^{[\nwarrow]} \qquad [5.110(1)]$$

$$W_{i-j-3}^* = \frac{\tau}{(i-j-3)} S_{i-1,j+1}^{[\nwarrow]} \qquad [5.110(3)]$$

$$W_{i-j-5}^* = \frac{\tau}{(i-j-5)} S_{i-2,j+2}^{[\nwarrow]} \qquad [5.110(5)]$$

Insertion of [5.110(1)], [5.110(3)] and [5.110(5)] at [5.109] conducts to,

$$\frac{\rho^2\tau(i-j-4)}{(i-j-1)} S_{i,j}^{[\nwarrow]} - \left[2\rho\tau^2 + \frac{(i-j-2)(i-j-3)(i-j-4)}{(i-j-3)}\right] S_{i-1,j+1}^{[\nwarrow]} + \tau^2 \frac{(i-j-2)}{(i-j-5)} S_{i-2,j+2}^{[\nwarrow]} = 0 \qquad [5.111]$$

Definition for $S_{i,j}^{[\nwarrow]}$,depending on $Q_{i,j}$ and $Q_{i-1,j+1}$, [5.103] is back,

$$S_{i,j}^{[\nwarrow]} = Q_{i-1,j+1} - \rho \cdot Q_{i,j} \qquad [5.103]$$

... and the corresponding expressions for $S^{[\nwarrow]}_{i-1,j+1}$ and $S^{[\nwarrow]}_{i-2,j+2}$ can also straightforwardly be written down,

$$S^{[\nwarrow]}_{i-1,j+1} = Q_{i-2,j+2} - \rho \cdot Q_{i-1,j+1} \qquad [5.112(1)]$$

$$S^{[\nwarrow]}_{i-2,j+2} = Q_{i-3,j+3} - \rho \cdot Q_{i-2,j+2} \qquad [5.112(2)]$$

As usual, now [5.103], [5.112(1)] and [5.112(2)] are included in [5.111], them the common terms in $Q_{i,j}$, $Q_{i-1,j+1}$, $Q_{i-2,j+2}$ and $Q_{i-3,j+3}$ are collected together and the desired homogeneous and contiguous straight line recurrence, located at the negative diagonal, is obtained.
The final result, already for functions $P_{i,j}$, reads,

$$\begin{aligned}&\left\{\rho^3\tau^2 \cdot \frac{(i-j-4)}{(i-j-1)}\right\} \cdot P_{i,j} + \\ &+\left\{-\rho \cdot (i-j-2) \cdot \left[\frac{3\rho\tau^2}{(i-j-1)} + (i-j-4)\right]\right\} \cdot P_{i-1,j+1} + \\ &+\left\{(i-j-4) \cdot \left[\frac{3\rho\tau^2}{(i-j-5)} + (i-j-2)\right]\right\} \cdot P_{i-2,j+2} + \\ &+\left\{-\tau^2 \cdot \frac{(i-j-2)}{(i-j-5)}\right\} \cdot P_{i-3,j+3} = 0 \qquad [5.113]\end{aligned}$$

Alternatively the notation using the parameter $\sigma^{[-]}_k$, defined at [5.93], allows this shortened representation,

$$\begin{aligned}&\left\{\rho^3\tau^2 \cdot \frac{\sigma^{[-]}_4}{\sigma^{[-]}_1}\right\} \cdot P_{i,j} + \left\{-\rho \cdot \sigma^{[-]}_2 \cdot \left[\frac{3\rho\tau^2}{\sigma^{[-]}_1} + \sigma^{[-]}_4\right]\right\} \cdot P_{i-1,j+1} + \\ &+\left\{\sigma^{[-]}_4 \cdot \left[\frac{3\rho\tau^2}{\sigma^{[-]}_5} + \sigma^{[-]}_2\right]\right\} \cdot P_{i-2,j+2} + \left\{-\tau^2 \cdot \frac{\sigma^{[-]}_2}{\sigma^{[-]}_5}\right\} \cdot P_{i-3,j+3} = 0 \qquad [5.114]\end{aligned}$$

4.6. Deduction of the positive diagonal recurrence for functions $Q_{i,j}$ and $P_{i,j}$

Our next deductive stage concerns homogeneous and contiguous straight line recurrence, located at the positive diagonal. Contrarily to negative diagonal case, just solved, in the case of recurrence located at the positive diagonal a complete reorganization of the parameters, accordingly with our "unifying" notation, will be applied, as also was the case for the vertical located recurrence.
This was not necessary from the very start point, for the negative diagonal location, as consequence of the formal simplicity of recurrence [5.73].
The recurrence to achieve is of form $F\left[Q_{i,j}(\tau), Q_{i+1,j+1}(\tau), Q_{i+2,j+2}(\tau), \ldots\right] = 0$ or, what is the same, of form $F\left[P_{i,j}(\tau), P_{i+1,j+1}(\tau), P_{i+2,j+2}(\tau), \ldots\right] = 0$.
Till now we have already the recurrence $F\left[R_{i,j}(\tau), R_{i+1,j+1}(\tau), R_{i+2,j+2}(\tau), \ldots\right] = 0$, given at [5.85] and now recalled, in the next page,

$$\left\{\frac{\rho\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}\right\}\cdot R_{i,j}+$$

$$+\left\{-\frac{\tau^2}{[(1-\rho)\cdot\tau+(i+j+2)]\cdot(i+j+3)}-\frac{\rho\cdot\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}\right\}\cdot R_{i+1,j+1}+$$

$$+\left\{\frac{\tau^2\cdot\left[\frac{1}{(i+j+3)}+\frac{(i+j+4)}{\rho\cdot\tau^2}+\frac{1}{(i+j+5)}\right]}{[(1-\rho)\cdot\tau+(i+j+4)]}+\frac{\rho\cdot\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i+2,j+2}+$$

$$+\left\{-\frac{\tau^2}{[(1-\rho)\cdot\tau+(i+j+6)]\cdot(i+j+5)}\right\}\cdot R_{i+3,j+3}=0 \qquad [5.85]$$

The suitable set of parameter definitions, to apply from now on, seems to be, by simple inspection of [5.85], quite easy to write down. In a similar way to the set up of definitions [5.90], [5.91(A)], [5.91(B)], [5.91(C)] and [5.91(D)], we now adopt the following ones,

$$\sigma_k^{[+]}=(i+j+k) \qquad [5.90]$$

$$A_{i,j}^{[\nearrow]}=\left\{\frac{\rho\cdot\tau^2}{\left[(1-\rho)\cdot\tau+\sigma_2^{[+]}\right]\cdot\sigma_3^{[+]}}\right\} \qquad [5.115(A)]$$

$$B_{i,j}^{[\nearrow]}=\left\{-\frac{\tau^2}{\left[(1-\rho)\cdot\tau+\sigma_2^{[+]}\right]\cdot\sigma_3^{[+]}}-\frac{\rho\cdot\tau^2\cdot\left[\frac{1}{\sigma_3^{[+]}}+\frac{\sigma_4^{[+]}}{\rho\cdot\tau^2}+\frac{1}{\sigma_5^{[+]}}\right]}{\left[(1-\rho)\cdot\tau+\sigma_4^{[+]}\right]}\right\} \qquad [5.115(B)]$$

$$C_{i,j}^{[\nearrow]}=\left\{\frac{\tau^2\cdot\left[\frac{1}{\sigma_3^{[+]}}+\frac{\sigma_4^{[+]}}{\rho\cdot\tau^2}+\frac{1}{\sigma_5^{[+]}}\right]}{\left[(1-\rho)\cdot\tau+\sigma_4^{[+]}\right]}+\frac{\rho\cdot\tau^2}{\left[(1-\rho)\cdot\tau+\sigma_6^{[+]}\right]\cdot\sigma_5^{[+]}}\right\} \qquad [5.115(C)]$$

$$D_{i,j}^{[\nearrow]}=\left\{-\frac{\tau^2}{\left[(1-\rho)\cdot\tau+\sigma_6^{[+]}\right]\cdot\sigma_5^{[+]}}\right\} \qquad [5.115(D)]$$

Remark: Observe that there is no need of changing the parameter definition of $\sigma_k^{[+]}$ adopting a similar one, albeit different, involving $(i\pm j\pm k)$, and therefore the relation [5.90] remains suitable for render concrete the next deduction.
The meaning of the arrows is obvious considering the already done observations about the pretended intuitiveness of our notation.

Lastly note that now $A_{i,j}^{[\nearrow]}$, $B_{i,j}^{[\nearrow]}$, $C_{i,j}^{[\nearrow]}$ and $D_{i,j}^{[\nearrow]}$ have the two sub-indexes i and j. This is due to the adopted criterion that the sub-indexes changing along the direction of the recurrence under deduction must be explicit, because they will vary over the deductive reasoning.

Let start the deduction writing down the abbreviated version of the recurrence [5.85] attending to [5.115(A)], [5.115(B)], [5.115(C)], and [5.115(D)],

$$A_{i,j}^{[\nearrow]} \cdot R_{i,j} + B_{i,j}^{[\nearrow]} \cdot R_{i+1,j+1} + C_{i,j}^{[\nearrow]} \cdot R_{i+2,j+2} + D_{i,j}^{[\nearrow]} \cdot R_{i+3,j+3} = 0 \qquad [5.116]$$

At [5.101 (0)] the usual relation between functions $R_{i,j}$ and $Q_{i,j}$ is expressed.
And the same is valid for functions $R_{i+1,j+1}$ and $Q_{i+1,j+1}$, for functions $R_{i+2,j+2}$ and $Q_{i+2,j+2}$ and for functions $R_{i+3,j+3}$ and $Q_{i+3,j+3}$ respectively at [5.101 (1)], [5.101 (2)] and [5.101 (3)].

$$R_{i,j} = Q_{i,j} - W_{i-j}^{*} \qquad [5.101\ (0)]$$

$$R_{i+1,j+1} = Q_{i+1,j+1} - W_{i-j}^{*} \qquad [5.101\ (1)]$$

$$R_{i+2,j+2} = Q_{i+2,j+2} - W_{i-j}^{*} \qquad [5.101\ (2)]$$

$$R_{i+3,j+3} = Q_{i+3,j+3} - W_{i-j}^{*} \qquad [5.101\ (3)]$$

<u>Observation:</u> considering $(i+k)$ and $(j+k)$ the value of $(i+k)-(j+k)=(i-j)$ don´t depends on k and, consequently, the function W_{i-j}^{*} appears without changing his order at all three relations 5.101 (1)], [5.101 (2)] and [5.101 (3)], remaining unchanged and equal as appears in [5.101 (0)].
Insertion of [5.101 (0)], [5.101 (1)], [5.101 (2)] and [5.101 (3)] in [5.116] is the logic next step.
After separate functions $Q_{i+k,j+k}$ with $(k = 0, 1, 2 \text{ and } 3)$ at the left hand side and function W_{i-j}^{*} at the right hand side the result reads,

$$A_{i,j}^{[\nearrow]} \cdot Q_{i,j} + B_{i,j}^{[\nearrow]} \cdot Q_{i+1,j+1} + C_{i,j}^{[\nearrow]} \cdot Q_{i+2,j+2} + D_{i,j}^{[\nearrow]} \cdot Q_{i+3,j+3} =$$

$$= \left\{A_{i,j}^{[\nearrow]} + B_{i,j}^{[\nearrow]} + C_{i,j}^{[\nearrow]} + D_{i,j}^{[\nearrow]}\right\} \cdot W_{i-j}^{*} \qquad [5.117]$$

As expected, now we define an auxiliary function,

$$S_{i,j}^{[\nearrow]} = A_{i,j}^{[\nearrow]} \cdot Q_{i,j} + B_{i,j}^{[\nearrow]} \cdot Q_{i+1,j+1} + C_{i,j}^{[\nearrow]} \cdot Q_{i+2,j+2} + D_{i,j}^{[\nearrow]} \cdot Q_{i+3,j+3} \qquad [5.118\ (0)]$$

Including definition [5.118(0)] in [5.117],

$$S_{i,j}^{[\nearrow]} = \left\{A_{i,j}^{[\nearrow]} + B_{i,j}^{[\nearrow]} + C_{i,j}^{[\nearrow]} + D_{i,j}^{[\nearrow]}\right\} \cdot W_{i-j}^{*} \qquad [5.119(0)]$$

This time our work is easier than in former deductions because we can, right now, profit the fact that if an addition of 1 is made over i and also over j the function W_{i-j}^{*} remains unchangeable.
Doing that at [5.119(0)] the new similar relation [5.119(1)] is written as,

$$S_{i+1,j+1}^{[\nearrow]} = \left\{A_{i+1,j+1}^{[\nearrow]} + B_{i+1,j+1}^{[\nearrow]} + C_{i+1,j+1}^{[\nearrow]} + D_{i+1,j+1}^{[\nearrow]}\right\} \cdot W_{i-j}^{*} \qquad [5.119(1)]$$

For notation economy and also coherence with all before notation, described and defined, now the auxiliary parameters,

$$E_{i,j}^{[\nearrow]} = \left\{A_{i,j}^{[\nearrow]} + B_{i,j}^{[\nearrow]} + C_{i,j}^{[\nearrow]} + D_{i,j}^{[\nearrow]}\right\} \qquad [5.120(0)]$$

… and,

$$E_{i+1,j+1}^{[\nearrow]} = \left\{A_{i+1,j+1}^{[\nearrow]} + B_{i+1,j+1}^{[\nearrow]} + C_{i+1,j+1}^{[\nearrow]} + D_{i+1,j+1}^{[\nearrow]}\right\} \qquad [5.120(1)]$$

… can be inserted respectively in [5.119(0)] and [5.119(1)],

$$S_{i,j}^{[\nearrow]} = E_{i,j}^{[\nearrow]} \cdot W_{i-j}^{*} \qquad [5.121(0)]$$

$$S_{i+1,j+1}^{[\nearrow]} = E_{i+1,j+1}^{[\nearrow]} \cdot W_{i-j}^{*} \qquad [5.121(1)]$$

Desired elimination of function W_{i-j}^{*} using [5.121(0)] and [5.121(1)] allows write,

$$S_{i,j}^{[\nearrow]} \cdot E_{i+1,j+1}^{[\nearrow]} = S_{i+1,j+1}^{[\nearrow]} \cdot E_{i,j}^{[\nearrow]} \qquad [5.122]$$

Going back to the definition [5.118(0)] and adding 1 to i and also 1 to j we reach,

$$S_{i+1,j+1}^{[\nearrow]} = A_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+1,j+1} + B_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+2,j+2} + C_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+3,j+3} + D_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+4,j+4}$$

[5.118(1)]

The two relations [5.118(0)] and [5.118(1)] introduced into [5.122] directs to an equation where the only existing functions are $Q_{i+k,j+k}$ with $(k = 0, 1, 2, \dots)$ as wanted,

$$\left\{A_{i,j}^{[\nearrow]} \cdot Q_{i,j} + B_{i,j}^{[\nearrow]} \cdot Q_{i+1,j+1} + C_{i,j}^{[\nearrow]} \cdot Q_{i+2,j+2} + D_{i,j}^{[\nearrow]} \cdot Q_{i+3,j+3}\right\} \cdot E_{i+1,j+1}^{[\nearrow]} =$$

$$= \left\{A_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+1,j+1} + B_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+2,j+2} + C_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+3,j+3} + D_{i+1,j+1}^{[\nearrow]} \cdot Q_{i+4,j+4}\right\} \cdot E_{i,j}^{[\nearrow]}$$

[5.123 − A]

Obvious algebraic work, namely collection of terms with common function $Q_{i+k,j+k}$ leads to the final desired result,

$$\left\{\frac{A_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}} \cdot Q_{i,j} + \frac{B_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}} \cdot Q_{i+1,j+1} + \frac{C_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}} \cdot Q_{i+2,j+2} + \frac{D_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}} \cdot Q_{i+3,j+3}\right\} =$$

$$= \left\{\frac{A_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}} \cdot Q_{i+1,j+1} + \frac{B_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}} \cdot Q_{i+2,j+2} + \frac{C_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}} \cdot Q_{i+3,j+3} + \frac{D_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}} \cdot Q_{i+4,j+4}\right\}$$

[5.123 − B]

… and, at last, the homogeneous positive diagonal recurrence which, as we can see, has five terms,

$$\left\{\frac{A_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}}\right\} \cdot P_{i,j} + \left\{\frac{B_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}} - \frac{A_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}}\right\} \cdot P_{i+1,j+1} + \left\{\frac{C_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}} - \frac{B_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}}\right\} \cdot P_{i+2,j+2} +$$

$$+ \left\{\frac{D_{i,j}^{[\nearrow]}}{E_{i,j}^{[\nearrow]}} - \frac{C_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}}\right\} \cdot P_{i+3,j+3} + \left\{-\frac{D_{i+1,j+1}^{[\nearrow]}}{E_{i+1,j+1}^{[\nearrow]}}\right\} \cdot P_{i+4,j+4} = 0$$

[5.124]

At [5.124] functions $Q_{i+k,j+k}$ have been, as usual, simply substituted by functions $P_{i+k,j+k}$.

4.7. Deduction of the horizontal recurrence for functions $Q_{i,j}$ and $P_{i,j}$

The last one, still remaining for being deduced, is the homogeneous horizontal recurrence which, before they rigorous deduction, we indicate simply by $F\left[Q_{i,j}(\tau), Q_{i+1,j}(\tau), Q_{i+2,j}(\tau), \ldots\right] = 0$ or, what is the same, by $F\left[P_{i,j}(\tau), P_{i+1,j}(\tau), P_{i+2,j}(\tau), \ldots\right] = 0$.
As expected the deduction of the horizontal case reveals remarkable similarities with the already done deduction of the vertical one.
However, regarding the complexity of all the subject is better to go on details signalling the adequate adaptations and the consequent differentiated aspects.

In the framework of our systematic notation is advisable introduce a set of new definitions. For such goal let recall the recurrence for functions $R_{i+k,j}(\tau)$ in the same horizontal direction expressed at [5.75],

$$\left\{\frac{\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i,j}+\left\{-1-\frac{\rho\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i+1,j}+ +\left\{1-\frac{\tau}{[(1-\rho)\tau+(i+j+2)]}\right\}\cdot R_{i+2,j}+\left\{\frac{\rho\tau}{[(1-\rho)\tau+(i+j+3)]}\right\}\cdot R_{i+3,j}=0 \qquad [5.75]$$

A simple inspection leaves to the following set of new parameter definitions, expressed by [5.90], as before, and now also by [5.125 (A)], [5.125 (B)], [5.125 (C)] and [5.125 (D)],

$$\sigma_k^{[+]} = (i+j+k) \qquad [5.90]$$

$$A_i^{[\rightarrow]} = \left\{\frac{\tau}{\left[(1-\rho)\tau+\sigma_2^{[+]}\right]}\right\} \qquad [5.125(A)]$$

$$B_i^{[\rightarrow]} = \left\{-1-\frac{\rho\cdot\tau}{\left[(1-\rho)\tau+\sigma_3^{[+]}\right]}\right\} \qquad [5.125(B)]$$

$$C_i^{[\rightarrow]} = \left\{1-\frac{\tau}{\left[(1-\rho)\tau+\sigma_2^{[+]}\right]}\right\} \qquad [5.125(C)]$$

$$D_i^{[\rightarrow]} = \left\{\frac{\rho\cdot\tau}{\left[(1-\rho)\tau+\sigma_3^{[+]}\right]}\right\} \qquad [5.125(D)]$$

Remark: In $A_i^{[\rightarrow]}$, $B_i^{[\rightarrow]}$, $C_i^{[\rightarrow]}$ and $D_i^{[\rightarrow]}$ the sub-index is not anymore j but, instead is i. This is done according to the notation criterion that we must explicit the index (or indexes) which will vary over the deduction we are just before in our way to realize. Just like criterion expressed already in the remarks about parameter definitions [5.115 (A)], [5.115 (B)], [5.115 (C)] and [5.115 (D)], before the former deduction for the five terms positive diagonal recurrence [5.124].
With [5.125 (A)], [5.125 (B)], [5.125 (C)] and [5.125 (D)] , and paying attention to [5.90], recurrence [5.75] is now rewritten as,

$$A_i^{[\rightarrow]}\cdot R_{i,j}+B_i^{[\rightarrow]}\cdot R_{i+1,j}+C_i^{[\rightarrow]}\cdot R_{i+2,j}+D_i^{[\rightarrow]}\cdot R_{i+3,j}=0 \qquad [5.126]$$

Once again, let recall [5.59],

$$R_{i,j} = Q_{i,j} - W_{i-j}^{*} \qquad [5.59]$$

… and add successively 1 to i, 2 to i and 3 to i,

$$R_{i+1,j} = Q_{i+1,j} - W^*_{i-j+1} \qquad [5.127(1)]$$

$$R_{i+2,j} = Q_{i+2,j} - W^*_{i-j+2} \qquad [5.127(2)]$$

$$R_{i+3,j} = Q_{i+3,j} - W^*_{i-j+3} \qquad [5.127(3)]$$

The definition of an auxiliary function $S^{[\rightarrow]}_{i,j}(\tau)$,

$$S^{[\rightarrow]}_{i,j} = A^{[\rightarrow]}_i \cdot Q_{i,j} + B^{[\rightarrow]}_i \cdot Q_{i+1,j} + C^{[\rightarrow]}_i \cdot Q_{i+2,j} + D^{[\rightarrow]}_i \cdot Q_{i+3,j} \qquad [5.128]$$

… directs, considering also [5.59], [5.127(1)] , [5.127(2)] and [5.127(3)] to write [5.126] this transformed way,

$$S^{[\rightarrow]}_{i,j} = A^{[\rightarrow]}_i \cdot W^*_{i-j} + B^{[\rightarrow]}_i \cdot W^*_{i-j+1} + C^{[\rightarrow]}_i \cdot W^*_{i-j+2} + D^{[\rightarrow]}_i \cdot W^*_{i-j+3} \qquad [5.129]$$

Attending [5.41], writing it backwards and forwards gives respectively,,

$$W^*_{n+1} = \frac{1}{\rho}\, W^*_{n-1} - \frac{n}{\rho\tau} W^*_n \qquad [5.130(\text{Bw})]$$

… and,

$$W^*_{n-1} = \rho W^*_{n+1} + \frac{n}{\tau} W^*_n \qquad [5.130(\text{Fw})]$$

Making $(n = i - j + 2)$ in [5.130(Bw)] and $(n = i - j + 1)$ in [5.130(Fw)] gives,

$$W^*_{i-j+3} = \frac{1}{\rho}\, W^*_{i-j+1} - \frac{(i-j+2)}{\rho\tau} W^*_{i-j+2} \qquad [5.131(\text{Bw})]$$

… and,

$$W^*_{i-j} = \rho W^*_{i-j+2} + \frac{(i-j+1)}{\tau} W^*_{i-j+1} \qquad [5.131(\text{Fw})]$$

These two last relations allows eliminate W^*_{i-j+3} and W^*_{i-j}at [5.129].
After a few algebraic rearrangement, collecting common terms in W^*_{i-j+1} and W^*_{i-j+2}, we reach,

$$S^{[\rightarrow]}_{i,j} = \left\{ B^{[\rightarrow]}_i + \frac{1}{\rho}\, D^{[\rightarrow]}_i + \frac{(i-j+1)}{\tau}\, A^{[\rightarrow]}_i \right\} \cdot W^*_{i-j+1} + \\ + \left\{ C^{[\rightarrow]}_i + \rho\, A^{[\rightarrow]}_i - \frac{(i-j+2)}{\rho\tau} D^{[\rightarrow]}_i \right\} \cdot W^*_{i-j+2} \qquad [5.132]$$

Following a similar way to the vertical recurrence deduction, now is the turn to define parameters $E^{[\rightarrow]}_i$ and $F^{[\rightarrow]}_i$.
Their notations are similar to those at the refereed foregoing vertical deduction ($E^{[\uparrow]}_j$ and $F^{[\uparrow]}_j$) except in two details, for disentangling purposes,

… one concerning the change in the arrow,
… and the other concerning the sub-index that now is i and not j, paying attention that now i, and not j, is the sub-index that changes all over the deduction.

The definitions for parameters $E_i^{[\rightarrow]}$ and $F_i^{[\rightarrow]}$ reads,

$$E_i^{[\rightarrow]} = \left[\ B_i^{[\rightarrow]} + \frac{1}{\rho}\ D_i^{[\rightarrow]} + \frac{(i-j+1)}{\tau}\ \ A_i^{[\rightarrow]}\right] \tag{5.133(E)}$$

$$F_i^{[\rightarrow]} = \left[\ C_{i+1}^{[\rightarrow]} + \rho\ A_{i+1}^{[\rightarrow]} - \frac{(i-j+3)}{\rho\tau} D_{i+1}^{[\rightarrow]}\right] \tag{5.133(F)}$$

With [5.133(E)] and [5.133(F)] one gets, from [5.132],

$$S_{i,j}^{[\rightarrow]} = E_i^{[\rightarrow]} \cdot W_{i-j+1}^{*} + F_{i-1}^{[\rightarrow]} \cdot W_{i-j+2}^{*} \tag{5.134(0)}$$

... and also, adding 1 to i,

$$S_{i+1,j}^{[\rightarrow]} = E_{i+1}^{[\rightarrow]} \cdot W_{i-j+2}^{*} + F_i^{[\rightarrow]} \cdot W_{i-j+3}^{*} \tag{5.134(1)}$$

With a little algebraic manipulation consisting in ...
... divide [5.134(0)] by $E_i^{[\rightarrow]}$ getting,

$$\frac{1}{E_i^{[\rightarrow]}} \cdot S_{i,j}^{[\rightarrow]} = W_{i-j+1}^{*} + \frac{F_{i-1}^{[\rightarrow]}}{E_i^{[\rightarrow]}} \cdot W_{i-j+2}^{*} \tag{5.135(0)}$$

... and also multiply [5.134(1)] by $\dfrac{\rho}{F_i^{[\rightarrow]}}$ getting,

$$\frac{\rho}{F_i^{[\rightarrow]}} \cdot S_{i+1,j}^{[\rightarrow]} = \rho \cdot \frac{E_{i+1}^{[\rightarrow]}}{F_i^{[\rightarrow]}} \cdot W_{i-j+2}^{*} + \rho \cdot W_{i-j+3}^{*} \tag{5.135(1)}$$

... and lastly subtract [5.135(1)] from [5.135(0)] reaching,

$$\left\{\frac{1}{E_i^{[\rightarrow]}}\right\} \cdot S_{i,j}^{[\rightarrow]} - \left\{\frac{\rho}{F_i^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} = \left[W_{i-j+1}^{*} - \rho \cdot W_{i-j+3}^{*}\right] + \left\{\frac{F_{i-1}^{[\rightarrow]}}{E_i^{[\rightarrow]}} - \rho \cdot \frac{E_{i+1}^{[\rightarrow]}}{F_i^{[\rightarrow]}}\right\} \cdot W_{i-j+2}^{*} \tag{5.136}$$

Recalling [5.131(Bw)], the quantity $\left[W_{i-j+1}^{*} - \rho \cdot W_{i-j+3}^{*}\right]$ can be written down as,

$$\left[W_{i-j+1}^{*} - \rho \cdot W_{i-j+3}^{*}\right] = \frac{(i-j+2)}{\tau} \cdot W_{i-j+2}^{*} \tag{5.137}$$

Inserting now [5.137] in [5.136] we reach,

$$\left\{\frac{1}{E_i^{[\rightarrow]}}\right\} \cdot S_{i,j}^{[\rightarrow]} - \left\{\frac{\rho}{F_i^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} = \left\{\frac{F_{i-1}^{[\rightarrow]}}{E_i^{[\rightarrow]}} - \rho \cdot \frac{E_{i+1}^{[\rightarrow]}}{F_i^{[\rightarrow]}} + \frac{(i-j+2)}{\tau}\right\} \cdot W_{i-j+2}^{*} \tag{5.138}$$

Once more, another auxiliary parameter definition must come to play,

$$G_i^{[\rightarrow]} = \left\{\rho \cdot \frac{E_{i+1}^{[\rightarrow]}}{F_i^{[\rightarrow]}} - \frac{F_{i-1}^{[\rightarrow]}}{E_i^{[\rightarrow]}} - \frac{(i-j+2)}{\tau}\right\} \tag{5.139(G)}$$

This one is the homologue of $G_j^{[\uparrow]}$ already defined in the deduction for vertical oriented recurrence.

According to [5.139(G)] the relation [5.138] can transforms to,

$$\left\{\frac{\rho}{F_i^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} - \left\{\frac{1}{E_i^{[\rightarrow]}}\right\} \cdot S_{i,j}^{[\rightarrow]} = G_i^{[\rightarrow]} \cdot W_{i-j+2}^{*} \qquad [5.140]$$

… or, equivalently,

$$W_{i-j+2}^{*} = \left\{\frac{\rho}{F_i^{[\rightarrow]} \cdot G_i^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} - \left\{\frac{1}{E_i^{[\rightarrow]} \cdot G_i^{[\rightarrow]}}\right\} \cdot S_{i,j}^{[\rightarrow]} \qquad [5.141(2)]$$

… in which, adding 1 to i,

$$W_{i-j+3}^{*} = \left\{\frac{\rho}{F_{i+1}^{[\rightarrow]} \cdot G_{i+1}^{[\rightarrow]}}\right\} \cdot S_{i+2,j}^{[\rightarrow]} - \left\{\frac{1}{E_{i+1}^{[\rightarrow]} \cdot G_{i+1}^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} \qquad [5.141(3)]$$

Lastly the insertion, of [5.141(2)] and [5.141(3)], in [5.134(1)] allows to eliminate the functions W_{i-j+2}^{*} and W_{i-j+3}^{*} from it,

$$S_{i+1,j}^{[\rightarrow]} = \left\{\frac{\rho \cdot E_{i+1}^{[\rightarrow]}}{F_i^{[\rightarrow]} \cdot G_i^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} - \left\{\frac{E_{i+1}^{[\rightarrow]}}{E_i^{[\rightarrow]} \cdot G_i^{[\rightarrow]}}\right\} \cdot S_{i,j}^{[\rightarrow]} + $$

$$+ \left\{\frac{\rho \cdot F_i^{[\rightarrow]}}{F_{i+1}^{[\rightarrow]} \cdot G_{i+1}^{[\rightarrow]}}\right\} \cdot S_{i+2,j}^{[\rightarrow]} - \left\{\frac{F_i^{[\rightarrow]}}{E_{i+1}^{[\rightarrow]} \cdot G_{i+1}^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} \qquad [5.142]$$

As usual, collection of terms with common functions $S_{i+k,j}^{[\rightarrow]}$ where $(k = 0, 1, 2)$ is the next logic step,

$$\left\{\frac{E_{i+1}^{[\rightarrow]}}{E_i^{[\rightarrow]} \cdot G_i^{[\rightarrow]}}\right\} \cdot S_{i,j}^{[\rightarrow]} + \left\{1 - \frac{\rho \cdot E_{i+1}^{[\rightarrow]}}{F_i^{[\rightarrow]} \cdot G_i^{[\rightarrow]}} + \frac{F_i^{[\rightarrow]}}{E_{i+1}^{[\rightarrow]} \cdot G_{i+1}^{[\rightarrow]}}\right\} \cdot S_{i+1,j}^{[\rightarrow]} + \left\{- \frac{\rho \cdot F_i^{[\rightarrow]}}{F_{i+1}^{[\rightarrow]} \cdot G_{i+1}^{[\rightarrow]}}\right\} \cdot S_{i+2,j}^{[\rightarrow]} = 0 \qquad [5.143]$$

Now we rewrite [5.128], and rename it to [5.128(0)],

$$S_{i,j}^{[\rightarrow]} = A_i^{[\rightarrow]} \cdot Q_{i,j} + B_i^{[\rightarrow]} \cdot Q_{i+1,j} + C_i^{[\rightarrow]} \cdot Q_{i+2,j} + D_i^{[\rightarrow]} \cdot Q_{i+3,j} \qquad [5.128(0)]$$

… and add 1 to i, twice and successively.

We get,

$$S_{i+1,j}^{[\rightarrow]} = A_{i+1}^{[\rightarrow]} \cdot Q_{i+1,j} + B_{i+1}^{[\rightarrow]} \cdot Q_{i+2,j} + C_{i+1}^{[\rightarrow]} \cdot Q_{i+3,j} + D_{i+1}^{[\rightarrow]} \cdot Q_{i+4,j} \qquad [5.128(1)]$$

$$S_{i+2,j}^{[\rightarrow]} = A_{i+2}^{[\rightarrow]} \cdot Q_{i+2,j} + B_{i+2}^{[\rightarrow]} \cdot Q_{i+3,j} + C_{i+2}^{[\rightarrow]} \cdot Q_{i+4,j} + D_{i+2}^{[\rightarrow]} \cdot Q_{i+5,j} \qquad [5.128(2)]$$

Finally, inclusion of [5.128(0)], [5.128(1)] and [5.128(2)] in [5.143], and collection of all the common terms with the same function $Q_{i+k,j}(\tau)$ where $(k = 0, 1, 2, 3, 4, 5)$ allows to get the desired homogeneous horizontal straight line recurrence for the functions $Q_{i,j}(\tau)$, what is the same to say the desired homogeneous horizontal straight line recurrence for the functions $P_{i,j}(\tau)$.

We verify that it is a six terms recurrence.

The final result reads,

$$\left[A_i^{[\to]}\cdot\frac{E_{i+1}^{[\to]}}{E_i^{[\to]}\cdot G_i^{[\to]}}\right]\cdot P_{i,j}+$$

$$+\left[A_{i+1}^{[\to]}\cdot\left(1-\frac{\rho\cdot E_{i+1}^{[\to]}}{F_i^{[\to]}\cdot G_i^{[\to]}}+\frac{F_i^{[\to]}}{E_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}\right)+B_i^{[\to]}\cdot\frac{E_{i+1}^{[\to]}}{E_i^{[\to]}\cdot G_i^{[\to]}}\right]\cdot P_{i+1,j}+$$

$$+\left[-A_{i+2}^{[\to]}\cdot\frac{\rho\cdot F_i^{[\to]}}{F_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}+B_{i+1}^{[\to]}\cdot\left(1-\frac{\rho\cdot E_{i+1}^{[\to]}}{F_i^{[\to]}\cdot G_i^{[\to]}}+\frac{F_i^{[\to]}}{E_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}\right)+C_i^{[\to]}\cdot\frac{E_{i+1}^{[\to]}}{E_i^{[\to]}\cdot G_i^{[\to]}}\right]\cdot P_{i+2,j}+$$

$$+\left[-B_{i+2}^{[\to]}\cdot\frac{\rho\cdot F_i^{[\to]}}{F_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}+C_{i+1}^{[\to]}\cdot\left(1-\frac{\rho\cdot E_{i+1}^{[\to]}}{F_i^{[\to]}\cdot G_i^{[\to]}}+\frac{F_i^{[\to]}}{E_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}\right)+D_i^{[\to]}\cdot\frac{E_{i+1}^{[\to]}}{E_i^{[\to]}\cdot G_i^{[\to]}}\right]\cdot P_{i+3,j}+$$

$$+\left[-C_{i+2}^{[\to]}\cdot\frac{\rho\cdot F_i^{[\to]}}{F_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}+D_{i+1}^{[\to]}\cdot\left(1-\frac{\rho\cdot E_{i+1}^{[\to]}}{F_i^{[\to]}\cdot G_i^{[\to]}}+\frac{F_i^{[\to]}}{E_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}\right)\right]\cdot P_{i+4,j}+$$

$$+\left[-D_{i+2}^{[\to]}\cdot\frac{\rho\cdot F_i^{[\to]}}{F_{i+1}^{[\to]}\cdot G_{i+1}^{[\to]}}\right]\cdot P_{i+5,j}=0 \qquad [5.144]$$

This recurrence can be further rearranged,

$$+\left\{\frac{E_{i+1}^{[\to]}}{G_i^{[\to]}}\cdot\left[\frac{A_i^{[\to]}}{E_i^{[\to]}}\right]\right\}\cdot P_{i,j}+$$

$$+\left\{A_{i+1}^{[\to]}+\frac{E_{i+1}^{[\to]}}{G_i^{[\to]}}\cdot\left[\frac{B_i^{[\to]}}{E_i^{[\to]}}-\frac{\rho\cdot A_{i+1}^{[\to]}}{F_i^{[\to]}}\right]+\frac{F_i^{[\to]}}{G_{i+1}^{[\to]}}\cdot\left[\frac{A_{i+1}^{[\to]}}{E_{i+1}^{[\to]}}\right]\right\}\cdot P_{i+1,j}+$$

$$+\left\{B_{i+1}^{[\to]}+\frac{E_{i+1}^{[\to]}}{G_i^{[\to]}}\cdot\left[\frac{C_i^{[\to]}}{E_i^{[\to]}}-\frac{\rho\cdot B_{i+1}^{[\to]}}{F_i^{[\to]}}\right]+\frac{F_i^{[\to]}}{G_{i+1}^{[\to]}}\cdot\left[\frac{B_{i+1}^{[\to]}}{E_{i+1}^{[\to]}}-\frac{\rho\cdot A_{i+2}^{[\to]}}{F_{i+1}^{[\to]}}\right]\right\}\cdot P_{i+2,j}+$$

$$+\left\{C_{i+1}^{[\to]}+\frac{E_{i+1}^{[\to]}}{G_i^{[\to]}}\cdot\left[\frac{D_i^{[\to]}}{E_i^{[\to]}}-\frac{\rho\cdot C_{i+1}^{[\to]}}{F_i^{[\to]}}\right]+\frac{F_i^{[\to]}}{G_{i+1}^{[\to]}}\cdot\left[\frac{C_{i+1}^{[\to]}}{E_{i+1}^{[\to]}}-\frac{\rho\cdot B_{i+2}^{[\to]}}{F_{i+1}^{[\to]}}\right]\right\}\cdot P_{i+3,j}+$$

$$+\left\{D_{i+1}^{[\to]}+\frac{E_{i+1}^{[\to]}}{G_i^{[\to]}}\cdot\left[-\frac{\rho\cdot D_{i+1}^{[\to]}}{F_i^{[\to]}}\right]+\frac{F_i^{[\to]}}{G_{i+1}^{[\to]}}\cdot\left[\frac{D_{i+1}^{[\to]}}{E_{i+1}^{[\to]}}-\frac{\rho\cdot C_{i+2}^{[\to]}}{F_{i+1}^{[\to]}}\right]\right\}\cdot P_{i+4,j}+$$

$$+\left\{\frac{F_i^{[\to]}}{G_{i+1}^{[\to]}}\cdot\left[-\frac{\rho\cdot D_{i+2}^{[\to]}}{F_{i+1}^{[\to]}}\right]\right\}\cdot P_{i+5,j}=0$$

[5.145]

We can also substitute $G_i^{[\to]}$, applying [5.139(G)] and $G_{i+1}^{[\to]}$ applying also [5.139(G)] but, in this case, adding 1 to i. This way we obtain the following alternative expression,

$$
\begin{aligned}
&\left[\frac{\tau E_{i+1}^{[\to]}\left(A_i^{[\to]}F_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - (i-j+2)\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} \right] \cdot P_{i,j} + \\
&+ \left[A_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(B_i^{[\to]}F_i^{[\to]} - \rho A_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - (i-j+2)\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(A_{i+1}^{[\to]}\, F_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - (i-j+3)\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+1,j} + \\
&+ \left[B_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(C_i^{[\to]}F_i^{[\to]} - \rho B_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - (i-j+2)\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(B_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \rho A_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - (i-j+3)\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+2,j} + \\
&+ \left[C_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(D_i^{[\to]}F_i^{[\to]} - \rho C_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - (i-j+2)\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(C_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \rho B_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - (i-j+3)\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+3,j} + \\
&+ \left[D_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(- \rho D_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - (i-j+2)\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(D_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \rho C_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - (i-j+3)\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+4,j} + \\
&+ \left[\frac{\tau F_i^{[\to]}\left(- \rho D_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - (i-j+3)\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+5,j} = \\
&= 0
\end{aligned}
$$

[5.146]

Recalling [5.93], $\sigma_k^{[-]} = (i-j+k)$… recurrence [5.146] can take one more form, simpler, accounting that, $(i-j+2) = \sigma_2^{[-]}$ and $(i-j+3) = \sigma_3^{[-]}$. That additional form reads,

$$
\begin{aligned}
&\left[\frac{\tau E_{i+1}^{[\to]}\left(A_i^{[\to]}F_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - \sigma_2^{[-]}\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} \right] \cdot P_{i,j} + \\
&+ \left[A_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(B_i^{[\to]}F_i^{[\to]} - \rho A_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - \sigma_2^{[-]}\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(A_{i+1}^{[\to]}\, F_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - \sigma_3^{[-]}\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+1,j} + \\
&+ \left[B_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(C_i^{[\to]}F_i^{[\to]} - \rho B_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - \sigma_2^{[-]}\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(B_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \rho A_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - \sigma_3^{[-]}\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+2,j} + \\
&+ \left[C_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(D_i^{[\to]}F_i^{[\to]} - \rho C_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - \sigma_2^{[-]}\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(C_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \rho B_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - \sigma_3^{[-]}\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+3,j} + \\
&+ \left[D_{i+1}^{[\to]} + \frac{\tau\, E_{i+1}^{[\to]}\left(- \rho D_{i+1}^{[\to]}\, E_i^{[\to]}\right)}{\rho\tau\, E_i^{[\to]}\, E_{i+1}^{[\to]} - \sigma_2^{[-]}\, E_i^{[\to]}\, F_i^{[\to]} - \tau F_{i-1}^{[\to]}\, F_i^{[\to]}} + \frac{\tau F_i^{[\to]}\left(D_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \rho C_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - \sigma_3^{[-]}\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+4,j} + \\
&+ \left[\frac{\tau F_i^{[\to]}\left(- \rho D_{i+2}^{[\to]}\, E_{i+1}^{[\to]}\right)}{\rho\tau\, E_{i+1}^{[\to]}\, E_{i+2}^{[\to]} - \sigma_3^{[-]}\, E_{i+1}^{[\to]}\, F_{i+1}^{[\to]} - \tau F_i^{[\to]}\, F_{i+1}^{[\to]}} \right] \cdot P_{i+5,j} = 0
\end{aligned}
\qquad [5.147]
$$

4.8. Plane chart (i , j) for $R_{i,j}$, and for $Q_{i,j}$ and $P_{i,j}$ Functions deduced till now.

In this last section of Chapter V we limit ourselves to present the location and size (number of terms) for all the contiguous and straight line recurrence relations for functions $R_{i,j}$, and then also for functions $Q_{i,j}$ and $P_{i,j}$, thus observing the increase in number of terms when one moves from Plane Chart of $R_{i,j}(\tau)$, at TABLE 8, to Plane Chart of $Q_{i,j}(\tau)$ or, what is the same, to Plane Chart of $P_{i,j}(\tau)$, at TABLE 9 (in next page).

TABLE 8

Plane chart (i , j) for R i , j Functions

	0,6							
	0,5	1,5						
	0,4	1,4	2,4					
	0,3	1,3	2,3	3,3				
	0,2	1,2	2,2	3,2	4,2			
-1,1 ↑	0,1	1,1	2,1	3,1	4,1	5,1		
↑	0,0	1,0	2,0	3,0	4,0	5,0	6,0	
Δj Δi →								

LEGEND - Contiguous straight line homogeneous recurrences for R i , j Functions

Orientation	Number of terms	Equation
Vertical	4	$F(R_{i,j}, R_{i,j+1}, R_{i,j+2}, R_{i,j+3}) = 0$ [5.46]
Horizontal	4	$F(R_{i,j}, R_{i+1,j}, R_{i+2,j}, R_{i+3,j}) = 0$ [5.75]
Diagonal positive	4	$F(R_{i,j}, R_{i+1,j+1}, R_{i+2,j+2}, R_{i+3,j+3})$ [5.85]
Diagonal negative	2	$F(R_{i,j}, R_{i-1,j+1}) = 0$ [5.73]

TABLE 9

Plane chart (i , j) for $Q_{i,j}$ and $P_{i,j}$ Functions										
			0,6							
			0,5	1,5						
			0,4	1,4	2,4		4,4			
-3,3			0,3	1,3	2,3	3,3				
	-2,2		0,2	1,2	2,2	3,2	4,2			
		-1,1	0,1	1,1	2,1	3,1	4,1	5,1		
		↑	0,0	1,0	2,0	3,0	4,0	5,0	6,0	
		Δj Δi →								

LEGEND - Contiguous straight line homogeneous recurrences for $Q_{i,j}$ or $P_{i,j}$ *Functions*		
Orientation	**Number of terms**	**Equation**
Vertical	**6**	$F(P_{i,j}, P_{i,j+1}, P_{i,j+2}, P_{i,j+3}, P_{i,j+4}, P_{i,j+5}) = 0$ [5.70], [5.71], [5.98], [5.99]
Horizontal	**6**	$F(P_{i,j}, P_{i,j+1}, P_{i,j+2}, P_{i,j+3}\ P_{i,j+4}, P_{i,j+5}) = 0$ [5.144], [5.145], [5.146], [5.147]
Diagonal positive	**5**	$F(P_{i,j}, P_{i,j+1}, P_{i,j+2}, P_{i,j+3}, P_{i,j+4}) = 0$ [5.124]
Diagonal negative	**4**	$F(P_{i,j}, P_{i,j+1}, P_{i,j+2}, P_{i,j+3}) = 0$ [5.113], [5.114]

CHAPTER VI

Analytic solution for the general model version (near at hand mathematical intractability borderline)

1. Introductory observations about "Mono-layered concentrated" growth kinetics version resolution

In this Chapter the analytic solution for the most general version of our model will be deduced in detail. This deduction is similar to, and inspired in, an old one of ours [50] but then done only for the zero order kinetics case. Now we present a more general deduction, valid for all the kinetics under analysis.
To avoid the inconvenient, facing the cumbersome formulae in analysis, whenever possible a description illustrated by diagrams and (or) boxes will be used.
Also, to make the explanation easier, all needed formulas of Chapter IV will be signed with a new numeration whose format is [6.XXX]. Whenever this occurs the enumeration change will be clearly signalized. This way all the deduction becomes more self-contained.
For start, returning to the solution found for the most simplified version,

$$\theta_n(\tau) = \alpha_n \cdot e^y \cdot \sum_{k=0}^{\infty} \beta_k \cdot J_{n+k}(x) \qquad [6.1] \text{ ([4.117] in Chapter IV)}$$

and recalling the meaning of (x) and (y),

$$x = 2 \cdot \tau \cdot \sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta)} \qquad [6.2]$$

$$y = (1 - \rho - 2 \cdot \Phi - \delta) \cdot \tau \qquad [6.3]$$

… we observe that time (τ) dependence is confined in the exponential factor, which is the same for all functions $\theta_n(\tau)$, and, in a more complex way, inside Bessel functions arguments.
Coefficients α_n and β_k are, although complicated, merely constants.
We also notice, as an important feature, that in each function $\theta_n(\tau)$ the order in Bessel functions infinite series start also in n.
Consequently one can adopt, as a guess solution, a function like,

$$\theta_n(\tau) = e^{a \cdot \tau} \cdot \sum_{k=n}^{\infty} d_{n,k} \cdot J_k(2 \cdot b \cdot \tau) \qquad [6.4]$$

where,

$$a = (1 - \rho - 2 \cdot \Phi - \delta_1) \qquad [6.5]$$

$$b = \sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta_1)} \qquad [6.6]$$

At this point a remark must be done about the fact that we are now searching the most general version and, coherently with such purpose, a differentiation in the detachment parameters δ_n according to the fraction $\theta_n(\tau)$ from where the detachment happens, becomes compulsory.
We chose, in the definitions of a and b, [6.5] and [6.6], the detachment from fraction "mono" covered $\theta_1(\tau)$ as the most logic option and, later in the sequel, all the other δ_n for $(n>1)$ will appear but not inserted in Bessel functions arguments.
As a useful mathematical tool, the following derivation rules must be applied all over the deduction,

$$\frac{dJ_k(2 \cdot b \cdot \tau)}{d\tau} = b \cdot \{J_{k-1}(2 \cdot b \cdot \tau) - J_{k+1}(2 \cdot b \cdot \tau)\} \qquad [6.7]$$

$$\frac{dJ_0(2\cdot b\cdot \tau)}{d\tau} = -2\cdot b\cdot J_1(2\cdot b\cdot \tau) \qquad [6.8]$$

The first rule for all the Bessel functions $J_k(2\cdot b\cdot \tau)$ with $(k\geq 1)$.
And the second rule, specific for $J_0(2\cdot b\cdot \tau)$, avoids negative order Bessel functions which would come into play if the first rule is applied to this function.
The important advantage of these rules is the fact that, after application, time (τ) remains exclusively inside Bessel functions argument.
Lastly let emphasize the criterion leading to the whole solution:

-Whenever an equality between two expressions must be satisfied and, in those two expressions, time (τ) figures only inside Bessel functions argument, the identity holds if an equality, term by term, is set into play. Here the terms that must be equal, one to each other, are those with the same Bessel function order in both sides of the main equality.

Applying all the aforesaid rules and criteria the general solution for the most complete version of the early stages in biofilm formation and growth model is obtained solving an algebra problem, although extensive and rather complex.

2. New dimensionless parameters.

Going now into calculus, first we remember, once again, the system, in dimensionless form, that must be solved,

$$\left(\frac{d\theta_0}{d\tau}\right) = -\rho\cdot\theta_0 + [\delta_1-(1-\Phi)]\cdot\theta_1 - (1-\Phi)\cdot\left(\sum_{j=2}^{L}\psi_j\cdot\theta_j\right) \qquad [6.9]\ ([4.17] \text{ in Chapter IV})$$

$$\left(\frac{d\theta_1}{d\tau}\right) = \rho\cdot\theta_0 + [(1-2\cdot\Phi)-\rho-\delta_1]\cdot\theta_1 + [\delta_2+(1-\Phi)\cdot(\psi_2-1)]\cdot\theta_2 +$$
$$+(1-\Phi)\cdot\left[\sum_{j=3}^{L}(\psi_j-\psi_{j-1})\cdot\theta_j\right] - (1-\Phi)\cdot\psi_L\cdot\theta_{L+1} \qquad [6.10]\ ([4.18] \text{ in Chapter IV})$$

$$\left(\frac{d\theta_n}{d\tau}\right) = \left[\rho + \Phi\cdot\left(\sum_{j=1}^{n-1}\psi_j\right)\right]\cdot\theta_{n-1} + \left[(1-\Phi)-\rho-\Phi\cdot\left(\sum_{j=1}^{n}\psi_j\right)-\delta_n\right]\cdot\theta_n +$$
$$+[(1-\Phi)\cdot(\psi_2-1)+\delta_{n+1}]\cdot\theta_{n+1} + (1-\Phi)\cdot\left[\sum_{j=2}^{L-1}(\psi_{j+1}-\psi_j)\cdot\theta_{n+j}\right] - (1-\Phi)\cdot\psi_L\cdot\theta_{n+L}$$
$$\dots\ if\ (2\leq n\leq L) \qquad [6.11]\ ([4.19] \text{ in Chapter IV})$$

$$\left(\frac{d\theta_n}{d\tau}\right) = \left[\rho + \Phi\cdot\left(\sum_{j=1}^{L}\psi_j\right)\right]\cdot\theta_{n-1} + \left[(1-\Phi)-\rho-\Phi\cdot\left(\sum_{j=1}^{L}\psi_j\right)-\delta_n\right]\cdot\theta_n +$$
$$+[(1-\Phi)\cdot(\psi_2-1)+\delta_{n+1}]\cdot\theta_{n+1} + (1-\Phi)\cdot\left[\sum_{j=2}^{L-1}(\psi_{j+1}-\psi_j)\cdot\theta_{n+j}\right] - (1-\Phi)\cdot\psi_L\cdot\theta_{n+L}$$
$$\dots\ if\ (n\geq L+1) \qquad [6.12]\ ([4.20] \text{ in Chapter IV})$$

The initial condition must, of course, be:

$$\theta_0(0) = 1, \;\; and \;\; \theta_n(0) = 0 \;\; for\ n{>}0 \qquad [6.13]\ ([4.21]\ \text{in Chapter IV})$$

(at the beginning (τ=0) al the solid support is bare)

Before any further step is convenient, for sake of notation economy, to rewrite the system this way,

$$\left(\frac{d\theta_0}{d\tau}\right) = -A_0 \cdot \theta_0 + B_1 \cdot \theta_1 - \left(\sum_{j=2}^{L} D_j \cdot \theta_j\right) \qquad [6.14(0)]$$

$$\left(\frac{d\theta_1}{d\tau}\right) = A_0 \cdot \theta_0 + (B_1 - A_1) \cdot \theta_1 + (D_2 - B_2) \cdot \theta_2 + \left[\sum_{j=2}^{L-1} (D_{j+1} - D_j) \cdot \theta_{j+1}\right] - D_L \cdot \theta_{L+1} \qquad [6.14(1)]$$

$$\left(\frac{d\theta_n}{d\tau}\right) = A_{n-1} \cdot \theta_{n-1} + (B_n - A_n) \cdot \theta_n + +(D_2 - B_{n+1}) \cdot \theta_{n+1} + \left[\sum_{j=2}^{L-1} (D_{j+1} - D_j) \cdot \theta_{n+j}\right] - D_L \cdot \theta_{L+n}$$

$$\ldots\ if\ (2 \le n \le L) \qquad [6.14(L)]$$

$$\left(\frac{d\theta_n}{d\tau}\right) = A_L \cdot \theta_{n-1} + (B_n - A_L) \cdot \theta_n + +(D_2 - B_{n+1}) \cdot \theta_{n+1} + \left[\sum_{j=2}^{L-1} (D_{j+1} - D_j) \cdot \theta_{n+j}\right] - D_L \cdot \theta_{L+n}$$

$$\ldots\ if\ (n \ge L + 1) \qquad [6.14(n)]$$

The initial condition must, of course, be:

$$\theta_0(0) = 1, \;\; and \;\; \theta_n(0) = 0 \;\; for\ n{>}0 \qquad [6.14, \tau, 0]$$

(at the beginning (τ=0) al the solid support is bare)

Where the meaning of the new parameters are,

$$A_0 = \rho \qquad [6.15(0)]$$

$$A_k = \rho + \Phi \cdot \sum_{j=1}^{k} \psi_j \;\; \ldots for\ (k \ge 1) \qquad [6.15(k)]$$

$$B_k = 1 - \Phi - \delta_k \qquad [6.16]$$

$$D_k = (1 - \Phi) \cdot \psi_k \qquad [6.17]$$

Latter on the sequel, we will see that it is also convenient introduce the corresponding asterisk versions this way,

$$A_k^* = \frac{A_k}{b} \qquad [6.18]$$

$$B_k^* = \frac{B_k}{b} \quad [6.19]$$

$$D_k^* = \frac{D_k}{b} \quad [6.20]$$

… where b has already been defined in [6.6].
Observe the additional information in the enumeration, inserting (k) in the equations of the system, which indicate the maximum order k of functions $\theta_k(\tau)$ relatively to each equation must be applied. In the case of $k = n$, in general, n goes to infinite.
In the same way, for A_0 and A_k we also make a similar differentiation, not adding the main enumeration.
To the particular case of the initial condition also additional information is $[6.14, \tau, 0]$
We notice, by simple inspection, that this rewritten system allows to cluster equations $[6.14(1)]$ and $[6.14(L)]$ into only one, valid for the range $(1 \leq n \leq L)$.
Equation $[6.14(L)]$ broads is range from $(2 \leq n \leq L)$ to $(1 \leq n \leq L)$ and equation $[6.14(1)]$ disappears.
Accordingly, the new system in a more economic form, reads,

$$\left(\frac{d\theta_0}{d\tau}\right) = -A_0 \cdot \theta_0 + B_1 \cdot \theta_1 - \left(\sum_{j=2}^{L} D_j \cdot \theta_j\right) \quad [6.21(0)]$$

$$\left(\frac{d\theta_n}{d\tau}\right) = A_{n-1} \cdot \theta_{n-1} + (B_n - A_n) \cdot \theta_n + +(D_2 - B_{n+1}) \cdot \theta_{n+1} + \left[\sum_{j=2}^{L-1}(D_{j+1} - D_j) \cdot \theta_{n+j}\right] - D_L \cdot \theta_{L+n}$$

$$\dots \; if \; (1 \leq n \leq L) \quad [6.21(L)]$$

$$\left(\frac{d\theta_n}{d\tau}\right) = A_L \cdot \theta_{n-1} + (B_n - A_L) \cdot \theta_n + +(D_2 - B_{n+1}) \cdot \theta_{n+1} + \left[\sum_{j=2}^{L-1}(D_{j+1} - D_j) \cdot \theta_{n+j}\right] - D_L \cdot \theta_{L+n}$$

$$\dots \; if \; (n \geq L + 1) \quad [6.21(n)]$$

The initial condition must, of course, be:

$$\theta_0(0) = 1, \;\; and \;\; \theta_n(0) = 0 \;\; for \; n > 0 \quad [6.21, \tau, 0]$$

(at the beginning (τ=0) al the solid support is bare)

Observe that we have to retake the enumeration order to avoid confusion between $[6.14(L)]$ valid in range $(2 \leq n \leq L)$ and $[6.14(L)]$ valid in range $(1 \leq n \leq L)$.
The equations $[6.14(0)]$ and $[6.14(n)]$ are respectively equals to equations $[6.21(0)]$ and $[6.21(n)]$, as well as the initial condition $[6.14, \tau, 0]$ is equal to $[6.21, \tau, 0]$.
But, from now on, we will adopt, for these cases only the enumeration $[6.21(0)]$, $[6.21(n)]$ and $[6.21, \tau, 0]$.

3. First steps: Left and right hand sides worked separately

Left hand sides (L.H.S.) and right hand sides (R.H.S.) in $[6.21(0)]$, $[6.21(L)]$ and $[6.21(n)]$, are worked separately. From now on Bessel functions $J_k(2 \cdot b \cdot \tau)$ will be written without the

argument $(2 \cdot b \cdot \tau)$ because this will not originate confusion or misinterpretations and lights the notation.
First of all the two distinct (L.H.S.) can be reached applying rules [6.7] and [6.8] over the guess solution [6.4].
Rule [6.8] is needed only in the (L.H.S.) of [6.21(0)] because $\theta_0(\tau)$ is the only function $\theta_n(\tau)$ where, according to the guess solution, exists a term in J_0
After applying the rules, the two (L.H.S.) are [6.22(0)] and [6.22(n)].

Left Hand Side (L.H.S.) of [6.21(0)]

$$e^{-a\cdot\tau}\cdot\frac{d\theta_0(\tau)}{d\tau}=\left[a\cdot d_{0,0}+b\cdot d_{0,1}\right]\cdot J_0+\left[a\cdot d_{0,1}+b\cdot\left(d_{0,2}-2\cdot d_{0,0}\right)\right]\cdot J_1+$$
$$+\sum_{k=2}^{\infty}\left[a\cdot d_{0,k}+b\cdot\left(d_{0,k+1}-d_{0,k-1}\right)\right]\cdot J_k \qquad [6.22(0)]$$

Common Left Hand Side (L.H.S.) of [6.21(L)] and [6.21(n)],

$$e^{-a\cdot\tau}\cdot\frac{d\theta_n(\tau)}{d\tau}=\left[b\cdot d_{n,n}\right]\cdot J_{n-1}+\left[a\cdot d_{n,n}+b\cdot d_{n,n+1}\right]\cdot J_n+$$
$$+\sum_{k=n+1}^{\infty}\left[a\cdot d_{n,k}+b\cdot\left(d_{n,k+1}-d_{n,k-1}\right)\right]\cdot J_k$$

... *with* $(1\le n\le L)$ *for* [6.21(L)] *and* $(n\ge L+1)$ *for* [6.21(n)] **[6.22(n)]**

The (R.H.S.) of [6.21(0)], [6.21(L)] and [6.21(n)] are merely obtained by substitution of [6.4] and reads ([6.23(0)], [6.23(L)] and [6.23(n)]),

Right Hand Side (R.H.S.) of [6.21(0)]

$$e^{-a\cdot\tau}\cdot\left\{-A_0\cdot\theta_0+B_1\cdot\theta_1-\left[\sum_{j=2}^{L}D_j\cdot\theta_j\right]\right\}=$$
$$=-A_0\cdot\left[\sum_{k=0}^{\infty}d_{0,k}\cdot J_k\right]-B_1\cdot\left[\sum_{k=1}^{\infty}d_{1,k}\cdot J_k\right]-\left\{\sum_{j=2}^{L}D_j\cdot\left[\sum_{k=j}^{\infty}d_{j,k}\cdot J_k\right]\right\} \qquad [6.23(0)]$$

Right Hand Side (R.H.S.) of [6.21(L)]

$$e^{-a\cdot\tau}\cdot\left\{A_{n-1}\cdot\theta_{n-1}+(B_n-A_n)\cdot\theta_n+(D_2-B_{n+1})\cdot\theta_{n+1}+\left[\sum_{j=2}^{L-1}(D_{j+1}-D_j)\cdot\theta_{n+j}\right]-D_L\cdot\theta_{L+n}\right\}=$$
$$=A_{n-1}\cdot\left[\sum_{k=n-1}^{\infty}d_{n-1,k}\cdot J_k\right]+(B_n-A_n)\cdot\left[\sum_{k=n}^{\infty}d_{n,k}\cdot J_k\right]+$$
$$+(D_2-B_{n+1})\cdot\left[\sum_{k=n+1}^{\infty}d_{n+1,k}\cdot J_k\right]+\left\{\sum_{j=2}^{L-1}(D_{j+1}-D_j)\cdot\left[\sum_{k=n+j}^{\infty}d_{n+j,k}\cdot J_k\right]\right\}+$$

$$+(-D_L)\cdot\left[\sum_{k=L+n}^{\infty} d_{L+n,k}\cdot J_k\right] \qquad \ldots if\ (1\le n\le L)$$

[6.23(L)]

Right Hand Side (R.H.S.) of [6.21(n)]

$$e^{-a\cdot\tau}\cdot\left\{A_L\cdot\theta_{n-1}+(B_n-A_L)\cdot\theta_n++(D_2-B_{n+1})\cdot\theta_{n+1}+\left[\sum_{j=2}^{L-1}(D_{j+1}-D_j)\cdot\theta_{n+j}\right]-D_L\cdot\theta_{L+n}\right\}=$$

$$=A_L\cdot\left[\sum_{k=n-1}^{\infty} d_{n-1,k}\cdot J_k\right]+(B_n-A_L)\cdot\left[\sum_{k=n}^{\infty} d_{n,k}\cdot J_k\right]+$$

$$+(D_2-B_{n+1})\cdot\left[\sum_{k=n+1}^{\infty} d_{n+1,k}\cdot J_k\right]+\left\{\sum_{j=2}^{L-1}(D_{j+1}-D_j)\cdot\left[\sum_{k=n+j}^{\infty} d_{n+j,k}\cdot J_k\right]\right\}+$$

$$+(-D_L)\cdot\left[\sum_{k=L+n}^{\infty} d_{L+n,k}\cdot J_k\right] \qquad \ldots\ if\ (n\ge L+1)$$

[6.23(n)]

Now equations **[6.22(0)]**, **[6.22(n)]**, **[6.23(0)]**, **[6.23(L)]** and **[6.23(n)]** are divided by b.
We must not only recall [6.18], [6.19] and [6.20], but also notice that,

$$a=1-\rho-2\cdot\Phi-\delta_1$$

$$1-\rho-2\cdot\Phi-\delta_1=(1-\Phi-\delta_1)-(\rho+\Phi)$$

$$(1-\Phi-\delta_1)-(\rho+\Phi)=B_1-A_1$$

So,

$$a=B_1-A_1$$

... and,

$$\frac{a}{b}=B_1^*-A_1^* \qquad [6.24]$$

Then the result, of dividing by b, equations **[6.22(0)]**, **[6.22(n)]**, **[6.23(0)]**, **[6.23(L)]** and **[6.23(n)]**, reads, respectively, like **[6.25(0)]**, **[6.25(n)]**, **[6.26(0)]**, **[6.26(L)]** and **[6.26(n)]**,

Left Hand Side (L.H.S.) of [6.21(0)]

$$\left(\frac{e^{-a\cdot\tau}}{b}\right)\cdot\frac{d\theta_0(\tau)}{d\tau}=\left[(B_1^*-A_1^*)\cdot d_{0,0}+d_{0,1}\right]\cdot J_0+\left[(B_1^*-A_1^*)\cdot d_{0,1}+d_{0,2}-2\cdot d_{0,0}\right]\cdot J_1+$$

$$+\sum_{k=2}^{\infty}\left[(B_1^*-A_1^*)\cdot d_{0,k}+d_{0,k+1}-d_{0,k-1}\right]\cdot J_k$$

[6.25(0)]

Common Left Hand Side (L.H.S.) of [6.21(L)] and [6.21(n)],

$$\left(\frac{e^{-a\cdot\tau}}{b}\right)\cdot\frac{d\theta_n(\tau)}{d\tau}=\left[d_{n,n}\right]\cdot J_{n-1}+\left[(B_1^*-A_1^*)\cdot d_{n,n}+d_{n,n+1}\right]\cdot J_n+$$

$$+\sum_{k=n+1}^{\infty}\left[\left(B_1^*-A_1^*\right)\cdot d_{n,k}+d_{n,k+1}-d_{n,k-1}\right]\cdot J_k$$

$$\dots\ with\ (1\le n\le L)\ for\ [6.21(L)]\ and\ (n\ge L+1)\ for\ [6.21(n)] \qquad [6.25(n)]$$

Right Hand Side (R.H.S.) of [6.21(0)]

$$e^{-a\cdot\tau}\cdot\left\{-A_0^*\cdot\theta_0+B_1^*\cdot\theta_1-\left[\sum_{j=2}^{L}D_j^*\cdot\theta_j\right]\right\}=$$

$$=-A_0^*\cdot\left[\sum_{k=0}^{\infty}d_{0,k}\cdot J_k\right]-B_1^*\cdot\left[\sum_{k=1}^{\infty}d_{1,k}\cdot J_k\right]-\left\{\sum_{j=2}^{L}D_j^*\cdot\left[\sum_{k=j}^{\infty}d_{j,k}\cdot J_k\right]\right\}$$

$$[6.26(0)]$$

Right Hand Side (R.H.S.) of [6.21(L)]

$$e^{-a\cdot\tau}\cdot\left\{A_{n-1}^*\cdot\theta_{n-1}+(B_n^*-A_n^*)\cdot\theta_n+(D_2^*-B_{n+1}^*)\cdot\theta_{n+1}+\left[\sum_{j=2}^{L-1}(D_{j+1}^*-D_j^*)\cdot\theta_{n+j}\right]-D_L^*\cdot\theta_{L+n}\right\}=$$

$$=A_{n-1}^*\cdot\left[\sum_{k=n-1}^{\infty}d_{n-1,k}\cdot J_k\right]+(B_n^*-A_n^*)\cdot\left[\sum_{k=n}^{\infty}d_{n,k}\cdot J_k\right]+$$

$$+(D_2^*-B_{n+1}^*)\cdot\left[\sum_{k=n+1}^{\infty}d_{n+1,k}\cdot J_k\right]+\left\{\sum_{j=2}^{L-1}\left(D_{j+1}^*-D_j^*\right)\cdot\left[\sum_{k=n+j}^{\infty}d_{n+j,k}\cdot J_k\right]\right\}$$

$$+(-D_L^*)\cdot\left[\sum_{k=L+n}^{\infty}d_{L+n,k}\cdot J_k\right] \qquad \dots if\ (1\le n\le L)$$

$$[6.26(L)]$$

Right Hand Side (R.H.S.) of [6.21(n)]

$$e^{-a\cdot\tau}\cdot\left\{A_L^*\cdot\theta_{n-1}+(B_n^*-A_L^*)\cdot\theta_n++(D_2^*-B_{n+1}^*)\cdot\theta_{n+1}+\left[\sum_{j=2}^{L-1}(D_{j+1}^*-D_j^*)\cdot\theta_{n+j}\right]-D_L^*\cdot\theta_{L+n}\right\}=$$

$$=A_L^*\cdot\left[\sum_{k=n-1}^{\infty}d_{n-1,k}\cdot J_k\right]+(B_n^*-A_L^*)\cdot\left[\sum_{k=n}^{\infty}d_{n,k}\cdot J_k\right]+$$

$$+(D_2^*-B_{n+1}^*)\cdot\left[\sum_{k=n+1}^{\infty}d_{n+1,k}\cdot J_k\right]+\left\{\sum_{j=2}^{L-1}\left(D_{j+1}^*-D_j^*\right)\cdot\left[\sum_{k=n+j}^{\infty}d_{n+j,k}\cdot J_k\right]\right\}$$

$$+(-D_L^*)\cdot\left[\sum_{k=L+n}^{\infty}d_{L+n,k}\cdot J_k\right] \qquad \dots\ if\ (n\ge L+1)$$

$$[6.26(n)]$$

4. Right hand sides (R. H. S.) double summation transformation

Right hand sides have, all of their, a double summation which form is not adequate for to pursuit before a suitable needed transformation is done.
Let collect those three double summations,

Right Hand Side Sums (R.H.S.S.) for modification purpose

$$\ldots from\ [6.26(0)] \qquad \left\{\sum_{j=2}^{L} D_j^* \cdot \left[\sum_{k=j}^{\infty} d_{j,k} \cdot J_k\right]\right\}$$

[6.27(0)]

$$\ldots from\ [6.26(L)] \qquad \left\{\sum_{j=2}^{L-1} (D_{j+1}^* - D_j^*) \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\} \ldots\ with\ (1 \le n \le L)$$

[6.27(L)]

$$\ldots from\ [6.26(n)] \qquad \left\{\sum_{j=2}^{L-1} (D_{j+1}^* - D_j^*) \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\} \ldots\ with\ (n \ge L+1)$$

[6.27(n)]

It is easy recognizable that they are very much alike.
In fact it is possible to define a common form, transform it and, at the end, differentiate back again the result.
This way an economy of effort is achieved.
Such common form is defined by [6.28],

(R.H.S.S.) for modification purpose: Common Form

$$\textbf{Common Form:} \qquad \left\{\sum_{j=2}^{L^*} D_j^{**} \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\}$$

$Where\ L^* = (L-1)\ \{[6.26(L)]\ and\ [6.26(n)]\}\ or\ L\ \{[6.26(0)]\}$

$\ldots\ D_j^{**} = D_j^*\ \{[6.26(0)]\}\ or\ (D_{j+1}^* - D_j^*)\ \{[6.26(L)]\ and\ [6.26(n)]\}$

$\ldots\ and\ n = 0\ \{[6.26(0)]\}, or\ (1 \le n \le L)\ \{[6.26(L)]\}, or\ (n \ge L+1)\ \{[6.26(n)]\}$

[6.28]

The modification of this common double summation is a little cumbersome.
So we will explain it by successive steps,

(R.H.S.S.) Common Form modification (Step I)

$$\textbf{Common Form:} \qquad \left\{\sum_{j=2}^{L^*} D_j^{**} \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\}$$

$$\left\{\sum_{j=2}^{L^*} D_j^{**} \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\} = \left\{\sum_{j=2}^{L^*} \left[\sum_{k=n+j}^{\infty} (D_j^{**} \cdot d_{n+j,k}) \cdot J_k\right]\right\}$$

$$\left\{\sum_{j=2}^{L^*}\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]\right\}=\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=2}+$$

$$+\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=3}+\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=4}+\cdots$$

$$\ldots+\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=L^*-2}+\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=L^*-1}+\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=L^*}$$

$[\mathbf{6.29(\mathit{I})}]$

Detailing these summations,

$(\boldsymbol{R.H.S.S.})$ ***Common Form modification (Step II)***

$$[\boldsymbol{Sum\ 1}]\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=2}=\left[\sum_{k=n+2}^{\infty}\left(D_2^{**}\cdot d_{n+2,k}\right)\cdot J_k\right]$$

$\boldsymbol{Bessel\ terms\ in\ [Sum\ 1]: J_{n+2}, J_{n+3}, J_{n+4}, \ldots\ J_{n+i}\ \ldots i\rightarrow\infty}$

$$[\boldsymbol{Sum\ 2}]\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=3}=\left[\sum_{k=n+3}^{\infty}\left(D_3^{**}\cdot d_{n+3,k}\right)\cdot J_k\right]$$

$\boldsymbol{Bessel\ terms\ in\ [Sum\ 2]: J_{n+3}, J_{n+4}, J_{n+5}, \ldots\ J_{n+i}\ \ldots i\rightarrow\infty}$

$$[\boldsymbol{Sum\ 3}]\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=4}=\left[\sum_{k=n+4}^{\infty}\left(D_4^{**}\cdot d_{n+4,k}\right)\cdot J_k\right]$$

$\boldsymbol{Bessel\ terms\ in\ [Sum\ 3]: J_{n+4}, J_{n+5}, J_{n+6}, \ldots\ J_{n+i}\ \ldots i\rightarrow\infty}$

… ….

… ….

$$[\mathbf{Sum}\ \boldsymbol{L^*}-\mathbf{3}]\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=L^*-2}=\left[\sum_{k=n+L^*-2}^{\infty}\left(D_{L^*-2}^{**}\cdot d_{n+L^*-2,k}\right)\cdot J_k\right]$$

$\boldsymbol{Bessel\ terms\ in\ [Sum\ L^*-3]: J_{n+L^*-2}, J_{n+L^*-1}, J_{n+L^*}, \ldots\ J_{n+i}\ \ldots i\rightarrow\infty}$

$$[\mathbf{Sum}\ \boldsymbol{L^*}-\mathbf{2}]\left[\sum_{k=n+j}^{\infty}\left(D_j^{**}\cdot d_{n+j,k}\right)\cdot J_k\right]_{j=L^*-1}=\left[\sum_{k=n+L^*-1}^{\infty}\left(D_{L^*-1}^{**}\cdot d_{n+L^*-1,k}\right)\cdot J_k\right]$$

$\boldsymbol{Bessel\ terms\ in\ [Sum\ L^*-2]: J_{n+L^*-1}, J_{n+L^*}, J_{n+L^*+1}, \ldots\ J_{n+i}\ \ldots i\rightarrow\infty}$

$$[\textbf{Sum } L^* - 1] \left[\sum_{k=n+j}^{\infty} \left(D_j^{**} \cdot d_{n+j,k}\right) \cdot J_k\right]_{j=L^*} = \left[\sum_{k=n+L^*}^{\infty} \left(D_{L^*}^{**} \cdot d_{n+L^*,k}\right) \cdot J_k\right]$$

$$\textit{Bessel terms in } [\text{Sum } L^* - 1]: J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$$

[6.29(*II*)]

Accounting for the whole set of Bessel terms,

(R.H.S.S.) Common Form modification (Step III)

We conclude that the whole set has terms in,

$$J_{n+2}, J_{n+3}, J_{n+4}, J_{n+5}, J_{n+6}, \cdots J_{n+L^*-2}, J_{n+L^*-1}, J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$$

***Expliciting by Sums*:,**

$[\text{Sum } 1]: J_{n+2}, J_{n+3}, J_{n+4}, J_{n+5}, J_{n+6}, \cdots J_{n+L^*-2}, J_{n+L^*-1}, J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$
$[\text{Sum } 2]: J_{n+3}, J_{n+4}, J_{n+5}, J_{n+6}, \cdots J_{n+L^*-2}, J_{n+L^*-1}, J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$
$[\text{Sum } 3]: J_{n+4}, J_{n+5}, J_{n+6}, \cdots J_{n+L^*-2}, J_{n+L^*-1}, J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$
… …
… …
$[\text{Sum } L^* - 3]: J_{n+L^*-2}, J_{n+L^*-1}, J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$
$[\text{Sum } L^* - 2]: J_{n+L^*-1}, J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$
$[\text{Sum } L^* - 1]: J_{n+L^*}, J_{n+L^*+1}, J_{n+L^*+2}, \cdots J_{n+i} \dots i \to \infty$

[6.29(*III*)]

Reorganizing terms,

(R.H.S.S.) Common Form modification (Step IV)

***Accordingly, the inventory of terms in J_{n+i} functions reads*:**
− ***One term in J_{n+2}, from*** [Sum 1]
− ***Two terms in J_{n+3}, from*** [Sum 1] ***and*** [Sum 2]
− ***Tree terms in J_{n+4}, from*** [Sum 1], [Sum 2] ***and*** [Sum 3]
… …
… …

− $(L^* - 3)$ ***terms in J_{n+L^*-2}, from*** [Sum 1], [Sum 2], [Sum 3], …, [Sum $L^* - 4$] ***and*** [Sum $L^* - 3$]
− $(L^* - 2)$ ***terms in J_{n+L^*-1}, from*** [Sum 1], [Sum 2], [Sum 3], …, [Sum $L^* - 3$] ***and*** [Sum $L^* - 2$]

− $(L^* - 1)$ ***terms in J_{n+L^*}, from*** [Sum 1], [Sum 2], [Sum 3], …,
…, [Sum $L^* - 3$], [Sum $L^* - 2$] ***and*** [Sum $L^* - 1$]

… ***and, for the terms in J_{n+i}, where*** (i>L^*) ***always*:**

− $(L^* - 1)$ ***terms in J_{n+i}, with $i = (L^* + 1), (L^* + 2), (L^* + 3), \dots$, from all the sums,*** [Sum 1],
[Sum 2], [Sum 3], …, [Sum $L^* - 3$], [Sum $L^* - 2$] ***and*** [Sum $L^* - 1$]

[6.29(*IV*)]

Including now, in the terms, theirs explicit expressions,

$(\boldsymbol{R.H.S.S.})$ ***Common Form modification*** $(\boldsymbol{Step\ V})$

The cataloguing of those terms is,

$\boldsymbol{J_{n+2}}$ ***, from*** **[Sum 1]**

$$\left(D_2^{**}\cdot d_{n+2,n+2}\right)\cdot J_{n+2}$$

$\boldsymbol{J_{n+3}}$ ***, from*** **[Sum 1]** ***and*** **[Sum 2]**

$$\left(D_2^{**}\cdot d_{n+2,n+3}\right)\cdot J_{n+3}$$
$$\left(D_3^{**}\cdot d_{n+3,n+3}\right)\cdot J_{n+3}$$

$\boldsymbol{J_{n+4}}$ ***, from*** **[Sum 1], [Sum 2]** ***and*** **[Sum 3]**

$$\left(D_2^{**}\cdot d_{n+2,n+4}\right)\cdot J_{n+4}$$
$$\left(D_3^{**}\cdot d_{n+3,n+4}\right)\cdot J_{n+4}$$
$$\left(D_4^{**}\cdot d_{n+4,n+4}\right)\cdot J_{n+4}$$

… …

… ….

$\boldsymbol{J_{n+L^*-2}}$ ***, from*** **[Sum 1], [Sum 2], [Sum 3], … , [Sum $\boldsymbol{L^*-5}$], [Sum $\boldsymbol{L^*-4}$]** ***and*** **[Sum $\boldsymbol{L^*-3}$]**

$$\left(D_2^{**}\cdot d_{n+2,n+L^*-2}\right)\cdot J_{n+L^*-2}$$
$$\left(D_3^{**}\cdot d_{n+3,n+L^*-2}\right)\cdot J_{n+L^*-2}$$
$$\left(D_4^{**}\cdot d_{n+4,n+L^*-2}\right)\cdot J_{n+L^*-2}$$
… … … … … … … … … … … … … … … …
$$\left(D_{L^*-4}^{**}\cdot d_{n+L^*-3,n+L^*-2}\right)\cdot J_{n+L^*-2}$$
$$\left(D_{L^*-3}^{**}\cdot d_{n+L^*-2,n+L^*-2}\right)\cdot J_{n+L^*-2}$$
$$\left(D_{L^*-2}^{**}\cdot d_{n+L^*-1,n+L^*-2}\right)\cdot J_{n+L^*-2}$$

$\boldsymbol{J_{n+L^*-1}}$ ***, from*** **[Sum 1], [Sum 2], [Sum 3], … , [Sum $\boldsymbol{L^*-4}$], [Sum $\boldsymbol{L^*-3}$]** ***and*** **[Sum $\boldsymbol{L^*-2}$]**

$$\left(D_2^{**}\cdot d_{n+2,n+L^*-1}\right)\cdot J_{n+L^*-1}$$
$$\left(D_3^{**}\cdot d_{n+3,n+L^*-1}\right)\cdot J_{n+L^*-1}$$
$$\left(D_4^{**}\cdot d_{n+4,n+L^*-1}\right)\cdot J_{n+L^*-1}$$
… … … … … … … … … … … … … … … …
$$\left(D_{L^*-3}^{**}\cdot d_{n+L^*-3,n+L^*-1}\right)\cdot J_{n+L^*-1}$$
$$\left(D_{L^*-2}^{**}\cdot d_{n+L^*-2,n+L^*-1}\right)\cdot J_{n+L^*-1}$$
$$\left(D_{L^*-1}^{**}\cdot d_{n+L^*-1,n+L^*-1}\right)\cdot J_{n+L^*-1}$$

$\boldsymbol{J_{n+L^*}}$ ***, from*** **[Sum 1], [Sum 2], [Sum 3], … , [Sum $\boldsymbol{L^*-3}$], [Sum $\boldsymbol{L^*-2}$]** ***and*** **[Sum $\boldsymbol{L^*-1}$]**

$$\left(D_2^{**}\cdot d_{n+2,n+L^*}\right)\cdot J_{n+L^*}$$
$$\left(D_3^{**}\cdot d_{n+3,n+L^*}\right)\cdot J_{n+L^*}$$
$$\left(D_4^{**}\cdot d_{n+4,n+L^*}\right)\cdot J_{n+L^*}$$
… … … … … … … … … … … … … … … …
$$\left(D_{L^*-2}^{**}\cdot d_{n+L^*-2,n+L^*}\right)\cdot J_{n+L^*}$$
$$\left(D_{L^*-1}^{**}\cdot d_{n+L^*-1,n+L^*}\right)\cdot J_{n+L^*}$$
$$\left(D_{L^*}^{**}\cdot d_{n+L^*,n+L^*}\right)\cdot J_{n+L^*}$$

… …

… ….

$\boldsymbol{J_{n+i}}$ ***, whith*** $\boldsymbol{i=(L^*+1),(L^*+2),(L^*+3),\ldots}$ ***, from all the sums,*** **[Sum 1], [Sum 2], [Sum 3], …, …, [Sum $\boldsymbol{L^*-3}$], [Sum $\boldsymbol{L^*-2}$]** ***and*** **[Sum $\boldsymbol{L^*-1}$]** $\quad\boldsymbol{[i\geq(L^*+1)]}$

$$\left(D_2^{**}\cdot d_{n+2,n+i}\right)\cdot J_{n+i}$$

$$(D_3^{**} \cdot d_{n+3,n+i}) \cdot J_{n+i}$$
$$(D_4^{**} \cdot d_{n+4,n+i}) \cdot J_{n+i}$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$(D_{L^*-2}^{**} \cdot d_{n+L^*-2,n+i}) \cdot J_{n+i}$$
$$(D_{L^*-1}^{**} \cdot d_{n+L^*-1,n+i}) \cdot J_{n+i}$$
$$(D_{L^*}^{**} \cdot d_{n+L^*,n+i}) \cdot J_{n+i}$$

$[6.29(V)]$

New associated sums are now indicated,

(R.H.S.S.) Common Form modification (Step VI)

Computing now the new remodelled sums,

J_{n+2}, ***from*** **[Sum 1]**

$$(D_2^{**} \cdot d_{n+2,n+2}) \cdot J_{n+2}$$

J_{n+3}, ***from*** **[Sum 1]** ***and*** **[Sum 2]**

$$[(D_2^{**} \cdot d_{n+2,n+3}) + (D_3^{**} \cdot d_{n+3,n+3})] \cdot J_{n+3}$$

J_{n+4}, ***from*** **[Sum 1], [Sum 2]** ***and*** **[Sum 3]**

$$[(D_2^{**} \cdot d_{n+2,n+4}) + (D_3^{**} \cdot d_{n+3,n+4}) + (D_4^{**} \cdot d_{n+4,n+4})] \cdot J_{n+4}$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

J_{n+L^*-2}, ***from*** **[Sum 1], [Sum 2], [Sum 3], …, [Sum L^*-5], [Sum L^*-4]** ***and*** **[Sum L^*-3]**

$$\begin{bmatrix} (D_2^{**} \cdot d_{n+2,n+L^*-2}) + (D_3^{**} \cdot d_{n+3,n+L^*-2}) + (D_4^{**} \cdot d_{n+4,n+L^*-2}) + \dots \dots \dots \\ \dots \dots \dots + (D_{L^*-4}^{**} \cdot d_{n+L^*-3,n+L^*-2}) + (D_{L^*-3}^{**} \cdot d_{n+L^*-2,n+L^*-2}) + (D_{L^*-2}^{**} \cdot d_{n+L^*-1,n+L^*-2}) \end{bmatrix} \cdot J_{n+L^*-2}$$

J_{n+L^*-1}, ***from*** **[Sum 1], [Sum 2], [Sum 3], …, [Sum L^*-4], [Sum L^*-3]** ***and*** **[Sum L^*-2]**

$$\begin{bmatrix} (D_2^{**} \cdot d_{n+2,n+L^*-1}) + (D_3^{**} \cdot d_{n+3,n+L^*-1}) + (D_4^{**} \cdot d_{n+4,n+L^*-1}) + \dots \dots \dots \\ \dots \dots \dots + (D_{L^*-3}^{**} \cdot d_{n+L^*-3,n+L^*-1}) + (D_{L^*-2}^{**} \cdot d_{n+L^*-2,n+L^*-1}) + (D_{L^*-1}^{**} \cdot d_{n+L^*-1,n+L^*-1}) \end{bmatrix} \cdot J_{n+L^*-1}$$

J_{n+L^*}, ***from*** **[Sum 1], [Sum 2], [Sum 3], …, [Sum L^*-3], [Sum L^*-2]** ***and*** **[Sum L^*-1]**

$$\begin{bmatrix} (D_2^{**} \cdot d_{n+2,n+L^*}) + (D_3^{**} \cdot d_{n+3,n+L^*}) + (D_4^{**} \cdot d_{n+4,n+L^*}) + \dots \dots \dots \\ \dots \dots \dots + (D_{L^*-2}^{**} \cdot d_{n+L^*-2,n+L^*}) + (D_{L^*-1}^{**} \cdot d_{n+L^*-1,n+L^*}) + (D_{L^*}^{**} \cdot d_{n+L^*,n+L^*}) \end{bmatrix} \cdot J_{n+L^*}$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

J_{n+i}, ***whith*** $i \geq (L^*+1)$, …, ***from all the sums***, **[Sum 1], [Sum 2], [Sum 3], …,**
…, [Sum L^*-3], [Sum L^*-2] ***and*** **[Sum L^*-1]**

$$\begin{bmatrix} (D_2^{**} \cdot d_{n+2,n+i}) + (D_3^{**} \cdot d_{n+3,n+i}) + (D_4^{**} \cdot d_{n+4,n+i}) + \dots \dots \dots \\ \dots \dots \dots + (D_{L^*-2}^{**} \cdot d_{n+L^*-2,n+i}) + (D_{L^*-1}^{**} \cdot d_{n+L^*-1,n+i}) + (D_{L^*}^{**} \cdot d_{n+L^*,n+i}) \end{bmatrix} \cdot J_{n+i}$$

$[6.29(VI)]$

… and using the summation notation,

$(\boldsymbol{R.H.S.S.})$ ***Common Form modification (Step VII)***

Introducing conventional summation sigma symbol, the terms in J_k reads,

$\boldsymbol{J_{n+2}\ (k = n + 2)\ , from\ [\text{Sum } 1]}$

$$\left[\sum_{j=2}^{2}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k = \left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k$$

$\boldsymbol{J_{n+3}\ (k = n + 3)\ , from\ [\text{Sum } 1]\ and\ [\text{Sum } 2]}$

$$\left[\sum_{j=2}^{3}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k = \left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k$$

$\boldsymbol{J_{n+4}\ (k = n + 4)\ \ , from\ [\text{Sum } 1], [\text{Sum } 2]\ and\ [\text{Sum } 3]}$

$$\left[\sum_{j=2}^{4}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k = \left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k$$

… …

… ….

$\boldsymbol{J_{n+L^*-2}\ (k = n + L^* - 2)\ , from\ [\text{Sum } 1], [\text{Sum } 2], [\text{Sum } 3], \ldots\ldots\ldots}$

$\boldsymbol{\ldots\ldots\ldots, [\text{ Sum } L^* - 5], [\text{Sum } L^* - 4]\ and\ [\text{Sum } L^* - 3]}$

$$\left[\sum_{j=2}^{L^*-2}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k = \left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k$$

$\boldsymbol{J_{n+L^*-1}\ (k = n + L^* - 1)\ \ , from\ [\text{Sum } 1], [\text{Sum } 2], [\text{Sum } 3], \ldots\ldots\ldots}$

$\boldsymbol{\ldots\ldots\ldots, [\text{Sum } L^* - 4], [\text{Sum } L^* - 3]\ and\ [\text{Sum } L^* - 2]}$

$$\left[\sum_{j=2}^{L^*-1}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k = \left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k$$

$\boldsymbol{J_{n+L^*}\ (k = n + L^*)\ , from\ [\text{Sum } 1], [\text{Sum } 2], [\text{Sum } 3], \ldots, [\text{Sum } L^* - 3], [\text{Sum } L^* - 2]\ and\ [\text{Sum } L^* - 1]}$

$$\left[\sum_{j=2}^{L^*}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k = \left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k$$

… …

… ….

$\boldsymbol{J_{n+i}\ (k = n + i)\ \ , whith\ i \geq (L^* + 1), \ldots, from\ all\ the\ sums, [\text{Sum } 1], [\text{Sum } 2], [\text{Sum } 3], \ldots\ldots\ldots}$

$\boldsymbol{\ldots\ldots\ldots, [\text{Sum } L^* - 3], [\text{Sum } L^* - 2]\ and\ [\text{Sum } L^* - 1]}$

$$\left[\sum_{j=2}^{L^*}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k$$

$[\boldsymbol{6.29(VII)}]$

Making the differentiation between the ranges $(n+2 \leq k \leq n+L^*)$ and $(k \geq n+L^*)$

(R.H.S.S.) Common Form modification (Step VIII)

We can now summarize all terms in J_k with $(k \geq n+2)$,

$$\left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k \ldots\ldots\ldots for\ (n+2 \leq k \leq n+L^*)$$

… and,

$$\left[\sum_{j=2}^{L^*}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k \ldots\ldots\ldots for\ (k \geq n+L^*)$$

Remark: For the border value, when $(k = n+L^*)$, both formulas are valid because they lead to the same result.

[6.29(VIII)]

Obtaining the final common form remodelled,

(R.H.S.S.) Common Form modification (Step IX)

Let now recall that we are calculating the aforesaid and defined "Common Form",

$$\left\{\sum_{j=2}^{L^*} D_j^{**} \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\}$$

… which is a finite sum of infinite sums where j is the dummy label, and now we reach an equivalent representation, more suitable for our purposes, which is an infinite sum of finite sums, where k is the correspondent dummy label.

Common Form:

$$\left\{\sum_{j=2}^{L^*} D_j^{**} \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\} = \left[\sum_{k=n+j}^{\infty}\left(D_j^{**} \cdot d_{n+j,k}\right) \cdot J_k\right]_{j=2} + \left[\sum_{k=n+j}^{\infty}\left(D_j^{**} \cdot d_{n+j,k}\right) \cdot J_k\right]_{j=3} +$$

$$+\left[\sum_{k=n+j}^{\infty}\left(D_j^{**} \cdot d_{n+j,k}\right) \cdot J_k\right]_{j=4} + \ldots\ldots\ldots + \left[\sum_{k=n+j}^{\infty}\left(D_j^{**} \cdot d_{n+j,k}\right) \cdot J_k\right]_{j=L^*-2} +$$

$$+\left[\sum_{k=n+j}^{\infty}\left(D_j^{**} \cdot d_{n+j,k}\right) \cdot J_k\right]_{j=L^*-1} + \left[\sum_{k=n+j}^{\infty}\left(D_j^{**} \cdot d_{n+j,k}\right) \cdot J_k\right]_{j=L^*}$$

"Common Form" new representation:

$$\left\{\sum_{j=2}^{L^*} D_j^{**} \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\} =$$

$$= \left\{\left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k\right\}_{k=n+2} + \left\{\left[\sum_{j=2}^{k-n}\left(D_j^{**} \cdot d_{n+j,k}\right)\right] \cdot J_k\right\}_{k=n+3} + \ldots\ldots\ldots$$

$$\dots\dots\dots \; + \left\{\left[\sum_{j=2}^{k-n}\left(D_j^{**}\cdot d_{n+j,k}\right)\right]\cdot J_k\right\}_{k=n+L^*-1} + \left\{\left[\sum_{j=2}^{k-n}\left(D_j^{**}\cdot d_{n+j,k}\right)\right]\cdot J_k\right\}_{k=n+L^*} +$$

$$+\left\{\left[\sum_{j=2}^{L^*}\left(D_j^{**}\cdot d_{n+j,k}\right)\right]\cdot J_k\right\}_{k=n+L^*+1} + \left\{\left[\sum_{j=2}^{L^*}\left(D_j^{**}\cdot d_{n+j,k}\right)\right]\cdot J_k\right\}_{k=n+L^*+1} + \dots\dots\dots$$

Lastly the required infinite sum of finite sums is straightaway obtained,

$$\left\{\sum_{j=2}^{L^*} D_j^{**}\cdot\left[\sum_{k=n+j}^{\infty} d_{n+j,k}\cdot J_k\right]\right\} = \left\{\sum_{k=n+2}^{n+L^*}\left[\sum_{j=2}^{k-n}\left(D_j^{**}\cdot d_{n+j,k}\right)\right]\cdot J_k\right\} + \left\{\sum_{k=n+L^*+1}^{\infty}\left[\sum_{j=2}^{L^*}\left(D_j^{**}\cdot d_{n+j,k}\right)\right]\cdot J_k\right\}$$

$[6.29(IX)]$

Discriminating the three right hand side summations (R.H.S.S.) under calculation,

(R.H.S.S.) Common Form modification (Step X)

After reach this identity we must recall that it holds for all the tree double summations in the differential-difference equations system we will solve.
Discerning in each equation,

In $[6.26(0)]$ $$\left\{\sum_{j=2}^{L^*} D_j^{**}\cdot\left[\sum_{k=n+j}^{\infty} d_{n+j,k}\cdot J_k\right]\right\} = \left\{\sum_{j=2}^{L} D_j^{*}\cdot\left[\sum_{k=j}^{\infty} d_{j,k}\cdot J_k\right]\right\}$$

In $[6.26(L)]$ $$\left\{\sum_{j=2}^{L^*} D_j^{**}\cdot\left[\sum_{k=n+j}^{\infty} d_{n+j,k}\cdot J_k\right]\right\} = \left\{\sum_{j=2}^{L-1}\left(D_{j+1}^{*}-D_j^{*}\right)\cdot\left[\sum_{k=n+j}^{\infty} d_{n+j,k}\cdot J_k\right]\right\}$$

In $[6.26(n)]$ $$\left\{\sum_{j=2}^{L^*} D_j^{**}\cdot\left[\sum_{k=n+j}^{\infty} d_{n+j,k}\cdot J_k\right]\right\} = \left\{\sum_{j=2}^{L-1}\left(D_{j+1}^{*}-D_j^{*}\right)\cdot\left[\sum_{k=n+j}^{\infty} d_{n+j,k}\cdot J_k\right]\right\}$$

Consequently,

...in $[6.26(0)]$,

$$L^* = L, \qquad D_j^{**} = D_j^{*}, \qquad and\ n = 0$$

...in $[6.26(L)]$,

$$L^* = (L-1), \qquad D_j^{**} = \left(D_{j+1}^{*}-D_j^{*}\right), \qquad and\ (1 \le n \le L)$$

...in $[6.26(n)]$,

$$L^* = (L-1), \qquad D_j^{**} = \left(D_{j+1}^{*}-D_j^{*}\right), \qquad and\ (n \ge L+1)$$

$[6.29(X)]$

Writing the three complete transformed right hand sides summations (R. H. S. S.),

(R.H.S.S.) Common Form modification (Step XI)

...and the Right Hand Side Sums (R.H.S.S.) modifications are,

...in $[6.26(0)]$,

$$\left\{\sum_{j=2}^{L} D_j^* \cdot \left[\sum_{k=j}^{\infty} d_{j,k} \cdot J_k\right]\right\} =$$

$$= \left\{\sum_{k=2}^{L}\left[\sum_{j=2}^{k}\left[D_j^* \cdot d_{j,k}\right]\right] \cdot J_k\right\} + \left\{\sum_{k=L+1}^{\infty}\left[\sum_{j=2}^{L}\left[D_j^* \cdot d_{j,k}\right]\right] \cdot J_k\right\}$$

...in $[6.26(L)]$,

$$\left\{\sum_{j=2}^{L-1}\left(D_{j+1}^* - D_j^*\right) \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\} =$$

$$= \left\{\sum_{k=n+2}^{n+L-1}\left[\sum_{j=2}^{k-n}\left[\left(D_{j+1}^* - D_j^*\right) \cdot d_{n+j,k}\right]\right] \cdot J_k\right\} + \left\{\sum_{k=n+L}^{\infty}\left[\sum_{j=2}^{L-1}\left[\left(D_{j+1}^* - D_j^*\right) \cdot d_{n+j,k}\right]\right] \cdot J_k\right\}$$

$$\ldots being\ (1 \le n \le L)$$

...in $[6.26(n)]$,

$$\left\{\sum_{j=2}^{L-1}\left(D_{j+1}^* - D_j^*\right) \cdot \left[\sum_{k=n+j}^{\infty} d_{n+j,k} \cdot J_k\right]\right\} =$$

$$= \left\{\sum_{k=n+2}^{n+L-1}\left[\sum_{j=2}^{k-n}\left[\left(D_{j+1}^* - D_j^*\right) \cdot d_{n+j,k}\right]\right] \cdot J_k\right\} + \left\{\sum_{k=n+L}^{\infty}\left[\sum_{j=2}^{L-1}\left[\left(D_{j+1}^* - D_j^*\right) \cdot d_{n+j,k}\right]\right] \cdot J_k\right\}$$

$$\ldots being\ (n \ge L+1)$$

$[6.29(XI)]$

At this point all the tree right hand sides can be written down again according to these modifications,

Right Hand Side (R.H.S.) of $[6.21(0)]$

$$e^{-a\cdot\tau} \cdot \left\{-A_0^* \cdot \theta_0 + B_1^* \cdot \theta_1 - \left[\sum_{j=2}^{L} D_j^* \cdot \theta_j\right]\right\} =$$

$$= -A_0^* \cdot \left[\sum_{k=0}^{\infty} d_{0,k} \cdot J_k\right] - B_1^* \cdot \left[\sum_{k=1}^{\infty} d_{1,k} \cdot J_k\right] - \left\{\sum_{k=2}^{L}\left[\sum_{j=2}^{k}\left[D_j^* \cdot d_{j,k}\right]\right] \cdot J_k\right\} -$$

$$-\left\{\sum_{k=L+1}^{\infty}\left[\sum_{j=2}^{L}\left[D_j^* \cdot d_{j,k}\right]\right] \cdot J_k\right\}$$

$[6.30(0)]$

Right Hand Side (R.H.S.) of [6.21(*L*)]

$$e^{-a\cdot\tau}\cdot\left\{A_{n-1}^{*}\cdot\theta_{n-1}+(B_n^{*}-A_n^{*})\cdot\theta_n++(D_2^{*}-B_{n+1}^{*})\cdot\theta_{n+1}+\left[\sum_{j=2}^{L-1}(D_{j+1}^{*}-D_j^{*})\cdot\theta_{n+j}\right]-D_L^{*}\cdot\theta_{L+n}\right\}=$$

$$=A_{n-1}^{*}\cdot\left[\sum_{k=n-1}^{\infty}d_{n-1,k}\cdot J_k\right]+(B_n^{*}-A_n^{*})\cdot\left[\sum_{k=n}^{\infty}d_{n,k}\cdot J_k\right]+$$

$$+(D_2^{*}-B_{n+1}^{*})\cdot\left[\sum_{k=n+1}^{\infty}d_{n+1,k}\cdot J_k\right]+\left\{\sum_{k=n+2}^{n+L-1}\left[\sum_{j=2}^{k-n}\left[(D_{j+1}^{*}-D_j^{*})\cdot d_{n+j,k}\right]\right]\cdot J_k\right\}+$$

$$+\left\{\sum_{k=n+L}^{\infty}\left[\sum_{j=2}^{L-1}\left[(D_{j+1}^{*}-D_j^{*})\cdot d_{n+j,k}\right]\right]\cdot J_k\right\}+(-D_L^{*})\cdot\left[\sum_{k=L+n}^{\infty}d_{n+L,k}\cdot J_k\right]$$

$\ldots if\ (1\le n\le L)$ **[6.30(*L*)]**

Right Hand Side (R.H.S.) of [6.21(*n*)]

$$e^{-a\cdot\tau}\cdot\left\{A_L^{*}\cdot\theta_{n-1}+(B_n^{*}-A_L^{*})\cdot\theta_n++(D_2^{*}-B_{n+1}^{*})\cdot\theta_{n+1}+\left[\sum_{j=2}^{L-1}(D_{j+1}^{*}-D_j^{*})\cdot\theta_{n+j}\right]-D_L^{*}\cdot\theta_{L+n}\right\}=$$

$$=A_L^{*}\cdot\left[\sum_{k=n-1}^{\infty}d_{n-1,k}\cdot J_k\right]+(B_n^{*}-A_L^{*})\cdot\left[\sum_{k=n}^{\infty}d_{n,k}\cdot J_k\right]+$$

$$+(D_2^{*}-B_{n+1}^{*})\cdot\left[\sum_{k=n+1}^{\infty}d_{n+1,k}\cdot J_k\right]+\left\{\sum_{k=n+2}^{n+L-1}\left[\sum_{j=2}^{k-n}\left[(D_{j+1}^{*}-D_j^{*})\cdot d_{n+j,k}\right]\right]\cdot J_k\right\}+$$

$$+\left\{\sum_{k=n+L}^{\infty}\left[\sum_{j=2}^{L-1}\left[(D_{j+1}^{*}-D_j^{*})\cdot d_{n+j,k}\right]\right]\cdot J_k\right\}+(-D_L^{*})\cdot\left[\sum_{k=L+n}^{\infty}d_{n+L,k}\cdot J_k\right]$$

$\ldots if\ (n\ge L+1)$ **[6.30(*n*)]**

5. Partitions in the set of J_k terms.

Next step consists in doing a suitable partition in the set of k values, for all the functions J_k, in these three (R.H.S.) expressions. The purpose is to collect together the terms with the same function J_k in one single term for each.
Such partitions are easily made by a simple inspection of the three (R.H.S.).
However observe that in each equation [6.30(0)], [6.30(*L*)] and [6.30(*n*)] the (R.H.S.) must be inevitably partitioned in a finer way than that already obtained in the Left Hand Sides (L.H.S.).
Inspecting [6.25(0)] and [6.25(*n*)] one easily concludes that in the (L.H.S.) the partition for $(n=0)$ is $\{[J_0\ term]+[J_1\ term]+[Sum\ of\ J_k\ terms\ where\ (k\ \ge\ 2)]\}$ and the common partition for $(n\ge 1)$ is $\{[J_{n-1}\ term]+[J_n\ term]+[Sum\ of\ J_k\ terms\ where\ (k\ \ge\ n+1)]\}$.
It is from those coarser (L.H.S.) partitions, considered as the starting point, that (R.H.S.) partitions must be constructed.

Specifying, by a simple inspection, as refereed,

In $[6.30(0)]$ ***the partition must be,***

$$\{[J_0\ term] + [J_1\ term] + [Sum\ of\ J_k\ terms\ (2 \le k \le L)] + [Sum\ of\ J_k\ terms\ (k \ge L+1)]\}$$

In $[6.30(L)]$ ***the partition must be,***

$$\left\{\begin{matrix}[J_{n-1}\ term] + [J_n\ term] + [J_{n+1}\ term] + [Sum\ of\ J_k\ terms\ (n+2 \le k \le n+L-1)] + \\ + [Sum\ of\ J_k\ terms\ (k \ge n+L)]\end{matrix}\right\}$$

In $[6.30(n)]$ ***the partition must be,***

$$\left\{\begin{matrix}[J_{n-1}\ term] + [J_n\ term] + [J_{n+1}\ term] + [Sum\ of\ J_k\ terms\ (n+2 \le k \le n+L-1)] + \\ + [Sum\ of\ J_k\ terms\ (k \ge n+L)]\end{matrix}\right\}$$

Remark: ***in*** $[6.30(L)]$ ***and*** $[6.30(n)]$ ***the two partitions are the same.***

$[6.31]$

Consequently now the three (R.H.S.) will become written like,

Right Hand Side $(R.H.S.)$ ***of*** $[6.21(0)]$

$$e^{-a\cdot\tau}\cdot\left\{-A_0^*\cdot\theta_0 + B_1^*\cdot\theta_1 - \left[\sum_{j=2}^{L} D_j^*\cdot\theta_j\right]\right\} =$$

$$= -A_0^*\cdot d_{0,0}\cdot J_0 + \left[-A_0^*\cdot d_{0,1} - B_1^*\cdot d_{1,1}\right]\cdot J_1 +$$

$$+\sum_{k=2}^{L}\left\{-A_0^*\cdot d_{0,k} - B_1^*\cdot d_{1,k} - \left[\sum_{j=2}^{k} D_j^*\cdot d_{j,k}\right]\right\}\cdot J_k +$$

$$+\sum_{k=L+1}^{\infty}\left\{-A_0^*\cdot d_{0,k} - B_1^*\cdot d_{1,k} - \left[\sum_{j=2}^{L} D_j^*\cdot d_{j,k}\right]\right\}\cdot J_k$$

$[6.32(0)]$

Right Hand Side $(R.H.S.)$ ***of*** $[6.21(L)]$

$$e^{-a\cdot\tau}\cdot\left\{A_{n-1}^*\cdot\theta_{n-1} + (B_n^* - A_n^*)\cdot\theta_n + +(D_2^* - B_{n+1}^*)\cdot\theta_{n+1} + \left[\sum_{j=2}^{L-1}(D_{j+1}^* - D_j^*)\cdot\theta_{n+j}\right] - D_L^*\cdot\theta_{L+n}\right\} =$$

$$= A_{n-1}^*\cdot d_{n-1,n-1}\cdot J_{n-1} + \left[A_{n-1}^*\cdot d_{n-1,n} + (B_n^* - A_n^*)\cdot d_{n,n}\right]\cdot J_n +$$

$$+\left[A_{n-1}^*\cdot d_{n-1,n+1} + (B_n^* - A_n^*)\cdot d_{n,n+1} + (D_2^* - B_{n+1}^*)\cdot d_{n+1,n+1}\right]\cdot J_{n+1} +$$

$$+\sum_{k=n+2}^{n+L-1}\left\{A_{n-1}^*\cdot d_{n-1,k} + (B_n^* - A_n^*)\cdot d_{n,k} + (D_2^* - B_{n+1}^*)\cdot d_{n+1,k} + \left[\sum_{j=2}^{k-n}\left(D_{j+1}^* - D_j^*\right)\cdot d_{n+j,k}\right]\right\}\cdot J_k +$$

$$+\sum_{k=n+L}^{\infty}\left\{A_{n-1}^{*}\cdot d_{n-1,k}+(B_{n}^{*}-A_{n}^{*})\cdot d_{n,k}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,k}+\left[\sum_{j=2}^{L-1}\left(D_{j+1}^{*}-D_{j}^{*}\right)\cdot d_{n+j,k}\right]-D_{L}^{*}\cdot d_{n+L,k}\right\}\cdot J_{k}$$

$$\ldots if\ (1\leq n\leq L) \qquad [6.32(L)]$$

Right Hand Side (R.H.S.) of [6.21(n)]

$$e^{-a\cdot\tau}\cdot\left\{A_{L}^{*}\cdot\theta_{n-1}+(B_{n}^{*}-A_{L}^{*})\cdot\theta_{n}++(D_{2}^{*}-B_{n+1}^{*})\cdot\theta_{n+1}+\left[\sum_{j=2}^{L-1}(D_{j+1}^{*}-D_{j}^{*})\cdot\theta_{n+j}\right]-D_{L}^{*}\cdot\theta_{n+L}\right\}=$$

$$=A_{L}^{*}\cdot d_{n-1,n-1}\cdot J_{n-1}+\left[A_{L}^{*}\cdot d_{n-1,n}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,n}\right]\cdot J_{n}+$$

$$+\left[A_{L}^{*}\cdot d_{n-1,n+1}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,n+1}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,n+1}\right]\cdot J_{n+1}+$$

$$+\sum_{k=n+2}^{n+L-1}\left\{A_{L}^{*}\cdot d_{n-1,k}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,k}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,k}+\left[\sum_{j=2}^{k-n}\left(D_{j+1}^{*}-D_{j}^{*}\right)\cdot d_{n+j,k}\right]\right\}\cdot J_{k}+$$

$$+\sum_{k=n+L}^{\infty}\left\{A_{L}^{*}\cdot d_{n-1,k}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,k}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,k}+\left[\sum_{j=2}^{L-1}\left(D_{j+1}^{*}-D_{j}^{*}\right)\cdot d_{n+j,k}\right]-D_{L}^{*}\cdot d_{n+L,k}\right\}\cdot J_{k}$$

$$\ldots if\ (n\geq L+1) \qquad [6.32(n)]$$

Observing fully this same conformity, the coarser partitions in the (L.H.S.) must now be subjected to a suitable finer partition as follows.
The (L. H. S.) and the (R. H. S.) of [6.21(0)] provided respectively by [6.25(0)], and [6.32(0)] must match.
The same is truth for the (L. H. S.) and the (R. H. S.) of $[6.21(L)]$ and $[6.21(n)]$, given by $[6.25(n)]$ and $[6.32(L)]$, $\{for\ [6.21(L)]\}$ and by $[6.25(n)]$ and $[6.32(n)]$ $\{for\ [6.21(n)]\}$.

(Remark: as we have already see the (L. H. S.) of $[6.21(L)]$ and $[6.21(n)]$ are common and given by $[6.25(n)]$)
However (L. H. S.) [6.25(0)] and $[6.25(n)]$ must be rewritten before match with their respective (R. H. S.).
The goal now is to achieve in both hand sides the same partition in the set of k values, for all the functions J_k.
For that purpose a finer partition must be made in both (L. H. S.), specifically the summations must be adequately partitioned, from $(k=2)$ to $(k=L)$ and for $(k\geq L+1)$ in [6.25(0)] ; and from $(k=n+2)$ to $(k=n+L-1)$ and for $(k\geq n+L)$ in $[6.25(n)]$.
After will be possible to equalize the terms with the same function J_k in both sides. Each one of such equalizations will constitute a difference equation in the coefficients $d_{n,n+k}$, as unknowns that must be determined.

Suitable partition in the Left Hand Side (L.H.S.) of [6.21(0)]

$$\left(\frac{e^{-a\cdot\tau}}{b}\right)\cdot\frac{d\theta_{0}(\tau)}{d\tau}=\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{0,0}+d_{0,1}\right]\cdot J_{0}+\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{0,1}+d_{0,2}-2\cdot d_{0,0}\right]\cdot J_{1}+$$

$$+\sum_{k=2}^{L}\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{0,k}+d_{0,k+1}-d_{0,k-1}\right]\cdot J_{k}+$$

$$+\sum_{k=L+1}^{\infty}\left[(B_1^*-A_1^*)\cdot d_{0,k}+d_{0,k+1}-d_{0,k-1}\right]\cdot J_k$$

[6.33(0)]

Suitable common partition in the Left Hand Sides (L.H.S.S.) of [6.21(L)] and [6.21(n)]

$$\left(\frac{e^{-a\cdot\tau}}{b}\right)\cdot\frac{d\theta_n(\tau)}{d\tau}=\left[d_{n,n}\right]\cdot J_{n-1}+\left[(B_1^*-A_1^*)\cdot d_{n,n}+d_{n,n+1}\right]\cdot J_n+$$

$$+\left[(B_1^*-A_1^*)\cdot d_{n,n+1}+d_{n,n+2}-d_{n,n}\right]\cdot J_{n+1}+\sum_{k=n+2}^{n+L-1}\left[(B_1^*-A_1^*)\cdot d_{n,k}+d_{n,k+1}-d_{n,k-1}\right]\cdot J_k+$$

$$+\sum_{k=n+L}^{\infty}\left[(B_1^*-A_1^*)\cdot d_{n,k}+d_{n,k+1}-d_{n,k-1}\right]\cdot J_k$$

... *with* $(1\le n\le L)$ *for* [6.21(L)] *and* $(n\ge L+1)$ *for* [6.21(n)] **[6.33(n)]**

6. Match between (L. H. S.) and (R. H. S.)

And, as required, the match between the (L. H. S.) and their corresponding (R. H. S.) follows immediately,

Match between (L.H.S.) [6.33(0)] and (R.H.S.) [6.32(0)] refered to equation [6.21(0)]

$$\left[(B_1^*-A_1^*)\cdot d_{0,0}+d_{0,1}\right]\cdot J_0+\left[(B_1^*-A_1^*)\cdot d_{0,1}+d_{0,2}-2\cdot d_{0,0}\right]\cdot J_1+$$

$$+\sum_{k=2}^{L}\left[(B_1^*-A_1^*)\cdot d_{0,k}+d_{0,k+1}-d_{0,k-1}\right]\cdot J_k+\sum_{k=L+1}^{\infty}\left[(B_1^*-A_1^*)\cdot d_{0,k}+d_{0,k+1}-d_{0,k-1}\right]\cdot J_k=$$

$$=-A_0^*\cdot d_{0,0}\cdot J_0+\left[-A_0^*\cdot d_{0,1}-B_1^*\cdot d_{1,1}\right]\cdot J_1+$$

$$+\sum_{k=2}^{L}\left\{-A_0^*\cdot d_{0,k}-B_1^*\cdot d_{1,k}-\left[\sum_{j=2}^{k}D_j^*\cdot d_{j,k}\right]\right\}\cdot J_k+$$

$$+\sum_{k=L+1}^{\infty}\left\{-A_0^*\cdot d_{0,k}-B_1^*\cdot d_{1,k}-\left[\sum_{j=2}^{L}D_j^*\cdot d_{j,k}\right]\right\}\cdot J_k$$

[6.34(0)]

Match between (L.H.S.) [6.33(n)] and (R.H.S.) [6.32(L)] refered to equation [6.21(L)]

$$\left[d_{n,n}\right]\cdot J_{n-1}+\left[(B_1^*-A_1^*)\cdot d_{n,n}+d_{n,n+1}\right]\cdot J_n+\left[(B_1^*-A_1^*)\cdot d_{n,n+1}+d_{n,n+2}-d_{n,n}\right]\cdot J_{n+1}+$$

$$+\sum_{k=n+2}^{n+L-1}\left[(B_1^*-A_1^*)\cdot d_{n,k}+d_{n,k+1}-d_{n,k-1}\right]\cdot J_k+\sum_{k=n+L}^{\infty}\left[(B_1^*-A_1^*)\cdot d_{n,k}+d_{n,k+1}-d_{n,k-1}\right]\cdot J_k=$$

$$=A_{n-1}^*\cdot d_{n-1,n-1}\cdot J_{n-1}+\left[A_{n-1}^*\cdot d_{n-1,n}+(B_n^*-A_n^*)\cdot d_{n,n}\right]\cdot J_n+$$

$$+\left[A_{n-1}^{*}\cdot d_{n-1,n+1}+(B_{n}^{*}-A_{n}^{*})\cdot d_{n,n+1}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,n+1}\right]\cdot J_{n+1}+$$

$$+\sum_{k=n+2}^{n+L-1}\left\{A_{n-1}^{*}\cdot d_{n-1,k}+(B_{n}^{*}-A_{n}^{*})\cdot d_{n,k}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,k}+\left[\sum_{j=2}^{k-n}(D_{j+1}^{*}-D_{j}^{*})\cdot d_{n+j,k}\right]\right\}\cdot J_{k}+$$

$$+\sum_{k=n+L}^{\infty}\left\{A_{n-1}^{*}\cdot d_{n-1,k}+(B_{n}^{*}-A_{n}^{*})\cdot d_{n,k}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,k}+\left[\sum_{j=2}^{L-1}(D_{j+1}^{*}-D_{j}^{*})\cdot d_{n+j,k}\right]-D_{L}^{*}\cdot d_{n+L,k}\right\}\cdot J_{k}$$

$$\ldots if\ (1\le n\le L)\qquad [6.34(L)]$$

Match between (L.H.S.) [6.33(n)] and (R.H.S.) [6.32(n)] refered to equation [6.21(n)]

$$[d_{n,n}]\cdot J_{n-1}+\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{n,n}+d_{n,n+1}\right]\cdot J_{n}+\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{n,n+1}+d_{n,n+2}-d_{n,n}\right]\cdot J_{n+1}+$$

$$+\sum_{k=n+2}^{n+L-1}\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{n,k}+d_{n,k+1}-d_{n,k-1}\right]\cdot J_{k}+\sum_{k=n+L}^{\infty}\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{n,k}+d_{n,k+1}-d_{n,k-1}\right]\cdot J_{k}=$$

$$=A_{L}^{*}\cdot d_{n-1,n-1}\cdot J_{n-1}+\left[A_{L}^{*}\cdot d_{n-1,n}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,n}\right]\cdot J_{n}+$$

$$+\left[A_{L}^{*}\cdot d_{n-1,n+1}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,n+1}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,n+1}\right]\cdot J_{n+1}+$$

$$+\sum_{k=n+2}^{n+L-1}\left\{A_{L}^{*}\cdot d_{n-1,k}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,k}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,k}+\left[\sum_{j=2}^{k-n}(D_{j+1}^{*}-D_{j}^{*})\cdot d_{n+j,k}\right]\right\}\cdot J_{k}+$$

$$+\sum_{k=n+L}^{\infty}\left\{A_{L}^{*}\cdot d_{n-1,k}+(B_{n}^{*}-A_{L}^{*})\cdot d_{n,k}+(D_{2}^{*}-B_{n+1}^{*})\cdot d_{n+1,k}+\left[\sum_{j=2}^{L-1}(D_{j+1}^{*}-D_{j}^{*})\cdot d_{n+j,k}\right]-D_{L}^{*}\cdot d_{n+L,k}\right\}\cdot J_{k}$$

$$\ldots if\ (n\ge L+1)\qquad [6.34(n)]$$

Lastly let emphasize the criterion leading to the whole solution: whenever an equality between two expressions must be satisfied and, in those two expressions, time (τ) figures only inside Bessel functions argument, the equality holds if an equality, term by term, is put into play. Here the terms that must be equal, one to each other, are those with the same Bessel function in both side of the main equality.

Equations obtained from the match between (L.H.S.) and (R.H.S.) in [6.34(0)]

{These equations are the conditions for solve [6.21(0)]}

$$\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{0,0}+d_{0,1}\right]=-A_{0}^{*}\cdot d_{0,0}\qquad (terms\ in\ J_{0}\ at\ [6.21(0)])\qquad [6.35(0)-I]$$

$$\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{0,1}+d_{0,2}-2\cdot d_{0,0}\right]=\left[-A_{0}^{*}\cdot d_{0,1}-B_{1}^{*}\cdot d_{1,1}\right]\qquad (terms\ in\ J_{1}\ at\ [6.21(0)])\qquad [6.35(0)-II]$$

$$\left[(B_{1}^{*}-A_{1}^{*})\cdot d_{0,k}+d_{0,k+1}-d_{0,k-1}\right]=-A_{0}^{*}\cdot d_{0,k}-B_{1}^{*}\cdot d_{1,k}-\left[\sum_{j=2}^{k}D_{j}^{*}\cdot d_{j,k}\right]$$

$$(terms\ in\ J_{k}\ at\ [6.21(0)])\ where\ (2\le k\le L)\qquad [6.35(0)-III]$$

$$\left[(B_1^* - A_1^*) \cdot d_{0,k} + d_{0,k+1} - d_{0,k-1}\right] = -A_0^* \cdot d_{0,k} - B_1^* \cdot d_{1,k} - \left[\sum_{j=2}^{L} D_j^* \cdot d_{j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(0)])\ where\ (k \geq L+1)$ **[6.35(0) − IV]**

Equations obtained from the match between (L.H.S.) and (R.H.S.) in [6.34(L)]

{These equations are the conditions for solve [6.21(L)]}

$d_{n,n} = A_{n-1}^* \cdot d_{n-1,n-1}$ $(terms\ in\ J_{n-1}\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)$ **[6.35(L) − I]**

$$\left[(B_1^* - A_1^*) \cdot d_{n,n} + d_{n,n+1}\right] = \left[A_{n-1}^* \cdot d_{n-1,n} + (B_n^* - A_n^*) \cdot d_{n,n}\right]$$

$(terms\ in\ J_n\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)$ **[6.35(L) − II]**

$$\left[(B_1^* - A_1^*) \cdot d_{n,n+1} + d_{n,n+2} - d_{n,n}\right] = \left[A_{n-1}^* \cdot d_{n-1,n+1} + (B_n^* - A_n^*) \cdot d_{n,n+1} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,n+1}\right]$$

$(terms\ in\ J_{n+1}\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)$ **[6.35(L) − III]**

$$\left[(B_1^* - A_1^*) \cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_{n-1}^* \cdot d_{n-1,k} + (B_n^* - A_n^*) \cdot d_{n,k} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{k-n} \left(D_{j+1}^* - D_j^*\right) \cdot d_{n+j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)\ and\ (n+2 \leq k \leq n+L-1)$ **[6.35(L) − IV]**

$$\left[(B_1^* - A_1^*) \cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_{n-1}^* \cdot d_{n-1,k} + (B_n^* - A_n^*) \cdot d_{n,k} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{L-1} \left(D_{j+1}^* - D_j^*\right) \cdot d_{n+j,k}\right] - D_L^* \cdot d_{n+L,k}$$

$(terms\ in\ J_k\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)\ and\ (k \geq n+L)$ **[6.35(L) − V]**

Equations obtained from the match between (L.H.S.) and (R.H.S.) in [6.34(n)]

{These equations are the conditions for solve [6.21(n)]}

$d_{n,n} = A_L^* \cdot d_{n-1,n-1}$ $(terms\ in\ J_{n-1}\ at\ [6.21(n)])\ where\ (n \geq L+1)$ **[6.35(n) − I]**

$$\left[(B_1^* - A_1^*) \cdot d_{n,n} + d_{n,n+1}\right] = \left[A_L^* \cdot d_{n-1,n} + (B_n^* - A_L^*) \cdot d_{n,n}\right]$$

$(terms\ in\ J_n\ at\ [6.21(n)])\ where\ (n \geq L+1)$ **[6.35(n) − II]**

$$\left[(B_1^* - A_1^*) \cdot d_{n,n+1} + d_{n,n+2} - d_{n,n}\right] = \left[A_L^* \cdot d_{n-1,n+1} + (B_n^* - A_L^*) \cdot d_{n,n+1} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,n+1}\right]$$

$(terms\ in\ J_{n+1}\ at\ [6.21(n)])\ where\ (n \geq L+1)$ **[6.35(n) − III]**

$$\left[(B_1^* - A_1^*) \cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_L^* \cdot d_{n-1,k} + (B_n^* - A_L^*) \cdot d_{n,k} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{k-n} \left(D_{j+1}^* - D_j^*\right) \cdot d_{n+j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(n)])\ where\ (n \geq L+1)\ and\ (n+2 \leq k \leq n+L-1)$ **[6.35(n) − IV]**

$$\left[(B_1^* - A_1^*)\cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_L^* \cdot d_{n-1,k} + (B_n^* - A_L^*)\cdot d_{n,k} + (D_2^* - B_{n+1}^*)\cdot d_{n+1,k} + \left[\sum_{j=2}^{L-1}\left(D_{j+1}^* - D_j^*\right)\cdot d_{n+j,k}\right] - D_L^* \cdot d_{n+L,k}$$

$(terms\ in\ J_k\ at\ [6.21(n)])\ where\ (n \geq L+1)\ and\ (k \geq n+L)$ **[6.35(n) − V]**

We are now faced to solve a totality of 14 difference equations.
For sake of systematization let write all in a more suitable ordination (including the initial time condition).
We start with the initial time condition and afterwards setting those equations not containing summations and lastly these ones. Also going through [6.35(0) −∗∗], then [6.35(L) −∗∗] and finally [6.35(n) −∗∗].
The new, equivalent but more rationally ordered system reads,

Initial time condition

$d_{0,0} = 1$

$(initial\ time\ condition)$ **[τ(0)]**

Terms in J_0 *from* **[6.35(0)]**

$\left[(B_1^* - A_1^*)\cdot d_{0,0} + d_{0,1}\right] = -A_0^* \cdot d_{0,0}$ $(terms\ in\ J_0\ at\ [6.21(0)])$ **[6.35(0) − I]**

Now→[*A*]

Terms in J_1 *from* **[6.35(0)]**

$\left[(B_1^* - A_1^*)\cdot d_{0,1} + d_{0,2} - 2\cdot d_{0,0}\right] = \left[-A_0^* \cdot d_{0,1} - B_1^* \cdot d_{1,1}\right]$ $(terms\ in\ J_1\ at\ [6.21(0]))$ **[6.35(0) − II]**

Now→[***B***]

Terms in J_{n-1} *from* **[6.35(L)]**

$d_{n,n} = A_{n-1}^* \cdot d_{n-1,n-1}$ $(terms\ in\ J_{n-1}\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)$**[6.35(L) − I]**

Now→[*C*]

Terms in J_n *from* **[6.35(L)]**

$\left[(B_1^* - A_1^*)\cdot d_{n,n} + d_{n,n+1}\right] = \left[A_{n-1}^* \cdot d_{n-1,n} + (B_n^* - A_n^*)\cdot d_{n,n}\right]$

$(terms\ in\ J_n\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)$**[6.35(L) − II]**

Now→[***D***]

Terms in J_{n+1} *from* **[6.35(L)]**

$\left[(B_1^* - A_1^*)\cdot d_{n,n+1} + d_{n,n+2} - d_{n,n}\right] = \left[A_{n-1}^* \cdot d_{n-1,n+1} + (B_n^* - A_n^*)\cdot d_{n,n+1} + (D_2^* - B_{n+1}^*)\cdot d_{n+1,n+1}\right]$

$(terms\ in\ J_{n+1}\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)$**[6.35(L) − III]**

Now→[***E***]

Terms in J_{n-1} *from* **[6.35(n)]**

$d_{n,n} = A_L^* \cdot d_{n-1,n-1}$ $(terms\ in\ J_{n-1}\ at\ [6.21(n)])\ where\ (n \geq L+1)$ **[6.35(n) − I]**

Now→[***F***]

$Terms\ in\ J_n\ \ from$ **[6.35(n)]**

$$\left[(B_1^* - A_1^*)\cdot d_{n,n} + d_{n,n+1}\right] = \left[A_L^*\cdot d_{n-1,n} + (B_n^* - A_L^*)\cdot d_{n,n}\right]$$

$(terms\ in\ J_n\ at\ [6.21(n)])\ where\ (n \geq L+1)$ **[6.35(n) − II]**

$Now \rightarrow$ **[G]**

$Terms\ in\ J_{n+1}\ \ from$ **[6.35(n)]**

$$\left[(B_1^* - A_1^*)\cdot d_{n,n+1} + d_{n,n+2} - d_{n,n}\right] = \left[A_L^*\cdot d_{n-1,n+1} + (B_n^* - A_L^*)\cdot d_{n,n+1} + (D_2^* - B_{n+1}^*)\cdot d_{n+1,n+1}\right]$$

$(terms\ in\ J_{n+1}\ at\ [6.21(n)])\ where\ (n \geq L+1)$ **[6.35(n) − III]**

$Now \rightarrow$ **[H]**

$Terms\ in\ J_k with\ (2 \leq k \leq L)\ from$ **[6.35(0)]**

$$\left[(B_1^* - A_1^*)\cdot d_{0,k} + d_{0,k+1} - d_{0,k-1}\right] = -A_0^*\cdot d_{0,k} - B_1^*\cdot d_{1,k} - \left[\sum_{j=2}^{k} D_j^*\cdot d_{j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(0)])\ where\ (2 \leq k \leq L)$ **[6.35(0) − III]**

$Now \rightarrow$ **[I]**

$Terms\ in\ J_k with\ (k \geq L+1)\ from$ **[6.35(0)]**

$$\left[(B_1^* - A_1^*)\cdot d_{0,k} + d_{0,k+1} - d_{0,k-1}\right] = -A_0^*\cdot d_{0,k} - B_1^*\cdot d_{1,k} - \left[\sum_{j=2}^{L} D_j^*\cdot d_{j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(0)])\ where\ (k \geq L+1)$ **[6.35(0) − IV]**

$Now \rightarrow$ **[J]**

$Terms\ in\ J_k\ with\ (n+2 \leq k \leq n+L-1)\ from$ **[6.35(L)]**

$$\left[(B_1^* - A_1^*)\cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_{n-1}^*\cdot d_{n-1,k} + (B_n^* - A_n^*)\cdot d_{n,k} + (D_2^* - B_{n+1}^*)\cdot d_{n+1,k} + \left[\sum_{j=2}^{k-n}\left(D_{j+1}^* - D_j^*\right)\cdot d_{n+j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)\ and\ (n+2 \leq k \leq n+L-1)$ **[6.35(L) − IV]**

$Now \rightarrow$ **[K]**

$Terms\ in\ J_k with\ (k \geq n+L)\ from$ **[6.35(L)]**

$$\left[(B_1^* - A_1^*)\cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_{n-1}^*\cdot d_{n-1,k} + (B_n^* - A_n^*)\cdot d_{n,k} + (D_2^* - B_{n+1}^*)\cdot d_{n+1,k} + \left[\sum_{j=2}^{L-1}\left(D_{j+1}^* - D_j^*\right)\cdot d_{n+j,k}\right] - D_L^*\cdot d_{n+L,k}$$

$(terms\ in\ J_k\ at\ [6.21(L)])\ where\ (1 \leq n \leq L)\ and\ (k \geq n+L)$ **[6.35(L) − V]**

$Now \rightarrow$ **[L]**

$Terms\ in\ J_k with\ (n+2 \leq k \leq n+L-1)\ from\ [\mathbf{6.35(n)}]$

$$\left[(B_1^* - A_1^*) \cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_L^* \cdot d_{n-1,k} + (B_n^* - A_L^*) \cdot d_{n,k} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{k-n} (D_{j+1}^* - D_j^*) \cdot d_{n+j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(n)])\ where\ (n \geq L+1)\ and\ (n+2 \leq k \leq n+L-1)$ $[\mathbf{6.35(n) - IV}]$

$Now \rightarrow [\boldsymbol{M}]$

$Terms\ in\ J_k\ with\ (k \geq n+L)\ from\ [\mathbf{6.35(n)}]$

$$\left[(B_1^* - A_1^*) \cdot d_{n,k} + d_{n,k+1} - d_{n,k-1}\right] =$$

$$= A_L^* \cdot d_{n-1,k} + (B_n^* - A_L^*) \cdot d_{n,k} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{L-1} (D_{j+1}^* - D_j^*) \cdot d_{n+j,k}\right] - D_L^* \cdot d_{n+L,k}$$

$(terms\ in\ J_k\ at\ [6.21(n)])\ where\ (n \geq L+1)\ and\ (k \geq n+L)$ $[\mathbf{6.35(n) - V}]$

$Now \rightarrow [\boldsymbol{N}]$

The enumeration of equations has been modified adopting a simple alphabetic ordination for convenient systematization facing the next recurrent routine.
Those new designations are,

TABLE 10

Equation $[6.35(?)-?]$	**Is now:**
$[6.35(0) - I]$	[A]
$[6.35(0) - II]$	[B]
$[6.35(L) - I]$	[C]
$[6.35(L) - II]$	[D]
$[6.35(L) - III]$	[E]
$[6.35(n) - I]$	[F]
$[6.35(n) - II]$	[G]
$[6.35(n) - III]$	[H]
$[6.35(0) - III]$	[I]
$[6.35(0) - IV]$	[J]
$[6.35(L) - IV]$	[K]
$[6.35(L) - V]$	[L]
$[6.35(n) - IV]$	[M]
$[6.35(n) - V]$	[N]

7. Nomenclature of initial system equations.

The suitable step now, before establish the precedence of the recurrent next stages, seems to be rewrite all equations, making explicit the coefficient which is calculated in each.
The system now suitably arranged for immediate recurrent calculations, is given by the following equations,

$d_{0,0} = 1$

$(initial\ time\ condition)$ $[\boldsymbol{\tau}, \mathbf{0}, \mathbf{0}]$

$d_{0,1} = [A_1^* - A_0^* - B_1^*] \cdot d_{0,0}$

$(terms\ in\ J_0\ at\ [6.21(0)])$ $[\boldsymbol{A}, \mathbf{0}, \mathbf{1}]$

$$d_{0,2} = [A_1^* - A_0^* - B_1^*] \cdot d_{0,1} + 2 \cdot d_{0,0} - B_1^* \cdot d_{1,1}$$

$(terms\ in\ J_1\ at\ [6.21(0)])$ $\quad$ $[\boldsymbol{B}, \boldsymbol{0}, \boldsymbol{2}]$

$$d_{n,n} = A_{n-1}^* \cdot d_{n-1,n-1}$$

$(terms\ in\ J_{n-1}\ at\ [6.21(L)])\ where\ (1 \le n \le L)$ $\quad$ $[\boldsymbol{C}, \boldsymbol{n}, \boldsymbol{0}]$

$$d_{n,n+1} = A_{n-1}^* \cdot d_{n-1,n} + [(B_n^* - B_1^*) - (A_n^* - A_1^*)] \cdot d_{n,n}$$

$(terms\ in\ J_n\ at\ [6.21(L)])\ where\ (1 \le n \le L)$ $\quad$ $[\boldsymbol{D}, \boldsymbol{n}, \boldsymbol{1}]$

$$d_{n,n+2} = A_{n-1}^* \cdot d_{n-1,n+1} + [(B_n^* - B_1^*) - (A_n^* - A_1^*)] \cdot d_{n,n+1} + d_{n,n} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,n+1}$$

$(terms\ in\ J_{n+1}\ at\ [6.21(L)])\ where\ (1 \le n \le L)$ $\quad$ $[\boldsymbol{E}, \boldsymbol{n}, \boldsymbol{2}]$

$$d_{n,n} = A_L^* \cdot d_{n-1,n-1}$$

$(terms\ in\ J_{n-1}\ at\ [6.21(n)])\ where\ (n \ge L+1)$ $\quad$ $[\boldsymbol{F}, \boldsymbol{n}, \boldsymbol{0}]$

$$d_{n,n+1} = A_L^* \cdot d_{n-1,n} + [(B_n^* - B_1^*) - (A_L^* - A_1^*)] \cdot d_{n,n}$$

$(terms\ in\ J_n\ at\ [6.21(n)])\ where\ (n \ge L+1)$ $\quad$ $[\boldsymbol{G}, \boldsymbol{n}, \boldsymbol{1}]$

$$d_{n,n+2} = A_L^* \cdot d_{n-1,n+1} + [(B_n^* - B_1^*) - (A_L^* - A_1^*)] \cdot d_{n,n+1} + d_{n,n} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,n+1}$$

$(terms\ in\ J_{n+1}\ at\ [6.21(n)])\ where\ (n \ge L+1)$ $\quad$ $[\boldsymbol{H}, \boldsymbol{n}, \boldsymbol{2}]$

$$d_{0,k+1} = [A_1^* - A_0^* - B_1^*] \cdot d_{0,k} + d_{0,k-1} - B_1^* \cdot d_{1,k} - \left[\sum_{j=2}^{k} D_j^* \cdot d_{j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(0)])\ where\ (2 \le k \le L)$ $\quad$ $[\boldsymbol{I}, \boldsymbol{0}, \boldsymbol{k+1}]$

$$d_{0,k+1} = [A_1^* - A_0^* - B_1^*] \cdot d_{0,k} + d_{0,k-1} - B_1^* \cdot d_{1,k} - \left[\sum_{j=2}^{L} D_j^* \cdot d_{j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(0)])\ where\ (k \ge L+1)$ $\quad$ $[\boldsymbol{J}, \boldsymbol{0}, \boldsymbol{k+1}]$

$$d_{n,k+1} = A_{n-1}^* \cdot d_{n-1,k} + [(B_n^* - B_1^*) - (A_n^* - A_1^*)] \cdot d_{n,k} + \\ + d_{n,k-1} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{k-n} (D_{j+1}^* - D_j^*) \cdot d_{n+j,k}\right]$$

$(terms\ in\ J_k\ at\ [6.21(L)])\ where\ (1 \le n \le L)\ and\ (n+2 \le k \le n+L-1)$

$[\boldsymbol{K}, \boldsymbol{n}, \boldsymbol{k-n+1}]$

$$d_{n,k+1} = A_{n-1}^* \cdot d_{n-1,k} + [(B_n^* - B_1^*) - (A_n^* - A_1^*)] \cdot d_{n,k} + d_{n,k-1} + \\ + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{L-1} (D_{j+1}^* - D_j^*) \cdot d_{n+j,k}\right] - D_L^* \cdot d_{n+L,k}$$

$(terms\ in\ J_k\ at\ [6.21(L)])\ where\ (1 \le n \le L)\ and\ (k \ge n+L)$

$[\boldsymbol{L}, \boldsymbol{n}, \boldsymbol{k-n+1}]$

$$d_{n,k+1} = A_L^* \cdot d_{n-1,k} + [(B_n^* - B_1^*) - (A_L^* - A_1^*)] \cdot d_{n,k} + \\ + d_{n,k-1} + (D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{k-n} (D_{j+1}^* - D_j^*) \cdot d_{n+j,k}\right]$$

$\left(terms\ in\ J_k\ at\ \left[6.21(n)\right]\right)\ where\ (n \ge L+1) and\ (n+2 \le k \le n+L-1)$

$[\boldsymbol{M}, \boldsymbol{n}, \boldsymbol{k-n+1}]$

$$d_{n,k+1} = A_L^* \cdot d_{n-1,k} + [(B_n^* - B_1^*) - (A_L^* - A_1^*)] \cdot d_{n,k} + d_{n,k-1} + \\ +(D_2^* - B_{n+1}^*) \cdot d_{n+1,k} + \left[\sum_{j=2}^{L-1} (D_{j+1}^* - D_j^*) \cdot d_{n+j,k}\right] - D_L^* \cdot d_{n+L,k}$$

$(terms\ in\ J_k\ at\ [6.21(n)])\ where\ (n \geq L+1)\ and\ (k \geq n+L)$ $\quad [\boldsymbol{N, n, k-n+1}]$

In this system the equations are referred to the indices n and k according to the coefficient that is obtained when its respective application is made. Solving an equation with the indices (n,k) means obtaining the coefficient $d_{n,n+k}$.
As an example, solving $[L, n, k-n+1]$ directs to reach $d_{n,k+1}$ where $(1 \leq n \leq L)$ and $(k \geq n+L)$.
The first capital letter has only the significance of alphabetic order to designate the equations in all this system. After the first comma the combination of indexes identify, in this nomenclature, what coefficients are calculated by using the equation choose.
The resolution of an equation, for achieve a given coefficient, implies having obtained previously other coefficients. So it is necessary to state a suitable calculation ordering for solving all this system of equations.
For orientation and visualization purposes let first define a tableau in which we set all these coefficients in a triangular configuration,

TABLE 11

Triangular table of coefficients $d_{n,k}$									
$\theta_0(\tau)$	$\theta_1(\tau)$	$\theta_2(\tau)$	$\theta_3(\tau)$	$\theta_4(\tau)$	$\theta_5(\tau)$	$\theta_6(\tau)$	$\theta_7(\tau)$	$\theta_8(\tau)$	……….
$d_{0,0}$									
$d_{0,1}$	$d_{1,1}$								
$d_{0,2}$	$d_{1,2}$	$d_{2,2}$							
$d_{0,3}$	$d_{1,3}$	$d_{2,3}$	$d_{3,3}$						
$d_{0,4}$	$d_{1,4}$	$d_{2,4}$	$d_{3,4}$	$d_{4,4}$					
$d_{0,5}$	$d_{1,5}$	$d_{2,5}$	$d_{3,5}$	$d_{4,5}$	$d_{5,5}$				
$d_{0,6}$	$d_{1,6}$	$d_{2,6}$	$d_{3,6}$	$d_{4,6}$	$d_{5,6}$	$d_{6,6}$			
$d_{0,7}$	$d_{1,7}$	$d_{2,7}$	$d_{3,7}$	$d_{4,7}$	$d_{5,7}$	$d_{6,7}$	$d_{7,7}$		
$d_{0,8}$	$d_{1,8}$	$d_{2,8}$	$d_{3,8}$	$d_{4,8}$	$d_{5,8}$	$d_{6,8}$	$d_{7,8}$	$d_{8,8}$	
……….	……….	……….	……….	……….	……….	……….	……….	……….	……….

Besides this tableau we also define, as useful tools, "diagonal", "column" and "computing sequence", as follows,
A diagonal is a set of coefficients of the form $d_{n,n+k}$ where $n = 1,2,\ldots$ and index k remains constant. Such set has an infinite number of elements and one expects that, as n increases, above a certain finite order, the absolute value of $d_{n,n+k}$ decreases, so decreasing as well the relative value of the function $\theta_n(\tau)$ in which it is included. In lower orders n, functions $\theta_n(\tau)$ can, in a given time τ, increase or decrease.
We also adopt often the ordination to design a diagonal: that of coefficients $d_{n,n}$ $(k = 0)$ is the first diagonal, that of $d_{n,n+1}$ $(k = 1)$ the second diagonal, … , that of $d_{n,n+k}$ $(k\ in\ general)$ is the $(k+1)$-nth diagonal.
Consequently, the order of a diagonal $d_{n,n+k}$ is defined as $(k+1)$.
A column is a set of coefficients of the form $d_{n,n+k}$ where $k = 1,2,\ldots$ and index n remains constant. Each column is associated to a function $\theta_n(\tau)$, because is in each column where we can find all the coefficients that define the infinite series of Bessel functions terms constituting a particular function $\theta_n(\tau)$.
Such set has an infinite number of elements and one also expects a decrease of absolute values as index k increases, at least above a certain finite value of k.
Lastly we define a computing sequence as a suitable set of tree equations chosen among those fifteen just written (we include also the initial time condition).

Each of those tree equations allows to compute one, and only one, specific diagonal whose elements are $d_{n,n+k}$, where k is fixed.
Clarifying better, the first one conducts to the coefficient $d_{0,k}$, the second one, which consists really in a system of L equations, allows to reach all the coefficients $d_{n,n+k}$ between $(n = 1)$ and $(n = L)$, and the third, which is also a system of equations, in this case infinite in number, whose utilization directs to all the coefficients $d_{n,n+k}$ where $(n \geq L + 1)$.
Solving such a computing sequence means to determine an entire diagonal.
For reasons of compulsory precedence a specific diagonal can be calculated only if all the others, of inferior order, are already obtained.

8. Recurrent solving steps.

Accordingly, and also in a exhaustively detailed way, now we must state the following calculus sequence,

Sequence 1. (is composed by steps 1, 2 and 3)

Step 1.

The initial condition $[\tau, 0, 0]$, in $\tau = 0$, implies that $d_{0,0} = 1$.

Step 2.

Solving equations $[C, n, 0]$, from $n = 1$ to $n = L$, $[C, 1, 0]$, $[C, 2, 0]$, … , $[C, L, 0]$, provides the coefficients, $d_{1,1}$, $d_{2,2}$, … , $d_{L,L}$

Step 3.

Solving equations $[F, n, 0]$, for $n = (L + 1),(L + 2),(L + 3)\ldots\ldots$ $[F, L + 1, 0]$, $[F, L + 2, 0]$, $[F, L + 3, 0]$, … … provides the coefficients, $d_{L+1,L+1}$, $d_{L+2,L+2}$, $d_{L+3,L+3}$ … …

Sequence 2. (is composed by steps 4, 5 and 6)

Step 4.

The equation $[A, 0, 1]$, allows to get $d_{0,1}$.

Step 5.

Solving equations $[D, n, 1]$, from $n = 1$ to $n = L$, $[D, 1, 1]$, $[D, 2, 1]$, … , $[D, L, 1]$, provides the coefficients, $d_{1,2}$, $d_{2,3}$, … , $d_{L,L+1}$

Step 6.

Solving equations $[G, n, 1]$, for $n = (L + 1),(L + 2),(L + 3)\ldots\ldots$ $[G, L + 1, 1]$, $[G, L + 2, 1]$, $[G, L + 3, 1]$, … … provides the coefficients, $d_{L+1,L+2}$, $d_{L+2,L+3}$, $d_{L+3,L+4}$ … …

Sequence 3. (is composed by steps 7, 8 and 9)

Step 7.

The equation $[B, 0, 2]$, allows to get $d_{0,2}$.

Step 8.

Solving equations $[E, n, 2]$, from $n = 1$ to $n = L$, $[E, 1, 2]$, $[E, 2, 2]$, … , $[E, L, 2]$, provides the coefficients, $d_{1,3}$, $d_{2,4}$, … , $d_{L,L+2}$

Step 9.

Solving equations $[H, n, 2]$, for $n = (L + 1),(L + 2),(L + 3)\ldots\ldots$ $[H, L + 1, 2]$, $[H, L + 2, 2]$, $[H, L + 3, 2]$, … … provides the coefficients, $d_{L+1,L+3}$, $d_{L+2,L+4}$, $d_{L+3,L+5}$ … …

Sequence 4. **(is composed by steps 10, 11 and 12)**

Step 10.

The equation $[I,0,k+1]$, is valid for $k=2$ to $k=L$ but in this step we only use it for $k=2$. Consequently, solving $[I,0,3]$, one gets $d_{0,3}$.

Step 11.

The equation $[K,n,k-n+1]$, is valid for $n=1$ to $n=L$ and for $k=(n+2)$ to $k=(n+L-1)$ but in this step we only use it for $k=(n+2)$. And for all the range of n, what is the same to say that we will solve the equations $[K,1,3]$, $[K,2,3]$, … , $[K,L,3]$. And the values of the coefficients $d_{1,4}$, $d_{2,5}$, … , $d_{L,L+3}$ are obtained.

Step 12.

The equation $[M,n,k-n+1]$, is valid for $n=(L+1)$,$(L+2)$,$(L+3)$… … and for $k=(n+2)$ to $k=(n+L-1)$ but in this step we only use it for $k=(n+2)$. And for all the range of n, in infinite number, what is the same to say that we will solve the equations $[M,L+1,3]$, $[M,L+2,3]$, $[M,L+3,3]$, … … . This way the values of the coefficients $d_{L+1,L+4}$, $d_{L+2L+,5}$, $d_{L+3,L+,6}$, … … , are obtained.

Sequence 5. **(is composed by steps 13, 14 and 15)**

Step 13.

Now we return to the equation $[I,0,k+1]$, valid for $k=2$ to $k=L$ but in this step we only use it for $k=3$. So, solving $[I,0,4]$, one gets $d_{0,4}$.

Step 14.

Back again to the equation $[K,n,k-n+1]$, valid for $n=1$ to $n=L$ and for $k=(n+2)$ to $k=(n+L-1)$ we now use it for $k=(n+3)$. And, as usual, for all the range of n, what is the same to say that we will solve the equations $[K,1,4]$, $[K,2,4]$, … , $[K,L,4]$. And, with this procedure, the values of the coefficients $d_{1,5}$, $d_{2,6}$, … , $d_{L,L+4}$ are obtained.

Step 15.

We revisit, as well, the equation $[M,n,k-n+1]$, which is valid for $n=(L+1)$, $(L+2)$,$(L+3)$… … and for $k=(n+2)$ to $k=(n+L-1)$ and this time we only use it for $k=(n+3)$, and also for all the range of n, in infinite number. Consequently we will solve the equations $[M,L+1,4]$, $[M,L+2,4]$, $[M,L+3,4]$, … … . This way one reaches the values of the coefficients $d_{L+1,L+5}$, $d_{L+2L+,6}$, $d_{L+3,L+,7}$, … … .

FIRST INTERLUDE REMARK:

The last two sequences, numbered 4 and 5, have the same repetitive routine, and in it we have inserted the two first values of k , belonging to the range from $k=2$ to $k=L$ in the case of equation $[I,0,k+1]$, and belonging to the range from $k=(n+2)$ to $k=(n+L-1)$ in the cases of equations $[K,n,k-n+1]$ and $[M,n,k-n+1]$.
The range of $[I,0,k+1]$ has $(L-1)$ values and those of $[K,n,k-n+1]$ and $[M,n,k-n+1]$ have $(L-2)$ values.
What follows now is the same repetitive application of the equations sequence $[I,0,k+1]$, plus $[K,n,k-n+1]$, plus $[M,n,k-n+1]$ by $(L-4)$ more times, till reach sequence number $(L+1)$.
In a more resumed fashion, sequences 6 to $(L+1)$ can be described as follows.

Sequence 6. **(is composed by steps 16, 17 and 18)**

Step 16.

Equation $[I, 0, k+1]$ **where** $k = 4$ **is** $[I, 0, 5]$ **and provide coefficient** $d_{0,5}$

Step 17.

Equations $[K, n, k-n+1]$ **where** $k = (n+4)$ **and** $n = 1, 2, \dots, L$ **are** $[K, 1, 5]$, $[K, 2, 5]$, … … , $[K, L, 5]$ **and provides coefficients** $d_{1,6}$, $d_{2,7}$, … , $d_{L,L+5}$

Step 18.

Equations $[M, n, k-n+1]$ **where** $k = (n+4)$ **and** $n = (L+1), (L+2), (L+3), \dots \dots$ **are** $[M, L+1, 5]$, $[M, L+2, 5]$, $[M, L+3, 5]$, … **and provides coefficients** $d_{L+1,L+6}$, $d_{L+2L+,7}$, $d_{L+3,L+,8}$, … ….

Sequence 7. **(is composed by steps 19, 20 and 21)**

Step 19.

Equation $[I, 0, k+1]$ **where** $k = 5$ **is** $[I, 0, 6]$ **and provide coefficient** $d_{0,6}$

Step 20.

Equations $[K, n, k-n+1]$ **where** $k = (n+5)$ **and** $n = 1, 2, \dots, L$ **are** $[K, 1, 6]$, $[K, 2, 6]$, … … , $[K, L, 6]$ **and provides coefficients** $d_{1,7}$, $d_{2,8}$, … , $d_{L,L+6}$

Step 21.

Equations $[M, n, k-n+1]$ **where** $k = (n+5)$ **and** $n = (L+1), (L+2), (L+3), \dots \dots$ **are** $[M, L+1, 6]$, $[M, L+2, 6]$, $[M, L+3, 6]$, … **and provides coefficients** $d_{L+1,L+7}$, $d_{L+2L+,8}$, $d_{L+3,L+,9}$, … … .

Sequence 8. **(is composed by steps 22, 23 and 24)**

Step 22.

Equation $[I, 0, k+1]$ **where** $k = 6$ **is** $[I, 0, 7]$ **and provide coefficient** $d_{0,7}$

Step 23.

Equations $[K, n, k-n+1]$ **where** $k = (n+6)$ **and** $n = 1, 2, \dots, L$ **are** $[K, 1, 7]$, $[K, 2, 7]$, … … , $[K, L, 7]$ **and provides coefficients** $d_{1,8}$, $d_{2,9}$, … , $d_{L,L+7}$

Step 24.

Equations $[M, n, k-n+1]$ **where** $k = (n+6)$ **and** $n = (L+1), (L+2), (L+3), \dots \dots$ **are** $[M, L+1, 7]$, $[M, L+2, 7]$, $[M, L+3, 7]$, … **and provides coefficients** $d_{L+1,L+8}$, $d_{L+2L+,9}$, $d_{L+3,L+,10}$, … ….

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Sequence L. **[is composed by steps (3L-2), (3L-1), and (3L)]**

Step (3L-2).

Equation $[I, 0, k+1]$ **where** $k = (L-2)$ **is** $[I, 0, L-1]$ **and provide coefficient** $d_{0,L-1}$

Step (3L-1).

Equations $[K, n, k-n+1]$ **where** $k = (n+L-2)$ **and** $n = 1, 2, \dots, L$ **are** $[K, 1, L-1]$, $[K, 2, L-1]$, … , $[K, L, L-1]$ **and provides coefficients** $d_{1,L}$, $d_{2,L+1}$, … , $d_{L,2L-1}$

Step (3L).

Equations $[M, n, k-n+1]$ **where** $k = (n+L-2)$ **and** $n = (L+1), (L+2), (L+3), \dots \dots$ **are** $[M, L+1, L-1]$, $[M, L+2, L-1]$, $[M, L+3, L-1]$, … **and provides coefficients** $d_{L+1,2L}$, $d_{L+2,2L+,1}$, $d_{L+3,2L+,2}$, … ….

Sequence (L+1). **[is composed by steps (3L+1), (3L+2), and (3L+3)]**

Step (3L+1).

Equation $[I, 0, k+1]$ **where** $k = L - 1$ **is** $[I, 0, L]$ **and provide coefficient** $d_{0,L}$

Step (3L+2).

Equations $[K, n, k-n+1]$ **where** $k = (n + L - 1)$ **and** $n = 1, 2, \dots, L$ **are** $[K, 1, L]$, $[K, 2, L]$, … , $[K, L, L]$ **and provides coefficients** $d_{1,L+1}$, $d_{2,L+2}$, … , $d_{L,2L}$

Step (3L+3).

Equations $[M, n, k-n+1]$ **where** $k = (n + L - 1)$ **and** $n = (L+1), (L+2), (L+3), \dots \dots$ **are** $[M, L+1, L]$, $[M, L+2, L]$, $[M, L+3, L]$, … **and provides coefficients** $d_{L+1,2L+1}$, $d_{L+2,2L+,2}$, $d_{L+3,2L+,3}$, … ….

SECOND INTERLUDE REMARK:

Notice that we have not yet exhaust all the possible applications in equation $[I, 0, k+1]$. **As a matter of fact we still did not used it for** $k = L$.

That application is done in the immediately next sequence, numbered (L+2).

Sequence (L+2). **[is composed by steps (3L+4), (3L+5), and (3L+6)]**

Step (3L+4).

Equation $[I, 0, k+1]$ **where** $k = L$ **is** $[I, 0, L+1]$ **and provide coefficient** $d_{0,L+1}$

Step (3L+5).

Equations $[L, n, k-n+1]$ **where** $k = (n + L)$ **and** $n = 1, 2, \dots, L$ **are** $[L, 1, L+1]$, $[L, 2, L+1]$, … , $[L, L, L+1]$ **and provides coefficients** $d_{1,L+2}$, $d_{2,L+3}$, … , $d_{L,2L+1}$

Step (3L+6).

Equations $[N, n, k-n+1]$ **where** $k = (n + L\ )$ **and** $n = (L+1), (L+2), (L+3), \dots \dots$ **are** $[N, L+1, L+1]$, $[N, L+2, L+1]$, $[N, L+3, L+1]$, … **and provides coefficients** $d_{L+1,2L+2}$, $d_{L+2,2L+,3}$, $d_{L+3,2L+,4}$, … ….

Sequence (L+3). **[is composed by steps (3L+7), (3L+8), and (3L+9)]**

Step (3L+7).

Equation $[J, 0, k+1]$ **where** $k = (L+1\ )$ **is** $[J, 0, L+2]$ **and provide coefficient** $d_{0,L+2}$

Step (3L+8).

Equations $[L, n, k-n+1]$ **where** $k = (n + L + 1)$ **and** $n = 1, 2, \dots, L$ **are** $[L, 1, L+2]$, $[L, 2, L+2]$, … , $[L, L, L+2]$ **and provides coefficients** $d_{1,L+3}$, $d_{2,L+4}$, … , $d_{L,2L+2}$

Step (3L+9).

Equations $[N, n, k-n+1]$ **where** $k = (n + L + 1)$ **and** $n = (L+1), (L+2), (L+3), \dots \dots$ **are** $[N, L+1, L+2]$, $[N, L+2, L+2]$, $[N, L+3, L+2]$, … **and provides coefficients** $d_{L+1,2L+3}$, $d_{L+2,2L+,4}$, $d_{L+3,2L+,5}$, … ….

THIRD INTERLUDE REMARK:

This last sequence has the number (L+3) and, after reach its corresponding calculation, the entire sequel consists in repeat once and once again the sequence of equations $[J, 0, k+1]$ **with** $k = (L + m\ )$, **plus** $[L, n, k-n+1]$ **with** $k = (n + L + m\ )$, **plus** $[N, n, k-n+1]$ **with** $k = (n + L + m\ )$, **being** $(m \geq 2)$.

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In general,

Sequence (L+m+2). [is composed by steps (3L+3m+4), (3L+3m+5), and (3L+3m+6)]

Step (3L+3m+4).

Equation $[J,0,k+1]$ where $k=(L+m)$ is $[J,0,L+m+1]$ and provide coefficient $d_{0,L+m+1}$

Step (3L+3m+5).

Equations $[L,n,k-n+1]$ where $k=(n+L+m)$ and $n=1,2,\ldots,L$ are $[L,1,L+m+1]$, $[L,2,L+m+1]$, … , $[L,L,L+m+1]$ and provides coefficients $d_{1,L+m+2}$, $d_{2,L+m+3}$, … … , $d_{L,2L+m+1}$

Step (3L+3m+6).

Equations $[N,n,k-n+1]$ where $k=(n+L+m)$ and $n=(L+1),(L+2),(L+3),\ldots\ldots$ are $[N,L+1,L+m+1]$, $[N,L+2,L+m+1]$, $[N,L+3,L+m+1]$, … and provides coefficients $d_{L+1,2L+m+2}$, $d_{L+2,2L+,m+3}$, $d_{L+3,2L+,m+4}$, … ….

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This way the exact analytical solution for the proposed mathematical model, describing the early stages of formation and growth of biofilm, is complete.

CHAPTER VII

A numerical result in M/M/1 queue transient phase and future perspectives. (seeding the basis of a future handbook)

1. Next future work (resumption of Chapters V and VI)

From the deductions made in Chapter V, namely in all the sections concerning M/M/1 queue transient phase, we can now establish the basis for construct gradually along next near future a sort of handbook containing tables and graphics accounting for transient regimes, not only for M/M/1 queue but also for much more other different queue models.
The main idea is to mimic Abramowitz and Stegun [51] celebrated handbook, incorporating three kind of sections:
- A deductive section also with a formulary as consequent result.
- A section of exhaustive accurate tables for the state probabilities during all transient phases.
- A section of graphic representation of data included in the afore mentioned tables.
Back in 1962 Descloux [52] published a handbook, reprinted many times (at least till 2013) of numerical tables in this context of traffic flow and service systems but not emphasizing transition and state probabilities, like we are proposing. Besides, all data refers there to steady state or averaged values over time intervals and all examples confine to a finite number of sources (finite storage room in our definitions).
In Chapter V we have constructed a very complete set of recurrence relations concerning the M/M/1 queue and in Chapter VI we have demonstrated the usefulness of applying a guess solution to a very complicated model, alike to those found in queuing transient regime, but rather more complicated.

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<u>Briefly reviewing Chapters V and VI we recall that,</u>
- A) First of all, a five terms recurrence [5.26(0) and 5.26(n)] or [5.27], has been obtained to handle the intricate solution reached in chapter IV for the "Mono-layered concentrated growth" kinetics, defined by formulas [4.117] for continuous biofilm (case 1) and by [4.127] for patchy biofilm (case 2). This has been done in the mathematical framework of old Three Terms Recurrence Miller´s algorithm but defining a new one algorithm, more general as a five (and not only three) terms recurrence requires. Here we were dealing with the Bessel functions of first kind, $J_k(\mathrm{x})$.
- B) Then an adaptation, leading to a similar Miller´s generalized method, has been established for classical M/M/1 queuing system with infinite storage room. For such finality a suitable normalization condition has been deduced in Section 3.5 at Chapter V. We don´t forget that now we are dealing with the Modified Bessel functions of first kind, $I_k(\mathrm{x})$.
- C) This last mentioned method has been validated computing the M/M/1 queuing system solution by direct calculation of the Modified Bessel functions at each term in the infinite summation and then inserting they in this infinite summation, affecting each one with their respective coefficients. The Bessel functions at each term have been computed using the TTR Miller Algorithm.
- D) Being more explicit, the values of the M/M/1 exact solution obtained by the two methods just recalled in B) and C) coincide one to each other in all the time range computed. So those two methods validate mutually, one to the other.
- E) The method at C) consists in made the get together of all the Bessel functions, in one side, with all the coefficients affecting each one of those functions in each term, on the other side. And, by this way construct and reach the complete infinite summation of terms solution.

- F) In the sequel we faced with a more complicated problem, described in Chapter VI, as consequence of the intricate structure of the initial system formed by the equations [6.21(0)], [6.21(L)] and [6.21(n)] and also the initial condition [6.21, τ, 0]. Then the exact analytic solution was accomplished adopting a guess solution which we now rewrite here,

$$\theta_n(\tau) = e^{a\cdot\tau} \cdot \sum_{k=n}^{\infty} d_{n,k} \cdot J_k(2 \cdot b \cdot \tau) \qquad [7.1-(n)]\ ([6.4] \text{ in Chapter VI})$$

For coherence purpose with the notation in so far analyzed queuing context it is preferable replace index n by index j. So this guess solution will, in the sequel, reads,

$$\theta_j(\tau) = e^{a\cdot\tau} \cdot \sum_{k=j}^{\infty} d_{j,k} \cdot J_k(2 \cdot b \cdot \tau) \qquad [7.1]$$

Let recall that this solution was obtained by a method consisting in the algebraic deduction and computation of the coefficients $d_{j,k}$, exhaustively explained in Chapter VI. This is the hard part of the task, and not the calculation of each Bessel function, which is easily done applying the TTR miller algorithm.

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From remarks A, B, C, D, E and F we can conclude, in a sequential way, that we handled the most simple version of biofilm model applying a generalization of Miller´s algorithm and, for such purpose, we deduced a Five Terms Recurrence.
Then, with the gained skills we applied a similar method in the classical M/M/1 queuing system, which is an easier model, and more interesting for most authors, than the biofilm growth model.
The logic continuation was to validate this last method, in the way like refereed in C. A task which has been done successively.
We observe now that the method at C is not only an accurate approach for M/M/1 queuing system but also consists in a building up of the infinite summation solution making the afore mentioned get together between the Bessel functions and their respective coefficients.

2. Recalling the two more simple analytic solutions

Let now take an attentive look at the biofilm "Mono-layered concentrated growth" kinetics model, for the continuous case (also replacing the index n by the index j coherently with the notation in queuing context),

SOLUTION FOR CASE 1: CONTINUOUS BIOFILM $[\boldsymbol{\delta}<(\mathbf{1}-\boldsymbol{\Phi})]$

$$\theta_j(\tau) = \alpha_j \cdot e^{y} \cdot \sum_{k=0}^{\infty} \beta_k \cdot J_{j+k}(x) \qquad [7.2]\ ([4.117] \text{ in Chapter IV})$$

Where:

$$x = 2 \cdot \tau \cdot \sqrt{(\rho + \Phi)\cdot(1 - \Phi - \delta)}$$

$$y = (1 - \rho - 2 \cdot \Phi - \delta) \cdot \tau$$

$$\alpha_0 = 1$$

$$\alpha_j = \frac{\rho \cdot (\rho + \Phi)^{j-1}}{\sqrt{(\rho + \Phi)^j \cdot (1 - \Phi - \delta)^j}} \ldots \textit{only for } j \geq 1$$

$$\beta_0 = 1$$

$$\beta_1 = \frac{(2 \cdot \Phi + \delta - 1)}{\sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta)}}$$

$$\beta_k = \frac{\{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^{k-1} + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^k\}}{(1 - \delta) \cdot \sqrt{(\rho + \Phi)^k \cdot (1 - \Phi - \delta)^k}}$$

$$\ldots only\ for\ k \geq 2$$

And, simultaneously, also a look at the classical analytic solution for the M/M/1 queuing system,

$$P_{i,j}(\tau) = e^{-(1+\rho)\tau} \cdot \{ \rho^{(j-i)/2}\, I_{j-i}(2\tau\sqrt{\rho}) + \rho^{(j-i-1)/2}\, I_{j+i+1}(2\tau\sqrt{\rho}) + $$

$$+(1-\rho)\, \rho^j \sum_{k=j+i+2}^{\infty} \rho^{-k/2}\, I_k(2\tau\sqrt{\rho})\}$$

[7.3] ([5.27] in Chapter V)

Both solutions [7.2] and [7.3] are analytic exact solutions. One, [7.2], deduced by ourselves and the other, [7.3], in a very different context, a classical one known many years before, since 1954.

Being analytic exact solutions the coefficients component is already completely known by direct inspection of their own mathematical expressions.

Now we can recall that [7.2] inspired us to construct a guess solution suitable for a more complicated problem, like that one concerning the most general version of biofilm model, described and deduced in detail in Chapter VI. In such way we adopted [7.1] as guess solution.

Applying a very similar criterion we adopt now a new guess solution with the purpose of also describe and deduce exact analytic solutions for transient regimes in others, more complex than M/M/1 system, queuing models.

That new guess solution reads,

$$P_{i,j}(\tau) = e^{a \cdot \tau} \cdot \sum_{k=0}^{\infty} d_{j,k} \cdot I_k(2 \cdot b \cdot \tau) \qquad [7.4 - A]$$

... where, $a = -(1+\rho)$ and $b = \sqrt{\rho}$.

However we rather prefer to replace a and b by $\alpha = \sqrt{\rho}$ and $\beta = (1-\rho)$. Accordingly, $a = (\beta - 2)$ and $b = \alpha$ and $[7.4 - A]$ now reads,

$$P_{i,j}(\tau) = e^{(\beta-2)\cdot\tau} \cdot \sum_{k=0}^{\infty} d_{j,k} \cdot I_k(2 \cdot \alpha \cdot \tau) \qquad [7.4 - B]$$

3. Tableau of coefficients $d_{j,k}$ ("Mono-layered concentrated biofilm growth" kinetics model)

Besides guess solutions, tableaux like the one at Chapter VI are also essential working tools. We reproduce now that afore mentioned tableau in the next page,

TABLE 12

Triangular table of coefficients $d_{j,k}$										
["Mono-layered concentrated biofilm growth" kinetics model/Chapter IV] or [Biofilm general model version/Chapter VI]										
$J_k(x)$	$\theta_0(\tau)$	$\theta_1(\tau)$	$\theta_2(\tau)$	$\theta_3(\tau)$	$\theta_4(\tau)$	$\theta_5(\tau)$	$\theta_6(\tau)$	$\theta_7(\tau)$	$\theta_8(\tau)$	……….
$J_0(x)$	$d_{0,0}$									
$J_1(x)$	$d_{0,1}$	$d_{1,1}$								
$J_2(x)$	$d_{0,2}$	$d_{1,2}$	$d_{2,2}$							
$J_3(x)$	$d_{0,3}$	$d_{1,3}$	$d_{2,3}$	$d_{3,3}$						
$J_4(x)$	$d_{0,4}$	$d_{1,4}$	$d_{2,4}$	$d_{3,4}$	$d_{4,4}$					
$J_5(x)$	$d_{0,5}$	$d_{1,5}$	$d_{2,5}$	$d_{3,5}$	$d_{4,5}$	$d_{5,5}$				
$J_6(x)$	$d_{0,6}$	$d_{1,6}$	$d_{2,6}$	$d_{3,6}$	$d_{4,6}$	$d_{5,6}$	$d_{6,6}$			
$J_7(x)$	$d_{0,7}$	$d_{1,7}$	$d_{2,7}$	$d_{3,7}$	$d_{4,7}$	$d_{5,7}$	$d_{6,7}$	$d_{7,7}$		
$J_8(x)$	$d_{0,8}$	$d_{1,8}$	$d_{2,8}$	$d_{3,8}$	$d_{4,8}$	$d_{5,8}$	$d_{6,8}$	$d_{7,8}$	$d_{8,8}$	
……….	……….	……….	……….	……….	……….	……….	……….	……….	……….	……….

This has a triangular shape as consequence of the location of null coefficients. Al null coefficients are located above the negative diagonal placed and defined by $d_{0,0}$, $d_{1,1}$, $d_{2,2}$, $d_{3,3}$, …
The yellow first column signals the bare fraction $\theta_0(\tau)$ as the only one non null at initial time.
This is true not only at the most simple version ("Mono-layered concentrated growth" kinetics model) but also at the general version deduced in chapter VI.
The triangular configuration is also true in these two cases.
In these two models, as in all the others one can conceive, in biofilm or in queuing theory, the locations of the null coefficients is straightforward obtained applying the initial condition, at time $\tau = 0$.
The most general version at Chapter VI wood direct us to close form expressions for all the coefficients in the non null triangle but their representations wood be cumbersome. We did not deduce such close forms in Chapter VI for this reason.
Even at the "Mono-layered concentrated growth" model such close forms are quit intricate as we can see at the formulas for α_0, α_j, β_0, β_1, and β_k, which define the solution [7.2].
Any way we can now, for orientation and visualization purposes, set all these non null coefficients in their respective places at the diagonal $d_{0,0}$, $d_{1,1}$, $d_{2,2}$, $d_{3,3}$, … and also at all the remainder triangle below.
We obtain,

TABLE 13

General table of coefficients $d_{j,k}$										
Specified for the ["Mono-layered concentrated biofilm growth" model/Chapter IV]										
$J_k(x)$	$\theta_0(\tau)$	$\theta_1(\tau)$	$\theta_2(\tau)$	$\theta_3(\tau)$	$\theta_4(\tau)$	$\theta_5(\tau)$	$\theta_6(\tau)$	$\theta_7(\tau)$	$\theta_8(\tau)$	$\theta_9(\tau)$	………
$J_0(x)$	1										
$J_1(x)$	β_1	α_1									
$J_2(x)$	β_2	$\alpha_1\beta_1$	α_2								
$J_3(x)$	β_3	$\alpha_1\beta_2$	$\alpha_2\beta_1$	α_3							
$J_4(x)$	β_4	$\alpha_1\beta_3$	$\alpha_2\beta_2$	$\alpha_3\beta_1$	α_4						
$J_5(x)$	β_5	$\alpha_1\beta_4$	$\alpha_2\beta_3$	$\alpha_3\beta_2$	$\alpha_4\beta_1$	α_5					
$J_6(x)$	β_6	$\alpha_1\beta_5$	$\alpha_2\beta_4$	$\alpha_3\beta_3$	$\alpha_4\beta_2$	$\alpha_5\beta_1$	α_6				
$J_7(x)$	β_7	$\alpha_1\beta_6$	$\alpha_2\beta_5$	$\alpha_3\beta_4$	$\alpha_4\beta_3$	$\alpha_5\beta_2$	$\alpha_6\beta_1$	α_7			
$J_8(x)$	β_8	$\alpha_1\beta_7$	$\alpha_2\beta_6$	$\alpha_3\beta_5$	$\alpha_4\beta_4$	$\alpha_5\beta_3$	$\alpha_6\beta_2$	$\alpha_7\beta_1$	α_8		
$J_9(x)$	β_9	$\alpha_1\beta_8$	$\alpha_2\beta_7$	$\alpha_3\beta_6$	$\alpha_4\beta_5$	$\alpha_5\beta_4$	$\alpha_6\beta_3$	$\alpha_7\beta_2$	$\alpha_8\beta_1$	α_9	
………	………	………	………	………	………	………	………	………	………	………	………

At this stage we also can notice the algebraic structure in all the set of non null determined coefficients. Each colour signs a subset where all the coefficients are alike.

<u>- Blue:</u> $d_{0,0} = 1$, as expected for the first term of the only non null function $\theta_j(\tau)$ at time $\tau = 0$.

- Yellow: $d_{0,1} = \beta_1$, the coefficient at the second term of the function $\theta_{\,0}(\tau)$. It is unique because

$$\beta_1 = \frac{(2 \cdot \Phi + \delta - 1)}{\sqrt{(\rho + \Phi) \cdot (1 - \Phi - \delta)}} \qquad [7.5(1)]\{From\ [7.2]\}$$

... don´t fit with the general formula for β_k $(k \geq 2)$ and besides in all the other places where β_1 occurs is multiplied by α_j $(j \geq 1)$.

- Light yellow: $d_{0,2} = \beta_2,\ d_{0,3} = \beta_3,\ d_{0,4} = \beta_4,\ d_{0,5}, = \beta_5\ \dots$, is the first column and a sequence of coefficients all of them observing the same general close formula which is

$$\beta_k = \frac{\{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^{k-1} + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^k\}}{(1 - \delta) \cdot \sqrt{(\rho + \Phi)^k \cdot (1 - \Phi - \delta)^k}}$$

$$[7.5(k)]\{From\ [7.2]\}$$

... has been achieved.

- Orange: $d_{1,1} = \alpha_1,\ d_{2,2} = \alpha_2,\ d_{3,3} = \alpha_3, d_{4,4} = \alpha_4,\ \dots$, is the first negative diagonal (except element $d_{0,0}$) and accounts for the coefficients at the first non null term in each function $\theta_{\,j}(\tau)$ with $(j \geq 1)$. Here we have also a close formula,

$$\alpha_j = \frac{\rho \cdot (\rho + \Phi)^{j-1}}{\sqrt{(\rho + \Phi)^j \cdot (1 - \Phi - \delta)^j}} \qquad [7.6(j)]\{From\ [7.2]\}$$

- Green: $d_{1,2} = \alpha_1\beta_1,\ d_{2,3} = \alpha_2\beta_1,\ d_{3,4} = \alpha_3\beta_1, d_{4,5} = \alpha_4\beta_1,\ \dots$, is the second negative diagonal (except element $d_{0,1}$) and accounts for the coefficients at the second non null term in each function $\theta_{\,j}(\tau)$ with $(j \geq 1)$. Here we have also a close formula,

$$\alpha_j\beta_1 = \rho \cdot (2 \cdot \Phi + \delta - 1) \cdot \sqrt{\frac{(\rho + \Phi)^{j-3}}{(1 - \Phi - \delta)^{j+1}}} \qquad [7.7(j+1)]\{From\ [7.2], after\ a\ little\ algebra\}$$

- Light blue: $d_{j,k} = \alpha_j\beta_{k-j}\ \ for\ (j \geq 1)\ and\ (k \geq j + 2)$. This light blue domain is the triangle below the second negative diagonal and at right hand side of the first column. Once again a close formula is possible. Let see,

$$\alpha_j = \frac{\rho \cdot (\rho + \Phi)^{j-1}}{\sqrt{(\rho + \Phi)^j \cdot (1 - \Phi - \delta)^j}} \qquad \{From\ [7.2]\}$$

$$\beta_{k-j} = \frac{\{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^{k-j-1} + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^{k-j}\}}{(1 - \delta) \cdot \sqrt{(\rho + \Phi)^{k-j} \cdot (1 - \Phi - \delta)^{k-j}}}$$

$$\{From\ [7.2], after\ replace\ k\ by\ (k - j)\}$$

... consequently,

$$\alpha_j\beta_{k-j} = \frac{\rho \cdot \{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^{k-j-1} + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^{k-j}\}}{(1 - \delta)} \cdot \sqrt{\frac{(\rho + \Phi)^{2(j-1)-k}}{(1 - \Phi - \delta)^k}}$$

$$[7.7(k)]\{From\ [7.2], after\ replace\ k\ by\ (k - j)\}$$

4. Tableau of coefficients $d_{j,k}$ (M/M/1 queuing system)

Also for orientation and visualization purposes, we will set now all the non null coefficients in their respective places for the M/M/1 queuing system solution defined by [7.3].
For start the Tableau where the non null coefficients take their places must be established.

An easy inspection at [7.3] leads to the following Tableau configuration, where the non null coefficients are partitioned in subsets only by their geometric shape and location: two diagonals (one positive and finite and the other negative and infinite) and also a triangular zone below a negative infinite diagonal,

TABLE 14

General table of coefficients $d_{j,k}$														
Specified for the M/M/1 queuing system														
I_k	$P_{i,0}(\tau)$	$P_{i,1}(\tau)$	$P_{i,2}(\tau)$	$P_{i,3}(\tau)$	$P_{i,4}(\tau)$	....	$P_{i,i-2}(\tau)$	$P_{i,i-1}(\tau)$	$P_{i,i}(\tau)$	$P_{i,i+1}(\tau)$	$P_{i,i+2}(\tau)$	$P_{i,i+3}(\tau)$	....	I_k
I_0									$d_{i,0}$					I_0
I_1								$d_{i-1,1}$		$d_{i+1,1}$				I_1
I_2							$d_{i-2,2}$				$d_{i+2,2}$			I_2
....						....						$d_{i+3,3}$		I_3
$I_{(i-4)}$					$d_{4,i-4}$								....	....
$I_{(i-3)}$				$d_{3,i-3}$										$I_{(i-3)}$
$I_{(i-2)}$			$d_{2,i-2}$											$I_{(i-2)}$
$I_{(i-1)}$		$d_{1,i-1}$												$I_{(i-1)}$
$I_{(i)}$	$d_{0,i}$													$I_{(i)}$
$I_{(i+1)}$	$d_{0,i+1}$													$I_{(i+1)}$
$I_{(i+2)}$	$d_{0,i+2}$	$d_{1,i+2}$												$I_{(i+2)}$
$I_{(i+3)}$	$d_{0,i+3}$	$d_{1,i+3}$	$d_{2,i+3}$											$I_{(i+3)}$
$I_{(i+4)}$	$d_{0,i+4}$	$d_{1,i+4}$	$d_{2,i+4}$	$d_{3,i+4}$										$I_{(i+4)}$
....	....	....	....	....	....									....
$I_{(2i-2)}$	$d_{0, 2i-2}$	$d_{1, 2i-2}$	$d_{2,2i-2}$	$d_{3,2i-2}$	$d_{4, 2i-2}$	....								$I_{(2i-2)}$
$I_{(2i-1)}$	$d_{0, 2i-1}$	$d_{1, 2i-1}$	$d_{2,2i-1}$	$d_{3,2i-1}$	$d_{4, 2i-1}$	....	$d_{i-2, 2i-1}$							$I_{(2i-1)}$
$I_{(2i)}$	$d_{0, 2i}$	$d_{1, 2i}$	$d_{2,2i}$	$d_{3, 2i}$	$d_{4, 2i}$	....	$d_{i-2, 2i}$	$d_{i-1, 2i}$						$I_{(2i)}$
$I_{(2i+1)}$	$d_{0, 2i+1}$	$d_{1, 2i+1}$	$d_{2, 2i+1}$	$d_{3, 2i+1}$	$d_{4, 2i+1}$	....	$d_{i-2, 2i+1}$	$d_{i-1, 2i+1}$	$d_{i, 2i+1}$					$I_{(2i+1)}$
$I_{(2i+2)}$	$d_{0, 2i+2}$	$d_{1, 2i+2}$	$d_{2, 2i+2}$	$d_{3, 2i+2}$	$d_{4, 2i+2}$	....	$d_{i-2, 2i+2}$	$d_{i-1, 2i+2}$	$d_{i, 2i+2}$	$d_{i+1, 2i+2}$				$I_{(2i+2)}$
$I_{(2i+3)}$	$d_{0, 2i+3}$	$d_{1, 2i+3}$	$d_{2, 2i+3}$	$d_{3, 2i+3}$	$d_{4, 2i+3}$	....	$d_{i-2, 2i+3}$	$d_{i-1, 2i+3}$	$d_{i, 2i+3}$	$d_{i+1, 2i+3}$	$d_{i+2, 2i+}$			$I_{(2i+3)}$
$I_{(2i+4)}$	$d_{0, 2i+4}$	$d_{1, 2i+4}$	$d_{2, 2i+4}$	$d_{3, 2i+4}$	$d_{4, 2i+4}$	....	$d_{i-2, 2i+4}$	$d_{i-1, 2i+4}$	$d_{i, 2i+4}$	$d_{i+1, 2i+4}$	$d_{i+2, 2i+}$	$d_{i+3, 2i+}$		$I_{(2i+4)}$
....	....	....	....	....	....	....	....	....	....	....	....	....	....	....

Analysing now the solution [7.3] one can define a more refined partition at those zones at this Tableau, according to the different close formulas.
As result of such refinement and, consequently, of setting all the corresponding explicit expressions, we easily obtain the following, more complete, Tableau, in next page,

TABLE 15

General table of coefficients $d_{j,k}$														
Specified for the M/M/1 queuing system														
Definitions: α = $\rho^{1/2}$ and β = (1-ρ)														
I_k	$P_{i,0}(\tau)$	$P_{i,1}(\tau)$	$P_{i,2}(\tau)$	$P_{i,3}(\tau)$	$P_{i,4}(\tau)$	....	$P_{i,i-2}(\tau)$	$P_{i,i-1}(\tau)$	$P_{i,i}(\tau)$	$P_{i,i+1}(\tau)$	$P_{i,i+2}(\tau)$	$P_{i,i+3}(\tau)$	....	I_k
I_0									1					I_0
I_1								$\alpha^{(-1)}$		$\alpha^{(+1)}$				I_1
I_2							$\alpha^{(-2)}$				$\alpha^{(+2)}$			I_2
....						....						$\alpha^{(+3)}$		I_3
$I_{(i-4)}$					$\alpha^{(4-i)}$								....	....
$I_{(i-3)}$				$\alpha^{(3-i)}$										$I_{(i-3)}$
$I_{(i-2)}$			$\alpha^{(2-i)}$											$I_{(i-2)}$
$I_{(i-1)}$		$\alpha^{(1-i)}$												$I_{(i-1)}$
$I_{(i)}$	$\alpha^{(-i)}$													$I_{(i)}$
$I_{(i+1)}$	$\alpha^{(-1-i)}$													$I_{(i+1)}$
$I_{(i+2)}$	$\beta\,\alpha^{(-2-i)}$	$\alpha^{(-i)}$												$I_{(i+2)}$
$I_{(i+3)}$	$\beta\,\alpha^{(-3-i)}$	$\beta\,\alpha^{(-1-i)}$	$\alpha^{(1-i)}$											$I_{(i+3)}$
$I_{(i+4)}$	$\beta\,\alpha^{(-4-i)}$	$\beta\,\alpha^{(-2-i)}$	$\beta\,\alpha^{(-i)}$	$\alpha^{(2-i)}$										$I_{(i+4)}$
....	....	....	....	....	....									....
$I_{(2i-2)}$	$\beta\,\alpha^{(2-2i)}$	$\beta\,\alpha^{(4-2i)}$	$\beta\,\alpha^{(6-2i)}$	$\beta\,\alpha^{(8-2i)}$	$\beta\,\alpha^{(10-2i}$	....								$I_{(2i-2)}$
$I_{(2i-1)}$	$\beta\,\alpha^{(1-2i)}$	$\beta\,\alpha^{(3-2i)}$	$\beta\,\alpha^{(5-2i)}$	$\beta\,\alpha^{(7-2i)}$	$\beta\,\alpha^{(9-2i)}$	....	$\alpha^{(-3)}$							$I_{(2i-1)}$
$I_{(2i)}$	$\beta\,\alpha^{(-2i)}$	$\beta\,\alpha^{(2-2i)}$	$\beta\,\alpha^{(4-2i)}$	$\beta\,\alpha^{(6-2i)}$	$\beta\,\alpha^{(8-2i)}$	....	$\beta\,\alpha^{(-4)}$	$\alpha^{(-2)}$						$I_{(2i)}$
$I_{(2i+1)}$	$\beta\,\alpha^{(-1-2i)}$	$\beta\,\alpha^{(1-2i)}$	$\beta\,\alpha^{(3-2i)}$	$\beta\,\alpha^{(5-2i)}$	$\beta\,\alpha^{(7-2i)}$	....	$\beta\,\alpha^{(-5)}$	$\beta\,\alpha^{(-3)}$	$\alpha^{(-1)}$					$I_{(2i+1)}$
$I_{(2i+2)}$	$\beta\,\alpha^{(-2-2i)}$	$\beta\,\alpha^{(-2i)}$	$\beta\,\alpha^{(2-2i)}$	$\beta\,\alpha^{(4-2i)}$	$\beta\,\alpha^{(6-2i)}$	....	$\beta\,\alpha^{(-6)}$	$\beta\,\alpha^{(-4)}$	$\beta\,\alpha^{(-2)}$	$\alpha^{(0)}$				$I_{(2i+2)}$
$I_{(2i+3)}$	$\beta\,\alpha^{(-3-2i)}$	$\beta\,\alpha^{(-1-2i)}$	$\beta\,\alpha^{(1-2i)}$	$\beta\,\alpha^{(3-2i)}$	$\beta\,\alpha^{(5-2i)}$	....	$\beta\,\alpha^{(-7)}$	$\beta\,\alpha^{(-5)}$	$\beta\,\alpha^{(-3)}$	$\beta\,\alpha^{(-1)}$	$\alpha^{(+1)}$			$I_{(2i+3)}$
$I_{(2i+4)}$	$\beta\,\alpha^{(-4-2i)}$	$\beta\,\alpha^{(-2-2i)}$	$\beta\,\alpha^{(-2i)}$	$\beta\,\alpha^{(2-2i)}$	$\beta\,\alpha^{(4-2i)}$	....	$\beta\,\alpha^{(-8)}$	$\beta\,\alpha^{(-6)}$	$\beta\,\alpha^{(-4)}$	$\beta\,\alpha^{(-2)}$	$\beta\,\alpha^{(0)}$	$\alpha^{(+2)}$		$I_{(2i+4)}$
....	....	....	....	....	....	....	....	....	....	....	....	....	....	....

- Blue: $d_{i,0} = 1$, as expected for the first term of the only non null function $P_{\,i,j}(\tau)$ at time $\tau = 0$. By definition that only non null function at $\tau = 0$ is $P_{\,i,i}(\tau)$. And the value is $P_{\,i,i}(0) = 1$.

- Orange: $d_{i+1,1} = \alpha^{+1}$, $d_{i+2,2} = \alpha^{+2}$, $d_{i+3,3} = \alpha^{+3}$, $d_{i+4,4} = \alpha^{+4}$, … , $d_{i+k,k} = \alpha^{+k}$, … is the negative diagonal that begins at element $d_{i,0} = \alpha^{0} = 1$ but excluding this one (already considered with blue colour). This set of coefficients are the first non nulls at the infinite summation of the functions $P_{\,i,j}(\tau)$ with $(j \geq i+1)$. A very simple close formula follows straightaway,

$$d_{i+k,k} = \alpha^{+k} = \rho^{+k/2} \qquad [7.8(+)]\{From\ [7.3]\}$$

- Green: $d_{i-1,1} = \alpha^{-1}$, $d_{i-2,2} = \alpha^{-2}$, $d_{i-3,3} = \alpha^{-3}$, … , $d_{i-k,k} = \alpha^{-k}$, … , $d_{1,i-1} = \alpha^{-(i-1)}$, $d_{0,i} = \alpha^{-i}$, is the positive diagonal that begins at element $d_{i,0} = \alpha^{0} = 1$, but excluding this one (already considered with blue colour), and ends downwards at $d_{0,i} = \alpha^{-i}$. These coefficients accounts for the first non null term in each function $P_{\,i,j}(\tau)$ with $(0 \leq j \leq i-1)$. The same simplicity as before is observed at the corresponding close formula,

$$d_{i-k,k} = \alpha^{-k} = \rho^{-k/2} \qquad [7.8(-)]\{From\ [7.3]\}$$

- Yellow: $d_{0,i+1} = \alpha^{+1}$, $d_{1,i+2} = \alpha^{+2}$, $d_{2,i+3} = \alpha^{+3}$, $d_{3,i+4} = \alpha^{+4}$, … , $d_{j,i+j+1} = \alpha^{+(j+1)}$, … is the complete negative diagonal starting at the tableau element $d_{0,i+1}$.
Once again we can achieve a corresponding close formula very easy,

$$d_{j,i+j+1} = \alpha^{+(j+1)} = \rho^{+(j+1)/2} \qquad [7.9]\{From\ [7.3]\}$$

- Light blue: $d_{j,k} = \beta\alpha^{(2j-k)}$ $for\ (j \geq 0)\ and\ (k \geq j+i+2)$. This light blue domain is the triangle below the complete second negative diagonal. As expected a close formula is also here possible. It reads,

$$d_{j,k} = \beta \cdot \alpha^{(2j-k)} = (1-\rho) \cdot \rho^{(2j-k)/2} \qquad [7.10]\{From\ [7.3]\}$$

- Light yellow: $d_{i,1} = 0$, $d_{i,2} = 0$, $d_{i,3} = 0$, … , $d_{i,2i-1} = 0$, $d_{i,2i} = 0$, this is the finite column of null coefficients between the first and the second terms, these ones not nulls, at the infinite summation series of function $P_{i,i}(\tau)$. Such column represents a vertical gap of $(2i)$ vacant places in the Tableau of $d_{j,k}$ coefficients. This gap refers to the function which assumes the total number of clients at initial time $\tau = 0$.

<u>- Grey:</u> This colour signals all the other gaps between the first non null and the second non null coefficient in the respective infinite summations series. The grey zone comprises two subsets (or two subzones).
… the subset on the left side of the light yellow column of holes, defined by,

$$d_{j,k} = 0 \text{ for } (1 \le j \le i-1) \text{ and } (i-j+1 \le k \le i+j).$$

In this subset the vertical gap of holes has, for each value of j, a number of $[(i+j)-(i-j)] = 2j$ holes.
… and the subset on the right side of the light yellow column of holes, defined by,

$$d_{j,k} = 0 \text{ for } (i+1 \le j) \text{ and } (j-i+1 \le k \le i+j).$$

In this subset the vertical gap of holes has, for each value of j, a number of $[(i+j)-(j-i)] = 2i$ holes.
So the number of holes between the first and the second coefficient (or terms) non nulls at the infinite summation series, raises at the Tableau of $d_{j,k}$, from left to right,
… from zero, at $P_{i,0}(\tau)$ to two at $P_{i,1}(\tau)$
… from two, at $P_{i,1}(\tau)$ to four at $P_{i,2}(\tau)$
………………………………………………………
… from $(2i-4)$, at $P_{i,i-2}(\tau)$ to $(2i-2)$ at $P_{i,i-1}(\tau)$
… from $(2i-2)$, at $P_{i,i-1}(\tau)$ to $(2i)$ at $P_{i,i}(\tau)$
… and then remains constant in $(2i)$ at $P_{i,i+1}(\tau)$, $P_{i,i+2}(\tau)$, $P_{i,i+3}(\tau)$, … , and so on.
As complement we provide, for the M/M/1 queuing system, also the Tableaus for the particular cases with $i = 0$, and $i = 3$.
The case for $i = 3$ allows eventually to understand better the case for a general value of i and the case for $i = 0$ illustrates the lack of vertical gaps in this singular situation.
Anyway, in both cases the initial condition corresponds to a very low value of i, and consequently this means that we are reporting two examples where the initial congestion is very low, (virtually we can consider that there is no congestion at all).
The example for $i = 0$ is for exhaustive description finality and the example for $i = 3$ for more easy and clear explanation of the M/M/1 queuing system, complementing the general i description just now described.

<u>TABLE 16</u>

General table of coefficients $d_{j,k}$											
Specified for the M/M/1 queuing system (Initial condition: i=0)											
$I_k(x)$	$P_{0,0}(\tau)$	$P_{0,1}(\tau)$	$P_{0,2}(\tau)$	$P_{0,3}(\tau)$	$P_{0,4}(\tau)$	$P_{0,5}(\tau)$	$P_{0,6}(\tau)$	$P_{0,7}(\tau)$	$P_{0,8}(\tau)$	$P_{0,9}(\tau)$	………
$I_0(x)$	1										
$I_1(x)$	$\rho^{-1/2}$	$\rho^{+1/2}$									
$I_2(x)$	$(1-\rho)\,\rho^{-1}$	1	ρ^{+1}								
$I_3(x)$	$(1-\rho)\,\rho^{-3/2}$	$(1-\rho)\,\rho^{-1/2}$	$\rho^{+1/2}$	$\rho^{+3/2}$							
$I_4(x)$	$(1-\rho)\,\rho^{-2}$	$(1-\rho)\,\rho^{-1}$	$(1-\rho)$	ρ^{+1}	ρ^{+2}						
$I_5(x)$	$(1-\rho)\,\rho^{-5/2}$	$(1-\rho)\,\rho^{-3/2}$	$(1-\rho)\,\rho^{-1/2}$	$(1-\rho)\,\rho^{+1/2}$	$\rho^{+3/2}$	$\rho^{+5/2}$					
$I_6(x)$	$(1-\rho)\,\rho^{-3}$	$(1-\rho)\,\rho^{-2}$	$(1-\rho)\,\rho^{-1}$	$(1-\rho)$	$(1-\rho)\,\rho^{+1}$	ρ^{+2}	ρ^{+3}				
$I_7(x)$	$(1-\rho)\,\rho^{-7/2}$	$(1-\rho)\,\rho^{-5/2}$	$(1-\rho)\,\rho^{-3/2}$	$(1-\rho)\,\rho^{-1/2}$	$(1-\rho)\,\rho^{+1/2}$	$(1-\rho)\,\rho^{+3/2}$	$\rho^{+5/2}$	$\rho^{+7/2}$			
$I_8(x)$	$(1-\rho)\,\rho^{-4}$	$(1-\rho)\,\rho^{-3}$	$(1-\rho)\,\rho^{-2}$	$(1-\rho)\,\rho^{-1}$	$(1-\rho)$	$(1-\rho)\,\rho^{+1}$	$(1-\rho)\,\rho^{+2}$	ρ^{+3}	ρ^{+4}		
$I_9(x)$	$(1-\rho)\,\rho^{-9/2}$	$(1-\rho)\,\rho^{-7/2}$	$(1-\rho)\,\rho^{-5/2}$	$(1-\rho)\,\rho^{-3/2}$	$(1-\rho)\,\rho^{-1/2}$	$(1-\rho)\,\rho^{+1/2}$	$(1-\rho)\,\rho^{+3/2}$	$(1-\rho)\,\rho^{+5/2}$	$\rho^{+7/2}$	$\rho^{+9/2}$	
………	………	………	………	………	………	………	………	………	………	………	………

TABLE 17

General table of coefficients $d_{j,k}$										
Specified for the M/M/1 queuing system (Initial condition: i=3)										
I_k	$P_{3,0}(\tau)$	$P_{3,1}(\tau)$	$P_{3,2}(\tau)$	$P_{3,3}(\tau)$	$P_{3,4}(\tau)$	$P_{3,5}(\tau)$	$P_{3,6}(\tau)$	$P_{3,7}(\tau)$	$P_{3,8}(\tau)$	....
I_0				$d_{3,0}$						
I_1			$d_{2,1}$		$d_{4,1}$					
I_2		$d_{1,2}$				$d_{5,2}$				
I_3	$d_{0,3}$						$d_{6,3}$			
I_4	$d_{0,4}$							$d_{7,4}$		
I_5	$d_{0,5}$	$d_{1,5}$							$d_{8,5}$	
I_6	$d_{0,6}$	$d_{1,6}$	$d_{2,6}$							....
I_7	$d_{0,7}$	$d_{1,7}$	$d_{2,7}$	$d_{3,7}$						
I_8	$d_{0,8}$	$d_{1,8}$	$d_{2,8}$	$d_{3,8}$	$d_{4,8}$					
I_9	$d_{0,9}$	$d_{1,9}$	$d_{2,9}$	$d_{3,9}$	$d_{4,9}$	$d_{5,9}$				
I_{10}	$d_{0,10}$	$d_{1,10}$	$d_{2,10}$	$d_{3,10}$	$d_{4,10}$	$d_{5,10}$	$d_{6,10}$			
I_{11}	$d_{0,11}$	$d_{1,11}$	$d_{2,11}$	$d_{3,11}$	$d_{4,11}$	$d_{5,11}$	$d_{6,11}$	$d_{7,11}$		
I_{12}	$d_{0,12}$	$d_{1,12}$	$d_{2,12}$	$d_{3,12}$	$d_{4,12}$	$d_{5,12}$	$d_{6,12}$	$d_{7,12}$	$d_{8,12}$	
....	....	....	....	....	....	....	....	....	....	....

TABLE 18

General table of coefficients $d_{j,k}$										
Specified for the M/M/1 queuing system (Initial condition: i=3)										
I_k	$P_{3,0}(\tau)$	$P_{3,1}(\tau)$	$P_{3,2}(\tau)$	$P_{3,3}(\tau)$	$P_{3,4}(\tau)$	$P_{3,5}(\tau)$	$P_{3,6}(\tau)$	$P_{3,7}(\tau)$	$P_{3,8}(\tau)$	....
I_0				1						
I_1			$\rho^{-1/2}$		$\rho^{1/2}$					
I_2		ρ^{-1}				ρ				
I_3	$\rho^{-3/2}$						$\rho^{3/2}$			
I_4	ρ^{-2}							ρ^{2}		
I_5	$(1-\rho)\rho^{-5/2}$	$\rho^{-3/2}$							$\rho^{5/2}$	
I_6	$(1-\rho)\rho^{-3}$	$(1-\rho)\rho^{-2}$	ρ^{-1}							....
I_7	$(1-\rho)\rho^{-7/2}$	$(1-\rho)\rho^{-5/2}$	$(1-\rho)\rho^{-3/2}$	$\rho^{-1/2}$						
I_8	$(1-\rho)\rho^{-4}$	$(1-\rho)\rho^{-3}$	$(1-\rho)\rho^{-2}$	$(1-\rho)\rho^{-1}$	1					
I_9	$(1-\rho)\rho^{-9/2}$	$(1-\rho)\rho^{-7/2}$	$(1-\rho)\rho^{-5/2}$	$(1-\rho)\rho^{-3/2}$	$(1-\rho)\rho^{-1/2}$	$\rho^{1/2}$				
I_{10}	$(1-\rho)\rho^{-5}$	$(1-\rho)\rho^{-4}$	$(1-\rho)\rho^{-3}$	$(1-\rho)\rho^{-2}$	$(1-\rho)\rho^{-1}$	$(1-\rho)$	ρ			
I_{11}	$(1-\rho)\rho^{-11/2}$	$(1-\rho)\rho^{-9/2}$	$(1-\rho)\rho^{-7/2}$	$(1-\rho)\rho^{-5/2}$	$(1-\rho)\rho^{-3/2}$	$(1-\rho)\rho^{-1/2}$	$(1-\rho)\rho^{1/2}$	$\rho^{3/2}$		
I_{12}	$(1-\rho)\rho^{--6}$	$(1-\rho)\rho^{-5}$	$(1-\rho)\rho^{-4}$	$(1-\rho)\rho^{-3}$	$(1-\rho)\rho^{-2}$	$(1-\rho)\rho^{-1}$	$(1-\rho)$	$(1-\rho)\rho$	ρ^{2}	
....	....	....	....	....	....	....	....	....	....	....

5. Complexity comparison between two models ("Mono-layered concentrated biofilm growth kinetics " versus M/M/1 queuing system)

Recalling the working Tableau analysis till now done let summarize the closed formulas at the two following boxes,

Closed formulas for the coefficients $d_{j,k}$ at the ...
... "Mono-layered concentrated biofilm growth" kinetics model

$$d_{j,k} = 1 \qquad \ldots for\ (j=0)\ and\ (k=0)$$

$$d_{j,k} = \frac{(2\cdot\Phi+\delta-1)}{\sqrt{(\rho+\Phi)\cdot(1-\Phi-\delta)}} \qquad \ldots for\ (j=0)\ and\ (k=1)$$

$$d_{j,k} = \frac{\{[\rho\cdot(1-\Phi-\delta)+\Phi\cdot(1-\delta)]\cdot\Phi^{k-1}+(\rho+1-\delta)\cdot(\Phi+\delta-1)^{k}\}}{(1-\delta)\cdot\sqrt{(\rho+\Phi)^{k}\cdot(1-\Phi-\delta)^{k}}}$$

$$\ldots for\ (j=0)\ and\ (k\geq 2)$$

$$d_{j,k} = \frac{\rho \cdot (\rho + \Phi)^{j-1}}{\sqrt{(\rho + \Phi)^j \cdot (1 - \Phi - \delta)^j}} \quad \ldots for\ (k = j)\ and\ (j \geq 1)$$

$$d_{j,k} = \rho \cdot (2 \cdot \Phi + \delta - 1) \cdot \sqrt{\frac{(\rho + \Phi)^{j-3}}{(1 - \Phi - \delta)^{j+1}}} \quad \ldots for\ (k = j + 1)\ and\ (j \geq 1)$$

$$d_{j,k} = \frac{\rho \cdot \{[\rho \cdot (1 - \Phi - \delta) + \Phi \cdot (1 - \delta)] \cdot \Phi^{k-j-1} + (\rho + 1 - \delta) \cdot (\Phi + \delta - 1)^{k-j}\}}{(1 - \delta)} \cdot \sqrt{\frac{(\rho + \Phi)^{2(j-1)-k}}{(1 - \Phi - \delta)^k}}$$

$$\ldots for\ (j \geq 1)\ and\ (k \geq j + 2)$$

Holes: $d_{j,k} = 0 \quad \ldots for\ (j \geq 1)\ and\ (0 \leq k \leq j - 1)$

$[7.11]\{Summary\ box\}$

[Closed formulas for the coefficients $d_{j,k}$ at the …](#)
[… M/M/1 queuing system, with infinite storage room.](#)

$d_{j,k} = 1 \ldots for\ (j = i)\ and\ (k = 0)$

$d_{j,k} = \rho^{+\frac{k}{2}} \ldots for\ (j = i + k)\ and\ (k \geq 1)$

$d_{j,k} = \rho^{+\frac{k}{2}} \ldots for\ (j = i - k)\ and\ (1 \leq k \leq i)$

$d_{j,k} = \rho^{+\frac{j+1}{2}} \ldots for\ (0 \leq j)\ and\ (k = j + i + 1)$

$d_{j,k} = (1 - \rho) \cdot \rho^{(2j-k)/2} \ldots for\ (0 \leq j)\ and\ (j + i + 2 \leq k)$

Holes: $d_{j,k} = 0 \quad \ldots for\ (0 \leq j \leq i - 1)\ and\ (0 \leq k \leq i - j - 1)$

$d_{j,k} = 0 \quad \ldots for\ (1 \leq j \leq i - 1)\ and\ (i - j + 1 \leq k \leq i + j)$

$d_{j,k} = 0 \quad \ldots for\ (i \leq j)\ and\ (j - i + 1 \leq k \leq i + j)$

$d_{j,k} = 0 \quad \ldots for\ (i + 1 \leq j)\ and\ (0 \leq k \leq j - i - 1)$

$[7.12]\{Summary\ box\}$

Now let´s make an …
… ***INTERLUDE REMARK:***
We have just now deduced the close formulas for two cases where such goal is not only possible (which is always the case) but also not prohibitively cumbersome:
… I) The most simplified version of biofilm formation and growth early stages, by us designed "Mono-layered concentrated growth" kinetics model.
… II) And the M/M/1 queuing system, described as a difficult to handle analytic solution in a vast bibliography.

… III) The close formulas for I are at [7.11]{$Summary\ box$}[7.11]{$Summary\ box$} and the close formulas for II are at [7.12]{$Summary\ box$}.
… III) Comparing the sets of close formulas at boxes [7.11] and [7.12] one immediately concludes that the first one is, by far, much more complex than the second.
In other words, the biofilm model, even in its more simplified version seems to be much more intricate, attending the mathematics involved, than the M/M/1 queuing system.
Now it´s turn to make, in a logic sequel a …
… ***CONCLUDING REMARK:***
It is wiser, at this moment, profit the skills gained in modelling the biofilm formation and growth, at their several and variegated versions, and with different degrees of difficulty, albeit always near intractability.
Biofilm formation and growth modelling is our current and foreseen task, and will provide, in the nearest future, more and more mathematical challenges.
Such experience can and must now be transferred into transient queuing context, where we found a more immediate general interest and a great number of still not deduced, and not so intricate to reach, exact solutions.

6. Starting a collection of queuing models for future resolution.

Well, our immediate future goal is to achieve a quantitative and accurate cognizance, as deep as we can get, about the transient regimes of, not only M/M/1 queuing system with infinite storage room, but also, and for start, of the following ones,

IV - M/M/1/K Queuing System (Finite storage room).

V - Discouraged Arrivals Queuing System (Infinite storage room).

VI - M/M/∞: Responsive Servers Queuing System (Infinite Number of Servers).

VII - M/M/m: m-Server Queuing System (Infinite storage room).

VIII- M/M/m/m: m-Server Loss Queuing System (Finite storage room).

IX- M/M/1//M: Single-Server Finite Customer Population Queuing System.

X- M/M/∞//M: Infinite Number of Servers and Finite Customer Population Queuing System.
XI- M/M/m/K/M: m-Server and Finite Customer Population Queuing System (Finite storage room).

… designed in roman from IV to XI.

We reserve the designations from I to III for,

I- Birth-Death Process /-/ General Case ; Infinite Storage Room

II- Birth-Death Process /-/ General Case ; Finite Storage Room

III- M/M/1 queuing system with infinite storage room

These models can be found in Kleinrock [53] and we will profit from their diagrammatic definition as a starting step with the purpose of get into light their corresponding transient regimes.
For such goal, and as future work, we will apply a guess solution, in a similar way like it has been done at Chapter VI for the general biofilm model version analytic solution.
Such guess solution is already defined by formula $[7.4 - B]$.
Additionally the easy derivative rules,

$$\frac{dI_k(2 \cdot b \cdot \tau)}{d\tau} = b \cdot \{I_{k-1}(2 \cdot b \cdot \tau) + I_{k+1}(2 \cdot b \cdot \tau)\} \qquad [7.13]$$

$$\frac{dI_0(2 \cdot b \cdot \tau)}{d\tau} = 2 \cdot b \cdot I_1(2 \cdot b \cdot \tau) \qquad [7.14]$$

… will be a useful mathematical tool, and must be applied all over the deductions.
Just like the rules,

$$\frac{dJ_k(2\cdot b\cdot\tau)}{d\tau} = b\cdot\{J_{k-1}(2\cdot b\cdot\tau) - J_{k+1}(2\cdot b\cdot\tau)\} \qquad [7.15]\ ([6.7] \text{ in Chapter VI})$$

$$\frac{dJ_0(2\cdot b\cdot\tau)}{d\tau} = -2\cdot b\cdot J_1(2\cdot b\cdot\tau) \qquad [7.16]\ ([6.8] \text{ in Chapter VI})$$

… where applied in Chapter VI at the analytical deduction of the solution for the general biofilm model version.
In the same fashion, first rule applies for all the Modified Bessel functions $I_k(2\cdot b\cdot\tau)$ with $(k \geq 1)$.
And the second rule, specific for $I_0(2\cdot b\cdot\tau)$, avoids negative order Modified Bessel functions which would come into play if the first rule is applied to this function.
Like in Chapter VI, the important advantage of these rules is the fact that, after application, time (τ) remains exclusively inside Bessel functions argument.

In other words, the experience and knowledge obtained in Chapter VI, bordering mathematical intractability, can came into play with the purpose of solve these queuing models, which we consider less difficult, in mathematical terms, than the model solved in Chapter VI. This last one corresponds, in fact, to the most difficult analytical solution obtained till now.
Besides the Tableau of coefficients $d_{j,k}$ and derivative rules [7.13] and [7.14], the other important working tool is, for each specific model, the respective state-transition rate diagram.
The state diagrams for these queuing models are, including the more simple M/M/1 queuing system with infinite storage room, all of they, particular cases of the afore mentioned cases I or II.
The state transition diagrams for these two cases are,

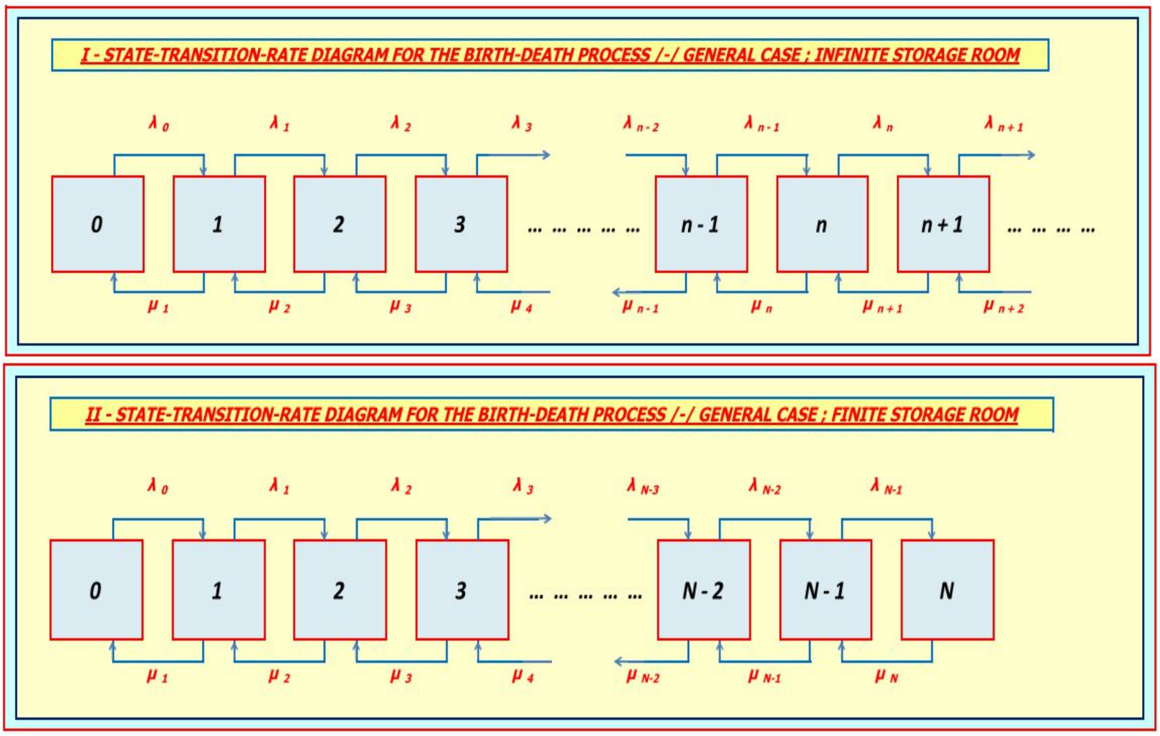


Figure 17 - State transition diagrams for the general birth-death process with infinite (case I) and finite (case II) storage room.

The particularities directing to all the others, from III to IX can be easily written down by simple inspection of the diagrams at Kleinrock book [53].

In the figures at next eleven pages all those diagram systems, including these two in Figure 17, can be related to their respective isomorphous chemical reaction scheme and also to those formal particularities, that must be inserted at I or II, directing this way to each case, from III to XI.

As a matter of fact all of they are particular cases of I because II is himself also a particular case of I. We obtain II from I imposing there,

$$\lambda_n = 0, \text{ and } \mu_{n+1} = 0 \text{ if } n > N$$

But this is only a formal detail without essential significance in the sequel analysis.
The state diagram for case I (Birth-Death Process /-/ General Case ; Infinite Storage Room) also can be found in [53], figure .
The state diagram for case II (Birth-Death Process /-/ General Case ; Finite Storage Room) is an obvious contribution of ours, for the sake of completeness and coherence, regarding our purposes.

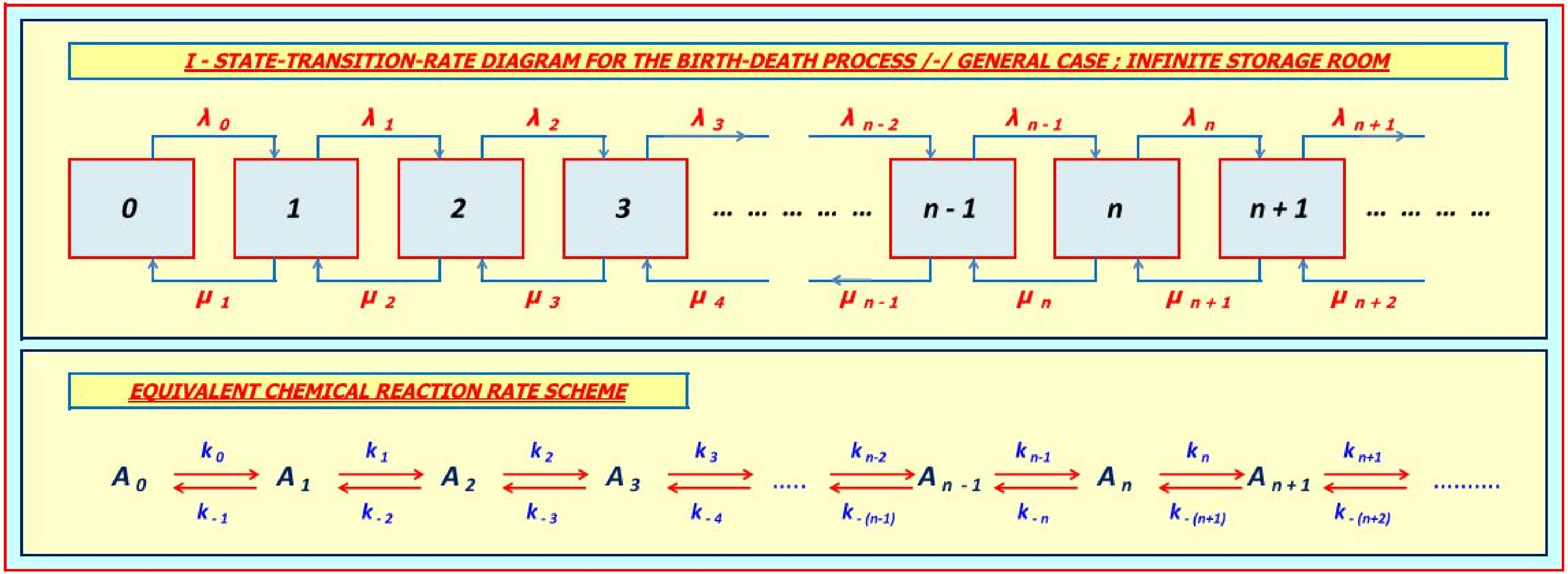


Figure 18 - I

SYSTEM I - [53] Kleinrock - Page 58 (Fig. 2.9)

Differential equations system,

$$\left(\frac{dP_{i,0}}{dt}\right) = -\lambda_0 \cdot P_{i,0} + \mu_1 \cdot P_{i,1}$$

$$\left(\frac{dP_{i,j}}{dt}\right) = \lambda_{j-1} \cdot P_{i,j-1} - (\lambda_j + \mu_j) \cdot P_{i,j} + \mu_{j+1} \cdot P_{i,j+1} \quad \ldots \text{if } (j \geq 1)$$

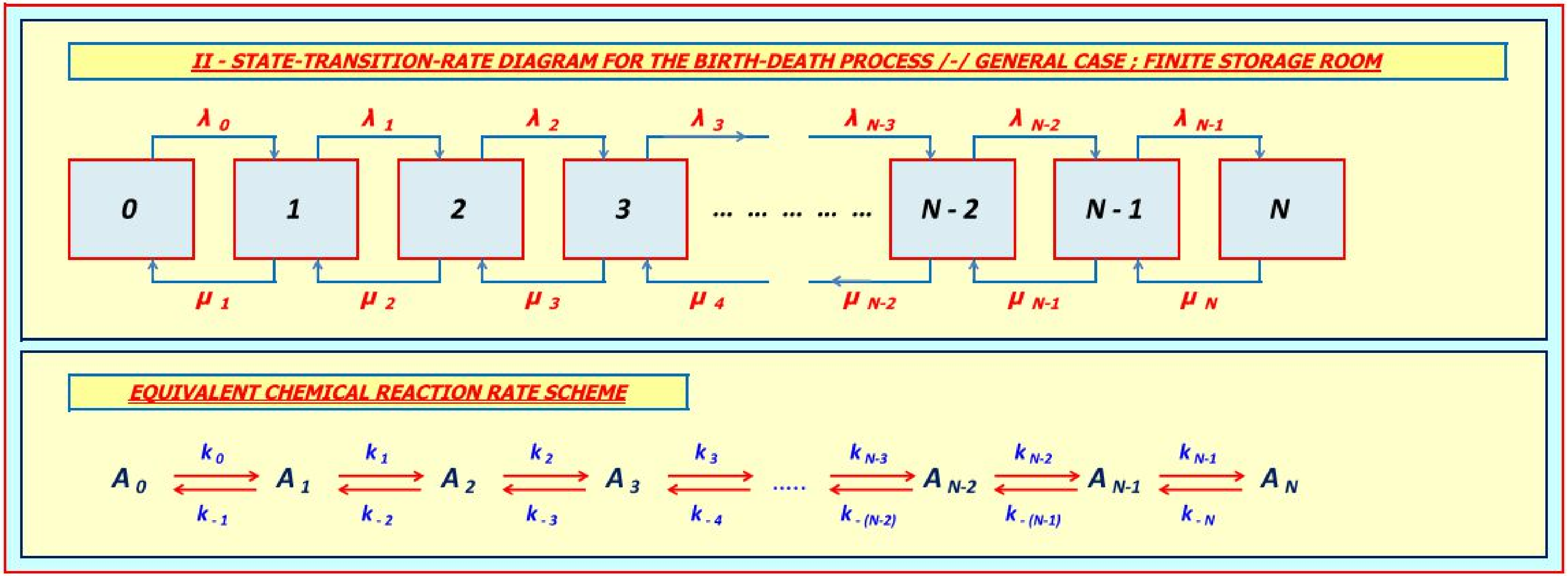


Figure 18 - II

SYSTEM II - Author contribution

Differential equations system,

$$\left(\frac{dP_{i,0}}{dt}\right) = -\lambda_0 \cdot P_{i,0} + \mu_1 \cdot P_{i,1}$$

$$\left(\frac{dP_{i,j}}{dt}\right) = \lambda_{j-1} \cdot P_{i,j-1} - (\lambda_j + \mu_j) \cdot P_{i,j} + \mu_{j+1} \cdot P_{i,j+1} \quad \ldots \text{if } (1 \le j \le \mathrm{N} - 1)$$

$$\left(\frac{dP_{i,N}}{dt}\right) = \lambda_{N-1} \cdot P_{i,N-1} - \mu_N \cdot P_{i,j}$$

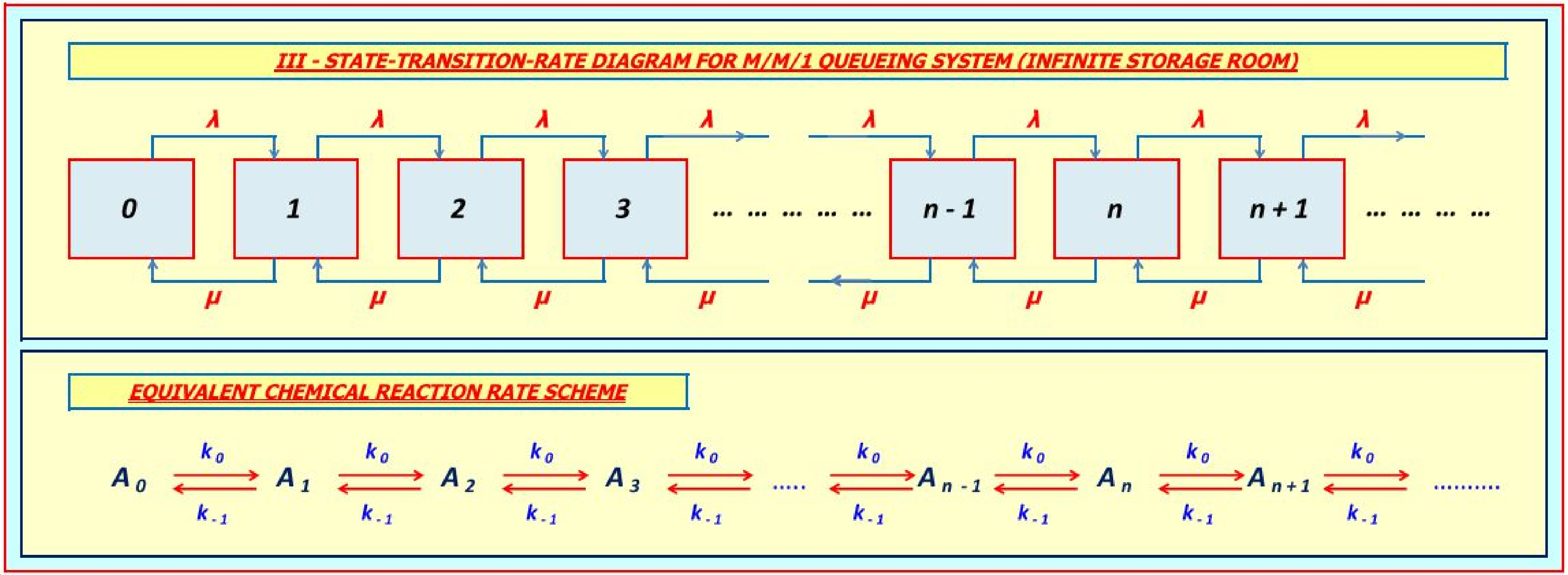


Figure 18 - III

SYSTEM III - [53] Kleinrock - Page 95 (Fig. 3.1)

M/M/1 queuing system with infinite storage room

Particular condition to apply at I:

$$\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda \qquad [7.17\text{-III. } \lambda]$$

$$\mu_1 = \mu_2 = \mu_3 = \cdots = \mu \qquad [7.17\text{-III. } \mu]$$

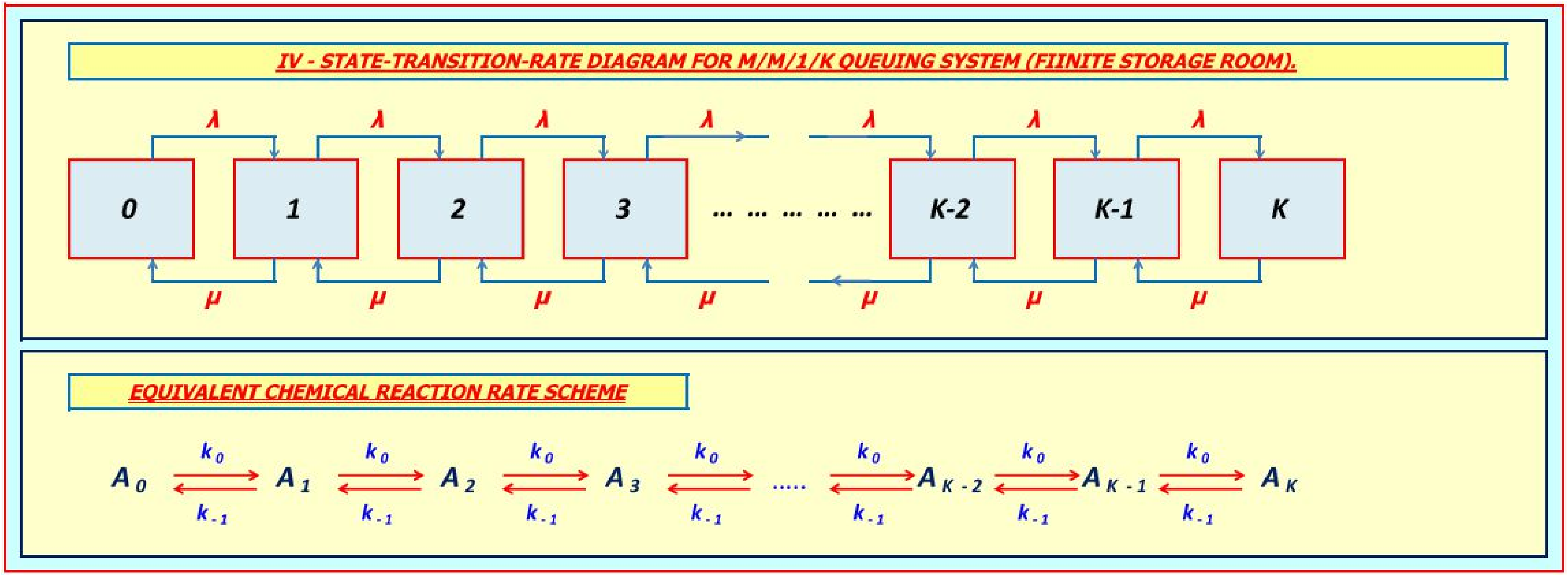


Figure 18 - IV

SYSTEM IV - [53] Kleinrock - Page 104 (Fig. 3.8)

M/M/1/K Queuing System (Finite storage room).

Particular condition to apply at II:

$$Put\ N = K\ and\ \dots$$

$$\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{K-1} = \lambda \qquad \textbf{[7.17-IV.}\ \lambda\textbf{]}$$

$$\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_K = \mu \qquad \textbf{[7.17-IV.}\ \mu\textbf{]}$$

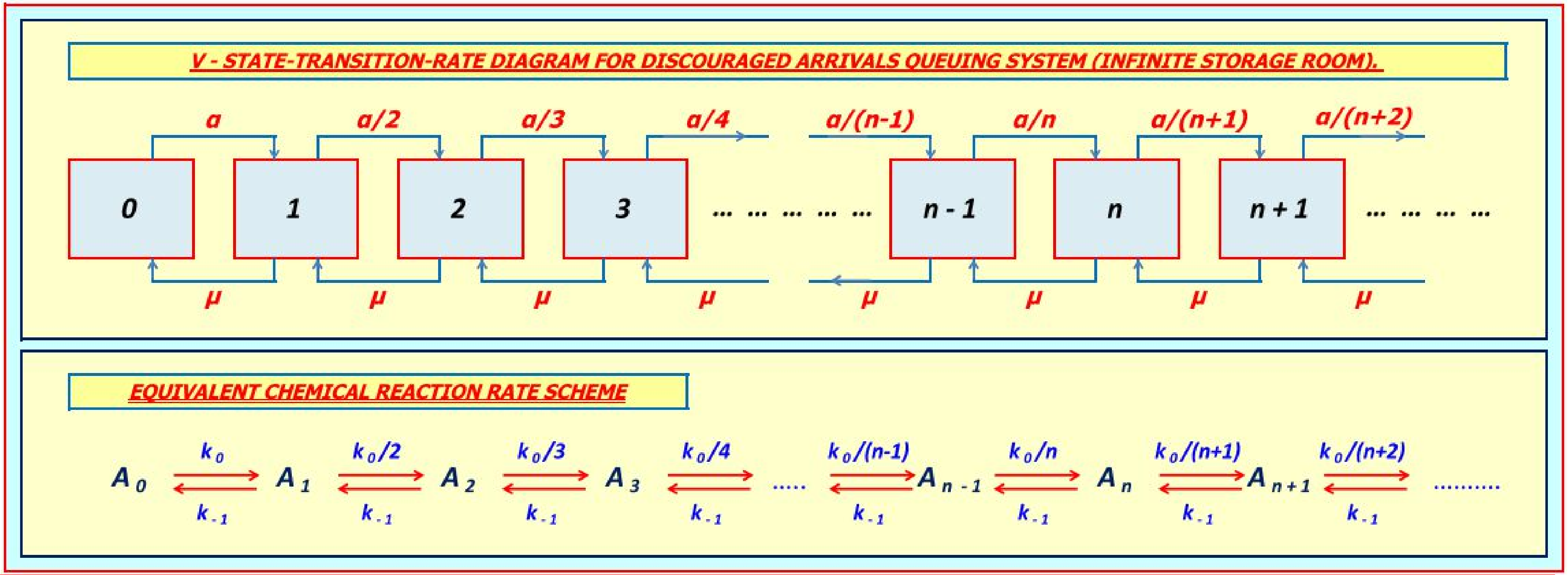


Figure 18 - V

SYSTEM V - [53] Kleinrock - Page 100 (Fig. 3.5)

Discouraged Arrivals Queuing System (Infinite storage room).

Particular condition to apply at I:

$$\lambda_0 = \alpha \,;\; \lambda_1 = \frac{\alpha}{2} \,;\; \lambda_2 = \frac{\alpha}{3} \,;\ldots;\; \lambda_n = \frac{\alpha}{(n+1)} \,;\ldots$$

[7.17-V. λ]

$$\mu_1 = \mu_2 = \mu_3 = \cdots = \mu$$

[7.17-V. μ]

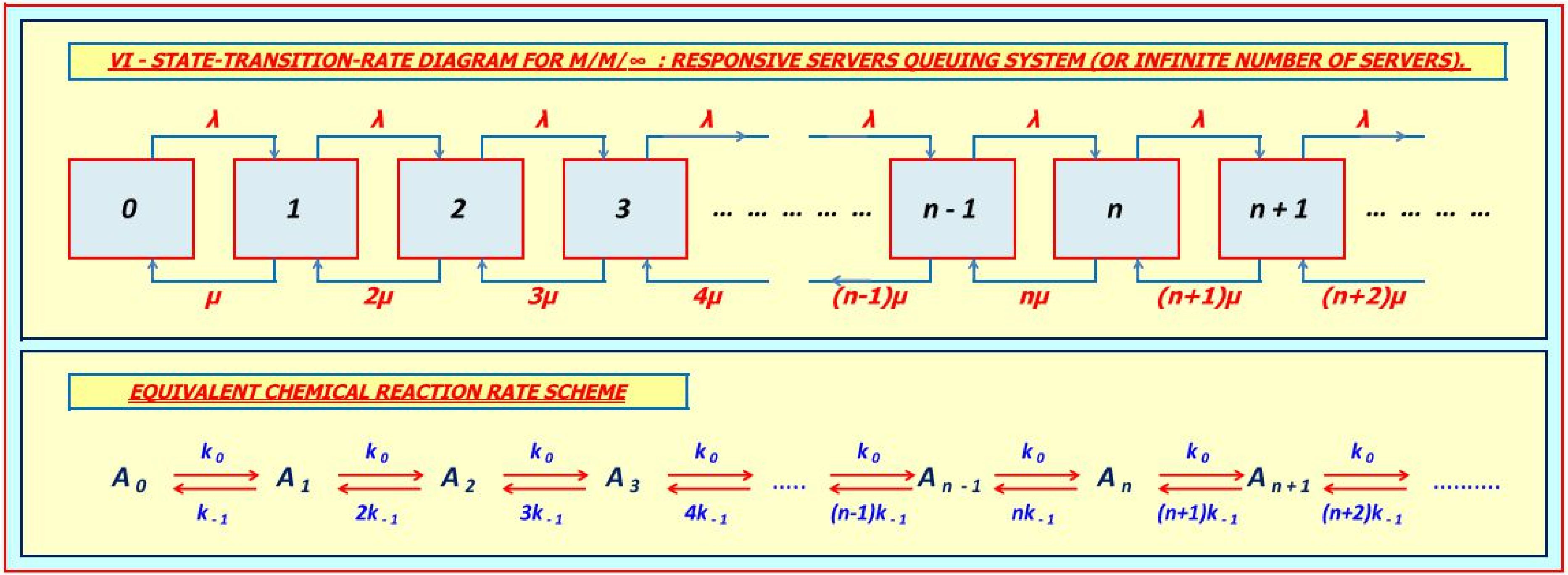


Figure 18 - VI

SYSTEM VI - [53] Kleinrock - Page 101 (Fig. 3.6)

M/M/∞ : Responsive Servers Queuing System (Infinite Number of Servers).

Particular condition to apply at I:

$$\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{N-1} = \lambda \qquad [7.17\text{-VI. } \lambda]$$

$$\mu_1 = \mu\ ;\ \mu_2 = 2\mu\ ;\ \mu_3 = 3\mu\ ;\ldots;\ \mu_n = \qquad [7.17\text{-VI. } \mu]$$

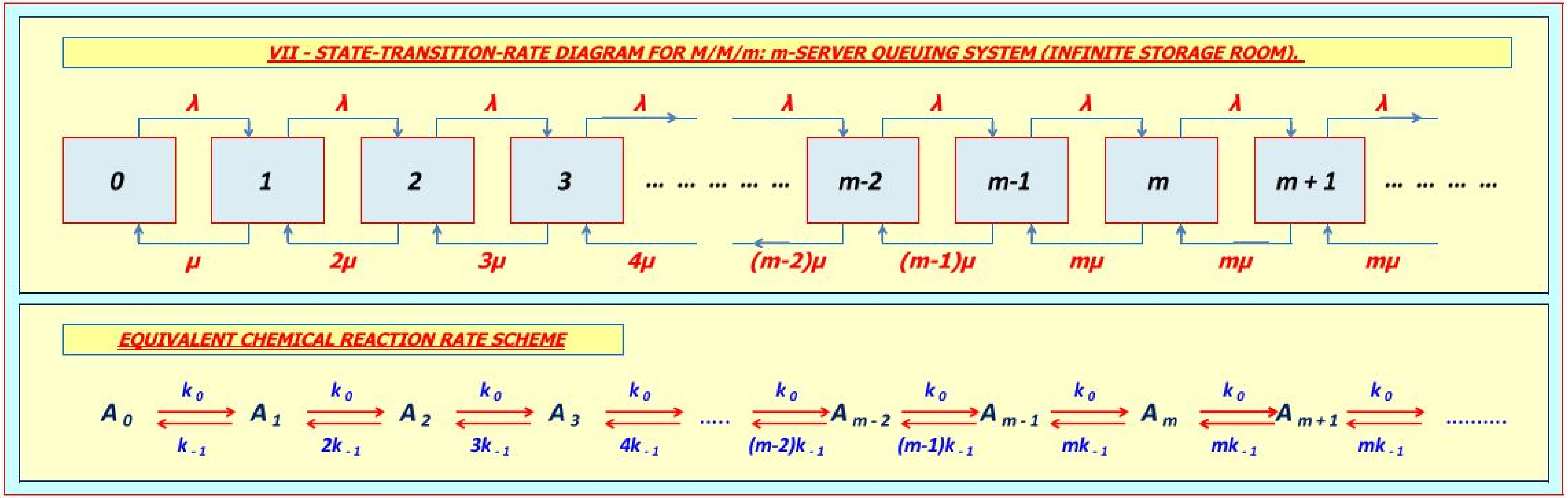


Figure 18 - VII

SYSTEM VII - [53] Kleinrock - Page 102 (Fig. 3.7)

M/M/m: m-Server Queuing System (Infinite storage room).

Particular condition to apply at I:

$$\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda \qquad \textbf{[7.17-VII. } \lambda \textbf{]}$$

$$\mu_1 = \mu\,;\ \mu_2 = 2\mu\,;\ \mu_3 = 3\mu\,;\ldots;\ \lambda_{m-1} = (m-1)\mu\,;\ \lambda_m = \lambda_{m+1} = \lambda_{m+2} = \cdots = m\mu \qquad \textbf{[7.17-VII. } \mu \textbf{]}$$

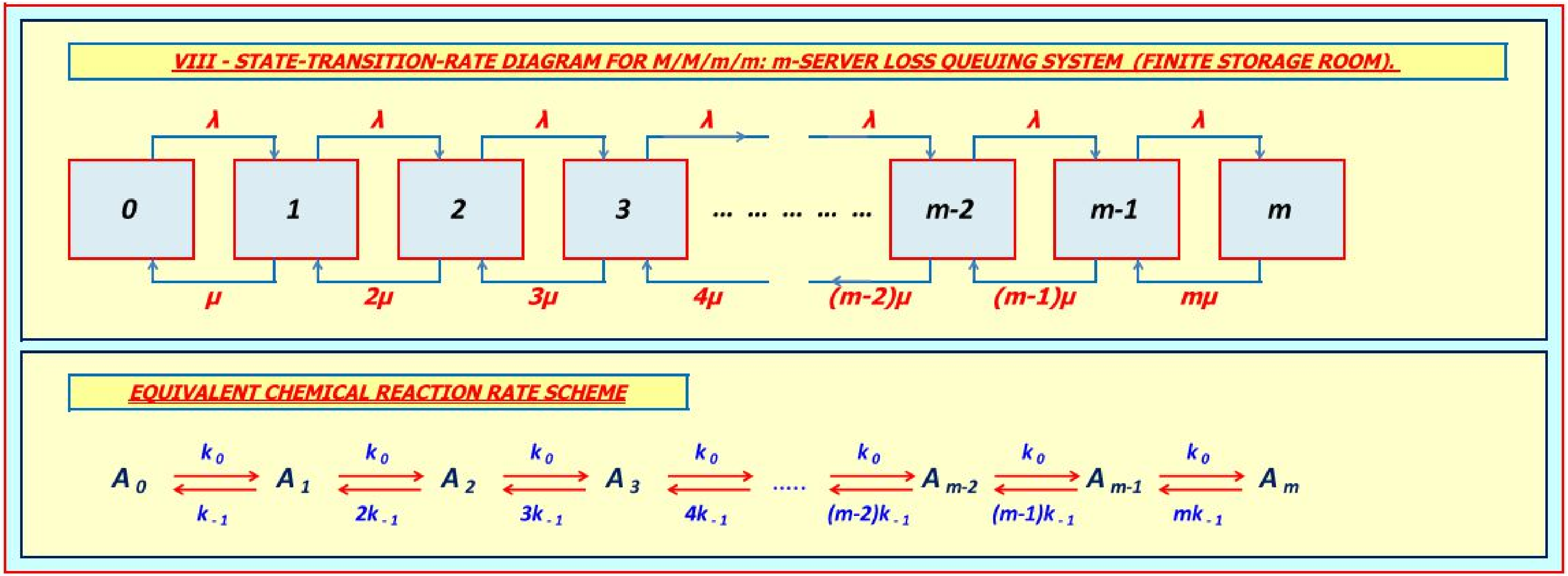


Figure 18 - VIII

SYSTEM VIII - [53] Kleinrock - Page 105 (Fig. 3.9)

M/M/m/m: m-Server Loss Queuing System (Finite storage room).

Particular condition to apply at II:

$$Put\ N = m\ and\ \ldots$$

$$\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = \lambda \qquad \textbf{[7.17-VIII. } \lambda\textbf{]}$$

$$\mu_1 = \mu\ ;\ \mu_2 = 2\mu\ ;\ \mu_3 = 3\mu\ ;\ \ldots\ ;\ \lambda_{m-1} = (m-1)\mu\ ;\ \lambda_m = m\mu \qquad \textbf{[7.17-VIII. } \mu\textbf{]}$$

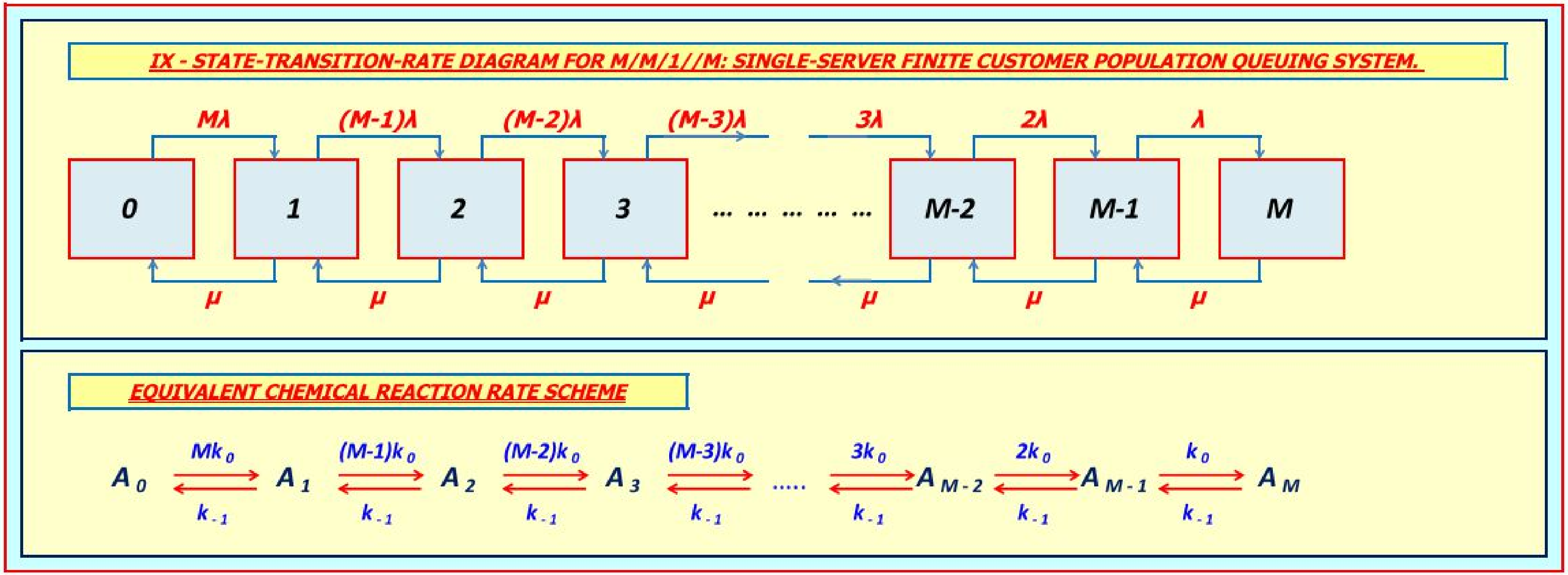


Figure 18 - IX

SYSTEM IX - [53] Kleinrock - Page 107 (Fig. 3.10)

M/M/1//M: Single-Server Finite Customer Population Queuing System.

Particular condition to apply at II:

$$Put\ N = M\ and\ \ldots$$

$$\lambda_0 = M\lambda\,;\ \lambda_1 = (M-1)\lambda\,;\ \lambda_2 = (M-2)\lambda\,;\ \ldots\,;\ \lambda_{M-2} = 2\lambda\,;\ \lambda_{M-1} = \lambda \qquad \textbf{[7.17-IX. }\lambda\textbf{]}$$

$$\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_{M-1} = \mu_M = \mu \qquad \textbf{[7.17-IX. }\mu\textbf{]}$$

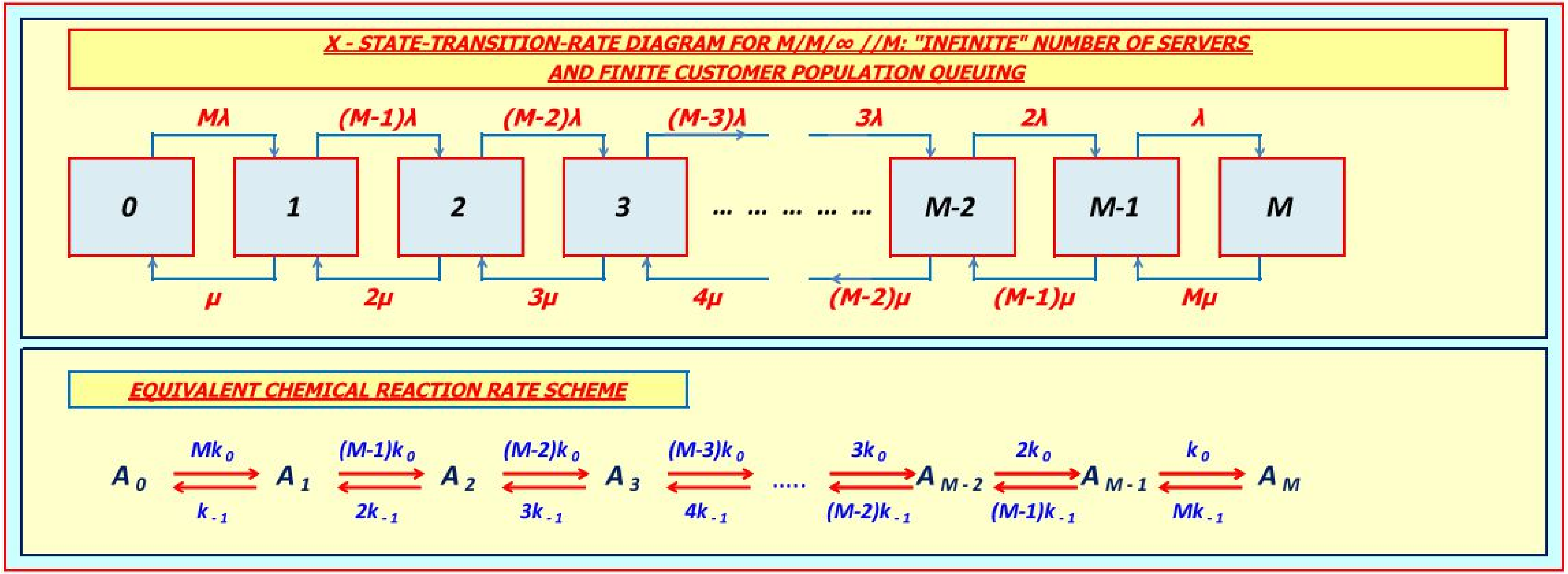


Figure 18 - X

SYSTEM X - [53] Kleinrock - Page 107 (Fig. 3.11)

M/M/∞//M: Infinite Number of Servers and Finite Customer Population Queuing System.

Particular condition to apply at II:

$$Put\ N = M\ and\ ...$$

$$\lambda_0 = M\lambda\ ;\ \lambda_1 = (M-1)\lambda\ ;\ \lambda_2 = (M-2)\lambda\ ;\ ...\ ;\ \lambda_{M-2} = 2\lambda\ ;\ \lambda_{M-1} = \lambda \qquad [7.17\text{-X}.\ \lambda]$$

$$\mu_1 = \mu\ ;\ \mu_2 = 2\mu\ ;\ \mu_3 = 3\mu\ ;\ ...\ ;\ \mu_{M-1} = (M-1)\mu\ ;\ \mu_M = M\mu \qquad [7.17\text{-X}.\ \mu]$$

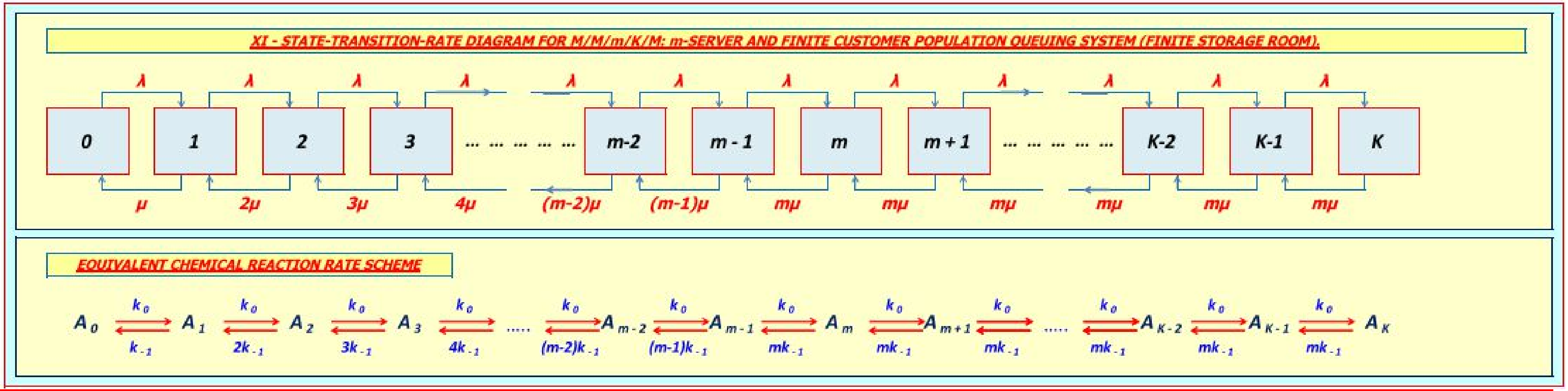


Figure 18 - XI

SYSTEM XI - [53] Kleinrock - Page 109 (Fig. 3.12)

M/M/m/K/M: m-Server and Finite Customer Population Queuing System (Finite storage room).

Particular condition to apply at II:

$$Put\ N = K\ and\ define\ M\ \ \mathbf{and}\ \mathbf{m}$$

$\lambda_0 = M\lambda\,;\ \lambda_1 = (M-1)\lambda\,;\ \lambda_2 = (M-2)\lambda\,; \dots$

$\dots;\ \lambda_{m-2} = (M-m+2)\lambda\,;\ \lambda_{m-1} = (M-m+1)\lambda;\ \lambda_m = (M-m)\lambda\,; \dots$

$\dots;\ \lambda_{K-3} = (M-K+3)\lambda\,;\ \lambda_{K-2} = (M-K+2)\lambda\,;\ \lambda_{K-1} = (M-K+1)\lambda$

[7.17-XI. λ]

$\mu_1 = \mu\,;\ \mu_2 = 2\mu\,;\ \mu_3 = 3\mu\,; \dots$

$\dots;\ \mu_{m-1} = (m-1)\mu\,;\ \mu_m = \ \mu_{m+1} = \ \mu_{m+2} = \cdots = \mu_{K-2} = \mu_{K-1} = \mu_K = m\mu$ **[7.17-XI. μ]**

7. Biofilm general model: state-transition rate diagram interpretation

The general version deduced at Chapter VI has such a diagram, like a State transition rate one, and we constructed it: is exposed in the next page (Figure 19).
Complexity associated with the corresponding equation system compels us to chose a value of the model parameter L high enough to capture such complexity and, for construct this State transition rate diagram, a value $L = 6$, was chosen.
The construction of a diagram for a parameter L, generic or greater would make that graphic construction prohibitive.
Any way the associated complexity is, observing Figure 19, evident comparing with the corresponding diagrams for the queuing models of birth-death processes kind, like M/M/1 system, in Figures 18 - I, 18 - II, 18 - III, 18 - IV, 18 - V, 18 - VI, 18 - VII, 18 - VIII, 18 - IX, 18 - X and 18 - XI.
In Figure 19 negative signals at many transition arrows can be observed. This is an appropriate adaptation taking in consideration partial processes that are proportional to a determined state and, simultaneously, have an effect of increase the value of that same state. As example: parallel growth of a fraction θ_n implies that θ_n increase, and θ_{n-1} decrease so, consequently, the arrow from θ_n to θ_{n-1} must have a negative signal.
Relatively to this important detail we recall now the system of differential-difference equations, for this biofilm general model, only to illustrate this essential partial rate transition signal analysis, relating it with Figure 19.

$$\left(\frac{d\theta_0}{d\tau}\right) = -A_0 \cdot \theta_0 + B_1 \cdot \theta_1 - \left(\sum_{j=2}^{L} D_j \cdot \theta_j\right) \qquad \text{[7.18(0)] ([6.21(0)] in Chapter VI)}$$

$$\left(\frac{d\theta_n}{d\tau}\right) = A_{n-1} \cdot \theta_{n-1} + (B_n - A_n) \cdot \theta_n + +(D_2 - B_{n+1}) \cdot \theta_{n+1} + \left[\sum_{j=2}^{L-1} (D_{j+1} - D_j) \cdot \theta_{n+j}\right] - D_L \cdot \theta_{L+n}$$

$$\ldots\ if\ (1 \le n \le L) \qquad \text{[7.18}(L)\text{] ([6.21}(L)\text{] in Chapter VI)}$$

$$\left(\frac{d\theta_n}{d\tau}\right) = A_L \cdot \theta_{n-1} + (B_n - A_L) \cdot \theta_n + +(D_2 - B_{n+1}) \cdot \theta_{n+1} + \left[\sum_{j=2}^{L-1} (D_{j+1} - D_j) \cdot \theta_{n+j}\right] - D_L \cdot \theta_{L+n}$$

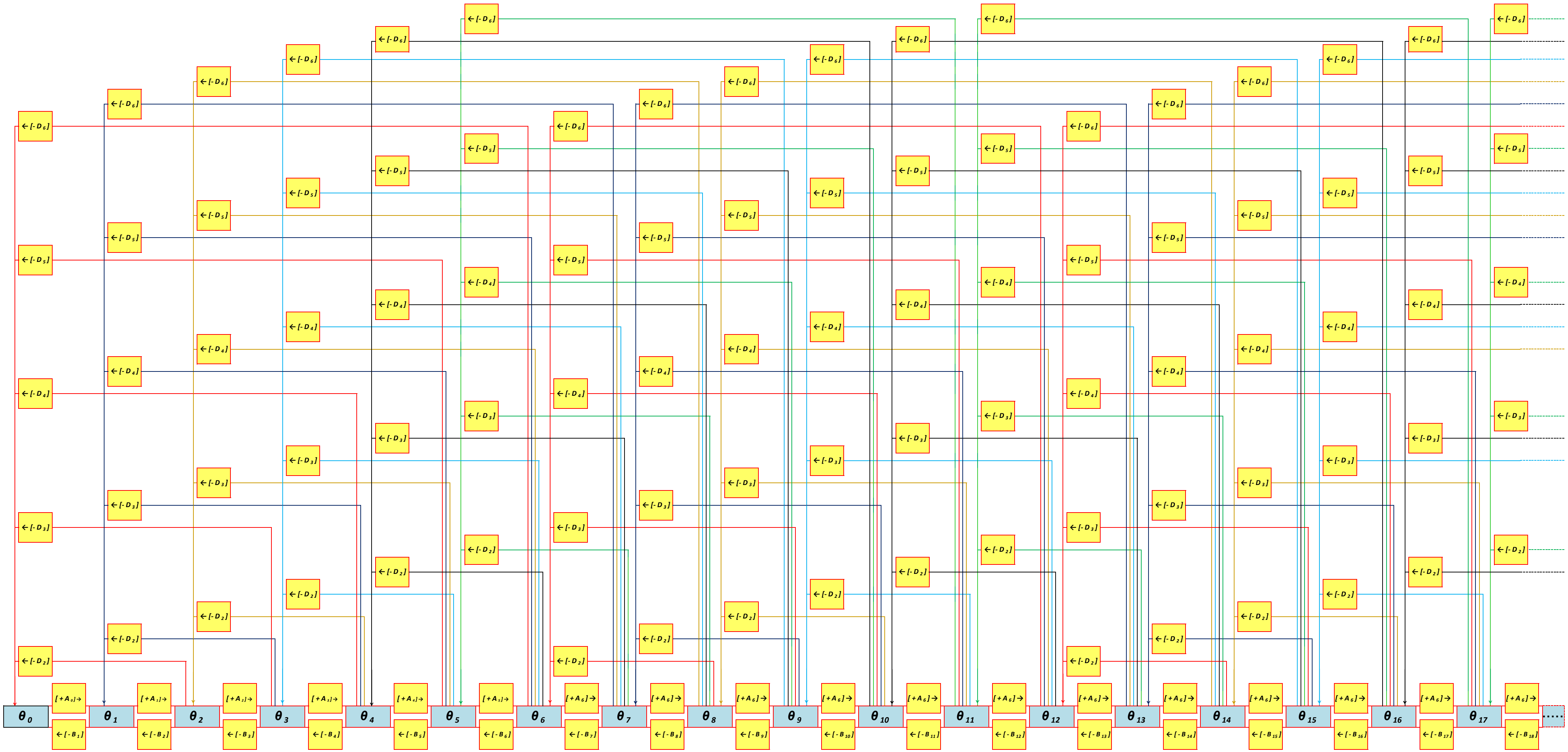
$$\ldots\ if\ (n \ge L + 1) \qquad \text{[7.18}(n)\text{] ([6.21}(n)\text{] in Chapter VI)}$$

The initial condition must, of course, be:

$$\theta_0(0) = 1,\ \ and\ \ \theta_n(0) = 0\ \ for\ n{>}0 \qquad [7.18, \tau, 0]\ \ ([6.21, \tau, 0] \text{ in Chapter VI})$$

(at the beginning (τ=0) al the solid support is bare)

FIGURE 19



8. Starting the handbook: M/M/1 Queuing System.

8.1. Brief introductory considerations

As previously referred, among all the options for next work, one of the most worth-while consists in achieve a quantitative and accurate cognizance, as deep as we can get, about the transient regimes of, for start, the above mentioned queuing systems labeled, in Section 6 of this Chapter, from II to XI, all of they being a particular case of I (General Birth-Death Process /-/ Infinite Storage Room) or II (General Birth-Death Process /-/ Finite Storage Room).
However we have already two independent computation methods for the M/M/1 Queuing System, one described in Chapter V, Section 3.5 with a four terms recurrence relation applying a Generalized variant of Miller´s algorithm and the method of calculus term by term at the infinite summation series, being each Modified Bessel function obtained by the classical TTR Miller´s Algorithm, and afterwards concretizing the direct computation of all the series.
Those two methods validate one to each other and this allows us to start right now the elaboration of our handbook of formulas, algorithms, tables and graphics, concerning the transient regimes of several queuing systems.
We have also an important and interesting set of recurrence relations that can also be used in variegated numerical confirmations, as a complementary method of validation (the third one).
Two achievements must be accomplished successfully.
One is the fact that we must dominate large degrees of congestion, which means that we must use not only low values of initial condition for parameter i values, like 1, 2, … , 10 at most, as is usually the case in the related literature, but reach, as first tentative a range between $i = 0, 1, 2, \dots$ and at least $i = 100$.
The other is that we must, once fixed the initial i value, try also to go through high values of $\rho = \frac{\lambda}{\mu}$ thus computing also the most numerical difficult choice, corresponding to slow variation of the system towards the steady state and, consequently, solving the need of computing long time transient regimes.
The next table registers our working range about the parameter values that must be chosen,

TABLE 19

PRELIMINARY CHOOSEN SCHEME

		i														
		0	1	2		5		10		20		50		100		
ρ	0,0														0,0	ρ
	0,1														0,1	
	0,2	X	X	X		X		X		X		X		X	0,2	
	0,3														0,3	
	0,4	X	X	X		X		X		X		X		X	0,4	
	0,5														0,5	
	0,6	X	X	X		X		X		X		X		X	0,6	
	0,7														0,7	
	0,8	X	X	X		X		X		X		X		V	0,8	
	0,9														0,9	
	1,0														1,0	
		0	1	2		5		10		20		50		100		
		i														

In this table ρ values are at the two columns, in both sides, and initial condition defined by i is at the lines.
From $i = 0$ to $i = 100$ it is more appropriate start linearly in low value (($i = 0, 1, 2, \dots$ then $i = 5$ and, further on, in a more exponential way, from $i = 1, 2, 5$ to $i = 10, 20, 50$ then $i = 100$.
The values of ρ $(0 < \rho < 1)$ are chosen linearly, let´s say, $\rho = 0,1$, $\rho = 0,2$, $\rho = 0,3$, … , $\rho = 0,8$, $\rho = 0,9$.
Those combinations chosen in the first computations are signaled by "X".
As been explained, greats value of i and values of ρ near the upper limit of 1, are the most difficult and those that require a bigger number of time values computations.
Accordingly, among the set of the first choices, the combination $i = 100$ and $\rho = 0,8$ is the more cumbersome at this working table.

We are therefore going now to illustrate the beginning of our handbook providing the tables computed and the corresponding graphic relative to this transient regime.

8.2. Tables description.

Now we describe how computed data, for all the transient regime at M/M/1 queuing system, are organized for the parameter combination $i = 100$ and $\rho = 0{,}8$.
Next page is a list of table pages herself also in a table format.
The left column was reserved for indicate all the time range computed: from $\tau = 0$ to $\tau = 1997$.
But the time values are not disposed linearly. We privileged a bigger density of points in the earliest time range because is at the very beginning when more faster changes in the state probabilities take place. Then, gradually, the time interval between two next near computations raises.:
… from $\tau = 0$ to $\tau = 4{,}9$ time interval between two computations is $\Delta\tau = 0{,}1$
… from $\tau = 5$ to $\tau = 49{,}5$ time interval between two computations is $\Delta\tau = 0{,}5$
… from $\tau = 50$ to $\tau = 199$ time interval between two computations is $\Delta\tau = 1$
… from $\tau = 200$ to $\tau = 498$ time interval between two computations is $\Delta\tau = 2$
… from $\tau = 5000$ to $\tau = 1997$ time interval between two computations is $\Delta\tau = 3$
In the third column the time ranges are indicated.
For example, from $\tau = 0$ to $\tau = 4{,}9$ there are five of these ranges:
… from $\tau = 0$ to $\tau = 0{,}9$
… from $\tau = 1$ to $\tau = 1{,}9$
… from $\tau = 2$ to $\tau = 2{,}9$
… from $\tau = 3$ to $\tau = 3{,}9$
… from $\tau = 4$ to $\tau = 4{,}9$
This complete column has 94 of such entries from the first,
… from $\tau = 0$ to $\tau = 0{,}9$
to the last,
… from $\tau = 1970$ to $\tau = 1997$
Those 94 table entries cross with five columns, each one relative to an ordered set of probability states:
… the first column for probability states from $P_{100,0}$ to $P_{100,41}$
… the second column for probability states from $P_{100,42}$ to $P_{100,83}$
… the third column for probability states from $P_{100,84}$ to $P_{100,125}$
… the fourth column for probability states from $P_{100,126}$ to $P_{100,167}$
… the fifth column for probability states from $P_{100,168}$ to $P_{100,209}$
In the intersection of horizontal lines with columns one gets the page number of the tables section at the handbook.
As an example,
Data for probability states from $P_{100,84}$ to $P_{100,125}$ during the time interval from $\tau = 340$ to $\tau = 358$ are registered in page [$84 - 125$ / $340 - 358$], and computed, from one time to the next applying an interval $\Delta\tau = 2$.
At the bottom of this page, which is an index of the section of table pages, it is indicated the degree of accuracy reached: all probability values equal or greater than $5 \cdot 10^{-12}$ are recorded. And all the digit figures recorded are correct.
All pages in blue have, in the respective table, at least one place equal or greater than $5 \cdot 10^{-12}$. Pages in white don´t have any value at least equal to $5 \cdot 10^{-12}$. They are all smaller than $5 \cdot 10^{-12}$ and they don´t are recorded.
In the Appendix we provide some of these table pages, namely, the pages at the column of $P_{100,0}$ to $P_{100,41}$ from page [0-41 / 25-29,5] to page [0-41 / 340-358], just has exemplification.

TABLE 20

= LIST OF PAGES =			FUNCTIONS: $P_{100,j}$				
GENERAL DENOMINATION: PAGE: [j -- (j + 41) / τ -- (τ + 9 · Δτ)]			From: $P_{100,0}$ to: $P_{100,41}$	From: $P_{100,42}$ to: $P_{100,83}$	From: $P_{100,84}$ to: $P_{100,125}$	From: $P_{100,126}$ to: $P_{100,167}$	From: $P_{100,168}$ to: $P_{100,209}$
TIME: τ	From 0 to 4,9: Δτ = 0,1	0 to 0,9	[0-41 / 0-0,9]	[42-83 / 0-0,9]	[84-125 / 0-0,9]	[126-167 / 0-0,9]	[168-209 / 0-0,9]
		1 to 1,9	[0-41 / 1-1,9]	[42-83 / 1-1,9]	[84-125 / 1-1,9]	[126-167 / 1-1,9]	[168-209 / 1-1,9]
		2 to 2,9	[0-41 / 2-2,9]	[42-83 / 2-2,9]	[84-125 / 2-2,9]	[126-167 / 2-2,9]	[168-209 / 2-2,9]
		3 to 3,9	[0-41 / 3-3,9]	[42-83 / 3-3,9]	[84-125 / 3-3,9]	[126-167 / 3-3,9]	[168-209 / 3-3,9]
		4 to 4,9	[0-41 / 4-4,9]	[42-83 / 4-4,9]	[84-125 / 4-4,9]	[126-167 / 4-4,9]	[168-209 / 4-4,9]
	From 5 to 49,5: Δτ = 0,5	5 to 9,5	[0-41 / 5-9,5]	[42-83 / 5-9,5]	[84-125 / 5-9,5]	[126-167 / 5-9,5]	[168-209 / 5-9,5]
		10 to 14,5	[0-41 / 10-14,5]	[42-83 / 10-14,5]	[84-125 / 10-14,5]	[126-167 / 10-14,5]	[168-209 / 10-14,5]
		15 to 19,5	[0-41 / 15-19,5]	[42-83 / 15-19,5]	[84-125 / 15-19,5]	[126-167 / 15-19,5]	[168-209 / 15-19,5]
		20 to 24,5	[0-41 / 20-24,5]	[42-83 / 20-24,5]	[84-125 / 20-24,5]	[126-167 / 20-24,5]	[168-209 / 20-24,5]
		25 to 29,5	[0-41 / 25-29,5]	[42-83 / 25-29,5]	[84-125 / 25-29,5]	[126-167 / 25-29,5]	[168-209 / 25-29,5]
		30 to 34,5	[0-41 / 30-34,5]	[42-83 / 30-34,5]	[84-125 / 30-34,5]	[126-167 / 30-34,5]	[168-209 / 30-34,5]
		35 to 39,5	[0-41 / 35-39,5]	[42-83 / 35-39,5]	[84-125 / 35-39,5]	[126-167 / 35-39,5]	[168-209 / 35-39,5]
		40 to 44,5	[0-41 / 40-44,5]	[42-83 / 40-44,5]	[84-125 / 40-44,5]	[126-167 / 40-44,5]	[168-209 / 40-44,5]
		45 to 49,5	[0-41 / 45-49,5]	[42-83 / 45-49,5]	[84-125 / 45-49,5]	[126-167 / 45-49,5]	[168-209 / 45-49,5]
	From 50 to 199: Δτ = 1	50 to 59	[0-41 / 50-59]	[42-83 / 50-59]	[84-125 / 50-59]	[126-167 / 50-59]	[168-209 / 50-59]
		60 to 69	[0-41 / 60-69]	[42-83 / 60-69]	[84-125 / 60-69]	[126-167 / 60-69]	[168-209 / 60-69]
		70 to 79	[0-41 / 70-79]	[42-83 / 70-79]	[84-125 / 70-79]	[126-167 / 70-79]	[168-209 / 70-79]
		80 to 89	[0-41 / 80-89]	[42-83 / 80-89]	[84-125 / 80-89]	[126-167 / 80-89]	[168-209 / 80-89]
		90 to 99	[0-41 / 90-99]	[42-83 / 90-99]	[84-125 / 90-99]	[126-167 / 90-99]	[168-209 / 90-99]
		100 to 109	[0-41 / 100-109]	[42-83 / 100-109]	[84-125 / 100-109]	[126-167 / 100-109]	[168-209 / 100-109]
		110 to 119	[0-41 / 110-119]	[42-83 / 110-119]	[84-125 / 110-119]	[126-167 / 110-119]	[168-209 / 110-119]
		120 to 129	[0-41 / 120-129]	[42-83 / 120-129]	[84-125 / 120-129]	[126-167 / 120-129]	[168-209 / 120-129]
		130 to 139	[0-41 / 130-139]	[42-83 / 130-139]	[84-125 / 130-139]	[126-167 / 130-139]	[168-209 / 130-139]
		140 to 149	[0-41 / 140-149]	[42-83 / 140-149]	[84-125 / 140-149]	[126-167 / 140-149]	[168-209 / 140-149]
		150 to 159	[0-41 / 150-159]	[42-83 / 150-159]	[84-125 / 150-159]	[126-167 / 150-159]	[168-209 / 150-159]
		160 to 169	[0-41 / 160-169]	[42-83 / 160-169]	[84-125 / 160-169]	[126-167 / 160-169]	[168-209 / 160-169]
		170 to 179	[0-41 / 170-179]	[42-83 / 170-179]	[84-125 / 170-179]	[126-167 / 170-179]	[168-209 / 170-179]
		180 to 189	[0-41 / 180-189]	[42-83 / 180-189]	[84-125 / 180-189]	[126-167 / 180-189]	[168-209 / 180-189]
		190 to 199	[0-41 / 190-199]	[42-83 / 190-199]	[84-125 / 190-199]	[126-167 / 190-199]	[168-209 / 190-199]
	From 200 to 498: Δτ = 2	200 to 218	[0-41 / 200-218]	[42-83 / 200-218]	[84-125 / 200-218]	[126-167 / 200-218]	[168-209 / 200-218]
		220 to 238	[0-41 / 220-238]	[42-83 / 220-238]	[84-125 / 220-238]	[126-167 / 220-238]	[168-209 / 220-238]
		240 to 258	[0-41 / 240-258]	[42-83 / 240-258]	[84-125 / 240-258]	[126-167 / 240-258]	[168-209 / 240-258]
		260 to 278	[0-41 / 260-278]	[42-83 / 260-278]	[84-125 / 260-278]	[126-167 / 260-278]	[168-209 / 260-278]
		280 to 298	[0-41 / 280-298]	[42-83 / 280-298]	[84-125 / 280-298]	[126-167 / 280-298]	[168-209 / 280-298]
		300 to 318	[0-41 / 300-318]	[42-83 / 300-318]	[84-125 / 300-318]	[126-167 / 300-318]	[168-209 / 300-318]
		320 to 338	[0-41 / 320-338]	[42-83 / 320-338]	[84-125 / 320-338]	[126-167 / 320-338]	[168-209 / 320-338]
		340 to 358	[0-41 / 340-358]	[42-83 / 340-358]	[84-125 / 340-358]	[126-167 / 340-358]	[168-209 / 340-358]
		360 to 378	[0-41 / 360-378]	[42-83 / 360-378]	[84-125 / 360-378]	[126-167 / 360-378]	[168-209 / 360-378]
		380 to 398	[0-41 / 380-398]	[42-83 / 380-398]	[84-125 / 380-398]	[126-167 / 380-398]	[168-209 / 380-398]
		400 to 418	[0-41 / 400-418]	[42-83 / 400-418]	[84-125 / 400-418]	[126-167 / 400-418]	[168-209 / 400-418]
		420 to 438	[0-41 / 420-438]	[42-83 / 420-438]	[84-125 / 420-438]	[126-167 / 420-438]	[168-209 / 420-438]
		440 to 458	[0-41 / 440-458]	[42-83 / 440-458]	[84-125 / 440-458]	[126-167 / 440-458]	[168-209 / 440-458]
		460 to 478	[0-41 / 460-478]	[42-83 / 460-478]	[84-125 / 460-478]	[126-167 / 460-478]	[168-209 / 460-478]
		480 to 498	[0-41 / 480-498]	[42-83 / 480-498]	[84-125 / 480-498]	[126-167 / 480-498]	[168-209 / 480-498]
	From 500 to 1997: Δτ = 3	500 to 527	[0-41 / 500-527]	[42-83 / 500-527]	[84-125 / 500-527]	[126-167 / 500-527]	[168-209 / 500-527]
		530 to 557	[0-41 / 530-557]	[42-83 / 530-557]	[84-125 / 530-557]	[126-167 / 530-557]	[168-209 / 530-557]
		560 to 587	[0-41 / 560-587]	[42-83 / 560-587]	[84-125 / 560-587]	[126-167 / 560-587]	[168-209 / 560-587]
		590 to 617	[0-41 / 590-617]	[42-83 / 590-617]	[84-125 / 590-617]	[126-167 / 590-617]	[168-209 / 590-617]
		620 to 647	[0-41 / 620-647]	[42-83 / 620-647]	[84-125 / 620-647]	[126-167 / 620-647]	[168-209 / 620-647]
		650 to 677	[0-41 / 650-677]	[42-83 / 650-677]	[84-125 / 650-677]	[126-167 / 650-677]	[168-209 / 650-677]
		680 to 707	[0-41 / 680-707]	[42-83 / 680-707]	[84-125 / 680-707]	[126-167 / 680-707]	[168-209 / 680-707]
		710 to 737	[0-41 / 710-737]	[42-83 / 710-737]	[84-125 / 710-737]	[126-167 / 710-737]	[168-209 / 710-737]
		740 to 767	[0-41 / 740-767]	[42-83 / 740-767]	[84-125 / 740-767]	[126-167 / 740-767]	[168-209 / 740-767]
		770 to 797	[0-41 / 770-797]	[42-83 / 770-797]	[84-125 / 770-797]	[126-167 / 770-797]	[168-209 / 770-797]
		800 to 827	[0-41 / 800-827]	[42-83 / 800-827]	[84-125 / 800-827]	[126-167 / 800-827]	[168-209 / 800-827]
		830 to 857	[0-41 / 830-857]	[42-83 / 830-857]	[84-125 / 830-857]	[126-167 / 830-857]	[168-209 / 830-857]
		860 to 887	[0-41 / 860-887]	[42-83 / 860-887]	[84-125 / 860-887]	[126-167 / 860-887]	[168-209 / 860-887]
		890 to 917	[0-41 / 890-917]	[42-83 / 890-917]	[84-125 / 890-917]	[126-167 / 890-917]	[168-209 / 890-917]
		920 to 947	[0-41 / 920-947]	[42-83 / 920-947]	[84-125 / 920-947]	[126-167 / 920-947]	[168-209 / 920-947]
		950 to 977	[0-41 / 950-977]	[42-83 / 950-977]	[84-125 / 950-977]	[126-167 / 950-977]	[168-209 / 950-977]
		980 to 1007	[0-41 / 980-1007]	[42-83 / 980-1007]	[84-125 / 980-1007]	[126-167 / 980-1007]	[168-209 / 980-1007]
		1010 to 1037	[0-41 / 1010-1037]	[42-83 / 1010-1037]	[84-125 / 1010-1037]	[126-167 / 1010-1037]	[168-209 / 1010-1037]
		1040 to 1067	[0-41 / 1040-1067]	[42-83 / 1040-1067]	[84-125 / 1040-1067]	[126-167 / 1040-1067]	[168-209 / 1040-1067]
		1070 to 1097	[0-41 / 1070-1097]	[42-83 / 1070-1097]	[84-125 / 1070-1097]	[126-167 / 1070-1097]	[168-209 / 1070-1097]
		1100 to 1127	[0-41 / 1100-1127]	[42-83 / 1100-1127]	[84-125 / 1100-1127]	[126-167 / 1100-1127]	[168-209 / 1100-1127]
		1130 to 1157	[0-41 / 1130-1157]	[42-83 / 1130-1157]	[84-125 / 1130-1157]	[126-167 / 1130-1157]	[168-209 / 1130-1157]
		1160 to 1187	[0-41 / 1160-1187]	[42-83 / 1160-1187]	[84-125 / 1160-1187]	[126-167 / 1160-1187]	[168-209 / 1160-1187]
		1190 to 1217	[0-41 / 1190-1217]	[42-83 / 1190-1217]	[84-125 / 1190-1217]	[126-167 / 1190-1217]	[168-209 / 1190-1217]
		1220 to 1247	[0-41 / 1220-1247]	[42-83 / 1220-1247]	[84-125 / 1220-1247]	[126-167 / 1220-1247]	[168-209 / 1220-1247]
		1250 to 1277	[0-41 / 1250-1277]	[42-83 / 1250-1277]	[84-125 / 1250-1277]	[126-167 / 1250-1277]	[168-209 / 1250-1277]
		1280 to 1307	[0-41 / 1280-1307]	[42-83 / 1280-1307]	[84-125 / 1280-1307]	[126-167 / 1280-1307]	[168-209 / 1280-1307]
		1310 to 1337	[0-41 / 1310-1337]	[42-83 / 1310-1337]	[84-125 / 1310-1337]	[126-167 / 1310-1337]	[168-209 / 1310-1337]
		1340 to 1367	[0-41 / 1340-1367]	[42-83 / 1340-1367]	[84-125 / 1340-1367]	[126-167 / 1340-1367]	[168-209 / 1340-1367]
		1370 to 1397	[0-41 / 1370-1397]	[42-83 / 1370-1397]	[84-125 / 1370-1397]	[126-167 / 1370-1397]	[168-209 / 1370-1397]
		1400 to 1427	[0-41 / 1400-1427]	[42-83 / 1400-1427]	[84-125 / 1400-1427]	[126-167 / 1400-1427]	[168-209 / 1400-1427]
		1430 to 1457	[0-41 / 1430-1457]	[42-83 / 1430-1457]	[84-125 / 1430-1457]	[126-167 / 1430-1457]	[168-209 / 1430-1457]
		1460 to 1487	[0-41 / 1460-1487]	[42-83 / 1460-1487]	[84-125 / 1460-1487]	[126-167 / 1460-1487]	[168-209 / 1460-1487]
		1490 to 1517	[0-41 / 1490-1517]	[42-83 / 1490-1517]	[84-125 / 1490-1517]	[126-167 / 1490-1517]	[168-209 / 1490-1517]
		1520 to 1547	[0-41 / 1520-1547]	[42-83 / 1520-1547]	[84-125 / 1520-1547]	[126-167 / 1520-1547]	[168-209 / 1520-1547]
		1550 to 1577	[0-41 / 1550-1577]	[42-83 / 1550-1577]	[84-125 / 1550-1577]	[126-167 / 1550-1577]	[168-209 / 1550-1577]
		1580 to 1607	[0-41 / 1580-1607]	[42-83 / 1580-1607]	[84-125 / 1580-1607]	[126-167 / 1580-1607]	[168-209 / 1580-1607]
		1610 to 1637	[0-41 / 1610-1637]	[42-83 / 1610-1637]	[84-125 / 1610-1637]	[126-167 / 1610-1637]	[168-209 / 1610-1637]
		1640 to 1667	[0-41 / 1640-1667]	[42-83 / 1640-1667]	[84-125 / 1640-1667]	[126-167 / 1640-1667]	[168-209 / 1640-1667]
		1670 to 1697	[0-41 / 1670-1697]	[42-83 / 1670-1697]	[84-125 / 1670-1697]	[126-167 / 1670-1697]	[168-209 / 1670-1697]
		1700 to 1727	[0-41 / 1700-1727]	[42-83 / 1700-1727]	[84-125 / 1700-1727]	[126-167 / 1700-1727]	[168-209 / 1700-1727]
		1730 to 1757	[0-41 / 1730-1757]	[42-83 / 1730-1757]	[84-125 / 1730-1757]	[126-167 / 1730-1757]	[168-209 / 1730-1757]
		1760 to 1787	[0-41 / 1760-1787]	[42-83 / 1760-1787]	[84-125 / 1760-1787]	[126-167 / 1760-1787]	[168-209 / 1760-1787]
		1790 to 1817	[0-41 / 1790-1817]	[42-83 / 1790-1817]	[84-125 / 1790-1817]	[126-167 / 1790-1817]	[168-209 / 1790-1817]
		1820 to 1847	[0-41 / 1820-1847]	[42-83 / 1820-1847]	[84-125 / 1820-1847]	[126-167 / 1820-1847]	[168-209 / 1820-1847]
		1850 to 1877	[0-41 / 1850-1877]	[42-83 / 1850-1877]	[84-125 / 1850-1877]	[126-167 / 1850-1877]	[168-209 / 1850-1877]
		1880 to 1907	[0-41 / 1880-1907]	[42-83 / 1880-1907]	[84-125 / 1880-1907]	[126-167 / 1880-1907]	[168-209 / 1880-1907]
		1910 to 1937	[0-41 / 1910-1937]	[42-83 / 1910-1937]	[84-125 / 1910-1937]	[126-167 / 1910-1937]	[168-209 / 1910-1937]
		1940 to 1967	[0-41 / 1940-1967]	[42-83 / 1940-1967]	[84-125 / 1940-1967]	[126-167 / 1940-1967]	[168-209 / 1940-1967]
		1970 to 1997	[0-41 / 1970-1997]	[42-83 / 1970-1997]	[84-125 / 1970-1997]	[126-167 / 1970-1997]	[168-209 / 1970-1997]
	Only 2000 ≈ Steady		[0-41 / 2000 ≈ Steady]	[42-83 / 2000 ≈ Steady]	[84-125 / 2000 ≈ Steady]	[126-167 / 2000 ≈ Steady]	[168-209 / 2000 ≈ Steady]

LEGEND:

(unshaded)	PAGE WITHOUT DATA (REAL VALUES ARE LESS THAM $5 \cdot 10^{-12}$)
(shaded)	PAGE WITH DATA (REAL VALUES ARE EQUAL OR MORE THAM $5 \cdot 10^{-12}$)

8.3. Graphic representation of the transient regime for $\rho = 0,8$ and $i = 100$ (M/M/1 Queuing System)

Now is time to show the graphic representation of this transient regime. Data for construct that graphic are those of the table pages in the Appendix.
The obvious graphic representation is simply to set state probabilities $P_{100,j}$ in function of τ.
This graphic is provided in the next page.
In fact we have already other graphic details concerning this same transient regime but unfortunately time lacks to present, here and now, that more complete graphic representation.
Only for reference purposes, such additional graphic details are,
- Localized close-ups.
- Transient evolution of only few selected state probabilities that are very close ones to each others, appreciating this way a sort of parallel evolutionary behavior.
- Histograms taken at successive regular times seeing the "animated movie" of a statistical distribution evolving from an initial single bar of value 1 to the steady state known distribution, transiting through a complete set of "intermediate hybrid" distributions.
- Graphic of state probabilities "normalized" relatively to the steady state, that´s to say, representation of,

$$\frac{P_{100,j}}{(1-\rho)\cdot\rho^j}$$

... in function of τ.
In such graphic representation all the functions converge asymptotically to 1 at infinite time, and not each one to a plateau, like in the present graphic.

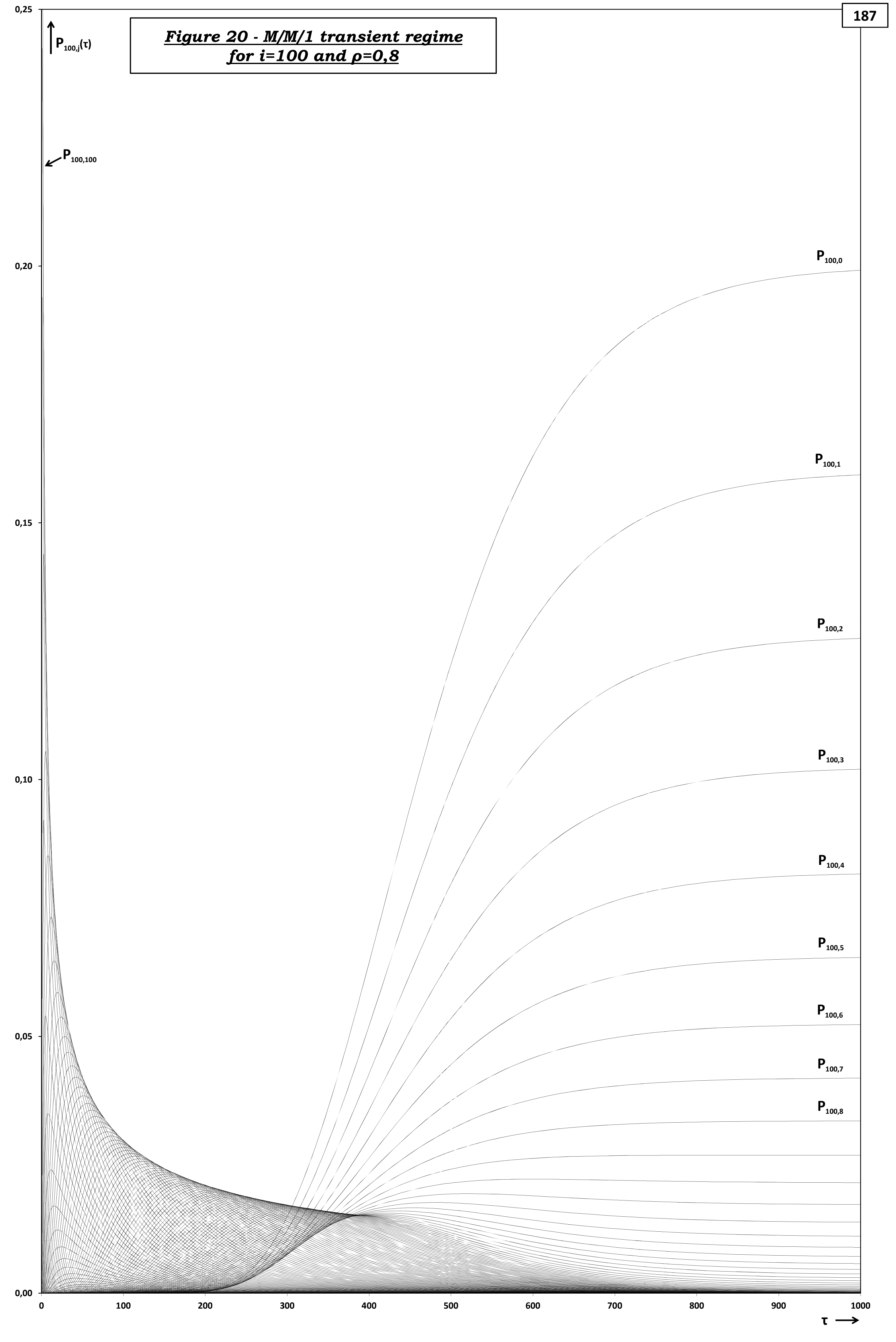


Figure 20 - M/M/1 transient regime
for i=100 and ρ=0,8
P100,j(τ)
P100,100
P100,0
P100,1
P100,2
P100,3
P100,4
P100,5
P100,6
P100,7
P100,8
0,25
0,20
0,15
0,10
0,05
0,00
0
100
200
300
400
500
600
700
800
900
1000
τ

9. Next work (actually is partially already current work)

The little handbook of queuing systems transient regimes, being a continuous not ending task, will extend our work for a long time.
M/M/1 queue transient solution is only the most easy case of the many more examples of birth and death processes (and not only processes in queuing theory).
Our goal, in this subject, is to deduce time-dependent evolution solution for the more general birth-death queue, with infinite storage room (example of Figure 18 - I).
After it is necessary compare with one or two queues having already an analytic transient solution for confirmation purposes of this general and unified approach.
However there are not so many exactly solved cases besides the M/M/1 system. M/M/∞ : Responsive Servers Queuing System (Infinite Number of Servers) is one of their (example of Figure 18 - VI). In fact the exact solution exists but not in form of an infinite summation series of Modified Bessel functions.
The logical next step relates with the idea that the guess solution method can be also applied to queuing systems even more complex than the ones mentioned in this Chapter VII . These have state transitions only between next neighbours states and there are a huge variety of queuing systems with transitions between not neighbours states, like happens at the diagram of Figure 19. See, for example [54].
In a recent work we deduced an "any distance recurrence" between the functions $R_{i,j}$, $R_{i+L,j}$ $R_{i+L+M,j}$ and $R_{i+L+M+N,j}$,defined in Chapter V, where L ≥ 1, M ≥ 1, and N ≥ 1.
Combining that recurrence with the simple identity $R_{i-1,j+1}(\tau) = \rho R_{i,j}(\tau)$ one can reach recurrences with at most 4 terms in any 4 functions $R_{i,j}$ a priori wanted choice in $(i\,,j)$ plane chart.
This work, that we will leave for future publication allows us to build an amplification of known recurrence formulae for this queue, representing also a continuation of what has been done in Chapter V.
All the recurrence "machinery" deduced in Chapter V, for functions $R_{i,j}(\tau)$ and $P_{i,j}(\tau)$ concerns only distances $\Delta i = \pm 1$ and $\Delta j = \pm 1$ but not "any distance recurrences", which is a more general concept.
The return to developed sequel of biofilm theoretical model is also an on going interesting subject. Revisitation of our old model also inspire a new contribution to Queue Transient analysis.
This time we recover a part of our old work were the mathematical description of linear phase global biomass accumulation in biofilm (macro scale experimental observation) is achieved from biomass balance differential-difference equations of bacteria dynamic multi-layers covering the solid support (micro scale theoretical description).
That biofilm model considers a scenery where, in an arbitrary time "t" (or "τ") of linear phase all the support area is covered with a variable number of bacteria layers.
Such local number of layers has a minimum value in the deepest "valleys" of this micro orography and also a maximum value in the highest "mountains".
As time evolves the areal fraction representing the deepest "valleys" diminishes his value and, finally, reaches zero. From that instant now on, one must stop and change the exact solution because next values of that fraction would be negative and consequently meaningless.
All the other values, of all the other fractions, with a bigger number of accumulated layers than that one just annulled, and that are still positive, will be the initial values of a new period of time.

Then, in this new period, the deepest "valley" now has a height of one more layer than the corresponding deepest "valley" in the immediately anterior time period. And in this new period all events repeat over again, with the exception that all fractions are one layer highest.
Calculus from period to period reach a situation where the time interval of each one have always the same value. So between two changes of initial conditions the global biomass increase by one complete layer, in the sense that such layer has an area equal to the area of the support (supposed plane).
Equal times elapsed, imply equal increase of biomass. Therefore we reached a linear growth global (or macro) beginning with assumptions in a micro scale scenery.
The variable orography maintains, over time, a nearly constant statistical distribution of covered fractions that, in global terms travels away from support area as consequence of biomass
So the result is that we have also built a traveling wave front ("valleys" and "mountains") beginning with micro scale assumptions.
Mathematical resolution in all described time periods was difficult due to so many not null initial values in the system of differential-differences equations. As a matter of fact in infinite number in this biofilm model. However everything was then achieved with very good accuracy.
Such task inspire us to a similar one with the purpose of generalize initial conditions in any queue, in general, and namely in M/M/1 model, in particular, including an initial start-up pure birth process period, for example. We will leave that idea also as future work.
All the aforesaid recurrence machinery and, in particular that we named above as “any distance recurrence”, is an interesting result to start a digression in Generalized Marcum and Nuttall functions literature.
Generalized Marcum function is closely related with $R_{i,j}$ functions described at Chapter V, in a way that one only needs modify conveniently parameters and variable combinations to connect ones with others.
Consequently general result for the recurrence between $R_{i,j}$, $R_{i+L,j}$, $R_{i+L+M,j}$ and $R_{i+L+M+N,j}$ allows us to obtain immediately all Generalized Marcum recurrences.
This function only has $(i+j+2)$ as an entire index when related with the two indexes of $R_{i,j}$ functions, and represents a positive diagonal in the two Nuttall function indexes plane.
Once solved a diagonal of such plane we can easily solve all the others diagonals in alternating position with that one.
Thus half plane of Nuttall function is completely solved (like in an infinite chess table, one half, black or white).
Of course, later, we must deduce the mathematical connexion between those two half of the plane defined by the two indexes of Nuttall function, and solve all the problem for more general initial conditions, profiting the mathematics biofilm linear growth associated to the traveling wave front, which is also a possible future work.

Bibliography

1. Preface

-[1] J. A. White, J. W. Schmidt and G. K. Bennett, Analysis of Queueing Systems, Academic Press, New York (1975)
-[2] Kleinrock, Leonard: Queueing Systems, vol.1: Theory, Wiley, New York (1975)
-[3] H. Akimaru and K. Kawashima, Teletraffic: Theory and Applications (2º ed), Springer, London (1999)
-[4] Stochastic Processes, Optimization, and Control Theory: Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems, Yan, Houmin, Yin, George, Zhang, Qing (Eds.), Springer (2006)
-[5] J. Abate an W. Whitt, Calculating Time-Dependent Performance Measures for the M/M/1 Queue. *IEEE Transactions on Communications*, vol. 37, No. 10, 1989, pp. 1102-1104
-[6] C.G. Cassandras and S. Lafortune, Introduction to Discrete Event Systems (2º edition), Springer, (2010)
-[7] Marvin K. Simon, Mohamed-Slim Alouini, Digital Communication over Fading Channels, 2nd Edition, Wiley-IEEE Press, (2005)
-[8] Thomas G. Robertazzi**,** Computer Networks and Systems: Queueyng Theory and Performance Evaluation, (3rd edition), Springer (2000)
-[9] Ryszard Syski, Introduction to congestion theory in telephone systems, 2ª edition (North-Holland, 1986)
-[10] F. C. Goodrich, Nucleation Rates and the Kinetics of Particle Growth. II. The Birth and Death Process, Proc.Roy.Soc.Lond. A 277 (1964) 167-182
-[11] J. M. Carvalho, Modelização da Cinética de Formação de Biofilmes, Capitulos I e II, Working Report, Lisboa (2003)
-[12] J. M. Carvalho, Modelização da Cinética de Formação de Biofilmes, Capitulo II and Apendice1, Working Report, Lisboa (2003)
-[13] J. M. Carvalho, Modelização da Cinética de Formação de Biofilmes, Capitulo IV, Working Report, Lisboa (2003)
-[14] Amparo Gil, Javier Segura and Nico M. Temme, Numerical Methods for Special Functions, SIAM, (2007)
-[15] H. C. Thacher, "New Backward Recurrences for Bessel Functions", Math. Comp., vol. 33, nº 146, 744-764, (1979)
-[16] A. B. Clarke, A Waiting Line Process of Markov Type, The Annals of Mathematical Statistics, Vol. 27, No. 2 (Jun., 1956), pp. 452-459.

2. Chapter I

2.1.Cited in the text

-[17] Marshall K. C. , *ASM News* Vol.58 , No.4 , pp.202-208 , 1992
-[18] J. D. Bryers, Biofilms, *An Introduction in Biofilms II: Process, Analysis and Applications*; Ed. J. D. Bryers; Wiley-Liss: New York, 2000.
-[19] Characklis W.G. and Marshall K.C., Biofilms: A basis for an interdisciplinary approach(1990) Chapter 1. in Biofilms. ed. / W.G. Characklis; K.C. Marshall. Wiley Series in Ecological and Applied Microbiology 1990. p. 3-15.
-[20a] Caldwell , D. E. , Brannan , D. K. , Morris , M. E. , and Betlach , M. R. , *Microb. Ecol.* (1981) 7 : 1-11
-[20b] D. K. Brannan, D. E. Caldwell, *Microb. Ecol.* 1982, *8*, 15-21.
-[20c] Caldwell , D. E. , Malone , J. A. and Kieft , T. L. , *Microb. Ecol.* (1983) 9 : 1-6
-[20d] Malone , J. A. and Caldwell , D.E., *Microb. Ecol.* (1983) 9 : 299-305

-[20e] Kieft , T. L. and Caldwell , D. E., *Microb. Ecol.* (1983) 9 : 7-13
-[21] Fletcher , M. , *Can. J. Microbiol.* , Vol. 23 , pp. 1-6 , (1977)
-[22] Picioreanu C., van Loosdrecht M.C.M. and Heijnen J.J. , *Biotechnol. Bioeng.* 72 : 205-218 , (2001)
-[23] Ohashi A., Koyama T., Syutsubo K. and Harada H. , A novel method for evaluation of biofilm tensile strength resisting erosion *Wat. Sci. Tech.* Vol.39 , No.7 , pp. 261-268 (1999)
-[24] van Loosdrecht , M.C.M., Picioreanu C. , Heijnen J.J. , A more unifying hypothesis for biofilm structures , *FEMS Microb. Ecol.* 24 (1997) 181-183
-[25] Capdeville , B. and Nguyen , K. M. , *Wat. Sci. Tech.* , Vol. 22 , N° 1/2 , pp. 149-170 , (1990)
-[26] Wanner O. and Gujer W. , *Biotechnol. Bioeng.* Vol.XXVIII , Pp. 314-328 (1986)
-[27] Picioreanu C., van Loosdrecht M.C.M. and Heijnen J.J., *Biotechnol. Bioeng.* 69 : 504-515,(2000)
-[28a] Kreft , J.-U. , Booth , G. , and Wimpenny , J.W.T. BacSim , a simulator for individual-based modelling of bacterial colony growth . *Microbiology* 144 , 3275-3287 (1998)
-[28b] Kreft , J.-U. , Picioreanu , C. , Wimpenny , J.W.T. , and van Loosdrecht , M.C.M., Individual-based modelling of biofilms , *Microbiology* 147 , 2897-2912 (2001)
-[29a] Picioreanu C., van Loosdrecht M.C.M. and Heijnen J.J., *Biotechnol. Bioeng.* 57 : 718-731,(1998)
-[29b] Picioreanu C., van Loosdrecht M.C.M. and Heijnen J.J., *Biotechnol. Bioeng.* 58 : 101-116, (1998)
-[30] Eberl , H. J. , Parker , D. F. and van Loosdrecht , M. C. M. A new deterministic spatiotemporal continuum model for biofilm development. *J.Theor.Med.* 3 , 161-175 (2001)
-[31] Dockery , J. and Klapper , I. , Finger formation in biofilm layers . *SIAM J. Appl. Math.* , 62 (3) , 853-869 (2001)
-[32] Picioreanu C. and van Loosdrecht M.C.M., Use of mathematical modelling to study biofilm development and morphology (2003) Chapter 22. Biofilms in Medicine, Industry and Environmental Biotechnology. Characteristics, analysis and control.. ed. / Piet Lens; Anthony P Moran; Therese Mahoney; Paul Stoodley; Vincent O'Flaherty. London : IWA Publishing, 2003. p. 413-438.

2.2. General (for further lecture on other models points of view)

1995
-[1.g], Oskar Wanner, NEW EXPERIMENTAL FINDINGS AND BIOFILM MODELLING CONCEPTS, *Welt. Sci. Tech.* Vol. 32. No.8. pp. 13l--140. 1995.
1998
-[2.g], Muhammad Arshad S. Chaudhry, and Shafkat A. Beg* A Review on the Mathematical Modelling of Biofilm Processes: Advances in Fundamentals of Biofilm Modeling Chem. Eng. Technol. 21 (1998) 9,701-710
1999
-[3.g], Daniel R. Noguera*, Satoshi Okabe and Cristian Picioreanu, *Waf Sci Tech Vol.* 39, No 7, PI' 273-278, 1999
- [4.g], Shafkat A. Beg and Muhammad Arshad S. Chaudhry, A Review of Mathematical Processes, Intern. J. Environ. Studies, (1999), Vol. 56, pp. 285-312

2003

- [5.g], *C. Picioreanu and M.C.M. van Loosdrecht*, Use of mathematical modelling to study biofilm development and morphology, Chapter 22, Biofilms in Medicine, Industry and Environmental Biotechnology. Characteristics, analysis and control.. ed. / Piet Lens; Anthony P Moran; Therese Mahoney; Paul Stoodley; Vincent O'Flaherty. London : IWA Publishing, 2003. p. 413-438.

2004

- [6.g], Wanner O., Eberl H. J., Morgenroth E., Noguera D. R., Picioreanu C., Rittmann B. E. and van Loosdrecht M. C.M., Mathematical Modeling of Biofilms This report was prepared by the IWA Task Group on Biofilm Modeling. Parts of this report were presented at the IWA Biofilm Specialists Conference in Cape Town, South Africa (September 2003) and have been published in modified form in *Water Science and Technology* Vol. 49, no. (11-12), 2004. Other parts of this report were presented at the IWA Biofilm Specialists Conference in Las Vegas, Nevada, USA (October 2004).
- [7.g], C. Picioreanu, J. B. Xavier and M. C. M. van Loosdrecht, Advances in mathematical modeling of biofilm structure, Biofilms / Volume 1 / Issue 04 / October 2004, pp. 337 - 349
- [8.g], D.R. Noguera* and E. Morgenroth, Introduction to the IWA Task Group on Biofilm Modeling, Water Science and Technology Vol 49 No 11–12 pp 131–136 © IWA Publishing 2004

2007

- [9.g], J.William Costerton, The Biofilm Primer (2007), Springer Series on Biofilms, Volume 1, Series Editor: J. William Costerton

2009

- [10.g], Andrew J. McBain, In Vitro Biofilm Models: An Overview, Chapter 4, 99-132, in Advances in Applied Microbiology, Volume 69, Elsevier, (2009).

2010

- [11.g], Klapper I. and Dockery J., Mathematical Description of Microbial Biofilms, SIAM REVIEW 2010 Society for Industrial and Applied MathematicsVol. 52, No. 2, pp. 221–265
- [12.g], Wang Q. and Zhang T., Review of mathematical models for biofilms, Solid State Communications 150 (2010) 1009_1022

2011

- [13.g], J. P. Boltz, E. Morgenroth, D. Brockmann, C. Bott, W. J. Gellner and P. A. Vanrolleghem, Systematic evaluation of biofilm models for engineering practice: components and critical assumptions, 930-944, and Supplementary material for 'Systematic evaluation of biofilm models for engineering practice: components and critical assumptions' I-VII © IWA Publishing 2011 Water Science & Technology | 64.4 | 2011

2012

- [14.g], ANIL KISHEN & MARKUS HAAPASALO, Biofilm models and methods of biofilm assessment, *Endodontic Topics 2012, 22, 58–78*

2014

- [15.g], Harald H. and Lackner S., Adv Biochem Eng Biotechnol (2014) 146: 53–76

2017

- [16.g] Zhang T., Modeling Biofilms: From Genes to Communities, Processes, 2017, 5, 5-22

3. Chapter II

- [33] Schügerl K., and Bellgardt K. H., Bioreaction Engineering (Modeling and Control) Springer-Verlag Berlin Heidelberg (2000)

4. Chapter III

- In Chapter III none additional reference has been made.

5. Chapter IV

- [34] Pinney, E., Ordinary Difference-Differential Equations, University of California Press (1958)
- [35] Bellman, R., and Cooke, K.L., Differential-Difference Equations, Academic Press (1963)
- [36] El′sgol′ts, L.E. e Norkin,S.B., Introduction to the Theory and Application of Differential Equations with Deviating Arguments, Academic Press. (1973)
- [37] Clarke,A.B. On time-dependent waiting line processes (Abst.) Ann.Math.Statist. 24, 491-492 (1953).
- [38] Clarke, A.B. A waiting time process of Markov Type. Ann. Math. Statist. 27, 452--459 (1956).
- [39] Saaty, T.L., Elements of Queueing Theory with applications , McGraw-Hill (1961)
- [40] Srivastava, H.M., and Kashyap, B.R.K., Special Functions in Queuing Theory, Academic Press (1982)

6. Chapter V

- [41] Abramowitz, M. and Stegun, I., Eds., *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards, (1964). Reprinted by Dover Publications, New York, (1968)
- [42] Miller J.C.P., *Bessel Functions*, vol.II in Mathematical Tables, Cambridge University Press, Londopn and New York (1952)
- [43] Zhang S. e Jin J.,*Computation of Special Functions* Wiley-Interscience (1996)
- [44] Clarke,A.B., On the Solution of the "Telephone Problem", Univ. Michigan Eng. Research Inst. Rept. R-32, March, 1952
- [45] W. Ledermann, G.G.H. Reuter, Spectral theory for the differential equations of simple birth and death processes, Philos. Trans. Roy. Soc. London, Ser. A, Vol. 246, pp. 321–369 (1954).
- [46] N. Bailey, A Continuous Time Treatment of a Simple Queue, Using Generating Functions, Journal of the Royal Statistical Society. Series B (Methodological), Vol. 16, No. 2, pp. 288-291, (1954).
- [47] P. M. Morse, Stochastic Properties of Waiting Lines, Journal of the Operations Research Society of America, Vol. 3, No. 3, pp. 255-261, (Aug., 1955),
- [48] Champernowne, An Elementary Method of Solution of the Queueing Problem with a Single Server and Constant Parameters, Journal of the Royal Statistical Society. Series B (Methodological), Vol. 18, No. 1, pp. 125-128, (1956).
-[49] A. B. Clarke, A Waiting Line Process of Markov Type, The Annals of Mathematical Statistics, Vol. 27, No. 2, pp. 452-459, (Jun., 1956).

7. Chapter VI

-[50] J. M. Carvalho, Modelização da Cinética de Formação de Biofilmes, Capitulo II and Apendice1, Working Report, Lisboa (2003)

8. Chapter VII

- [51] Abramowitz, M. and Stegun, I., Eds., *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards, (1964). Reprinted by Dover Publications, New York, (1968)

- [52] Descloux, A., Delay Table For Finite and Infinite Source Systems., Bell Telephone Laboratoires, Inc., McGraw-Hill Book Company (1962). Reprinted many times till (2013).
-[53] Kleinrock, Leonard: Queueing Systems, vol.1: Theory, Wiley, New York (1975). Pages and figures:
58 (Fig. 2.9), 95 (Fig. 3.1), 100 (Fig. 3.5), 101 (Fig. 3.6), 102 (Fig. 3.7), 104 (Fig. 3.8), 105 (Fig. 3.9), 107 (Fig. 3.10), 107 (Fig. 3.11), 109 (Fig. 3.12).
-[54] Sudhesh, R. : Transient analysis of a queue with system disasters and customer impatience, Queueing Systems Vol. 66, pp. 95-105, (2010).

APPENDIX

Mathematical tables

Transient regime at M/M/1 queuing system.

Parameters:

$$\rho = 0,8$$

$$i = 100$$

Pages:

From [0-41/25-29,5] to [0-41/340-358]

PAGE: [0-41 / 25-29,5]

τ =	25	25,5	26	26,5	27	27,5	28	28,5	29	29,5
$P_{100,0}$										
$P_{100,1}$										
$P_{100,2}$										
$P_{100,3}$										
$P_{100,4}$										
$P_{100,5}$										
$P_{100,6}$										
$P_{100,7}$										
$P_{100,8}$										
$P_{100,9}$										
$P_{100,10}$										
$P_{100,11}$										
$P_{100,12}$										
$P_{100,13}$										
$P_{100,14}$										
$P_{100,15}$										
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$P_{100,26}$										
$P_{100,27}$										
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$P_{100,30}$										
$P_{100,31}$										
$P_{100,32}$										
$P_{100,33}$										
$P_{100,34}$										
$P_{100,35}$										
$P_{100,36}$										
$P_{100,37}$										
$P_{100,38}$										
$P_{100,39}$										
$P_{100,40}$										0,000000000001
$P_{100,41}$								0,000000000001	0,000000000001	0,000000000001

PAGE: [0-41 / 30-34,5]

τ =	30	30,5	31	31,5	32	32,5	33	33,5	34	34,5
$P_{100,0}$										
$P_{100,1}$										
$P_{100,2}$										
$P_{100,3}$										
$P_{100,4}$										
$P_{100,5}$										
$P_{100,6}$										
$P_{100,7}$										
$P_{100,8}$										
$P_{100,9}$										
$P_{100,10}$										
$P_{100,11}$										
$P_{100,12}$										
$P_{100,13}$										
$P_{100,14}$										
$P_{100,15}$										
$P_{100,16}$										
$P_{100,17}$										
$P_{100,18}$										
$P_{100,19}$										
$P_{100,20}$										
$P_{100,21}$										
$P_{100,22}$										
$P_{100,23}$										
$P_{100,24}$										
$P_{100,25}$										
$P_{100,26}$										
$P_{100,27}$										
$P_{100,28}$										
$P_{100,29}$										
$P_{100,30}$										
$P_{100,31}$										
$P_{100,32}$										
$P_{100,33}$										
$P_{100,34}$										
$P_{100,35}$										0,000000000001
$P_{100,36}$								0,000000000001	0,000000000001	0,000000000001
$P_{100,37}$						0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003
$P_{100,38}$				0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000007
$P_{100,39}$		0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000007	0,000000000010	0,000000000015
$P_{100,40}$	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000007	0,000000000011	0,000000000016	0,000000000022	0,000000000032
$P_{100,41}$	0,000000000002	0,000000000003	0,000000000005	0,000000000008	0,000000000011	0,000000000016	0,000000000024	0,000000000034	0,000000000048	0,000000000067

PAGE: [0-41 / 35-39,5]

τ =	35	35,5	36	36,5	37	37,5	38	38,5	39	39,5
$P_{100,0}$										
$P_{100,1}$										
$P_{100,2}$										
$P_{100,3}$										
$P_{100,4}$										
$P_{100,5}$										
$P_{100,6}$										
$P_{100,7}$										
$P_{100,8}$										
$P_{100,9}$										
$P_{100,10}$										
$P_{100,11}$										
$P_{100,12}$										
$P_{100,13}$										
$P_{100,14}$										
$P_{100,15}$										
$P_{100,16}$										
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$P_{100,20}$										
$P_{100,21}$										
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$P_{100,23}$										
$P_{100,24}$										
$P_{100,25}$										
$P_{100,26}$										
$P_{100,27}$										
$P_{100,28}$										
$P_{100,29}$										
$P_{100,30}$										0,000000000001
$P_{100,31}$								0,000000000001	0,000000000001	0,000000000001
$P_{100,32}$						0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000002
$P_{100,33}$				0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000005
$P_{100,34}$		0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000005	0,000000000007	0,000000000010
$P_{100,35}$	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000006	0,000000000008	0,000000000011	0,000000000016	0,000000000021
$P_{100,36}$	0,000000000002	0,000000000003	0,000000000004	0,000000000006	0,000000000009	0,000000000012	0,000000000017	0,000000000023	0,000000000032	0,000000000043
$P_{100,37}$	0,000000000005	0,000000000007	0,000000000009	0,000000000013	0,000000000018	0,000000000025	0,000000000035	0,000000000048	0,000000000065	0,000000000087
$P_{100,38}$	0,000000000010	0,000000000014	0,000000000020	0,000000000027	0,000000000038	0,000000000052	0,000000000071	0,000000000097	0,000000000130	0,000000000174
$P_{100,39}$	0,000000000021	0,000000000030	0,000000000041	0,000000000057	0,000000000078	0,000000000107	0,000000000144	0,000000000194	0,000000000259	0,000000000343
$P_{100,40}$	0,000000000045	0,000000000062	0,000000000086	0,000000000118	0,000000000160	0,000000000216	0,000000000289	0,000000000385	0,000000000509	0,000000000668
$P_{100,41}$	0,000000000093	0,000000000129	0,000000000176	0,000000000239	0,000000000322	0,000000000431	0,000000000572	0,000000000755	0,000000000989	0,000000001289

PAGE: *[0-41 / 40-44,5]*

τ =	40	40,5	41	41,5	42	42,5	43	43,5	44	44,5
$P_{100,0}$										
$P_{100,1}$										
$P_{100,2}$										
$P_{100,3}$										
$P_{100,4}$										
$P_{100,5}$										
$P_{100,6}$										
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$P_{100,20}$										
$P_{100,21}$										
$P_{100,22}$										
$P_{100,23}$										
$P_{100,24}$										
$P_{100,25}$										
$P_{100,26}$									0,000000000001	0,000000000001
$P_{100,27}$							0,000000000001	0,000000000001	0,000000000001	0,000000000002
$P_{100,28}$					0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000002	0,000000000003
$P_{100,29}$			0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000005	0,000000000006
$P_{100,30}$	0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000005	0,000000000007	0,000000000010	0,000000000013
$P_{100,31}$	0,000000000002	0,000000000002	0,000000000003	0,000000000004	0,000000000006	0,000000000008	0,000000000011	0,000000000014	0,000000000019	0,000000000025
$P_{100,32}$	0,000000000003	0,000000000005	0,000000000006	0,000000000009	0,000000000012	0,000000000016	0,000000000021	0,000000000028	0,000000000038	0,000000000050
$P_{100,33}$	0,000000000007	0,000000000009	0,000000000013	0,000000000018	0,000000000024	0,000000000032	0,000000000042	0,000000000056	0,000000000074	0,000000000096
$P_{100,34}$	0,000000000014	0,000000000019	0,000000000026	0,000000000035	0,000000000047	0,000000000063	0,000000000083	0,000000000109	0,000000000142	0,000000000184
$P_{100,35}$	0,000000000029	0,000000000039	0,000000000053	0,000000000070	0,000000000093	0,000000000123	0,000000000161	0,000000000210	0,000000000272	0,000000000351
$P_{100,36}$	0,000000000058	0,000000000078	0,000000000104	0,000000000138	0,000000000182	0,000000000238	0,000000000310	0,000000000401	0,000000000515	0,000000000660
$P_{100,37}$	0,000000000117	0,000000000155	0,000000000205	0,000000000269	0,000000000352	0,000000000457	0,000000000590	0,000000000757	0,000000000967	0,000000001229
$P_{100,38}$	0,000000000231	0,000000000304	0,000000000399	0,000000000520	0,000000000673	0,000000000867	0,000000001112	0,000000001417	0,000000001797	0,000000002268
$P_{100,39}$	0,000000000451	0,000000000590	0,000000000767	0,000000000992	0,000000001275	0,000000001631	0,000000002074	0,000000002625	0,000000003306	0,000000004142
$P_{100,40}$	0,000000000872	0,000000001132	0,000000001461	0,000000001874	0,000000002391	0,000000003035	0,000000003832	0,000000004815	0,000000006020	0,000000007492
$P_{100,41}$	0,000000001669	0,000000002149	0,000000002751	0,000000003503	0,000000004436	0,000000005589	0,000000007007	0,000000008741	0,000000010854	0,000000013416

PAGE:	[0-41 / 45-49,5]

τ =	45	45,5	46	46,5	47	47,5	48	48,5	49	49,5
$P_{100,0}$										
$P_{100,1}$										
$P_{100,2}$										
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$P_{100,18}$										
$P_{100,19}$										
$P_{100,20}$										
$P_{100,21}$										
$P_{100,22}$								0,000000000001	0,000000000001	0,000000000001
$P_{100,23}$						0,000000000001	0,000000000001	0,000000000001	0,000000000001	0,000000000002
$P_{100,24}$				0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000002	0,000000000003	0,000000000004
$P_{100,25}$	0,000000000001	0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000002	0,000000000003	0,000000000004	0,000000000005	0,000000000007
$P_{100,26}$	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000005	0,000000000006	0,000000000008	0,000000000011	0,000000000014
$P_{100,27}$	0,000000000002	0,000000000003	0,000000000004	0,000000000005	0,000000000007	0,000000000009	0,000000000012	0,000000000016	0,000000000020	0,000000000027
$P_{100,28}$	0,000000000004	0,000000000006	0,000000000008	0,000000000010	0,000000000014	0,000000000018	0,000000000023	0,000000000030	0,000000000039	0,000000000050
$P_{100,29}$	0,000000000009	0,000000000012	0,000000000015	0,000000000020	0,000000000026	0,000000000034	0,000000000044	0,000000000057	0,000000000074	0,000000000094
$P_{100,30}$	0,000000000017	0,000000000023	0,000000000030	0,000000000039	0,000000000051	0,000000000065	0,000000000084	0,000000000108	0,000000000138	0,000000000175
$P_{100,31}$	0,000000000033	0,000000000044	0,000000000057	0,000000000075	0,000000000096	0,000000000124	0,000000000159	0,000000000202	0,000000000256	0,000000000323
$P_{100,32}$	0,000000000065	0,000000000085	0,000000000110	0,000000000142	0,000000000182	0,000000000232	0,000000000295	0,000000000374	0,000000000471	0,000000000591
$P_{100,33}$	0,000000000125	0,000000000162	0,000000000208	0,000000000267	0,000000000340	0,000000000432	0,000000000545	0,000000000686	0,000000000859	0,000000001070
$P_{100,34}$	0,000000000238	0,000000000306	0,000000000391	0,000000000498	0,000000000631	0,000000000795	0,000000000998	0,000000001247	0,000000001551	0,000000001922
$P_{100,35}$	0,000000000450	0,000000000574	0,000000000728	0,000000000921	0,000000001158	0,000000001450	0,000000001808	0,000000002245	0,000000002776	0,000000003419
$P_{100,36}$	0,000000000840	0,000000001064	0,000000001342	0,000000001685	0,000000002106	0,000000002620	0,000000003247	0,000000004006	0,000000004923	0,000000006026
$P_{100,37}$	0,000000001554	0,000000001956	0,000000002450	0,000000003056	0,000000003794	0,000000004691	0,000000005776	0,000000007082	0,000000008651	0,000000010526
$P_{100,38}$	0,000000002848	0,000000003560	0,000000004430	0,000000005488	0,000000006771	0,000000008318	0,000000010178	0,000000012405	0,000000015061	0,000000018218
$P_{100,39}$	0,000000005167	0,000000006415	0,000000007931	0,000000009762	0,000000011967	0,000000014610	0,000000017767	0,000000021523	0,000000025977	0,000000031240
$P_{100,40}$	0,000000009282	0,000000011449	0,000000014061	0,000000017197	0,000000020948	0,000000025416	0,000000030721	0,000000036994	0,000000044388	0,000000053072
$P_{100,41}$	0,000000016510	0,000000020230	0,000000024685	0,000000030000	0,000000036315	0,000000043793	0,000000052614	0,000000062984	0,000000075134	0,000000089322

PAGE: [0-41 / 50-59]

τ =	50	51	52	53	54	55	56	57	58	59
$P_{100,0}$										
$P_{100,1}$										
$P_{100,2}$										
$P_{100,3}$										
$P_{100,4}$										
$P_{100,5}$										
$P_{100,6}$										
$P_{100,7}$										
$P_{100,8}$										
$P_{100,9}$										
$P_{100,10}$										
$P_{100,11}$										
$P_{100,12}$										
$P_{100,13}$										
$P_{100,14}$										0,000000000001
$P_{100,15}$								0,000000000001	0,000000000001	0,000000000002
$P_{100,16}$							0,000000000001	0,000000000001	0,000000000002	0,000000000003
$P_{100,17}$						0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005
$P_{100,18}$					0,000000000001	0,000000000001	0,000000000002	0,000000000004	0,000000000006	0,000000000010
$P_{100,19}$			0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000007	0,000000000011	0,000000000018
$P_{100,20}$		0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000008	0,000000000013	0,000000000021	0,000000000032
$P_{100,21}$	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000006	0,000000000009	0,000000000015	0,000000000024	0,000000000037	0,000000000058
$P_{100,22}$	0,000000000001	0,000000000002	0,000000000004	0,000000000006	0,000000000011	0,000000000017	0,000000000028	0,000000000043	0,000000000067	0,000000000103
$P_{100,23}$	0,000000000003	0,000000000004	0,000000000007	0,000000000012	0,000000000020	0,000000000032	0,000000000050	0,000000000078	0,000000000120	0,000000000182
$P_{100,24}$	0,000000000005	0,000000000008	0,000000000014	0,000000000023	0,000000000037	0,000000000058	0,000000000091	0,000000000140	0,000000000213	0,000000000320
$P_{100,25}$	0,000000000009	0,000000000016	0,000000000026	0,000000000042	0,000000000067	0,000000000106	0,000000000164	0,000000000250	0,000000000376	0,000000000558
$P_{100,26}$	0,000000000018	0,000000000030	0,000000000049	0,000000000078	0,000000000123	0,000000000191	0,000000000292	0,000000000441	0,000000000657	0,000000000966
$P_{100,27}$	0,000000000034	0,000000000056	0,000000000090	0,000000000143	0,000000000223	0,000000000342	0,000000000518	0,000000000773	0,000000001139	0,000000001658
$P_{100,28}$	0,000000000064	0,000000000104	0,000000000166	0,000000000259	0,000000000399	0,000000000607	0,000000000909	0,000000001343	0,000000001959	0,000000002824
$P_{100,29}$	0,000000000120	0,000000000192	0,000000000301	0,000000000466	0,000000000711	0,000000001068	0,000000001582	0,000000002314	0,000000003343	0,000000004772
$P_{100,30}$	0,000000000222	0,000000000350	0,000000000544	0,000000000832	0,000000001254	0,000000001864	0,000000002733	0,000000003957	0,000000005660	0,000000008002
$P_{100,31}$	0,000000000406	0,000000000634	0,000000000973	0,000000001472	0,000000002194	0,000000003226	0,000000004682	0,000000006711	0,000000009505	0,000000013311
$P_{100,32}$	0,000000000738	0,000000001138	0,000000001727	0,000000002582	0,000000003807	0,000000005539	0,000000007956	0,000000011289	0,000000015836	0,000000021969
$P_{100,33}$	0,000000001329	0,000000002024	0,000000003037	0,000000004491	0,000000006550	0,000000009429	0,000000013406	0,000000018837	0,000000026170	0,000000035969
$P_{100,34}$	0,000000002371	0,000000003570	0,000000005294	0,000000007743	0,000000011172	0,000000015917	0,000000022404	0,000000031173	0,000000042899	0,000000058421
$P_{100,35}$	0,000000004193	0,000000006239	0,000000009149	0,000000013234	0,000000018895	0,000000026644	0,000000037130	0,000000051164	0,000000069752	0,000000094127
$P_{100,36}$	0,000000007349	0,000000010808	0,000000015672	0,000000022425	0,000000031681	0,000000044221	0,000000061020	0,000000083283	0,000000112490	0,000000150436
$P_{100,37}$	0,000000012761	0,000000018554	0,000000026609	0,000000037666	0,000000052664	0,000000072773	0,000000099441	0,000000134442	0,000000179929	0,000000238487
$P_{100,38}$	0,000000021958	0,000000031566	0,000000044775	0,000000062713	0,000000086786	0,000000118735	0,000000160687	0,000000215220	0,000000285432	0,000000375004
$P_{100,39}$	0,000000037437	0,000000053217	0,000000074673	0,000000103496	0,000000141775	0,000000192065	0,000000257455	0,000000341653	0,000000449059	0,000000584862
$P_{100,40}$	0,000000063239	0,000000088904	0,000000123417	0,000000169288	0,000000229582	0,000000308005	0,000000408992	0,000000537805	0,000000700632	0,000000904691
$P_{100,41}$	0,000000105836	0,000000147167	0,000000202143	0,000000274443	0,000000368512	0,000000489657	0,000000644170	0,000000839432	0,000001084039	0,000001387914

PAGE: [0-41 / 60-69]

τ =	60	61	62	63	64	65	66	67	68	69
$P_{100,0}$										
$P_{100,1}$										
$P_{100,2}$										
$P_{100,3}$										
$P_{100,4}$										
$P_{100,5}$										
$P_{100,6}$										0,000000000001
$P_{100,7}$									0,000000000001	0,000000000001
$P_{100,8}$							0,000000000001	0,000000000001	0,000000000001	0,000000000002
$P_{100,9}$						0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004
$P_{100,10}$					0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000007
$P_{100,11}$			0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000008	0,000000000012
$P_{100,12}$		0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000006	0,000000000009	0,000000000014	0,000000000021
$P_{100,13}$	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000007	0,000000000011	0,000000000017	0,000000000025	0,000000000036
$P_{100,14}$	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000008	0,000000000013	0,000000000019	0,000000000029	0,000000000043	0,000000000063
$P_{100,15}$	0,000000000002	0,000000000004	0,000000000006	0,000000000010	0,000000000015	0,000000000023	0,000000000034	0,000000000050	0,000000000074	0,000000000107
$P_{100,16}$	0,000000000005	0,000000000007	0,000000000011	0,000000000017	0,000000000027	0,000000000040	0,000000000059	0,000000000087	0,000000000127	0,000000000182
$P_{100,17}$	0,000000000008	0,000000000013	0,000000000020	0,000000000031	0,000000000047	0,000000000070	0,000000000103	0,000000000150	0,000000000216	0,000000000308
$P_{100,18}$	0,000000000015	0,000000000024	0,000000000036	0,000000000055	0,000000000082	0,000000000121	0,000000000176	0,000000000255	0,000000000364	0,000000000516
$P_{100,19}$	0,000000000028	0,000000000042	0,000000000064	0,000000000096	0,000000000142	0,000000000208	0,000000000301	0,000000000432	0,000000000612	0,000000000859
$P_{100,20}$	0,000000000049	0,000000000075	0,000000000113	0,000000000168	0,000000000246	0,000000000356	0,000000000511	0,000000000726	0,000000001020	0,000000001421
$P_{100,21}$	0,000000000088	0,000000000133	0,000000000197	0,000000000290	0,000000000421	0,000000000605	0,000000000861	0,000000001212	0,000000001690	0,000000002334
$P_{100,22}$	0,000000000155	0,000000000232	0,000000000342	0,000000000497	0,000000000716	0,000000001020	0,000000001439	0,000000002009	0,000000002779	0,000000003809
$P_{100,23}$	0,000000000273	0,000000000402	0,000000000587	0,000000000848	0,000000001210	0,000000001708	0,000000002389	0,000000003308	0,000000004539	0,000000006174
$P_{100,24}$	0,000000000474	0,000000000693	0,000000001003	0,000000001434	0,000000002028	0,000000002840	0,000000003938	0,000000005409	0,000000007364	0,000000009937
$P_{100,25}$	0,000000000819	0,000000001186	0,000000001699	0,000000002408	0,000000003376	0,000000004687	0,000000006446	0,000000008783	0,000000011861	0,000000015884
$P_{100,26}$	0,000000001402	0,000000002013	0,000000002857	0,000000004013	0,000000005579	0,000000007681	0,000000010475	0,000000014159	0,000000018973	0,000000025215
$P_{100,27}$	0,000000002384	0,000000003391	0,000000004770	0,000000006640	0,000000009152	0,000000012494	0,000000016902	0,000000022664	0,000000030137	0,000000039750
$P_{100,28}$	0,000000004023	0,000000005668	0,000000007902	0,000000010904	0,000000014902	0,000000020176	0,000000027075	0,000000036021	0,000000047531	0,000000062227
$P_{100,29}$	0,000000006735	0,000000009403	0,000000012992	0,000000017774	0,000000024086	0,000000032345	0,000000043057	0,000000056841	0,000000074437	0,000000096732
$P_{100,30}$	0,000000011188	0,000000015479	0,000000021200	0,000000028755	0,000000038643	0,000000051472	0,000000067979	0,000000089051	0,000000115746	0,000000149317
$P_{100,31}$	0,000000018441	0,000000025285	0,000000034329	0,000000046169	0,000000061534	0,000000081307	0,000000106545	0,000000138511	0,000000178700	0,000000228867
$P_{100,32}$	0,000000030158	0,000000040984	0,000000055164	0,000000073570	0,000000097256	0,000000127487	0,000000165771	0,000000213887	0,000000273924	0,000000348320
$P_{100,33}$	0,000000048931	0,000000065915	0,000000087966	0,000000116344	0,000000152562	0,000000198417	0,000000256030	0,000000327888	0,000000416884	0,000000526366
$P_{100,34}$	0,000000078766	0,000000105186	0,000000139191	0,000000182587	0,000000237518	0,000000306512	0,000000392526	0,000000498997	0,000000629893	0,000000789765
$P_{100,35}$	0,000000125788	0,000000166541	0,000000218544	0,000000284357	0,000000366990	0,000000469961	0,000000597350	0,000000753859	0,000000944873	0,000001176520
$P_{100,36}$	0,000000199283	0,000000261611	0,000000340471	0,000000439453	0,000000562738	0,000000715170	0,000000902320	0,000001130554	0,000001407102	0,000001740128
$P_{100,37}$	0,000000313199	0,000000407709	0,000000526289	0,000000673914	0,000000856330	0,000001080136	0,000001352857	0,000001683019	0,000002080230	0,000002555248
$P_{100,38}$	0,000000488286	0,000000630365	0,000000807155	0,000001025476	0,000001293140	0,000001619038	0,000002013220	0,000002486984	0,000003052953	0,000003725154
$P_{100,39}$	0,000000755121	0,000000966866	0,000001228187	0,000001548330	0,000001937794	0,000002408422	0,000002973492	0,000003647803	0,000004447759	0,000005391438
$P_{100,40}$	0,000001158335	0,000001471158	0,000001854102	0,000002319557	0,000002881472	0,000003555441	0,000004358803	0,000005310721	0,000006432264	0,000007746467
$P_{100,41}$	0,000001762432	0,000002220534	0,000002776841	0,000003447766	0,000004251615	0,000005208683	0,000006341342	0,000007674113	0,000009233734	0,000011049209

PAGE: [0-41 / 70-79]

τ =	70	71	72	73	74	75	76	77	78	79
$P_{100,0}$							0,000000000001	0,000000000001	0,000000000002	0,000000000002
$P_{100,1}$						0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003
$P_{100,2}$					0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004
$P_{100,3}$				0,000000000001	0,000000000001	0,000000000002	0,000000000002	0,000000000003	0,000000000005	0,000000000007
$P_{100,4}$		0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000004	0,000000000006	0,000000000008	0,000000000012
$P_{100,5}$	0,000000000001	0,000000000001	0,000000000001	0,000000000002	0,000000000003	0,000000000005	0,000000000007	0,000000000010	0,000000000014	0,000000000020
$P_{100,6}$	0,000000000001	0,000000000002	0,000000000002	0,000000000004	0,000000000005	0,000000000008	0,000000000012	0,000000000017	0,000000000024	0,000000000034
$P_{100,7}$	0,000000000002	0,000000000003	0,000000000004	0,000000000006	0,000000000009	0,000000000014	0,000000000020	0,000000000028	0,000000000040	0,000000000056
$P_{100,8}$	0,000000000003	0,000000000005	0,000000000007	0,000000000011	0,000000000016	0,000000000023	0,000000000033	0,000000000047	0,000000000066	0,000000000093
$P_{100,9}$	0,000000000006	0,000000000009	0,000000000013	0,000000000019	0,000000000027	0,000000000039	0,000000000056	0,000000000079	0,000000000110	0,000000000153
$P_{100,10}$	0,000000000010	0,000000000015	0,000000000022	0,000000000032	0,000000000046	0,000000000066	0,000000000093	0,000000000131	0,000000000182	0,000000000250
$P_{100,11}$	0,000000000018	0,000000000026	0,000000000038	0,000000000055	0,000000000078	0,000000000111	0,000000000155	0,000000000216	0,000000000298	0,000000000407
$P_{100,12}$	0,000000000031	0,000000000045	0,000000000065	0,000000000093	0,000000000131	0,000000000185	0,000000000257	0,000000000354	0,000000000485	0,000000000659
$P_{100,13}$	0,000000000053	0,000000000077	0,000000000110	0,000000000156	0,000000000219	0,000000000305	0,000000000422	0,000000000578	0,000000000786	0,000000001060
$P_{100,14}$	0,000000000091	0,000000000130	0,000000000185	0,000000000260	0,000000000363	0,000000000502	0,000000000688	0,000000000936	0,000000001264	0,000000001695
$P_{100,15}$	0,000000000154	0,000000000219	0,000000000309	0,000000000431	0,000000000597	0,000000000819	0,000000001116	0,000000001508	0,000000002023	0,000000002693
$P_{100,16}$	0,000000000260	0,000000000366	0,000000000512	0,000000000710	0,000000000976	0,000000001330	0,000000001798	0,000000002414	0,000000003216	0,000000004254
$P_{100,17}$	0,000000000435	0,000000000609	0,000000000844	0,000000001162	0,000000001585	0,000000002145	0,000000002881	0,000000003840	0,000000005081	0,000000006678
$P_{100,18}$	0,000000000723	0,000000001004	0,000000001383	0,000000001889	0,000000002558	0,000000003438	0,000000004585	0,000000006070	0,000000007980	0,000000010419
$P_{100,19}$	0,000000001195	0,000000001647	0,000000002251	0,000000003051	0,000000004103	0,000000005474	0,000000007251	0,000000009536	0,000000012454	0,000000016158
$P_{100,20}$	0,000000001961	0,000000002682	0,000000003639	0,000000004896	0,000000006537	0,000000008663	0,000000011396	0,000000014887	0,000000019317	0,000000024902
$P_{100,21}$	0,000000003197	0,000000004340	0,000000005844	0,000000007807	0,000000010349	0,000000013620	0,000000017796	0,000000023095	0,000000029775	0,000000038144
$P_{100,22}$	0,000000005176	0,000000006975	0,000000009323	0,000000012366	0,000000016278	0,000000021276	0,000000027615	0,000000035605	0,000000045611	0,000000058065
$P_{100,23}$	0,000000008325	0,000000011135	0,000000014775	0,000000019458	0,000000025438	0,000000033023	0,000000042579	0,000000054544	0,000000069431	0,000000087845
$P_{100,24}$	0,000000013299	0,000000017656	0,000000023260	0,000000030416	0,000000039492	0,000000050924	0,000000065232	0,000000083028	0,000000105029	0,000000132074
$P_{100,25}$	0,000000021099	0,000000027807	0,000000036372	0,000000047233	0,000000060910	0,000000078021	0,000000099295	0,000000125584	0,000000157881	0,000000197335
$P_{100,26}$	0,000000033245	0,000000043498	0,000000056496	0,000000072860	0,000000093326	0,000000118760	0,000000150174	0,000000188744	0,000000235832	0,000000293002
$P_{100,27}$	0,000000052024	0,000000067581	0,000000087163	0,000000111644	0,000000142054	0,000000179594	0,000000225658	0,000000281858	0,000000350042	0,000000432325
$P_{100,28}$	0,000000080850	0,000000104283	0,000000133570	0,000000169932	0,000000214796	0,000000269813	0,000000336890	0,000000418211	0,000000516269	0,000000633892
$P_{100,29}$	0,000000124779	0,000000159817	0,000000203300	0,000000256920	0,000000322634	0,000000402696	0,000000499686	0,000000616541	0,000000756590	0,000000923585
$P_{100,30}$	0,000000191241	0,000000243244	0,000000307332	0,000000385827	0,000000481390	0,000000597069	0,000000736324	0,000000903068	0,000001101706	0,000001337171
$P_{100,31}$	0,000000291064	0,000000367671	0,000000461436	0,000000575506	0,000000713476	0,000000879421	0,000001077941	0,000001314203	0,000001593982	0,000001923703
$P_{100,32}$	0,000000439898	0,000000551909	0,000000688075	0,000000852633	0,000001050381	0,000001286723	0,000001567716	0,000001900117	0,000002291426	0,000002749933
$P_{100,33}$	0,000000660181	0,000000822726	0,000001018997	0,000001254639	0,000001535996	0,000001870166	0,000002265047	0,000002729388	0,000003272839	0,000003905993
$P_{100,34}$	0,000000983806	0,000001217903	0,000001498693	0,000001833624	0,000002231004	0,000002700060	0,000003250991	0,000003895016	0,000004644425	0,000005512624
$P_{100,35}$	0,000001455733	0,000001790316	0,000002189000	0,000002661508	0,000003218611	0,000003872185	0,000004635266	0,000005522096	0,000006548170	0,000007730278
$P_{100,36}$	0,000002138797	0,000002613344	0,000003175135	0,000003836735	0,000004611961	0,000005515942	0,000006565165	0,000007777522	0,000009172348	0,000010770455
$P_{100,37}$	0,000003120058	0,000003787943	0,000004573542	0,000005492919	0,000006563612	0,000007804685	0,000009236773	0,000010882110	0,000012764565	0,000014909655
$P_{100,38}$	0,000004519090	0,000005451808	0,000006541958	0,000007809852	0,000009277509	0,000010968689	0,000012908930	0,000015125562	0,000017647720	0,000020506342
$P_{100,39}$	0,000006498670	0,000007791088	0,000009292186	0,000011027360	0,000013023937	0,000015311198	0,000017920387	0,000020884712	0,000024239327	0,000028021308
$P_{100,40}$	0,000009278399	0,000011055197	0,000013106113	0,000015462525	0,000018157955	0,000021228059	0,000024710613	0,000028645481	0,000033074572	0,000038041780
$P_{100,41}$	0,000013151849	0,000015575291	0,000018355510	0,000021530813	0,000025141816	0,000029231406	0,000033844689	0,000039028924	0,000044833440	0,000051309538

PAGE: *[0-41 / 80-89]*

τ =	80	81	82	83	84	85	86	87	88	89
$P_{100,0}$	0,000000000004	0,000000000005	0,000000000007	0,000000000011	0,000000000015	0,000000000021	0,000000000030	0,000000000042	0,000000000058	0,000000000079
$P_{100,1}$	0,000000000004	0,000000000006	0,000000000009	0,000000000012	0,000000000017	0,000000000024	0,000000000034	0,000000000047	0,000000000065	0,000000000088
$P_{100,2}$	0,000000000006	0,000000000009	0,000000000013	0,000000000018	0,000000000025	0,000000000035	0,000000000048	0,000000000066	0,000000000091	0,000000000122
$P_{100,3}$	0,000000000010	0,000000000015	0,000000000020	0,000000000029	0,000000000040	0,000000000055	0,000000000075	0,000000000103	0,000000000139	0,000000000186
$P_{100,4}$	0,000000000017	0,000000000024	0,000000000034	0,000000000047	0,000000000065	0,000000000089	0,000000000121	0,000000000163	0,000000000219	0,000000000292
$P_{100,5}$	0,000000000028	0,000000000040	0,000000000055	0,000000000077	0,000000000105	0,000000000143	0,000000000193	0,000000000260	0,000000000347	0,000000000460
$P_{100,6}$	0,000000000047	0,000000000066	0,000000000091	0,000000000125	0,000000000170	0,000000000230	0,000000000309	0,000000000413	0,000000000548	0,000000000722
$P_{100,7}$	0,000000000078	0,000000000108	0,000000000149	0,000000000203	0,000000000274	0,000000000369	0,000000000492	0,000000000653	0,000000000862	0,000000001129
$P_{100,8}$	0,000000000129	0,000000000177	0,000000000241	0,000000000327	0,000000000440	0,000000000587	0,000000000780	0,000000001029	0,000000001348	0,000000001757
$P_{100,9}$	0,000000000210	0,000000000287	0,000000000389	0,000000000524	0,000000000701	0,000000000931	0,000000001228	0,000000001610	0,000000002099	0,000000002720
$P_{100,10}$	0,000000000342	0,000000000464	0,000000000625	0,000000000836	0,000000001111	0,000000001466	0,000000001923	0,000000002507	0,000000003249	0,000000004186
$P_{100,11}$	0,000000000553	0,000000000746	0,000000000998	0,000000001326	0,000000001751	0,000000002297	0,000000002995	0,000000003882	0,000000005002	0,000000006409
$P_{100,12}$	0,000000000889	0,000000001190	0,000000001583	0,000000002091	0,000000002744	0,000000003578	0,000000004638	0,000000005977	0,000000007658	0,000000009758
$P_{100,13}$	0,000000001421	0,000000001890	0,000000002497	0,000000003278	0,000000004275	0,000000005542	0,000000007142	0,000000009151	0,000000011659	0,000000014775
$P_{100,14}$	0,000000002256	0,000000002982	0,000000003916	0,000000005108	0,000000006623	0,000000008535	0,000000010936	0,000000013932	0,000000017653	0,000000022249
$P_{100,15}$	0,000000003562	0,000000004678	0,000000006104	0,000000007915	0,000000010201	0,000000013070	0,000000016650	0,000000021093	0,000000026579	0,000000033319
$P_{100,16}$	0,000000005589	0,000000007295	0,000000009460	0,000000012193	0,000000015621	0,000000019898	0,000000025205	0,000000031754	0,000000039796	0,000000049621
$P_{100,17}$	0,000000008718	0,000000011307	0,000000014574	0,000000018672	0,000000023782	0,000000030120	0,000000037938	0,000000047533	0,000000059250	0,000000073489
$P_{100,18}$	0,000000013516	0,000000017422	0,000000022320	0,000000028426	0,000000035996	0,000000045330	0,000000056778	0,000000070751	0,000000087720	0,000000108232
$P_{100,19}$	0,000000020828	0,000000026683	0,000000033980	0,000000043021	0,000000054164	0,000000067825	0,000000084488	0,000000104711	0,000000129139	0,000000158511
$P_{100,20}$	0,000000031902	0,000000040621	0,000000051422	0,000000064726	0,000000081027	0,000000100897	0,000000124998	0,000000154090	0,000000189043	0,000000230848
$P_{100,21}$	0,000000048564	0,000000061466	0,000000077351	0,000000096803	0,000000120500	0,000000149223	0,000000183868	0,000000225461	0,000000275169	0,000000334312
$P_{100,22}$	0,000000073479	0,000000092446	0,000000115658	0,000000143918	0,000000178149	0,000000219407	0,000000268901	0,000000328001	0,000000398261	0,000000481427
$P_{100,23}$	0,000000110493	0,000000138195	0,000000171897	0,000000212691	0,000000261824	0,000000320716	0,000000390981	0,000000474439	0,000000573139	0,000000689375
$P_{100,24}$	0,000000165133	0,000000205327	0,000000253943	0,000000312451	0,000000382524	0,000000466057	0,000000565185	0,000000682307	0,000000820105	0,000000981567
$P_{100,25}$	0,000000245271	0,000000303208	0,000000372879	0,000000456252	0,000000555552	0,000000673282	0,000000812251	0,000000975590	0,000001166784	0,000001389692
$P_{100,26}$	0,000000362048	0,000000445010	0,000000544202	0,000000662237	0,000000802048	0,000000966918	0,000001160506	0,000001386871	0,000001650502	0,000001956342
$P_{100,27}$	0,000000531111	0,000000649118	0,000000789413	0,000000955430	0,000001151007	0,000001380413	0,000001648375	0,000001960109	0,000002321349	0,000002738375
$P_{100,28}$	0,000000774277	0,000000941016	0,000001138131	0,000001370107	0,000001641918	0,000001959066	0,000002327607	0,000002754186	0,000003246066	0,000003811157
$P_{100,29}$	0,000001121738	0,000001355753	0,000001630865	0,000001952870	0,000002328165	0,000002763775	0,000003267394	0,000003847414	0,000004512951	0,000005273883
$P_{100,30}$	0,000001614964	0,000001941189	0,000002322596	0,000002766613	0,000003281388	0,000003875818	0,000004559586	0,000005343192	0,000006237984	0,000007256181
$P_{100,31}$	0,000002310483	0,000002762173	0,000003287396	0,000003895585	0,000004597022	0,000005402870	0,000006325206	0,000007377051	0,000008572394	0,000009926215
$P_{100,32}$	0,000003284763	0,000003905913	0,000004624300	0,000005451789	0,000006401235	0,000007486513	0,000008722543	0,000010125318	0,000011711923	0,000013500550
$P_{100,33}$	0,000004640435	0,000005488777	0,000006464702	0,000007582994	0,000008859570	0,000010311506	0,000011957060	0,000013815683	0,000015908037	0,000018255999
$P_{100,34}$	0,000006514174	0,000007664830	0,000008981574	0,000010482643	0,000012187550	0,000014117104	0,000016293418	0,000018739918	0,000021481339	0,000024543720
$P_{100,35}$	0,000009086542	0,000010636439	0,000012400833	0,000014401991	0,000016663588	0,000019210721	0,000022069899	0,000025269036	0,000028837434	0,000032805760
$P_{100,36}$	0,000012594154	0,000014667278	0,000017015188	0,000019664782	0,000022644483	0,000025984234	0,000029715472	0,000033871099	0,000038485449	0,000043594239
$P_{100,37}$	0,000017344557	0,000020098111	0,000023200814	0,000026684797	0,000030583810	0,000034933177	0,000039769759	0,000045131900	0,000051059365	0,000057593275
$P_{100,38}$	0,000023734162	0,000027365687	0,000031437166	0,000035986551	0,000041053444	0,000046679036	0,000052906038	0,000059778600	0,000067342223	0,000075643664
$P_{100,39}$	0,000032269621	0,000037025064	0,000042330215	0,000048229358	0,000054768400	0,000061994780	0,000069957370	0,000078706361	0,000088293149	0,000098770206
$P_{100,40}$	0,000043592915	0,000049775622	0,000056639283	0,000064234914	0,000072615045	0,000081833600	0,000091945755	0,000103007800	0,000115076984	0,000128211365
$P_{100,41}$	0,000058510383	0,000066490882	0,000075307550	0,000085018364	0,000095682609	0,000107360718	0,000120114097	0,000134004951	0,000149096099	0,000165450793

PAGE:	[0-41 / 90-99]

τ =	90	91	92	93	94	95	96	97	98	99
$P_{100,0}$	0,000000000108	0,000000000146	0,000000000197	0,000000000263	0,000000000350	0,000000000462	0,000000000607	0,000000000793	0,000000001031	0,000000001333
$P_{100,1}$	0,000000000120	0,000000000161	0,000000000215	0,000000000286	0,000000000378	0,000000000497	0,000000000650	0,000000000845	0,000000001093	0,000000001406
$P_{100,2}$	0,000000000165	0,000000000220	0,000000000292	0,000000000385	0,000000000505	0,000000000660	0,000000000856	0,000000001105	0,000000001420	0,000000001816
$P_{100,3}$	0,000000000249	0,000000000330	0,000000000435	0,000000000570	0,000000000743	0,000000000963	0,000000001242	0,000000001594	0,000000002035	0,000000002585
$P_{100,4}$	0,000000000387	0,000000000510	0,000000000668	0,000000000871	0,000000001128	0,000000001454	0,000000001864	0,000000002378	0,000000003018	0,000000003813
$P_{100,5}$	0,000000000606	0,000000000794	0,000000001034	0,000000001339	0,000000001725	0,000000002211	0,000000002820	0,000000003578	0,000000004518	0,000000005678
$P_{100,6}$	0,000000000946	0,000000001232	0,000000001597	0,000000002057	0,000000002636	0,000000003360	0,000000004263	0,000000005381	0,000000006760	0,000000008455
$P_{100,7}$	0,000000001471	0,000000001906	0,000000002456	0,000000003147	0,000000004011	0,000000005087	0,000000006420	0,000000008064	0,000000010082	0,000000012548
$P_{100,8}$	0,000000002277	0,000000002933	0,000000003758	0,000000004790	0,000000006075	0,000000007665	0,000000009625	0,000000012030	0,000000014967	0,000000018540
$P_{100,9}$	0,000000003504	0,000000004490	0,000000005722	0,000000007255	0,000000009153	0,000000011491	0,000000014358	0,000000017858	0,000000022112	0,000000027261
$P_{100,10}$	0,000000005364	0,000000006836	0,000000008666	0,000000010931	0,000000013720	0,000000017138	0,000000021309	0,000000026376	0,000000032505	0,000000039887
$P_{100,11}$	0,000000008167	0,000000010352	0,000000013055	0,000000016382	0,000000020458	0,000000025429	0,000000031463	0,000000038758	0,000000047540	0,000000058069
$P_{100,12}$	0,000000012367	0,000000015593	0,000000019562	0,000000024422	0,000000030346	0,000000037533	0,000000046216	0,000000056662	0,000000069177	0,000000084112
$P_{100,13}$	0,000000018626	0,000000023361	0,000000029156	0,000000036215	0,000000044775	0,000000055110	0,000000067535	0,000000082411	0,000000100150	0,000000121221
$P_{100,14}$	0,000000027899	0,000000034809	0,000000043221	0,000000053416	0,000000065717	0,000000080496	0,000000098176	0,000000119244	0,000000144250	0,000000173817
$P_{100,15}$	0,000000041559	0,000000051585	0,000000063727	0,000000078368	0,000000095944	0,000000116958	0,000000141977	0,000000171650	0,000000206707	0,000000247970
$P_{100,16}$	0,000000061569	0,000000076031	0,000000093455	0,000000114360	0,000000139332	0,000000169043	0,000000204250	0,000000245810	0,000000294686	0,000000351957
$P_{100,17}$	0,000000090712	0,000000111450	0,000000136311	0,000000165987	0,000000201265	0,000000243036	0,000000292301	0,000000350187	0,000000417954	0,000000497006
$P_{100,18}$	0,000000132912	0,000000162477	0,000000197740	0,000000239626	0,000000289179	0,000000347571	0,000000416119	0,000000496293	0,000000589730	0,000000698248
$P_{100,19}$	0,000000193667	0,000000235568	0,000000285295	0,000000344073	0,000000413274	0,000000494437	0,000000589275	0,000000699696	0,000000827811	0,000000975952
$P_{100,20}$	0,000000280630	0,000000339662	0,000000409376	0,000000491378	0,000000587462	0,000000699629	0,000000830096	0,000000981317	0,000001155995	0,000001357104
$P_{100,21}$	0,000000404383	0,000000487058	0,000000584215	0,000000697949	0,000000830587	0,000000984711	0,000001163167	0,000001369090	0,000001605918	0,000001877411
$P_{100,22}$	0,000000579463	0,000000694561	0,000000829164	0,000000985983	0,000001168017	0,000001378568	0,000001621266	0,000001900085	0,000002219363	0,000002583820
$P_{100,23}$	0,000000825708	0,000000984985	0,000001170359	0,000001385313	0,000001633675	0,000001919646	0,000002247812	0,000002623176	0,000003051166	0,000003537666
$P_{100,24}$	0,000001170009	0,000001389097	0,000001642871	0,000001935766	0,000002272637	0,000002658778	0,000003099946	0,000003602383	0,000004172835	0,000004818574
$P_{100,25}$	0,000001648574	0,000001948113	0,000002293442	0,000002690167	0,000003144391	0,000003662739	0,000004252374	0,000004921026	0,000005677010	0,000006529240
$P_{100,26}$	0,000002309819	0,000002716865	0,000003183950	0,000003718103	0,000004326933	0,000005018658	0,000005802124	0,000006686824	0,000007682919	0,000008801253
$P_{100,27}$	0,000003218043	0,000003767809	0,000004395757	0,000005110624	0,000005921822	0,000006839461	0,000007874369	0,000009038109	0,000010342995	0,000011802106
$P_{100,28}$	0,000004458045	0,000005196020	0,000006035098	0,000006986047	0,000008060408	0,000009270512	0,000010629500	0,000012151335	0,000013850811	0,000015743565
$P_{100,29}$	0,000006140871	0,000007125384	0,000008239726	0,000009497052	0,000010911387	0,000012497641	0,000014271621	0,000016250035	0,000018450502	0,000020891551
$P_{100,30}$	0,000008410901	0,000009716182	0,000011186997	0,000012839275	0,000014689905	0,000016756752	0,000019058653	0,000021615423	0,000024447849	0,000027577682
$P_{100,31}$	0,000011454509	0,000013174293	0,000015103626	0,000017261611	0,000019668401	0,000022345193	0,000025314231	0,000028598787	0,000032223151	0,000036212611
$P_{100,32}$	0,000015510511	0,000017762245	0,000020277318	0,000023078422	0,000026189367	0,000029635068	0,000033441526	0,000037635806	0,000042246008	0,000047301238
$P_{100,33}$	0,000020882660	0,000023812321	0,000027070484	0,000030683834	0,000034680220	0,000039088625	0,000043939132	0,000049262893	0,000055092077	0,000061459828
$P_{100,34}$	0,000027954389	0,000031741944	0,000035936228	0,000040568296	0,000045670374	0,000051275821	0,000057419074	0,000064135591	0,000071461798	0,000079435016
$P_{100,35}$	0,000037206013	0,000042071486	0,000047436724	0,000053337471	0,000059810609	0,000066894102	0,000074626919	0,000083048964	0,000092200995	0,000102124540
$P_{100,36}$	0,000049234520	0,000055444617	0,000062264059	0,000069733514	0,000077894699	0,000086790306	0,000096463902	0,000106959841	0,000118323160	0,000130599478
$P_{100,37}$	0,000064776025	0,000072651205	0,000081263508	0,000090658631	0,000100883176	0,000111984540	0,000124010803	0,000137010613	0,000151033064	0,000166127578
$P_{100,38}$	0,000084730834	0,000094652682	0,000105459084	0,000117200722	0,000129928952	0,000143695677	0,000158553211	0,000174554147	0,000191751212	0,000210197132
$P_{100,39}$	0,000110190950	0,000122609604	0,000136081059	0,000150660720	0,000166404361	0,000183367969	0,000201607592	0,000221179182	0,000242138442	0,000264540670
$P_{100,40}$	0,000142469644	0,000157911001	0,000174594926	0,000192581050	0,000211928970	0,000232698079	0,000254947393	0,000278735380	0,000304119796	0,000331157512
$P_{100,41}$	0,000183132522	0,000202204827	0,000222731103	0,000244774415	0,000268397299	0,000293661583	0,000320628194	0,000349356983	0,000379906547	0,000412334059

PAGE:	[0-41 / 100-109]

τ =	100	101	102	103	104	105	106	107	108	109
$P_{100,0}$	0,000000001716	0,000000002197	0,000000002800	0,000000003552	0,000000004485	0,000000005639	0,000000007060	0,000000008802	0,000000010930	0,000000013519
$P_{100,1}$	0,000000001801	0,000000002295	0,000000002912	0,000000003678	0,000000004625	0,000000005790	0,000000007220	0,000000008965	0,000000011089	0,000000013663
$P_{100,2}$	0,000000002310	0,000000002926	0,000000003688	0,000000004630	0,000000005786	0,000000007201	0,000000008925	0,000000011019	0,000000013552	0,000000016604
$P_{100,3}$	0,000000003269	0,000000004114	0,000000005156	0,000000006433	0,000000007992	0,000000009890	0,000000012188	0,000000014963	0,000000018300	0,000000022300
$P_{100,4}$	0,000000004795	0,000000006003	0,000000007482	0,000000009285	0,000000011476	0,000000014127	0,000000017322	0,000000021159	0,000000025750	0,000000031223
$P_{100,5}$	0,000000007104	0,000000008848	0,000000010974	0,000000013553	0,000000016671	0,000000020425	0,000000024928	0,000000030309	0,000000036718	0,000000044324
$P_{100,6}$	0,000000010526	0,000000013049	0,000000016107	0,000000019801	0,000000024245	0,000000029572	0,000000035931	0,000000043498	0,000000052470	0,000000063070
$P_{100,7}$	0,000000015548	0,000000019184	0,000000023573	0,000000028848	0,000000035166	0,000000042705	0,000000051667	0,000000062283	0,000000074817	0,000000089565
$P_{100,8}$	0,000000022866	0,000000028084	0,000000034353	0,000000041855	0,000000050800	0,000000061425	0,000000074003	0,000000088840	0,000000106284	0,000000126725
$P_{100,9}$	0,000000033468	0,000000040919	0,000000049830	0,000000060447	0,000000073049	0,000000087955	0,000000105525	0,000000126164	0,000000150330	0,000000178534
$P_{100,10}$	0,000000048745	0,000000059330	0,000000071932	0,000000086880	0,000000104547	0,000000125353	0,000000149775	0,000000178346	0,000000211662	0,000000250391
$P_{100,11}$	0,000000070643	0,000000085601	0,000000103330	0,000000124268	0,000000148907	0,000000177803	0,000000211578	0,000000250929	0,000000296633	0,000000349551
$P_{100,12}$	0,000000101867	0,000000122893	0,000000147705	0,000000176879	0,000000211064	0,000000250988	0,000000297461	0,000000351387	0,000000413769	0,000000485714
$P_{100,13}$	0,000000146156	0,000000175556	0,000000210095	0,000000250533	0,000000297717	0,000000352592	0,000000416209	0,000000489732	0,000000574447	0,000000671771
$P_{100,14}$	0,000000208649	0,000000249536	0,000000297362	0,000000353118	0,000000417902	0,000000492937	0,000000579573	0,000000679301	0,000000793762	0,000000924753
$P_{100,15}$	0,000000296363	0,000000352921	0,000000418793	0,000000495262	0,000000583746	0,000000685812	0,000000803186	0,000000937765	0,000001091626	0,000001267037
$P_{100,16}$	0,000000418828	0,000000496641	0,000000586884	0,000000691206	0,000000811421	0,000000949530	0,000001107722	0,000001288394	0,000001494158	0,000001727858
$P_{100,17}$	0,000000588906	0,000000695384	0,000000818351	0,000000959911	0,000001122375	0,000001308273	0,000001520366	0,000001761661	0,000002035423	0,000002345190
$P_{100,18}$	0,000000823855	0,000000968767	0,000001135421	0,000001326485	0,000001544879	0,000001793783	0,000002076653	0,000002397237	0,000002759587	0,000003168073
$P_{100,19}$	0,000001146686	0,000001342831	0,000001567470	0,000001823966	0,000002115978	0,000002447477	0,000002822760	0,000003246464	0,000003723581	0,000004259473
$P_{100,20}$	0,000001587900	0,000001851938	0,000002153092	0,000002495569	0,000002883925	0,000003323080	0,000003818337	0,000004375393	0,000005000355	0,000005699755
$P_{100,21}$	0,000002187669	0,000002541146	0,000002942676	0,000003397479	0,000003911186	0,000004489852	0,000005139972	0,000005868493	0,000006682832	0,000007590885
$P_{100,22}$	0,000002998578	0,000003469180	0,000004001605	0,000004602288	0,000005278137	0,000006036544	0,000006885404	0,000007833127	0,000008888652	0,000010061455
$P_{100,23}$	0,000004089026	0,000004712086	0,000005414191	0,000006203211	0,000007087551	0,000008076171	0,000009178595	0,000010404924	0,000011765847	0,000013272649
$P_{100,24}$	0,000005547412	0,000006367726	0,000007288467	0,000008319183	0,000009470023	0,000010751761	0,000012175793	0,000013754158	0,000015499535	0,000017425255
$P_{100,25}$	0,000007487254	0,000008561224	0,000009761976	0,000011100995	0,000012590443	0,000014243164	0,000016072690	0,000018093248	0,000020319758	0,000022767838
$P_{100,26}$	0,000010053370	0,000011451525	0,000013008697	0,000014738595	0,000016655665	0,000018775097	0,000021112821	0,000023685509	0,000026510571	0,000029606148
$P_{100,27}$	0,000013429297	0,000015239206	0,000017247263	0,000019469689	0,000021923501	0,000024626504	0,000027597291	0,000030855231	0,000034420457	0,000038313857
$P_{100,28}$	0,000017846083	0,000020175696	0,000022750587	0,000025589782	0,000028713143	0,000032141361	0,000035895938	0,000039999172	0,000044474136	0,000049344657
$P_{100,29}$	0,000023592617	0,000026574039	0,000029857048	0,000033463754	0,000037417130	0,000041740994	0,000046459979	0,000051599515	0,000057185790	0,000063245722
$P_{100,30}$	0,000031027624	0,000034821317	0,000038983317	0,000043539073	0,000048514902	0,000053937951	0,000059836171	0,000066238268	0,000073173671	0,000080672480
$P_{100,31}$	0,000040593431	0,000045392817	0,000050638895	0,000056360665	0,000062587965	0,000069351429	0,000076682436	0,000084613059	0,000093176011	0,000102404592
$P_{100,32}$	0,000052831567	0,000058867988	0,000065442376	0,000072587432	0,000080336629	0,000088724158	0,000097784862	0,000107554176	0,000118068055	0,000129362910
$P_{100,33}$	0,000068400210	0,000075948147	0,000084139367	0,000093010331	0,000102598168	0,000112940598	0,000124075860	0,000136042633	0,000148879955	0,000162627140
$P_{100,34}$	0,000088093394	0,000097475835	0,000107621917	0,000118571812	0,000130366200	0,000143046184	0,000156653197	0,000171228915	0,000186815157	0,000203453799
$P_{100,35}$	0,000112861808	0,000124455598	0,000136949204	0,000150386319	0,000164810931	0,000180267222	0,000196799471	0,000214451942	0,000233268782	0,000253293921
$P_{100,36}$	0,000143834890	0,000158075854	0,000173369085	0,000189761437	0,000207299791	0,000226030942	0,000246001479	0,000267257676	0,000289845374	0,000313809873
$P_{100,37}$	0,000182343776	0,000199731356	0,000218339966	0,000238219074	0,000259417847	0,000281985019	0,000305968772	0,000331416612	0,000358375248	0,000386890475
$P_{100,38}$	0,000229944491	0,000251045589	0,000273552306	0,000297515962	0,000322987183	0,000350015765	0,000378650548	0,000408939286	0,000440928527	0,000474663494
$P_{100,39}$	0,000288440605	0,000313892278	0,000340948865	0,000369662538	0,000400084329	0,000432263990	0,000466249866	0,000502088763	0,000539825834	0,000579504459
$P_{100,40}$	0,000359904362	0,000390414976	0,000422742637	0,000456939127	0,000493054593	0,000531137406	0,000571234040	0,000613388948	0,000657644450	0,000704040630
$P_{100,41}$	0,000446695101	0,000483043511	0,000521431231	0,000561908163	0,000604522036	0,000649318282	0,000696339916	0,000745627430	0,000797218695	0,000851148868

PAGE: [0-41 / 110-119]

τ =	110	111	112	113	114	115	116	117	118	119
$P_{100,0}$	0,000000016657	0,000000020447	0,000000025008	0,000000030478	0,000000037015	0,000000044803	0,000000054051	0,000000064998	0,000000077917	0,000000093117
$P_{100,1}$	0,000000016771	0,000000020512	0,000000024997	0,000000030358	0,000000036743	0,000000044323	0,000000053296	0,000000063883	0,000000076338	0,000000090948
$P_{100,2}$	0,000000020270	0,000000024656	0,000000029888	0,000000036107	0,000000043477	0,000000052181	0,000000062432	0,000000074467	0,000000088556	0,000000105004
$P_{100,3}$	0,000000027075	0,000000032759	0,000000039501	0,000000047472	0,000000056868	0,000000067908	0,000000080841	0,000000095949	0,000000113546	0,000000133986
$P_{100,4}$	0,000000037726	0,000000045427	0,000000054518	0,000000065213	0,000000077759	0,000000092432	0,000000109540	0,000000129431	0,000000152494	0,000000179161
$P_{100,5}$	0,000000053319	0,000000063924	0,000000076385	0,000000090984	0,000000108033	0,000000127886	0,000000150937	0,000000177626	0,000000208442	0,000000243928
$P_{100,6}$	0,000000075555	0,000000090210	0,000000107360	0,000000127369	0,000000150641	0,000000177632	0,000000208847	0,000000244848	0,000000286255	0,000000333757
$P_{100,7}$	0,000000106862	0,000000127085	0,000000150656	0,000000178045	0,000000209780	0,000000246444	0,000000288685	0,000000337221	0,000000392841	0,000000456414
$P_{100,8}$	0,000000150602	0,000000178407	0,000000210688	0,000000248056	0,000000291187	0,000000340832	0,000000397818	0,000000463058	0,000000537554	0,000000622402
$P_{100,9}$	0,000000211349	0,000000249412	0,000000293432	0,000000344195	0,000000402570	0,000000469513	0,000000546079	0,000000633422	0,000000732805	0,000000845610
$P_{100,10}$	0,000000295273	0,000000347133	0,000000406881	0,000000475524	0,000000554168	0,000000644029	0,000000746440	0,000000862855	0,000000994860	0,000001144180
$P_{100,11}$	0,000000410639	0,000000480954	0,000000561660	0,000000654036	0,000000759487	0,000000879546	0,000001015889	0,000001170336	0,000001344867	0,000001541626
$P_{100,12}$	0,000000568448	0,000000663318	0,000000771801	0,000000895516	0,000001036230	0,000001195870	0,000001376526	0,000001580467	0,000001810147	0,000002068214
$P_{100,13}$	0,000000783263	0,000000910627	0,000001055731	0,000001220607	0,000001407467	0,000001618713	0,000001856941	0,000002124958	0,000002425786	0,000002762676
$P_{100,14}$	0,000001074245	0,000001244384	0,000001437511	0,000001656164	0,000001903094	0,000002181274	0,000002493910	0,000002844450	0,000003236595	0,000003674310
$P_{100,15}$	0,000001466472	0,000001692617	0,000001948384	0,000002236925	0,000002561636	0,000002926178	0,000003334478	0,000003790748	0,000004299489	0,000004865508
$P_{100,16}$	0,000001992578	0,000002291654	0,000002628692	0,000003007571	0,000003432464	0,000003907843	0,000004438491	0,000005029516	0,000005686359	0,000006414802
$P_{100,17}$	0,000002694782	0,000003088318	0,000003530225	0,000004025255	0,000004578492	0,000005195365	0,000005881660	0,000006643530	0,000007487504	0,000008420497
$P_{100,18}$	0,000003627397	0,000004142606	0,000004719102	0,000005362660	0,000006079434	0,000006875971	0,000007759221	0,000008736543	0,000009815719	0,000011004959
$P_{100,19}$	0,000004859885	0,000005530957	0,000006279237	0,000007111692	0,000008035719	0,000009059158	0,000010190293	0,000011437870	0,000012811095	0,000014319646
$P_{100,20}$	0,000006480561	0,000007350188	0,000008316515	0,000009387889	0,000010573137	0,000011881575	0,000013323012	0,000014907759	0,000016646631	0,000018550952
$P_{100,21}$	0,000008601043	0,000009722199	0,000010963757	0,000012335646	0,000013848321	0,000015512769	0,000017340519	0,000019343635	0,000021534728	0,000023926945
$P_{100,22}$	0,000011361561	0,000012799553	0,000014386577	0,000016134347	0,000018055153	0,000020161858	0,000022467901	0,000024987294	0,000027734623	0,000030725036
$P_{100,23}$	0,000014937214	0,000016772040	0,000018790229	0,000021005500	0,000023432183	0,000026085215	0,000028980141	0,000032133104	0,000035560838	0,000039280655
$P_{100,24}$	0,000019545299	0,000021874301	0,000024427544	0,000027220960	0,000030271123	0,000033595239	0,000037211138	0,000041137263	0,000045392652	0,000049996923
$P_{100,25}$	0,000025453796	0,000028394630	0,000031608016	0,000035112302	0,000038926495	0,000043070243	0,000047563824	0,000052428125	0,000057684615	0,000063355332
$P_{100,26}$	0,000032991101	0,000036685001	0,000040708117	0,000045081394	0,000049826434	0,000054965480	0,000060521383	0,000066517583	0,000072978072	0,000079927369
$P_{100,27}$	0,000042557050	0,000047172372	0,000052182848	0,000057612171	0,000063484671	0,000069825286	0,000076659528	0,000084013449	0,000091913601	0,000100386999
$P_{100,28}$	0,000054635287	0,000060371273	0,000066578529	0,000073283597	0,000080513612	0,000088296261	0,000096659741	0,000105632714	0,000115244262	0,000125523836
$P_{100,29}$	0,000069806919	0,000076897642	0,000084546761	0,000092783710	0,000101638439	0,000111141366	0,000121323324	0,000132215506	0,000143849414	0,000156256795
$P_{100,30}$	0,000088765422	0,000097483798	0,000106859430	0,000116924606	0,000127712023	0,000139254722	0,000151586034	0,000164739513	0,000178748870	0,000193647913
$P_{100,31}$	0,000112332622	0,000122994383	0,000134424553	0,000146658140	0,000159730414	0,000173676837	0,000188532990	0,000204334507	0,000221116996	0,000238915973
$P_{100,32}$	0,000141475528	0,000154443006	0,000168302670	0,000183092001	0,000198848553	0,000215609879	0,000233413451	0,000252296576	0,000272296323	0,000293449441
$P_{100,33}$	0,000177323697	0,000193009240	0,000209723406	0,000227505769	0,000246395749	0,000266432528	0,000287654967	0,000310101513	0,000333810122	0,000358818173
$P_{100,34}$	0,000221186674	0,000240055477	0,000260101674	0,000281366402	0,000303890379	0,000327713811	0,000352876302	0,000379416759	0,000407373314	0,000436783230
$P_{100,35}$	0,000274570962	0,000297143081	0,000321052926	0,000346342513	0,000373053134	0,000401225254	0,000430898421	0,000462111177	0,000494900968	0,000529304061
$P_{100,36}$	0,000339195817	0,000366047087	0,000394406698	0,000424316691	0,000455818037	0,000488950540	0,000523752743	0,000560261839	0,000598513591	0,000638542247
$P_{100,37}$	0,000417007063	0,000448768641	0,000482217596	0,000517394968	0,000554340353	0,000593091808	0,000633685767	0,000676156956	0,000720538315	0,000766860927
$P_{100,38}$	0,000510187971	0,000547544196	0,000586772756	0,000627912494	0,000671000414	0,000716071596	0,000763159119	0,000812293986	0,000863505058	0,000916818994
$P_{100,39}$	0,000621166143	0,000664850409	0,000710594710	0,000758434334	0,000808402332	0,000860529437	0,000914844005	0,000971371950	0,001030136699	0,001091159142
$P_{100,40}$	0,000752615238	0,000803403600	0,000856438538	0,000911750300	0,000969366492	0,001029312022	0,001091609057	0,001156276975	0,001223332340	0,001292788874
$P_{100,41}$	0,000907450317	0,000966152544	0,001027282128	0,001090862671	0,001156914756	0,001225455909	0,001296500578	0,001370060111	0,001446142753	0,001524753637

PAGE: [0-41 / 120-129]

τ =	120	121	122	123	124	125	126	127	128	129
$P_{100,0}$	0,000000110949	0,000000131810	0,000000156147	0,000000184461	0,000000217315	0,000000255339	0,000000299236	0,000000349787	0,000000407861	0,000000474419
$P_{100,1}$	0,000000108036	0,000000127968	0,000000151154	0,000000178053	0,000000209180	0,000000245108	0,000000286475	0,000000333990	0,000000388436	0,000000450682
$P_{100,2}$	0,000000124151	0,000000146381	0,000000172121	0,000000201850	0,000000236099	0,000000275459	0,000000320583	0,000000372195	0,000000431091	0,000000498150
$P_{100,3}$	0,000000157664	0,000000185021	0,000000216546	0,000000252784	0,000000294336	0,000000341867	0,000000396111	0,000000457874	0,000000528040	0,000000607579
$P_{100,4}$	0,000000209915	0,000000245288	0,000000285872	0,000000332320	0,000000385350	0,000000445753	0,000000514396	0,000000592227	0,000000680282	0,000000779692
$P_{100,5}$	0,000000284685	0,000000331376	0,000000384734	0,000000445562	0,000000514740	0,000000593233	0,000000682095	0,000000782472	0,000000895613	0,000001022870
$P_{100,6}$	0,000000388109	0,000000450145	0,000000520777	0,000000601005	0,000000691920	0,000000794712	0,000000910675	0,000001041211	0,000001187841	0,000001352208
$P_{100,7}$	0,000000528897	0,000000611334	0,000000704866	0,000000810741	0,000000930311	0,000001065048	0,000001216544	0,000001386522	0,000001576838	0,000001789494
$P_{100,8}$	0,000000718805	0,000000828071	0,000000951626	0,000001091017	0,000001247923	0,000001424158	0,000001621679	0,000001842596	0,000002089176	0,000002363851
$P_{100,9}$	0,000000973338	0,000001117623	0,000001280237	0,000001463096	0,000001668268	0,000001897984	0,000002154641	0,000002440812	0,000002759254	0,000003112913
$P_{100,10}$	0,000001312688	0,000001502408	0,000001715531	0,000001954417	0,000002221604	0,000002519818	0,000002851981	0,000003221217	0,000003630860	0,000004084463
$P_{100,11}$	0,000001762930	0,000002011279	0,000002289362	0,000002600071	0,000002946502	0,000003331970	0,000003760015	0,000004234406	0,000004759157	0,000005338527
$P_{100,12}$	0,000002357519	0,000002681127	0,000003042325	0,000003444630	0,000003891800	0,000004387840	0,000004937011	0,000005543839	0,000006213124	0,000006949942
$P_{100,13}$	0,000003139117	0,000003558842	0,000004025840	0,000004544369	0,000005118954	0,000005754408	0,000006455830	0,000007228620	0,000008078479	0,000009011424
$P_{100,14}$	0,000004161835	0,000004703688	0,000005304685	0,000005969938	0,000006704872	0,000007515226	0,000008407067	0,000009386793	0,000010461137	0,000011637180
$P_{100,15}$	0,000005493920	0,000006190164	0,000006960007	0,000007809557	0,000008745266	0,000009773940	0,000010902746	0,000012139214	0,000013491248	0,000014967123
$P_{100,16}$	0,000007220983	0,000008111399	0,000009092916	0,000010172777	0,000011358609	0,000012658427	0,000014080640	0,000015634055	0,000017327880	0,000019171727
$P_{100,17}$	0,000009449818	0,000010583177	0,000011828693	0,000013194896	0,000014690740	0,000016325597	0,000018109268	0,000020051979	0,000022164386	0,000024457572
$P_{100,18}$	0,000012312908	0,000013748650	0,000015321717	0,000017042088	0,000018920197	0,000020966928	0,000023193621	0,000025612066	0,000028234505	0,000031073627
$P_{100,19}$	0,000015973673	0,000017783807	0,000019761154	0,000021917306	0,000024264331	0,000026814778	0,000029581669	0,000032578499	0,000035819228	0,000039318274
$P_{100,20}$	0,000020632555	0,000022903781	0,000025377483	0,000028067017	0,000030986243	0,000034149516	0,000037571680	0,000041268061	0,000045254454	0,000049547113
$P_{100,21}$	0,000026533975	0,000029370040	0,000032449895	0,000035788814	0,000039402588	0,000043307511	0,000047520368	0,000052058426	0,000056939414	0,000062181507
$P_{100,22}$	0,000033974243	0,000037498505	0,000041314623	0,000045439925	0,000049892255	0,000054689958	0,000059851860	0,000065397248	0,000071345856	0,000077717836
$P_{100,23}$	0,000043310438	0,000047668622	0,000052374183	0,000057446613	0,000062905908	0,000068772542	0,000075067445	0,000081811979	0,000089027911	0,000096737385
$P_{100,24}$	0,000054970259	0,000060333382	0,000066107535	0,000072314453	0,000078976344	0,000086115853	0,000093756037	0,000101920337	0,000110632537	0,000119916738
$P_{100,25}$	0,000069462847	0,000076030245	0,000083081089	0,000090639392	0,000098729582	0,000107376472	0,000116605217	0,000126441281	0,000136910397	0,000148038529
$P_{100,26}$	0,000087390487	0,000095392891	0,000103960468	0,000113119484	0,000122896548	0,000133318565	0,000144412697	0,000156206317	0,000168726965	0,000182002303
$P_{100,27}$	0,000109461078	0,000119163650	0,000129522860	0,000140567138	0,000152325154	0,000164825768	0,000178097980	0,000192170879	0,000207073595	0,000222835244
$P_{100,28}$	0,000136501212	0,000148206432	0,000160669757	0,000173921612	0,000187992531	0,000202913101	0,000218713908	0,000235425480	0,000253078232	0,000271702407
$P_{100,29}$	0,000169469590	0,000183519871	0,000198439784	0,000214261483	0,000231017077	0,000248738560	0,000267457760	0,000287206267	0,000308015381	0,000329916051
$P_{100,30}$	0,000209470477	0,000226250359	0,000244021254	0,000262816683	0,000282669931	0,000303613983	0,000325681451	0,000348904517	0,000373314867	0,000398943628
$P_{100,31}$	0,000257766783	0,000277704535	0,000298764022	0,000320979656	0,000344385397	0,000369014679	0,000394900347	0,000422074590	0,000450568873	0,000480413879
$P_{100,32}$	0,000315792282	0,000339360725	0,000364190097	0,000390315105	0,000417769755	0,000446587289	0,000476800113	0,000508439730	0,000541536677	0,000576120465
$P_{100,33}$	0,000385162386	0,000412878746	0,000442002419	0,000472567685	0,000504607860	0,000538155228	0,000573240973	0,000609895118	0,000648146458	0,000688022511
$P_{100,34}$	0,000467682825	0,000500107389	0,000534091111	0,000569667002	0,000606866825	0,000645721033	0,000686258703	0,000728507476	0,000772493503	0,000818241397
$P_{100,35}$	0,000565355463	0,000603088845	0,000642536471	0,000683729125	0,000726696050	0,000771464888	0,000818061623	0,000866510530	0,000916834129	0,000969053142
$P_{100,36}$	0,000680380469	0,000724059258	0,000769607894	0,000817053871	0,000866422841	0,000917738569	0,000971022879	0,001026295622	0,001083574634	0,001142875709
$P_{100,37}$	0,000815153953	0,000865444568	0,000917757908	0,000972117021	0,001028542823	0,001087054058	0,001147667269	0,001210396769	0,001275254620	0,001342250617
$P_{100,38}$	0,000972260200	0,001029850777	0,001089610481	0,001151556693	0,001215704385	0,001282066100	0,001350651938	0,001421469540	0,001494524090	0,001569818310
$P_{100,39}$	0,001154457599	0,001220047788	0,001287942800	0,001358153088	0,001430686450	0,001505548032	0,001582740324	0,001662263175	0,001744113802	0,001828286813
$P_{100,40}$	0,001364657440	0,001438946035	0,001515659785	0,001594800951	0,001676368939	0,001760360320	0,001846768850	0,001935585504	0,002026798506	0,002120393374
$P_{100,41}$	0,001605894800	0,001689565190	0,001775760692	0,001864474154	0,001955695421	0,002049411379	0,002145605997	0,002244260384	0,002345352841	0,002448858927

PAGE: *[0-41 / 130-139]*

τ =	130	131	132	133	134	135	136	137	138	139
$P_{100,0}$	0,000000550527	0,000000637357	0,000000736203	0,000000848484	0,000000975757	0,000001119727	0,000001282256	0,000001465373	0,000001671287	0,000001902399
$P_{100,1}$	0,000000521685	0,000000602498	0,000000694278	0,000000798293	0,000000915931	0,000001048707	0,000001198275	0,000001366430	0,000001555126	0,000001766479
$P_{100,2}$	0,000000574336	0,000000660705	0,000000758414	0,000000868724	0,000000993013	0,000001132776	0,000001289639	0,000001465363	0,000001661855	0,000001881174
$P_{100,3}$	0,000000697550	0,000000799108	0,000000913513	0,000001042130	0,000001186445	0,000001348062	0,000001528718	0,000001730287	0,000001954787	0,000002204389
$P_{100,4}$	0,000000891684	0,000001017594	0,000001158865	0,000001317064	0,000001493878	0,000001691127	0,000001910771	0,000002154913	0,000002425809	0,000002725876
$P_{100,5}$	0,000001165712	0,000001325722	0,000001504612	0,000001704225	0,000001926541	0,000002173687	0,000002447942	0,000002751743	0,000003087691	0,000003458564
$P_{100,6}$	0,000001536085	0,000001741379	0,000001970142	0,000002224576	0,000002507036	0,000002820043	0,000003166287	0,000003548633	0,000003970131	0,000004434018
$P_{100,7}$	0,000002026638	0,000002290579	0,000002583785	0,000002908897	0,000003268732	0,000003666291	0,000004104765	0,000004587544	0,000005118219	0,000005700592
$P_{100,8}$	0,000002669227	0,000003008087	0,000003383405	0,000003798344	0,000004256270	0,000004760758	0,000005315593	0,000005924783	0,000006592561	0,000007323394
$P_{100,9}$	0,000003504936	0,000003938672	0,000004417684	0,000004945755	0,000005526894	0,000006165343	0,000006865581	0,000007632334	0,000008470577	0,000009385543
$P_{100,10}$	0,000004585806	0,000005138901	0,000005748000	0,000006417603	0,000007152461	0,000007957587	0,000008838258	0,000009800022	0,000010848701	0,000011990399
$P_{100,11}$	0,000005977033	0,000006679453	0,000007450833	0,000008296497	0,000009222049	0,000010233380	0,000011336671	0,000012538401	0,000013845348	0,000015264595
$P_{100,12}$	0,000007759659	0,000008647930	0,000009620711	0,000010684263	0,000011845155	0,000013110269	0,000014486807	0,000015982288	0,000017604557	0,000019361784
$P_{100,13}$	0,000010033785	0,000011152220	0,000012373712	0,000013705577	0,000015155469	0,000016731380	0,000018441644	0,000020294938	0,000022300283	0,000024467044
$P_{100,14}$	0,000012922348	0,000014324424	0,000015851543	0,000017512203	0,000019315262	0,000021269939	0,000023385818	0,000025672846	0,000028141328	0,000030801930
$P_{100,15}$	0,000016575495	0,000018325398	0,000020226252	0,000022287857	0,000024520397	0,000026934437	0,000029540923	0,000032351179	0,000035376898	0,000038630145
$P_{100,16}$	0,000021175611	0,000023349955	0,000025705583	0,000028253723	0,000031006002	0,000033974443	0,000037171460	0,000040609852	0,000044302797	0,000048263841
$P_{100,17}$	0,000026943049	0,000029632749	0,000032539027	0,000035674654	0,000039052809	0,000042687075	0,000046591429	0,000050780232	0,000055268222	0,000060070497
$P_{100,18}$	0,000034142562	0,000037454878	0,000041024572	0,000044866059	0,000048994169	0,000053424129	0,000058171558	0,000063252446	0,000068683148	0,000074480361
$P_{100,19}$	0,000043090502	0,000047151219	0,000051516160	0,000056201477	0,000061223721	0,000066599833	0,000072347123	0,000078483257	0,000085026235	0,000091994371
$P_{100,20}$	0,000054162739	0,000059118463	0,000064431835	0,000070120800	0,000076203686	0,000082699181	0,000089626315	0,000097004435	0,000104853183	0,000113192474
$P_{100,21}$	0,000067803313	0,000073823848	0,000080262518	0,000087139098	0,000094473704	0,000102286776	0,000110599046	0,000119431514	0,000128805421	0,000138742218
$P_{100,22}$	0,000084533741	0,000091814493	0,000099581365	0,000107855949	0,000116660127	0,000126016045	0,000135946078	0,000146472806	0,000157618971	0,000169407456
$P_{100,23}$	0,000104962894	0,000113727248	0,000123053547	0,000132965141	0,000143485602	0,000154638688	0,000166448307	0,000178938482	0,000192133314	0,000206056943
$P_{100,24}$	0,000129797321	0,000140298907	0,000151446327	0,000163264576	0,000175778782	0,000189014159	0,000202995975	0,000217749505	0,000233299993	0,000249672612
$P_{100,25}$	0,000159851825	0,000172376584	0,000185639207	0,000199666155	0,000214483909	0,000230118920	0,000246597572	0,000263946130	0,000282190703	0,000301357197
$P_{100,26}$	0,000196060063	0,000210928010	0,000226633883	0,000243205357	0,000260669989	0,000279055171	0,000298388087	0,000318695660	0,000340004506	0,000362340892
$P_{100,27}$	0,000239484881	0,000257051441	0,000275563694	0,000295050190	0,000315539210	0,000337058711	0,000359636281	0,000383299085	0,000408073819	0,000433986663
$P_{100,28}$	0,000291328022	0,000311984814	0,000333702180	0,000356509127	0,000380434217	0,000405505514	0,000431750532	0,000459196186	0,000487868741	0,000517793767
$P_{100,29}$	0,000352938809	0,000377113722	0,000402470326	0,000429037574	0,000456843783	0,000485916576	0,000516282836	0,000547968650	0,000580999268	0,000615399051
$P_{100,30}$	0,000425821306	0,000453977731	0,000483441995	0,000514242398	0,000546406393	0,000579960537	0,000614930435	0,000651340701	0,000689214906	0,000728575536
$P_{100,31}$	0,000511639443	0,000544274498	0,000578347016	0,000613883953	0,000650911202	0,000689453539	0,000729534581	0,000771176738	0,000814401178	0,000859227784
$P_{100,32}$	0,000612219518	0,000649861119	0,000689071354	0,000729875067	0,000772295808	0,000816355794	0,000862075864	0,000909475444	0,000958572513	0,001009383574
$P_{100,33}$	0,000729549456	0,000772752087	0,000817653764	0,000864276370	0,000912640267	0,000962764268	0,001014665596	0,001068359859	0,001123861021	0,001181181386
$P_{100,34}$	0,000865774180	0,000915113245	0,000966278312	0,001019287398	0,001074156780	0,001130900976	0,001189532713	0,001250062914	0,001312500680	0,001376853282
$P_{100,35}$	0,001023186456	0,001079251091	0,001137262168	0,001197232887	0,001259174509	0,001323096336	0,001389005705	0,001456907977	0,001526806534	0,001598702782
$P_{100,36}$	0,001204212573	0,001267596863	0,001333038111	0,001400543735	0,001470119029	0,001541767165	0,001615489191	0,001691284042	0,001769148549	0,001849077452
$P_{100,37}$	0,001411392278	0,001482684836	0,001556131243	0,001631732167	0,001709486009	0,001789388910	0,001871434772	0,001955615277	0,002041919917	0,002130336017
$P_{100,38}$	0,001647352470	0,001727124396	0,001809129491	0,001893360751	0,001979808795	0,002068461890	0,002159305990	0,002252324770	0,002347499671	0,002444809941
$P_{100,39}$	0,001914774231	0,002003565522	0,002094647634	0,002188005031	0,002283619742	0,002381471403	0,002481537313	0,002583792483	0,002688209698	0,002794759578
$P_{100,40}$	0,002216352961	0,002314657508	0,002415284696	0,002518209706	0,002623405277	0,002730841776	0,002840487263	0,002952307564	0,003066266343	0,003182325177
$P_{100,41}$	0,002554751521	0,002663000895	0,002773574787	0,002886438479	0,003001554873	0,003118884579	0,003238385996	0,003360015401	0,003483727038	0,003609473207

PAGE: [0-41 / 140-149]

τ =	140	141	142	143	144	145	146	147	148	149
$P_{100,0}$	0,000002161310	0,000002450839	0,000002774031	0,000003134172	0,000003534800	0,000003979725	0,000004473034	0,000005019111	0,000005622650	0,000006288671
$P_{100,1}$	0,000002002780	0,000002266505	0,000002560323	0,000002887110	0,000003249959	0,000003652191	0,000004097364	0,000004589288	0,000005132036	0,000005729955
$P_{100,2}$	0,000002125541	0,000002397347	0,000002699162	0,000003033745	0,000003404054	0,000003813252	0,000004264720	0,000004762065	0,000005309131	0,000005910006
$P_{100,3}$	0,000002481422	0,000002788387	0,000003127956	0,000003502990	0,000003916538	0,000004371852	0,000004872391	0,000005421831	0,000006024074	0,000006683254
$P_{100,4}$	0,000003057695	0,000003424025	0,000003827801	0,000004272152	0,000004760397	0,000005296064	0,000005882886	0,000006524816	0,000007226032	0,000007990940
$P_{100,5}$	0,000003867314	0,000004317083	0,000004811204	0,000005353209	0,000005946836	0,000006596037	0,000007304981	0,000008078062	0,000008919905	0,000009835372
$P_{100,6}$	0,000004943730	0,000005502903	0,000006115383	0,000006785230	0,000007516725	0,000008314375	0,000009182918	0,000010127330	0,000011152828	0,000012264876
$P_{100,7}$	0,000006338681	0,000007036726	0,000007799195	0,000008630786	0,000009536439	0,000010521333	0,000011590897	0,000012750810	0,000014007009	0,000015365686
$P_{100,8}$	0,000008121984	0,000008993278	0,000009942471	0,000010975010	0,000012096599	0,000013313202	0,000014631050	0,000016056639	0,000017596736	0,000019258380
$P_{100,9}$	0,000010382724	0,000011467877	0,000012647029	0,000013926480	0,000015312807	0,000016812863	0,000018433786	0,000020182995	0,000022068193	0,000024097368
$P_{100,10}$	0,000013231501	0,000014578681	0,000016038904	0,000017619426	0,000019327799	0,000021171869	0,000023159781	0,000025299976	0,000027601190	0,000030072456
$P_{100,11}$	0,000016803527	0,000018469841	0,000020271542	0,000022216947	0,000024314681	0,000026573683	0,000029003199	0,000031612784	0,000034412297	0,000037411901
$P_{100,12}$	0,000021262463	0,000023315417	0,000025529794	0,000027915069	0,000030481040	0,000033237826	0,000036195865	0,000039365911	0,000042759023	0,000046386570
$P_{100,13}$	0,000026804930	0,000029323991	0,000032034615	0,000034947530	0,000038073792	0,000041424787	0,000045012223	0,000048848122	0,000052944815	0,000057314933
$P_{100,14}$	0,000033665674	0,000036743933	0,000040048427	0,000043591216	0,000047384698	0,000051441594	0,000055774946	0,000060398106	0,000065324723	0,000070568738
$P_{100,15}$	0,000042123346	0,000045869281	0,000049881079	0,000054172208	0,000058756464	0,000063647964	0,000068861131	0,000074410682	0,000080311620	0,000086579211
$P_{100,16}$	0,000052506892	0,000057046209	0,000061896392	0,000067072367	0,000072589377	0,000078462964	0,000084708960	0,000091343465	0,000098382835	0,000105843663
$P_{100,17}$	0,000065202507	0,000070680041	0,000076519207	0,000082736421	0,000089348390	0,000096372095	0,000103824769	0,000111723882	0,000120087121	0,000128932368
$P_{100,18}$	0,000080661114	0,000087242745	0,000094242888	0,000101679448	0,000109570587	0,000117934696	0,000126790379	0,000136156427	0,000146051798	0,000156495589
$P_{100,19}$	0,000099406275	0,000107280833	0,000115637178	0,000124494673	0,000133872884	0,000143791555	0,000154270586	0,000165330001	0,000176989927	0,000189270563
$P_{100,20}$	0,000122042467	0,000131423546	0,000141356285	0,000151861428	0,000162959858	0,000174672566	0,000187020628	0,000200025170	0,000213707340	0,000228088280
$P_{100,21}$	0,000149263542	0,000160391181	0,000172147042	0,000184553129	0,000197631500	0,000211404243	0,000225893440	0,000241121135	0,000257109304	0,000273879817
$P_{100,22}$	0,000181861241	0,000195003378	0,000208856950	0,000223445038	0,000238790688	0,000254916875	0,000271846466	0,000289602185	0,000308206581	0,000327681991
$P_{100,23}$	0,000220733516	0,000236187142	0,000252441860	0,000269521598	0,000287450137	0,000306251069	0,000325947767	0,000346563341	0,000368120603	0,000390642032
$P_{100,24}$	0,000266892422	0,000284984327	0,000303973040	0,000323883036	0,000344738516	0,000366563365	0,000389381116	0,000413214908	0,000438087449	0,000464020979
$P_{100,25}$	0,000321471268	0,000342558284	0,000364643280	0,000387750915	0,000411905431	0,000437130612	0,000463449745	0,000490885581	0,000519460295	0,000549195453
$P_{100,26}$	0,000385730686	0,000410199313	0,000435771711	0,000462472290	0,000490324888	0,000519352729	0,000549578388	0,000581023748	0,000613709966	0,000647657439
$P_{100,27}$	0,000461063233	0,000489328535	0,000518806923	0,000549522055	0,000581496855	0,000614753466	0,000649313223	0,000685196606	0,000722423215	0,000761011730
$P_{100,28}$	0,000548996091	0,000581499755	0,000615327970	0,000650503076	0,000687046509	0,000724978754	0,000764319320	0,000805086699	0,000847298340	0,000890970622
$P_{100,29}$	0,000651191433	0,000688398873	0,000727042818	0,000767143668	0,000808720738	0,000851792223	0,000896375173	0,000942485463	0,000990137764	0,001039345522
$P_{100,30}$	0,000769443957	0,000811840371	0,000855783785	0,000901291975	0,000948381459	0,000997067471	0,001047363930	0,001099283426	0,001152837193	0,001208035101
$P_{100,31}$	0,000905675125	0,000953760417	0,001003499500	0,001054906810	0,001107995355	0,001162776695	0,001219260927	0,001277456668	0,001337371045	0,001399009686
$P_{100,32}$	0,001061923620	0,001116206118	0,001172242982	0,001230044558	0,001289619608	0,001350975300	0,001414117199	0,001479049258	0,001545773823	0,001614291626
$P_{100,33}$	0,001240331573	0,001301320504	0,001364155395	0,001428841742	0,001495383321	0,001563782187	0,001634038670	0,001706151386	0,001780117241	0,001855931444
$P_{100,34}$	0,001443126149	0,001511322864	0,001581445170	0,001653492966	0,001727464315	0,001803355458	0,001881160820	0,001960873031	0,002042482944	0,002125979655
$P_{100,35}$	0,001672596157	0,001748484127	0,001826362212	0,001906223993	0,001988061135	0,002071863406	0,002157618703	0,002245313082	0,002334930782	0,002426454264
$P_{100,36}$	0,001931063420	0,002015097073	0,002101167008	0,002189259824	0,002279360159	0,002371450720	0,002465512325	0,002561523937	0,002659462712	0,002759304040
$P_{100,37}$	0,002220848775	0,002313441292	0,002408094618	0,002504787787	0,002603497867	0,002704200008	0,002806867489	0,002911471771	0,003017982555	0,003126367831
$P_{100,38}$	0,002544232685	0,002645742917	0,002749313613	0,002854915765	0,002962518444	0,003072088856	0,003183592406	0,003296992766	0,003412251933	0,003529330302
$P_{100,39}$	0,002903410637	0,003014129352	0,003126880230	0,003241625876	0,003358327068	0,003476942827	0,003597430492	0,003719745795	0,003843842935	0,003969674659
$P_{100,40}$	0,003300443640	0,003420579372	0,003542688172	0,003666724070	0,003792639416	0,003920384964	0,004049909953	0,004181162196	0,004314088161	0,004448633062
$P_{100,41}$	0,003737204355	0,003866869172	0,003998414679	0,004131786324	0,004266928078	0,004403782523	0,004542290950	0,004682393450	0,004824029008	0,004967135591

PAGE: [0-41 / 150-159]

τ =	150	151	152	153	154	155	156	157	158	159
$P_{100,0}$	0,000007022531	0,000007829941	0,000008716984	0,000009690124	0,000010756227	0,000011922573	0,000013196871	0,000014587277	0,000016102405	0,000017751347
$P_{100,1}$	0,000006387677	0,000007110135	0,000007902568	0,000008770543	0,000009719957	0,000010757059	0,000011888452	0,000013121113	0,000014462404	0,000015920079
$P_{100,2}$	0,000006569037	0,000007290834	0,000008080285	0,000008942560	0,000009883129	0,000010907763	0,000012022549	0,000013233900	0,000014548560	0,000015973617
$P_{100,3}$	0,000007403748	0,000008190180	0,000009047435	0,000009980660	0,000010995279	0,000012096995	0,000013291800	0,000014585982	0,000015986133	0,000017499155
$P_{100,4}$	0,000008824187	0,000009730665	0,000010715514	0,000011784136	0,000012942192	0,000014195618	0,000015550620	0,000017013690	0,000018591602	0,000020291422
$P_{100,5}$	0,000010829565	0,000011907837	0,000013075791	0,000014339289	0,000015704452	0,000017177672	0,000018765608	0,000020475193	0,000022313640	0,000024288439
$P_{100,6}$	0,000013469188	0,000014771733	0,000016178740	0,000017696699	0,000019332365	0,000021092761	0,000022985180	0,000025017187	0,000027196622	0,000029531597
$P_{100,7}$	0,000016833298	0,000018416566	0,000020122478	0,000021958291	0,000023931533	0,000026050001	0,000028321767	0,000030755174	0,000033358837	0,000036141641
$P_{100,8}$	0,000021048884	0,000022975836	0,000025047099	0,000027270815	0,000029655399	0,000032209543	0,000034942211	0,000037862641	0,000040980339	0,000044305080
$P_{100,9}$	0,000026278793	0,000028621025	0,000031132906	0,000033823559	0,000036702385	0,000039779065	0,000043063552	0,000046566071	0,000050297110	0,000054267418
$P_{100,10}$	0,000032723099	0,000035562737	0,000038601275	0,000041848902	0,000045316090	0,000049013585	0,000052952402	0,000057143824	0,000061599386	0,000066330879
$P_{100,11}$	0,000040622057	0,000044053519	0,000047717330	0,000051624817	0,000055787582	0,000060217499	0,000064926700	0,000069927574	0,000075232751	0,000080855097
$P_{100,12}$	0,000050260216	0,000054391917	0,000058793912	0,000063478718	0,000068459116	0,000073748145	0,000079359091	0,000085305474	0,000091601040	0,000098259747
$P_{100,13}$	0,000061971400	0,000066927420	0,000072196470	0,000077792289	0,000083728862	0,000090020415	0,000096681395	0,000103726460	0,000111170466	0,000119028448
$P_{100,14}$	0,000076144366	0,000082066089	0,000088348640	0,000095006990	0,000102056333	0,000109512071	0,000117389801	0,000125705293	0,000134474479	0,000143713433
$P_{100,15}$	0,000093228980	0,000100276686	0,000107738312	0,000115630047	0,000123968266	0,000132769516	0,000142050494	0,000151828031	0,000162119070	0,000172940648
$P_{100,16}$	0,000113742763	0,000122097148	0,000130924016	0,000140240725	0,000150064776	0,000160413791	0,000171305491	0,000182757677	0,000194788205	0,000207414966
$P_{100,17}$	0,000138277680	0,000148141264	0,000158541462	0,000169496720	0,000181025571	0,000193146608	0,000205878460	0,000219239772	0,000233249174	0,000247925263
$P_{100,18}$	0,000167507015	0,000179105382	0,000191310065	0,000204140477	0,000217616048	0,000231756199	0,000246580310	0,000262107701	0,000278357600	0,000295349118
$P_{100,19}$	0,000202192156	0,000215774968	0,000230039256	0,000245005235	0,000260693057	0,000277122777	0,000294314328	0,000312287493	0,000331061875	0,000350656868
$P_{100,20}$	0,000243189093	0,000259030813	0,000275634377	0,000293020591	0,000311210101	0,000330223364	0,000350080618	0,000370801850	0,000392406768	0,000414914774
$P_{100,21}$	0,000291454411	0,000309854652	0,000329101907	0,000349217311	0,000370221731	0,000392135740	0,000414979584	0,000438773150	0,000463535938	0,000489287032
$P_{100,22}$	0,000348050504	0,000369333929	0,000391553760	0,000414731144	0,000438886846	0,000464041221	0,000490214178	0,000517425151	0,000545693071	0,000575036337
$P_{100,23}$	0,000414149737	0,000438665423	0,000464210355	0,000490805326	0,000518470622	0,000547225992	0,000577090620	0,000608083087	0,000640221350	0,000673522712
$P_{100,24}$	0,000491037238	0,000519157422	0,000548402159	0,000578791467	0,000610344730	0,000643080660	0,000677017274	0,000712171861	0,000748560958	0,000786200322
$P_{100,25}$	0,000580111973	0,000612230093	0,000645569339	0,000680148490	0,000715985551	0,000753097724	0,000791501379	0,000831212031	0,000872244314	0,000914611963
$P_{100,26}$	0,000682885767	0,000719413721	0,000757259216	0,000796439280	0,000836970027	0,000878866630	0,000922143302	0,000966813269	0,001012888753	0,001060380953
$P_{100,27}$	0,000800979884	0,000842344434	0,000885121135	0,000929324709	0,000974968833	0,001022066106	0,001070628039	0,001120665033	0,001172186366	0,001225200178
$P_{100,28}$	0,000936118820	0,000982757090	0,001030898440	0,001080554712	0,001131736566	0,001184453463	0,001238713651	0,001294524155	0,001351890767	0,001410818038
$P_{100,29}$	0,001090120939	0,001142474953	0,001196417219	0,001251956100	0,001309098652	0,001367850618	0,001428216419	0,001490199149	0,001553800575	0,001619021133
$P_{100,30}$	0,001264885633	0,001323395880	0,001383571530	0,001445416860	0,001508934733	0,001574126600	0,001640992494	0,001709531038	0,001779739450	0,001851613548
$P_{100,31}$	0,001462376713	0,001527474741	0,001594304874	0,001662866710	0,001733158345	0,001805176377	0,001878915919	0,001954370608	0,002031532619	0,002110392680
$P_{100,32}$	0,001684601794	0,001756701850	0,001830587725	0,001906253768	0,001983692759	0,002062895924	0,002143852958	0,002226552040	0,002310979855	0,002397121623
$P_{100,33}$	0,001933587517	0,002013077314	0,002094391039	0,002177517265	0,002262442958	0,002349153507	0,002437632743	0,002527862975	0,002619825019	0,002713498233
$P_{100,34}$	0,002211350526	0,002298581216	0,002387655704	0,002478556326	0,002571263805	0,002665757287	0,002762014378	0,002860011181	0,002959722341	0,003061121082
$P_{100,35}$	0,002519864245	0,002615139734	0,002712258071	0,002811194972	0,002911924569	0,003014419457	0,003118650738	0,003224588073	0,003332199727	0,003441452620
$P_{100,36}$	0,002861021594	0,002964587374	0,003069971764	0,003177143578	0,003286070116	0,003396717215	0,003509049310	0,003623029485	0,003738619532	0,003855780012
$P_{100,37}$	0,003236593943	0,003348625641	0,003462426147	0,003577957213	0,003695179184	0,003814051061	0,003934530564	0,004056574198	0,004180137314	0,004305174177
$P_{100,38}$	0,003648186731	0,003768778611	0,003891061933	0,004014991362	0,004140520304	0,004267600977	0,004396184483	0,004526220877	0,004657659238	0,004790447740
$P_{100,39}$	0,004097192336	0,004226346035	0,004357084601	0,004489355734	0,004623106067	0,004758281239	0,004894825976	0,005032684163	0,005171798919	0,005312112674
$P_{100,40}$	0,004584740935	0,004722354730	0,004861416393	0,005001866944	0,005143646568	0,005286694689	0,005430950052	0,005576350803	0,005722834568	0,005870338526
$P_{100,41}$	0,005111650243	0,005257509171	0,005404647836	0,005553001036	0,005702502997	0,005853087452	0,006004687724	0,006157236809	0,006310667453	0,006464912224

PAGE: [0-41 / 160-169]

τ =	160	161	162	163	164	165	166	167	168	169
$P_{100,0}$	0,000019543684	0,000021489501	0,000023599403	0,000025884530	0,000028356566	0,000031027760	0,000033910934	0,000037019495	0,000040367454	0,000043969430
$P_{100,1}$	0,000017502300	0,000019217649	0,000021075135	0,000023084211	0,000025254780	0,000027597207	0,000030122330	0,000032841471	0,000035766442	0,000038909558
$P_{100,2}$	0,000017516513	0,000019185047	0,000020987390	0,000022932090	0,000025028082	0,000027284693	0,000029711652	0,000032319098	0,000035117582	0,000038118079
$P_{100,3}$	0,000019132267	0,000020893010	0,000022789258	0,000024829217	0,000027021437	0,000029374811	0,000031898588	0,000034602370	0,000037496119	0,000040590163
$P_{100,4}$	0,000022120513	0,000024086537	0,000026197460	0,000028461555	0,000030887406	0,000033483911	0,000036260286	0,000039226062	0,000042391094	0,000045765557
$P_{100,5}$	0,000026407366	0,000028678479	0,000031110126	0,000033710941	0,000036489850	0,000039456069	0,000042619104	0,000045988752	0,000049575099	0,000053388522
$P_{100,6}$	0,000032030501	0,000034701997	0,000037555026	0,000040598801	0,000043842810	0,000047296814	0,000050970843	0,000054875195	0,000059020434	0,000063417385
$P_{100,7}$	0,000039112741	0,000042281561	0,000045657789	0,000049251377	0,000053072535	0,000057131732	0,000061439685	0,000066007363	0,000070845971	0,000075966958
$P_{100,8}$	0,000047846901	0,000051616100	0,000055623227	0,000059879087	0,000064394727	0,000069181434	0,000074250727	0,000079614352	0,000085284272	0,000091272663
$P_{100,9}$	0,000058488002	0,000062970114	0,000067725251	0,000072765144	0,000078101753	0,000083747256	0,000089714043	0,000096014704	0,000102662022	0,000109668964
$P_{100,10}$	0,000071350334	0,000076670014	0,000082302410	0,000088260229	0,000094556383	0,000101203977	0,000108216304	0,000115606829	0,000123389177	0,000131577123
$P_{100,11}$	0,000086807703	0,000093103872	0,000099757111	0,000106781115	0,000114189759	0,000121997084	0,000130217282	0,000138864682	0,000147953740	0,000157499022
$P_{100,12}$	0,000105295751	0,000112723395	0,000120557194	0,000128811821	0,000137502093	0,000146642954	0,000156249463	0,000166336774	0,000176920124	0,000188014813
$P_{100,13}$	0,000127315609	0,000136047303	0,000145239016	0,000154906355	0,000165065025	0,000175730816	0,000186919585	0,000198647234	0,000210929697	0,000223782918
$P_{100,14}$	0,000153438352	0,000163665540	0,000174411389	0,000185692361	0,000197524964	0,000209925739	0,000222911238	0,000236498000	0,000250702539	0,000265541317
$P_{100,15}$	0,000184309874	0,000196243910	0,000208759950	0,000221875198	0,000235606846	0,000249972056	0,000264987935	0,000280671516	0,000297039733	0,000314109405
$P_{100,16}$	0,000220655860	0,000234528778	0,000249051575	0,000264242052	0,000280117924	0,000296696808	0,000313996192	0,000332033414	0,000350825641	0,000370389845
$P_{100,17}$	0,000263286574	0,000279351557	0,000296138552	0,000313665764	0,000331951240	0,000351012841	0,000370868224	0,000391534811	0,000413029771	0,000435369994
$P_{100,18}$	0,000313101226	0,000331632724	0,000350962215	0,000371108085	0,000392088470	0,000413921236	0,000436623951	0,000460213864	0,000484707876	0,000510122523
$P_{100,19}$	0,000371091633	0,000392385068	0,000414555782	0,000437622067	0,000461601876	0,000486512790	0,000512372000	0,000539196281	0,000567001966	0,000595804924
$P_{100,20}$	0,000438344931	0,000462715942	0,000488046114	0,000514353338	0,000541655063	0,000569968264	0,000599309426	0,000629694514	0,000661138954	0,000693657609
$P_{100,21}$	0,000516045069	0,000543828214	0,000572654132	0,000602539960	0,000633502286	0,000665557120	0,000698719875	0,000733005340	0,000768427663	0,000805000329
$P_{100,22}$	0,000605472784	0,000637019663	0,000669693608	0,000703510617	0,000738486025	0,000774634482	0,000811969931	0,000850505591	0,000890253934	0,000931226668
$P_{100,23}$	0,000708003795	0,000743680513	0,000780568055	0,000818680852	0,000858032565	0,000898636060	0,000940503390	0,000983645778	0,001028073602	0,001073796379
$P_{100,24}$	0,000825104911	0,000865288855	0,000906765440	0,000949547084	0,000993645324	0,001039070795	0,001085833215	0,001133941375	0,001183403123	0,001234225355
$P_{100,25}$	0,000958327785	0,001003403650	0,001049850465	0,001097678164	0,001146895690	0,001197510985	0,001249530978	0,001302961578	0,001357807661	0,001414073069
$P_{100,26}$	0,001109300030	0,001159655089	0,001211454169	0,001264704232	0,001319411153	0,001375579711	0,001433213587	0,001492315354	0,001552886483	0,001614927333
$P_{100,27}$	0,001279713464	0,001335732059	0,001393260634	0,001452302691	0,001512860558	0,001574935388	0,001638527158	0,001703634671	0,001770255562	0,001838386299
$P_{100,28}$	0,001471309276	0,001533366538	0,001596990636	0,001662181130	0,001728936336	0,001797253331	0,001867127953	0,001938554819	0,002011527325	0,002086037664
$P_{100,29}$	0,001685859936	0,001754314771	0,001824382110	0,001896057115	0,001969333650	0,002044204289	0,002120660334	0,002198691824	0,002278287558	0,002359435109
$P_{100,30}$	0,001925147762	0,002000335142	0,002077167377	0,002155634804	0,002235726432	0,002317429953	0,002400731769	0,002485617010	0,002572069560	0,002660072079
$P_{100,31}$	0,002190940091	0,002273162745	0,002357047143	0,002442578426	0,002529740391	0,002618515526	0,002708885029	0,002800828844	0,002894325686	0,002989353078
$P_{100,32}$	0,002484961122	0,002574480716	0,002665661386	0,002758482757	0,002852923136	0,002948959542	0,003046567742	0,003145722287	0,003246396549	0,003348562759
$P_{100,33}$	0,002808860549	0,002905888510	0,003004557307	0,003104840822	0,003206711660	0,003310141200	0,003415099626	0,003521555979	0,003629478196	0,003738833154
$P_{100,34}$	0,003164179252	0,003268867368	0,003375154661	0,003483009119	0,003592397540	0,003703285574	0,003815637776	0,003929417652	0,004044587708	0,004161109503
$P_{100,35}$	0,003552312380	0,003664743393	0,003778708855	0,003894170828	0,004011090289	0,004129427189	0,004249140502	0,004370188285	0,004492527727	0,004616115204
$P_{100,36}$	0,003974470306	0,004094648681	0,004216272343	0,004339297502	0,004463679426	0,004589372503	0,004716330297	0,004844505609	0,004973850536	0,005104316526
$P_{100,37}$	0,004431638027	0,004559481146	0,004688654923	0,004819109914	0,004950795909	0,005083661994	0,005217656612	0,005352727628	0,005488822385	0,005625887770
$P_{100,38}$	0,004924533718	0,005059863741	0,005196383678	0,005334038766	0,005472773677	0,005612532582	0,005753259217	0,005894896946	0,006037388823	0,006180677655
$P_{100,39}$	0,005453567237	0,005596103873	0,005739663371	0,005884186112	0,006029612142	0,006175881232	0,006322932949	0,006470706717	0,006619141879	0,006768177758
$P_{100,40}$	0,006018799483	0,006168153951	0,006318338212	0,006469288392	0,006620940531	0,006773230642	0,006926094782	0,007079469112	0,007233289955	0,007387493859
$P_{100,41}$	0,006619903596	0,006775574012	0,006931855960	0,007088682041	0,007245985030	0,007403697946	0,007561754112	0,007720087210	0,007878631344	0,008037321091

PAGE: [0-41 / 170-179]

τ =	170	171	172	173	174	175	176	177	178	179
$P_{100,0}$	0,000047840667	0,000051997043	0,000056455081	0,000061231956	0,000066345510	0,000071814256	0,000077657386	0,000083894781	0,000090547016	0,000097635364
$P_{100,1}$	0,000042283641	0,000045902033	0,000049778603	0,000053927750	0,000058364416	0,000063104087	0,000068162801	0,000073557153	0,000079304300	0,000085421963
$P_{100,2}$	0,000041331990	0,000044771151	0,000048447835	0,000052374757	0,000056565082	0,000061032425	0,000065790853	0,000070854893	0,000076239531	0,000081960212
$P_{100,3}$	0,000043895197	0,000047422285	0,000051182867	0,000055188755	0,000059452142	0,000063985597	0,000068802069	0,000073914888	0,000079337762	0,000085084780
$P_{100,4}$	0,000049359948	0,000053185090	0,000057252127	0,000061572528	0,000066158084	0,000071020909	0,000076173435	0,000081628414	0,000087398914	0,000093498313
$P_{100,5}$	0,000057439685	0,000061739538	0,000066299315	0,000071130533	0,000076244983	0,000081654735	0,000087372129	0,000093409770	0,000099780528	0,000106497528
$P_{100,6}$	0,000068077134	0,000073011016	0,000078230621	0,000083747782	0,000089574569	0,000095723290	0,000102206478	0,000109036889	0,000116227490	0,000123791460
$P_{100,7}$	0,000081381997	0,000087102991	0,000093142056	0,000099511523	0,000106223922	0,000113291981	0,000120728613	0,000128546910	0,000136760132	0,000145381701
$P_{100,8}$	0,000097591903	0,000104254562	0,000111273399	0,000118661344	0,000126431497	0,000134597112	0,000143171589	0,000152168463	0,000161601392	0,000171484147
$P_{100,9}$	0,000117048666	0,000124814426	0,000132979693	0,000141558053	0,000150563219	0,000160009018	0,000169909379	0,000180278320	0,000191129938	0,000202478389
$P_{100,10}$	0,000140184578	0,000149225578	0,000158714268	0,000168664888	0,000179091766	0,000190009293	0,000201431919	0,000213374131	0,000225850444	0,000238875383
$P_{100,11}$	0,000167515187	0,000178016978	0,000189019201	0,000200536712	0,000212584401	0,000225177177	0,000238329952	0,000252057622	0,000266375055	0,000281297071
$P_{100,12}$	0,000199636190	0,000211799632	0,000224520535	0,000237814287	0,000251696257	0,000266181776	0,000281286120	0,000297024492	0,000313412002	0,000330463655
$P_{100,13}$	0,000237222836	0,000251265363	0,000265926365	0,000281221649	0,000297166935	0,000313777847	0,000331069886	0,000349058418	0,000367758650	0,000387185617
$P_{100,14}$	0,000281030727	0,000297187071	0,000314026545	0,000331565210	0,000349818982	0,000368803606	0,000388534636	0,000409027420	0,000430297078	0,000452358485
$P_{100,15}$	0,000331897208	0,000350419659	0,000369693092	0,000389733640	0,000410557209	0,000432179464	0,000454615803	0,000477881343	0,000501990896	0,000526958951
$P_{100,16}$	0,000390742781	0,000411900964	0,000433880650	0,000456697810	0,000480368113	0,000504906904	0,000530329182	0,000556649585	0,000583882367	0,000612041378
$P_{100,17}$	0,000458572068	0,000482652257	0,000507626482	0,000533510292	0,000560318853	0,000588066920	0,000616768818	0,000646438429	0,000677089165	0,000708733958
$P_{100,18}$	0,000536473946	0,000563777875	0,000592049604	0,000621303970	0,000651555335	0,000682817564	0,000715104007	0,000748427481	0,000782800254	0,000818234028
$P_{100,19}$	0,000625620539	0,000656463687	0,000688348716	0,000721289426	0,000755299049	0,000790390234	0,000826575025	0,000863864849	0,000902270497	0,000941802113
$P_{100,20}$	0,000727264759	0,000761974078	0,000797798621	0,000834750799	0,000872842366	0,000912084401	0,000952487295	0,000994060734	0,001036813687	0,001080754396
$P_{100,21}$	0,000842736141	0,000881647200	0,000921744891	0,000963039867	0,001005542031	0,001049260526	0,001094203722	0,001140379203	0,001187793758	0,001236453374
$P_{100,22}$	0,000973434724	0,001016888235	0,001061596527	0,001107568103	0,001154810633	0,001203330942	0,001253135004	0,001304227932	0,001356613970	0,001410296493
$P_{100,23}$	0,001120822752	0,001169160479	0,001218816422	0,001269796537	0,001322105867	0,001375748537	0,001430727745	0,001487045763	0,001544703931	0,001603702657
$P_{100,24}$	0,001286414006	0,001339974041	0,001394909450	0,001451223242	0,001508917444	0,001567993095	0,001628450249	0,001690287975	0,001753504357	0,001818096498
$P_{100,25}$	0,001471760606	0,001530872029	0,001591408053	0,001653368348	0,001716751541	0,001781555218	0,001847775931	0,001915409200	0,001984449522	0,002054890381
$P_{100,26}$	0,001678437159	0,001743414111	0,001809855237	0,001877756492	0,001947112742	0,002017917772	0,002090164298	0,002163843977	0,002238947416	0,002315464191
$P_{100,27}$	0,001908022191	0,001979157398	0,002051784940	0,002125896705	0,002201483468	0,002278534896	0,002357039574	0,002436985010	0,002518357661	0,002601142950
$P_{100,28}$	0,002162076835	0,002239634661	0,002318699801	0,002399259770	0,002481300957	0,002564808642	0,002649767020	0,002736159224	0,002823967343	0,002913172449
$P_{100,29}$	0,002442120842	0,002526329940	0,002612046420	0,002699253162	0,002787931928	0,002878063393	0,002969627168	0,003062601827	0,003156964938	0,003252693091
$P_{100,30}$	0,002749606031	0,002840651711	0,002933188273	0,003027193760	0,003122645132	0,003219518300	0,003317788159	0,003417428616	0,003518412628	0,003620712235
$P_{100,31}$	0,003085887379	0,003183903818	0,003283376531	0,003384278596	0,003486582065	0,003590258007	0,003695276538	0,003801606867	0,003909217325	0,004018075411
$P_{100,32}$	0,003452192047	0,003557254479	0,003663719100	0,003771553973	0,003880726220	0,003991202066	0,004102946878	0,004215925209	0,004330100839	0,004445436818
$P_{100,33}$	0,003849586717	0,003961703777	0,004075148304	0,004189883390	0,004305871293	0,004423073484	0,004541450695	0,004660962962	0,004781569671	0,004903229606
$P_{100,34}$	0,004278943694	0,004398050089	0,004518387697	0,004639914776	0,004762588883	0,004886366924	0,005011205204	0,005137059473	0,005263884977	0,005391636503
$P_{100,35}$	0,004740906335	0,004866856037	0,004993918572	0,005122047607	0,005251196263	0,005381317167	0,005512362504	0,005644284067	0,005777033307	0,005910561384
$P_{100,36}$	0,005235854435	0,005368414587	0,005501946826	0,005636400573	0,005771724881	0,005907868485	0,006044779858	0,006182407261	0,006320698793	0,006459602439
$P_{100,37}$	0,005763870267	0,005902716019	0,006042370885	0,006182780493	0,006323890299	0,006465645635	0,006607991768	0,006750873945	0,006894237446	0,007038027631
$P_{100,38}$	0,006324706058	0,006469416518	0,006614751447	0,006760653239	0,006907064323	0,007053927216	0,007201184575	0,007348779246	0,007496654309	0,007644753127
$P_{100,39}$	0,006917753717	0,007067809213	0,007218283857	0,007369117463	0,007520250101	0,007671622150	0,007823174344	0,007974847818	0,008126584154	0,008278325422
$P_{100,40}$	0,007542017651	0,007696798490	0,007851773921	0,008006881927	0,008162060976	0,008317250065	0,008472388768	0,008627417277	0,008782276443	0,008936907813
$P_{100,41}$	0,008196091556	0,008354878420	0,008513617988	0,008672247240	0,008830703865	0,008988926314	0,009146853828	0,009304426484	0,009461585225	0,009618271895

PAGE: [0-41 / 180-189]

τ =	180	181	182	183	184	185	186	187	188	189
$P_{100,0}$	0,000105181804	0,000113209023	0,000121740422	0,000130800117	0,000140412938	0,000150604435	0,000161400877	0,000172829248	0,000184917249	0,000197693292
$P_{100,1}$	0,000091928432	0,000098842568	0,000106183806	0,000113972153	0,000122228191	0,000130973080	0,000140228551	0,000150016908	0,000160361026	0,000171284347
$P_{100,2}$	0,000088032846	0,000094473805	0,000101299922	0,000108528496	0,000116177284	0,000124264506	0,000132808839	0,000141829413	0,000151345813	0,000161378070
$P_{100,3}$	0,000091170407	0,000097609487	0,000104417239	0,000111609253	0,000119201489	0,000127210274	0,000135652296	0,000144544603	0,000153904596	0,000163750021
$P_{100,4}$	0,000099940304	0,000106738882	0,000113908346	0,000121463293	0,000129418612	0,000137789480	0,000146591356	0,000155839976	0,000165551343	0,000175741726
$P_{100,5}$	0,000113574148	0,000121024011	0,000128860980	0,000137099152	0,000145752850	0,000154836617	0,000164365207	0,000174353578	0,000184816884	0,000195770467
$P_{100,6}$	0,000131742173	0,000140093199	0,000148858289	0,000158051369	0,000167686532	0,000177778029	0,000188340258	0,000199387757	0,000210935189	0,000222997341
$P_{100,7}$	0,000154425187	0,000163904301	0,000173832885	0,000184224901	0,000195094420	0,000206455610	0,000218322730	0,000230710111	0,000243632152	0,000257103306
$P_{100,8}$	0,000181830601	0,000192654714	0,000203970527	0,000215792141	0,000228133716	0,000241009447	0,000254433562	0,000268420300	0,000282983907	0,000298138615
$P_{100,9}$	0,000214337882	0,000226722662	0,000239646997	0,000253125164	0,000267171436	0,000281800067	0,000297025280	0,000312861251	0,000329322098	0,000346421862
$P_{100,10}$	0,000252463468	0,000266629199	0,000281387046	0,000296751425	0,000312736690	0,000329357118	0,000346626888	0,000364560074	0,000383170622	0,000402472344
$P_{100,11}$	0,000296838429	0,000313013810	0,000329837799	0,000347324869	0,000365489369	0,000384345503	0,000403907317	0,000424188684	0,000445203287	0,000466964604
$P_{100,12}$	0,000348194329	0,000366618761	0,000385751528	0,000405607030	0,000426199474	0,000447542858	0,000469650954	0,000492537294	0,000516215150	0,000540697524
$P_{100,13}$	0,000407354159	0,000428278906	0,000449974261	0,000472454380	0,000495733157	0,000519824206	0,000544740848	0,000570496090	0,000597102612	0,000624572755
$P_{100,14}$	0,000475226247	0,000498914693	0,000523437846	0,000548809415	0,000575042771	0,000602150936	0,000630146563	0,000659041923	0,000688848888	0,000719578918
$P_{100,15}$	0,000552799659	0,000579526810	0,000607153818	0,000635693706	0,000665159084	0,000695562139	0,000726914616	0,000759227806	0,000792512530	0,000826779126
$P_{100,16}$	0,000641140050	0,000671191380	0,000702207907	0,000734201704	0,000767184355	0,000801166947	0,000836160049	0,000872173707	0,000909217422	0,000947300147
$P_{100,17}$	0,000741385236	0,000775054913	0,000809754368	0,000845494437	0,000882285389	0,000920136922	0,000959058147	0,000999057576	0,001040143111	0,001082322035
$P_{100,18}$	0,000854739920	0,000892328454	0,000931009540	0,000970792468	0,001011685890	0,001053697812	0,001096835584	0,001141105887	0,001186514729	0,001233067437
$P_{100,19}$	0,000982469177	0,001024280496	0,001067244189	0,001111367680	0,001156657684	0,001203120202	0,001250760512	0,001299583164	0,001349591969	0,001400790001
$P_{100,20}$	0,001125890364	0,001172228341	0,001219774322	0,001268533536	0,001318510439	0,001369708709	0,001422131239	0,001475780140	0,001530656729	0,001586761538
$P_{100,21}$	0,001286363222	0,001337527660	0,001389950219	0,001443633602	0,001498579682	0,001554789497	0,001612263250	0,001671000311	0,001730999214	0,001792257663
$P_{100,22}$	0,001465277995	0,001521560094	0,001579143526	0,001638028144	0,001698212920	0,001759695947	0,001822474441	0,001886544744	0,001951902330	0,002018541812
$P_{100,23}$	0,001664041418	0,001725718758	0,001788732294	0,001853078719	0,001918753803	0,001985752405	0,002054068474	0,002123695062	0,002194624327	0,002266847549
$P_{100,24}$	0,001884060527	0,001951391602	0,002020083918	0,002090130714	0,002161524286	0,002234255991	0,002308316262	0,002383694622	0,002460379694	0,002538359214
$P_{100,25}$	0,002126724253	0,002199942620	0,002274535979	0,002350493858	0,002427804828	0,002506456518	0,002586435630	0,002667727957	0,002750318400	0,002834190986
$P_{100,26}$	0,002393382856	0,002472690961	0,002553375072	0,002635420781	0,002718812731	0,002803534633	0,002889569286	0,002976898600	0,003065503614	0,003155364523
$P_{100,27}$	0,002685325280	0,002770888064	0,002857813737	0,002946083787	0,003035678771	0,003126578344	0,003218761283	0,003312205509	0,003406888117	0,003502785400
$P_{100,28}$	0,003003754624	0,003095692981	0,003188965695	0,003283550025	0,003379422346	0,003476558179	0,003574932212	0,003674518339	0,003775289681	0,003877218624
$P_{100,29}$	0,003349761925	0,003448146165	0,003547819647	0,003648755351	0,003750925434	0,003854301265	0,003958853452	0,004064551879	0,004171365738	0,004279263564
$P_{100,30}$	0,003724298592	0,003829142006	0,003935211974	0,004042477213	0,004150905698	0,004260464703	0,004371120829	0,004482840044	0,004595587723	0,004709328678
$P_{100,31}$	0,004128147828	0,004239400520	0,004351798716	0,004465306962	0,004579889167	0,004695508639	0,004812128123	0,004929709843	0,005048215538	0,005167606500
$P_{100,32}$	0,004561895509	0,004679438629	0,004798027293	0,004917622053	0,005038182943	0,005159669519	0,005282040899	0,005405255805	0,005529272604	0,005654049346
$P_{100,33}$	0,005025900991	0,005149541537	0,005274108486	0,005399558654	0,005525848476	0,005652934049	0,005780771174	0,005909315394	0,006038522044	0,006168346282
$P_{100,34}$	0,005520268431	0,005649734774	0,005779989229	0,005910985220	0,006042675944	0,006175014412	0,006307953494	0,006441445961	0,006575444525	0,006709901880
$P_{100,35}$	0,006044819211	0,006179757504	0,006315326830	0,006451477648	0,006588160358	0,006725325341	0,006862923003	0,007000903815	0,007139218354	0,007277817343
$P_{100,36}$	0,006599066122	0,006739037747	0,006879465246	0,007020296624	0,007161480005	0,007302963668	0,007444696094	0,007586625999	0,007728702380	0,007870874546
$P_{100,37}$	0,007182189987	0,007326670176	0,007471414075	0,007616367820	0,007761477849	0,007906690943	0,008051954257	0,008197215366	0,008342422295	0,008487523554
$P_{100,38}$	0,007793019392	0,007941397163	0,008089830913	0,008238265563	0,008386646526	0,008534919739	0,008683031701	0,008830929502	0,008978560861	0,009125874150
$P_{100,39}$	0,008430014224	0,008581593732	0,008733007723	0,008884200620	0,009035117521	0,009185704233	0,009335907304	0,009485674051	0,009634952586	0,009783691842
$P_{100,40}$	0,009091253667	0,009245257054	0,009398861824	0,009552012657	0,009704655094	0,009856735565	0,010008201411	0,010159000911	0,010309083303	0,010458398802
$P_{100,41}$	0,009774429268	0,009930001081	0,010084932054	0,010239167924	0,010392655460	0,010545342488	0,010697177911	0,010848111725	0,010998095037	0,011147080077

PAGE: [0-41 / 190-199]

τ =	190	191	192	193	194	195	196	197	198	199
$P_{100,0}$	0,000211186501	0,000225426706	0,000240444435	0,000256270911	0,000272938046	0,000290478431	0,000308925328	0,000328312661	0,000348675004	0,000370047574
$P_{100,1}$	0,000182810878	0,000194965185	0,000207772389	0,000221258159	0,000235448709	0,000250370787	0,000266051669	0,000282519150	0,000299801538	0,000317927638
$P_{100,2}$	0,000171946658	0,000183072491	0,000194776916	0,000207081704	0,000220009051	0,000233581561	0,000247822247	0,000262754516	0,000278402166	0,000294789372
$P_{100,3}$	0,000174098971	0,000184969871	0,000196381479	0,000208352874	0,000220903451	0,000234052910	0,000247821253	0,000262228771	0,000277296036	0,000293043893
$P_{100,4}$	0,000186427647	0,000197625876	0,000209353426	0,000221627538	0,000234465680	0,000247885533	0,000261904983	0,000276542115	0,000291815198	0,000307742678
$P_{100,5}$	0,000207229847	0,000219210715	0,000231728921	0,000244800466	0,000258441496	0,000272668284	0,000287497229	0,000302944837	0,000319027719	0,000335762575
$P_{100,6}$	0,000235589103	0,000248725468	0,000262421513	0,000276692395	0,000291553335	0,000307019610	0,000323106541	0,000339829484	0,000357203815	0,000375244922
$P_{100,7}$	0,000271138065	0,000285750955	0,000300956519	0,000316769307	0,000333203863	0,000350274716	0,000367996367	0,000386383274	0,000405449846	0,000425210425
$P_{100,8}$	0,000313898637	0,000330278148	0,000347291275	0,000364952085	0,000383274571	0,000402272640	0,000421960101	0,000442350650	0,000463457861	0,000485295173
$P_{100,9}$	0,000364174502	0,000382593870	0,000401693709	0,000421487630	0,000441989105	0,000463211452	0,000485167820	0,000507871179	0,000531334307	0,000555569777
$P_{100,10}$	0,000422478897	0,000443203772	0,000464660278	0,000486861532	0,000509820437	0,000533549681	0,000558061712	0,000583368732	0,000609482684	0,000636415238
$P_{100,11}$	0,000489485893	0,000512780182	0,000536860245	0,000561738599	0,000587427483	0,000613938847	0,000641284339	0,000669475293	0,000698522717	0,000728437281
$P_{100,12}$	0,000565997129	0,000592126377	0,000619097364	0,000646921856	0,000675611276	0,000705176692	0,000735628802	0,000766977925	0,000799233988	0,000832406517
$P_{100,13}$	0,000652918501	0,000682151462	0,000712282866	0,000743323543	0,000775283915	0,000808173979	0,000842003301	0,000876781002	0,000912515748	0,000949215741
$P_{100,14}$	0,000751243049	0,000783851874	0,000817415537	0,000851943715	0,000887445612	0,000923929944	0,000961404929	0,000999878281	0,001039357197	0,001079848351
$P_{100,15}$	0,000862037436	0,000898296795	0,000935566020	0,000973853397	0,001013166673	0,001053513046	0,001094899157	0,001137331082	0,001180814323	0,001225353806
$P_{100,16}$	0,000986430268	0,001026615599	0,001067863371	0,001110180223	0,001153572192	0,001198044710	0,001243602593	0,001290250037	0,001337990614	0,001386827268
$P_{100,17}$	0,001125601005	0,001169986037	0,001215482507	0,001262095138	0,001309827997	0,001358684490	0,001408667357	0,001459778669	0,001512019825	0,001565391552
$P_{100,18}$	0,001280768646	0,001329622298	0,001379631634	0,001430799193	0,001483126805	0,001536615593	0,001591265968	0,001647077631	0,001704049570	0,001762180066
$P_{100,19}$	0,001453179590	0,001506762318	0,001561539019	0,001617509776	0,001674673924	0,001733030048	0,001792575986	0,001853308830	0,001915224930	0,001978319901
$P_{100,20}$	0,001644094303	0,001702653972	0,001762438704	0,001823445868	0,001885672053	0,001949113065	0,002013763937	0,002079618930	0,002146671546	0,002214914529
$P_{100,21}$	0,001854772531	0,001918539868	0,001983554903	0,002049812051	0,002117304919	0,002186026313	0,002255968250	0,002327121963	0,002399477910	0,002473025792
$P_{100,22}$	0,002086456944	0,002155640635	0,002226084953	0,002297781137	0,002370719607	0,002444889974	0,002520281052	0,002596880872	0,002674676696	0,002753655029
$P_{100,23}$	0,002340355137	0,002415136643	0,002491180773	0,002568475398	0,002647007574	0,002726763554	0,002807728802	0,002889888009	0,002973225115	0,003057723321
$P_{100,24}$	0,002617620052	0,002698148220	0,002779928895	0,002862946432	0,002947184384	0,003032625520	0,003119251843	0,003207044614	0,003295984367	0,003386050931
$P_{100,25}$	0,002919328888	0,003005714443	0,003093329174	0,003182153811	0,003272168312	0,003363351884	0,003455683009	0,003549139464	0,003643698345	0,003739336094
$P_{100,26}$	0,003246460698	0,003338770711	0,003432272358	0,003526942687	0,003622758017	0,003719693970	0,003817725493	0,003916826885	0,004016971823	0,004118133392
$P_{100,27}$	0,003599872876	0,003698125319	0,003797516780	0,003898020622	0,003999609543	0,004102255607	0,004205930273	0,004310604424	0,004416248394	0,004522832000
$P_{100,28}$	0,003980276845	0,004084435341	0,004189664465	0,004295933955	0,004403212963	0,004511470087	0,004620673406	0,004730790505	0,004841788511	0,004953634123
$P_{100,29}$	0,004388213266	0,004498182162	0,004609137014	0,004721044059	0,004833869043	0,004947577255	0,005062133562	0,005177502438	0,005293647998	0,005410534034
$P_{100,30}$	0,004824027196	0,004939647078	0,005056151671	0,005173503904	0,005291666327	0,005410601139	0,005530270230	0,005650635211	0,005771657446	0,005893298091
$P_{100,31}$	0,005287843613	0,005408887393	0,005530698021	0,005653235381	0,005776459098	0,005900328574	0,006024803021	0,006149841495	0,006275402936	0,006401446192
$P_{100,32}$	0,005779543804	0,005905713511	0,006032515803	0,006159907849	0,006287846695	0,006416289295	0,006545192548	0,006674513332	0,006804208539	0,006934235106
$P_{100,33}$	0,006298743134	0,006429667531	0,006561074347	0,006692918439	0,006825154677	0,006957737985	0,007090623376	0,007223765979	0,007357121081	0,007490644149
$P_{100,34}$	0,006844770740	0,006980003877	0,007115554161	0,007251374593	0,007387418341	0,007523638776	0,007659989501	0,007796424388	0,007932897604	0,008069363644
$P_{100,35}$	0,007416651687	0,007555672510	0,007694831190	0,007834079396	0,007973369118	0,008112652700	0,008251882872	0,008391012778	0,008529996005	0,008668786611
$P_{100,36}$	0,008013092154	0,008155305248	0,008297464286	0,008439520175	0,008581424302	0,008723128561	0,008864585381	0,009005747754	0,009146569260	0,009287004088
$P_{100,37}$	0,008632468172	0,008777205725	0,008921686370	0,009065860868	0,009209680617	0,009353097672	0,009496064771	0,009638535361	0,009780463613	0,009921804447
$P_{100,38}$	0,009272818428	0,009419343463	0,009565399762	0,009710938592	0,009855912005	0,010000272856	0,010143974827	0,010286972443	0,010429221089	0,010570677025
$P_{100,39}$	0,009931841600	0,010079352507	0,010226176098	0,010372264818	0,010517572036	0,010662052065	0,010805660174	0,010948352604	0,011090086576	0,011230820310
$P_{100,40}$	0,010606898623	0,010754534993	0,010901261172	0,011047031461	0,011191801220	0,011335526876	0,011478165933	0,011619676980	0,011760019700	0,011899154875
$P_{100,41}$	0,011295020213	0,011441869960	0,011587584993	0,011732122151	0,011875439448	0,012017496073	0,012158252402	0,012297669993	0,012435711592	0,012572341132

PAGE: [0-41 / 200-218]

τ =	200	202	204	206	208	210	212	214	216	218
$P_{100,0}$	0,000392466214	0,000440588143	0,000493339602	0,000551028553	0,000613971587	0,000682493364	0,000756926001	0,000837608424	0,000924885685	0,001019108246
$P_{100,1}$	0,000336926748	0,000377663571	0,000422255752	0,000470954110	0,000524016096	0,000581705324	0,000644291055	0,000712047668	0,000785254091	0,000864193215
$P_{100,2}$	0,000311940678	0,000348635551	0,000388690121	0,000432312922	0,000479717296	0,000531120970	0,000586745620	0,000646816403	0,000711561479	0,000781211508
$P_{100,3}$	0,000309493448	0,000344583328	0,000382739374	0,000424138797	0,000468961951	0,000517391942	0,000569614234	0,000625816231	0,000686186853	0,000750916102
$P_{100,4}$	0,000324343168	0,000359638403	0,000397852739	0,000439139991	0,000483655597	0,000531556247	0,000582999510	0,000638143452	0,000697146251	0,000760165816
$P_{100,5}$	0,000353166184	0,000390047115	0,000429805936	0,000472578649	0,000518501478	0,000567710512	0,000620341349	0,000676528737	0,000736406223	0,000800105806
$P_{100,6}$	0,000393968193	0,000433522707	0,000475990029	0,000521492108	0,000570149822	0,000622082621	0,000677408191	0,000736242113	0,000798697542	0,000864884886
$P_{100,7}$	0,000445679280	0,000488798442	0,000534919530	0,000584152805	0,000636606239	0,000692385176	0,000751592002	0,000814325832	0,000880682211	0,000950752829
$P_{100,8}$	0,000507875876	0,000555319817	0,000605892622	0,000659694145	0,000716820805	0,000777365260	0,000841416099	0,000909057553	0,000980369221	0,001055425826
$P_{100,9}$	0,000580589945	0,000633032648	0,000688756495	0,000747851380	0,000810402679	0,000876490945	0,000946191620	0,001019574776	0,001096704880	0,001177640581
$P_{100,10}$	0,000664177779	0,000722236879	0,000783744968	0,000848781794	0,000917421565	0,000989732661	0,001065777385	0,001145611737	0,001229285218	0,001316840664
$P_{100,11}$	0,000759229303	0,000823485192	0,000891365555	0,000962939330	0,001038268942	0,001117410056	0,001200411367	0,001287314414	0,001378153432	0,001472955236
$P_{100,12}$	0,000866504624	0,000937511895	0,001012320082	0,001090986281	0,001173560173	0,001260083816	0,001350591483	0,001445109526	0,001543656286	0,001646242025
$P_{100,13}$	0,000986888708	0,001065182061	0,001147447847	0,001233730039	0,001324064358	0,001418478123	0,001516990139	0,001619610624	0,001726341176	0,001837174768
$P_{100,14}$	0,001121357887	0,001207453975	0,001297683730	0,001392076544	0,001490652805	0,001593423804	0,001700391697	0,001811549489	0,001926881067	0,002046361273
$P_{100,15}$	0,001270953869	0,001365350144	0,001464026003	0,001566994696	0,001674259816	0,001785815274	0,001901645328	0,002021724644	0,002146018399	0,002274482422
$P_{100,16}$	0,001436762308	0,001539933608	0,001647510256	0,001759487775	0,001875851480	0,001996576541	0,002121628084	0,002250961329	0,002384521775	0,002522245406
$P_{100,17}$	0,001619893903	0,001732287313	0,001849187000	0,001970569192	0,002096399493	0,002226633027	0,002361214625	0,002500079042	0,002643151219	0,002790346573
$P_{100,18}$	0,001821466691	0,001943495086	0,002070101291	0,002201240754	0,002336858033	0,002476887020	0,002621251213	0,002769864018	0,002922629096	0,003079440720
$P_{100,19}$	0,002042588623	0,002174623209	0,002311273310	0,002452472259	0,002598142379	0,002748195311	0,002902532360	0,003061044896	0,003223614763	0,003390114734
$P_{100,20}$	0,002284339876	0,002426701969	0,002573679208	0,002725181653	0,002881108419	0,003041348083	0,003205779131	0,003374270430	0,003546681726	0,003722864163
$P_{100,21}$	0,002547754556	0,002700706839	0,002858231804	0,003020215325	0,003186532572	0,003357048514	0,003531618451	0,003710088562	0,003892296480	0,004078071879
$P_{100,22}$	0,002833801634	0,002997539099	0,003165760968	0,003338328291	0,003515091856	0,003695892777	0,003880563103	0,004068926447	0,004260798622	0,004455988299
$P_{100,23}$	0,003143365110	0,003318005878	0,003496993613	0,003680164233	0,003867344033	0,004058350357	0,004252992277	0,004451071291	0,004652382025	0,004856712931
$P_{100,24}$	0,003477223454	0,003662799683	0,003852533403	0,004046235451	0,004243707884	0,004444744720	0,004649132689	0,004856651987	0,005067077025	0,005280177176
$P_{100,25}$	0,003836028516	0,004032477594	0,004232840319	0,004436902889	0,004644443762	0,004855234468	0,005069040407	0,005285621650	0,005504733733	0,005726128433
$P_{100,26}$	0,004220284106	0,004427440348	0,004638210328	0,004852356463	0,005069634668	0,005289795214	0,005512583572	0,005737741253	0,005965006617	0,006194115678
$P_{100,27}$	0,004630324566	0,004847911612	0,005068755448	0,005292595998	0,005519168106	0,005748202425	0,005979426300	0,006212564610	0,006447340608	0,006683476716
$P_{100,28}$	0,005066293640	0,005293917783	0,005524384540	0,005757413076	0,005992719060	0,006230015581	0,006469014036	0,006709424984	0,006950958974	0,007193327338
$P_{100,29}$	0,005528124042	0,005765268679	0,006004785201	0,006246374185	0,006489734444	0,006734563949	0,006980560709	0,007227423618	0,007474853259	0,007722552673
$P_{100,30}$	0,006015518122	0,006261539541	0,006509407147	0,006758805532	0,007009419375	0,007260934353	0,007513038000	0,007765420525	0,008017775577	0,008269800977
$P_{100,31}$	0,006527930062	0,006782054751	0,007037447498	0,007293779916	0,007550725655	0,007807961273	0,008065167061	0,008322027811	0,008578233535	0,008833480133
$P_{100,32}$	0,007064550050	0,007325873706	0,007587838366	0,007850106051	0,008112342829	0,008374219628	0,008635413005	0,008895605857	0,009154488063	0,009411757089
$P_{100,33}$	0,007624290869	0,007891779261	0,008159237156	0,008426320714	0,008692692162	0,008958020556	0,009221982463	0,009484262594	0,009744554374	0,010002560454
$P_{100,34}$	0,008205777359	0,008478269163	0,008750019966	0,009020684070	0,009289923867	0,009557410498	0,009822824459	0,010085856134	0,010346206272	0,010603586400
$P_{100,35}$	0,008807339149	0,009083550864	0,009358278423	0,009631178511	0,009901917862	0,010170173821	0,010435634841	0,010698000913	0,010956983937	0,011212308025
$P_{100,36}$	0,009427007064	0,009705540056	0,009981820301	0,010255511262	0,010526288318	0,010793839212	0,011057864430	0,011318077515	0,011574205321	0,011825988206
$P_{100,37}$	0,010062513551	0,010341863248	0,010618174161	0,010891121011	0,011160392167	0,011425689973	0,011686730997	0,011943246222	0,012194981172	0,012441695992
$P_{100,38}$	0,010711297403	0,010989864620	0,011264598234	0,011535188703	0,011801341702	0,012062778311	0,012319235127	0,012570464329	0,012816233674	0,013056326452
$P_{100,39}$	0,011370513025	0,011646617350	0,011918093678	0,012184652611	0,012446021331	0,012701943639	0,012952179938	0,013196507156	0,013434718620	0,013666623880
$P_{100,40}$	0,012037044388	0,012308939502	0,012575422275	0,012836227694	0,013091108448	0,013339834829	0,013582194569	0,013817992632	0,014047050958	0,014269208167
$P_{100,41}$	0,012707523733	0,012973414519	0,013233128546	0,013486429176	0,013733098338	0,013972936279	0,014205761266	0,014431409249	0,014649733473	0,014860604077

PAGE: ***[0-41 / 220-238]***

τ =	220	222	224	226	228	230	232	234	236	238
$P_{100,0}$	0,001120631231	0,001229813654	0,001347017625	0,001472607536	0,001606949238	0,001750409196	0,001903353650	0,002066147762	0,002239154772	0,002422735150
$P_{100,1}$	0,000949151277	0,001040417237	0,001138282130	0,001243038405	0,001354979262	0,001474397970	0,001601587197	0,001736838322	0,001880440762	0,002032681298
$P_{100,2}$	0,000855999141	0,000936158494	0,001021924613	0,001113532936	0,001211218747	0,001315216627	0,001425759912	0,001543080141	0,001667406526	0,001798965410
$P_{100,3}$	0,000820194622	0,000894213254	0,000973162587	0,001057232507	0,001146611752	0,001241487465	0,001342044755	0,001448466261	0,001560931725	0,001679617579
$P_{100,4}$	0,000827359395	0,000898883194	0,000974891996	0,001055538786	0,001140974378	0,001231347056	0,001326802222	0,001427482051	0,001533525159	0,001645066289
$P_{100,5}$	0,000867757593	0,000939489465	0,001015426757	0,001095691936	0,001180404303	0,001269679702	0,001363630240	0,001462364022	0,001565984907	0,001674592270
$P_{100,6}$	0,000934911510	0,001008881444	0,001086895104	0,001169049040	0,001255435683	0,001346143124	0,001441254898	0,001540849794	0,001645001678	0,001753779336
$P_{100,7}$	0,001024625260	0,001102382714	0,001184103803	0,001269862343	0,001359727153	0,001453761898	0,001552024928	0,001654569160	0,001761441964	0,001872685074
$P_{100,8}$	0,001134296986	0,001217047011	0,001303734724	0,001394413300	0,001489130132	0,001587926722	0,001690838588	0,001797895199	0,001909119930	0,002024530042
$P_{100,9}$	0,001262434529	0,001351133214	0,001443776840	0,001540399212	0,001641027660	0,001745682987	0,001854379432	0,001967124671	0,002083919831	0,002204759527
$P_{100,10}$	0,001408314107	0,001503734670	0,001603124484	0,001706498640	0,001813865166	0,001925225029	0,002040572163	0,002159893527	0,002283169183	0,002410372393
$P_{100,11}$	0,001571739135	0,001674516876	0,001781292628	0,001892062979	0,002006816981	0,002125536209	0,002248194848	0,002374759814	0,002505190890	0,002639440886
$P_{100,12}$	0,001752868903	0,001863530984	0,001978214268	0,002096896764	0,002219548586	0,002346132075	0,002476601954	0,002610905499	0,002748982743	0,002890766690
$P_{100,13}$	0,001952095790	0,002071080112	0,002194095188	0,002321100188	0,002452046156	0,002586876197	0,002725525693	0,002867922538	0,003013987392	0,003163633961
$P_{100,14}$	0,002169955998	0,002297622324	0,002429308688	0,002564955083	0,002704493280	0,002847847079	0,002994932585	0,003145658500	0,003299926441	0,003457631270
$P_{100,15}$	0,002407063367	0,002543698918	0,002684318025	0,002828841169	0,002977180654	0,003129240916	0,003284918862	0,003444104220	0,003606679907	0,003772522420
$P_{100,16}$	0,002664058948	0,002809880136	0,002959618029	0,003113173337	0,003270438773	0,003431299435	0,003595633189	0,003763311087	0,003934197780	0,004108151954
$P_{100,17}$	0,002941571317	0,003096722813	0,003255689946	0,003418353521	0,003584586676	0,003754255322	0,003927218584	0,004103329264	0,004282434309	0,004464375284
$P_{100,18}$	0,003240184185	0,003404736215	0,003572965417	0,003744732731	0,003919891913	0,004098290019	0,004279767903	0,004464160723	0,004651298452	0,004841006388
$P_{100,19}$	0,003560408971	0,003734353522	0,003911796817	0,004092580198	0,004276538437	0,004463500282	0,004653288996	0,004845722901	0,005040615923	0,005237778139
$P_{100,20}$	0,003902660820	0,004085907263	0,004272432111	0,004462057605	0,004654600186	0,004849871076	0,005047676857	0,005247820050	0,005450099684	0,005654311866
$P_{100,21}$	0,004267237075	0,004459607636	0,004654992996	0,004853197073	0,005054018887	0,005257253175	0,005462691001	0,005670120355	0,005879326746	0,006090093782
$P_{100,22}$	0,004654297654	0,004855523036	0,005059455618	0,005265882057	0,005474585142	0,005685344433	0,005897936892	0,006112137498	0,006327719849	0,006544456748
$P_{100,23}$	0,005063846999	0,005273562448	0,005485633421	0,005699830672	0,005915922229	0,006133674056	0,006352850689	0,006573215861	0,006794533097	0,007016566299
$P_{100,24}$	0,005495717517	0,005713459556	0,005933161948	0,006154581193	0,006377472317	0,006601589535	0,006826686886	0,007052518853	0,007278840950	0,007505410285
$P_{100,25}$	0,005949554532	0,006174758565	0,006401485543	0,006629479661	0,006858484973	0,007088246048	0,007318508600	0,007549020080	0,007779530250	0,008009791723
$P_{100,26}$	0,006424802874	0,006656801819	0,006889846027	0,007123669611	0,007358007943	0,007592598294	0,007827180438	0,008061497217	0,008295295085	0,008528324611
$P_{100,27}$	0,006920695298	0,007158719404	0,007397273478	0,007636084033	0,007874880289	0,008113394781	0,008351363926	0,008588528554	0,008824634405	0,009059432589
$P_{100,28}$	0,007436242941	0,007679420905	0,007922579284	0,008165439710	0,008407727990	0,008649174670	0,008889515561	0,009128492215	0,009365852375	0,009601350378
$P_{100,29}$	0,007970228077	0,008217589549	0,008464351664	0,008710234089	0,008954962134	0,009198267259	0,009439887541	0,009679568101	0,009917061480	0,010152127989
$P_{100,30}$	0,008521199387	0,008771678943	0,009020953837	0,009268744853	0,009514779852	0,009758794221	0,010000531273	0,010239742597	0,010476188378	0,010709637669
$P_{100,31}$	0,009087470006	0,009339912622	0,009590525034	0,009839032338	0,010085168095	0,010328674698	0,010569303700	0,010806816090	0,011040982535	0,011271583577
$P_{100,32}$	0,009667118526	0,009920286574	0,010170984483	0,010418944933	0,010663910371	0,010905633296	0,011143876507	0,011378413295	0,011609027605	0,011835514148
$P_{100,33}$	0,010257993160	0,010510574901	0,010760038507	0,011006127529	0,011248596481	0,011487211041	0,011721748203	0,011951996392	0,012177755530	0,012398837075
$P_{100,34}$	0,010857719183	0,011108338721	0,011355190801	0,011598033094	0,011836635305	0,012070779277	0,012300259051	0,012524880893	0,012744463272	0,012958836809
$P_{100,35}$	0,011463709757	0,011710938374	0,011953755925	0,012191937360	0,012425270582	0,012653556455	0,012876608769	0,013094254170	0,013306332054	0,013512694436
$P_{100,36}$	0,012073180168	0,012315548931	0,012552875984	0,012784956571	0,013011599633	0,013232627726	0,013447876885	0,013657196467	0,013860448957	0,014057509750
$P_{100,37}$	0,012683165461	0,012919178966	0,013149540433	0,013374068204	0,013592594885	0,013804967161	0,014011045566	0,014210704241	0,014403830653	0,014590325294
$P_{100,38}$	0,013290541382	0,013518692475	0,013740608843	0,013956134483	0,014165128022	0,014367462429	0,014563024709	0,014751715562	0,014933449029	0,015108152119
$P_{100,39}$	0,013892048492	0,014110833763	0,014322836458	0,014527928479	0,014725996513	0,014916941656	0,015100679020	0,015277137315	0,015446258422	0,015607996946
$P_{100,40}$	0,014484319226	0,014692255085	0,014892902283	0,015086162526	0,015271952252	0,015450202165	0,015620856768	0,015783873873	0,015939224107	0,016086890406
$P_{100,41}$	0,015063907640	0,015259546723	0,015447439372	0,015627518617	0,015799731944	0,015964040763	0,016120419867	0,016268856881	0,016409351710	0,016541915983

PAGE: *[0-41 / 240-258]*

τ =	240	242	244	246	248	250	252	254	256	258
$P_{100,0}$	0,002617245756	0,002823039020	0,003040462121	0,003269856191	0,003511555538	0,003765886889	0,004033168652	0,004313710217	0,004607811273	0,004915761160
$P_{100,1}$	0,002193843409	0,002364206615	0,002544045833	0,002733630752	0,002933225212	0,003143086619	0,003363465364	0,003594604275	0,003836738089	0,004090092948
$P_{100,2}$	0,001937979748	0,002084668594	0,002239246597	0,002401923517	0,002572903751	0,002752385882	0,002940562241	0,003137618492	0,003343733236	0,003559077636
$P_{100,3}$	0,001804696536	0,001936337198	0,002074703674	0,002219955219	0,002372245878	0,002531724158	0,002698532711	0,002872808034	0,003054680194	0,003244272562
$P_{100,4}$	0,001762236002	0,001885160386	0,002013960782	0,002148753520	0,002289649678	0,002436754851	0,002590168945	0,002749985975	0,002916293896	0,003089174442
$P_{100,5}$	0,001788280785	0,001907140225	0,002031255272	0,002160705352	0,002295564484	0,002435901140	0,002581778128	0,002733252492	0,002890375427	0,003053192204
$P_{100,6}$	0,001867246329	0,001985460876	0,002108475750	0,002236338183	0,002369089805	0,002506766586	0,002649398801	0,002797011011	0,002949622054	0,003107245063
$P_{100,7}$	0,001988334522	0,002108420585	0,002232967755	0,002361994727	0,002495514401	0,002633533907	0,002776054638	0,002923072307	0,003074577013	0,003230553322
$P_{100,8}$	0,002144136672	0,002267944860	0,002395953578	0,002528155789	0,002664538519	0,002805082944	0,002949764499	0,003098552995	0,003251412748	0,003408302730
$P_{100,9}$	0,002329631931	0,002458518844	0,002591395804	0,002728232201	0,002868991411	0,003013630953	0,003162102649	0,003314352806	0,003470322407	0,003629947314
$P_{100,10}$	0,002541469746	0,002676421297	0,002815180728	0,002957695530	0,003103907193	0,003253751417	0,003407158331	0,003564052729	0,003724354315	0,003887977952
$P_{100,11}$	0,002777455821	0,002919175125	0,003064531855	0,003213452931	0,003365859385	0,003521666621	0,003680784688	0,003843118562	0,004008568437	0,004177030023
$P_{100,12}$	0,003036183558	0,003185153034	0,003337588547	0,003493397554	0,003652481839	0,003814737822	0,003980056882	0,004148325674	0,004319426471	0,004493237489
$P_{100,13}$	0,003316769292	0,003473294077	0,003633102984	0,003796084988	0,003962123716	0,004131097798	0,004302881233	0,004477343747	0,004654351160	0,004833765758
$P_{100,14}$	0,003618661439	0,003782899355	0,003950221743	0,004120500030	0,004293600733	0,004469385843	0,004647713227	0,004828437021	0,005011408028	0,005196474110
$P_{100,15}$	0,003941502221	0,004113484153	0,004288327844	0,004465888130	0,004646015479	0,004828556408	0,005013353917	0,005200247903	0,005389075585	0,005579671917
$P_{100,16}$	0,004285026768	0,004464670302	0,004646926005	0,004831633152	0,005018627289	0,005207740690	0,005398802803	0,005591640692	0,005786079473	0,005981942742
$P_{100,17}$	0,004648988852	0,004836107257	0,005025558804	0,005217168338	0,005410757724	0,005606146313	0,005803151413	0,006001588735	0,006201272848	0,006402017609
$P_{100,18}$	0,005033105665	0,005227413770	0,005423745041	0,005621911174	0,005821721712	0,006022984533	0,006225506318	0,006429093017	0,006633550298	0,006838683976
$P_{100,19}$	0,005437016310	0,005638134414	0,005840934171	0,006045215559	0,006250777311	0,006457417408	0,006664933553	0,006873123634	0,007081786162	0,007290720705
$P_{100,20}$	0,005860250332	0,006067706999	0,006276472491	0,006486336664	0,006697089106	0,006908519626	0,007120418722	0,007332578033	0,007544790770	0,007756852131
$P_{100,21}$	0,006302203736	0,006515438095	0,006729578094	0,006944405235	0,007159701778	0,007375251223	0,007590838763	0,007806251720	0,008021279955	0,008235716262
$P_{100,22}$	0,006762120766	0,006980484789	0,007199322547	0,007418409114	0,007637521391	0,007856438566	0,008074942545	0,008292818363	0,008509854574	0,008725843606
$P_{100,23}$	0,007239080300	0,007461841398	0,007684617866	0,007907180432	0,008129302740	0,008350761779	0,008571338292	0,008790817150	0,009008987708	0,009225644127
$P_{100,24}$	0,007731986103	0,007958330293	0,008184207871	0,008409387435	0,008633641590	0,008856747344	0,009078486480	0,009298645892	0,009517017904	0,009733400545
$P_{100,25}$	0,008239560467	0,008468596291	0,008696663284	0,008923530240	0,009148971036	0,009372764995	0,009594697204	0,009814558815	0,010032147310	0,010247266737
$P_{100,26}$	0,008760340946	0,008991104263	0,009220380163	0,009447940044	0,009673561437	0,009897028319	0,010118131383	0,010336668287	0,010552443865	0,010765270321
$P_{100,27}$	0,009292680007	0,009524139743	0,009753581412	0,009980781481	0,010205523553	0,010427598621	0,010646805285	0,010862949944	0,011075846956	0,011285318772
$P_{100,28}$	0,009834747522	0,010065812397	0,010294321180	0,010520057895	0,010742814635	0,010962391757	0,011178598040	0,011391250822	0,011600176093	0,011805208573
$P_{100,29}$	0,010384536010	0,010614062263	0,010840492035	0,011063619375	0,011283247254	0,011499187697	0,011711261874	0,011919300176	0,012123142248	0,012322637006
$P_{100,30}$	0,010939868623	0,011166668693	0,011389834790	0,011609173406	0,011824500715	0,012035642623	0,012242434810	0,012444722729	0,012642361583	0,012835216281
$P_{100,31}$	0,011498409790	0,011721261906	0,011939950895	0,012154298027	0,012364134888	0,012569303374	0,012769655658	0,012965054126	0,013155371294	0,013340489701
$P_{100,32}$	0,012057678486	0,012275337076	0,012488317275	0,012696457327	0,012899606310	0,013097624060	0,013290381068	0,013477758356	0,013659647331	0,013835949615
$P_{100,33}$	0,012615064016	0,012826270841	0,013032303468	0,013233019152	0,013428286365	0,013617984649	0,013802004449	0,013980246927	0,014152623751	0,014319056878
$P_{100,34}$	0,013167844197	0,013371340083	0,013569190926	0,013761274833	0,013947481364	0,014127711321	0,014301876519	0,014469899538	0,014631713457	0,014787261582
$P_{100,35}$	0,013713205778	0,013907742798	0,014096194258	0,014278460719	0,014454454289	0,014624098343	0,014787327239	0,014944086006	0,015094330037	0,015238024760
$P_{100,36}$	0,014248266901	0,014432620865	0,014610484204	0,014781781286	0,014946447958	0,015104431219	0,015255688874	0,015400189175	0,015537910462	0,015668840795
$P_{100,37}$	0,014770101364	0,014943084434	0,015109212086	0,015268433553	0,015420709337	0,015566010824	0,015704319890	0,015835628500	0,015959938305	0,016077260240
$P_{100,38}$	0,015275764416	0,015436237679	0,015589535425	0,015735632510	0,015874514699	0,016006178230	0,016130629382	0,016247884033	0,016357967228	0,016460912736
$P_{100,39}$	0,015762319771	0,015909205588	0,016048644438	0,016180637238	0,016305195306	0,016422339895	0,016532101713	0,016634520463	0,016729644374	0,016817529743
$P_{100,40}$	0,016226867509	0,016359161451	0,016483789043	0,016600777372	0,016710163284	0,016811992886	0,016906321049	0,016993210910	0,017072733395	0,017144966742
$P_{100,41}$	0,016666572501	0,016783354687	0,016892306035	0,016993479574	0,017086937328	0,017172749796	0,017250995429	0,017321760130	0,017385136757	0,017441224638

PAGE: [0-41 / 260-278]

τ =	260	262	264	266	268	270	272	274	276	278
$P_{100,0}$	0,005237838256	0,005574309388	0,005925429291	0,006291440088	0,006672570820	0,007069037003	0,007481040224	0,007908767786	0,008352392373	0,008812071771
$P_{100,1}$	0,004354885921	0,004631324559	0,004919606468	0,005219918923	0,005532438499	0,005857330743	0,006194749870	0,006544838492	0,006907727373	0,007283535227
$P_{100,2}$	0,003783815067	0,004018100789	0,004262081635	0,004515895738	0,004779672269	0,005053531207	0,005337583130	0,005631929034	0,005936660172	0,006251857923
$P_{100,3}$	0,003441701581	0,003647076537	0,003860499364	0,004082064464	0,004311858542	0,004549960475	0,004796441182	0,005051363533	0,005314782266	0,005586743925
$P_{100,4}$	0,003268702981	0,003454948397	0,003647972980	0,003847832337	0,004054575321	0,004268243972	0,004488873486	0,004716492183	0,004951121503	0,005192776016
$P_{100,5}$	0,003221742123	0,003396058472	0,003576168505	0,003762093433	0,003953848433	0,004151442666	0,004354879309	0,004564155606	0,004779262923	0,005000186814
$P_{100,6}$	0,003269887481	0,003437551108	0,003610232149	0,003787921278	0,003970603718	0,004158259322	0,004350862678	0,004548383212	0,004750785304	0,004958028418
$P_{100,7}$	0,003390980362	0,003555831931	0,003725076612	0,003898677904	0,004076594357	0,004258779723	0,004445183106	0,004635749127	0,004830418090	0,005029126155
$P_{100,8}$	0,003569176723	0,003733983485	0,003902666926	0,004075166296	0,004251416373	0,004431347661	0,004614886597	0,004801955756	0,004992474064	0,005186357012
$P_{100,9}$	0,003793158475	0,003959882150	0,004130040133	0,004303549989	0,004480325291	0,004660275862	0,004843308021	0,005029324829	0,005218226338	0,005409909838
$P_{100,10}$	0,004054833921	0,004224828193	0,004397862697	0,004573835595	0,004752641559	0,004934172053	0,005118315610	0,005304958115	0,005493983077	0,005685271909
$P_{100,11}$	0,004348394846	0,004522550557	0,004699381236	0,004878767711	0,005060587859	0,005244716922	0,005431027810	0,005619391409	0,005809676879	0,006001751950
$P_{100,12}$	0,004669633237	0,004848484845	0,005029660415	0,005213025351	0,005398442701	0,005585773485	0,005774877028	0,005965611280	0,006157833131	0,006351398724
$P_{100,13}$	0,005015446659	0,005199250180	0,005385030205	0,005572638537	0,005761925265	0,005952739103	0,006144927738	0,006338338164	0,006532817006	0,006728210836
$P_{100,14}$	0,005383480583	0,005572270604	0,005762685553	0,005954565407	0,006147749114	0,006342074943	0,006537380843	0,006733504777	0,006930285054	0,007127560641
$P_{100,15}$	0,005771869997	0,005965501468	0,006160396916	0,006356386249	0,006553299073	0,006750965056	0,006949214279	0,007147877572	0,007346786841	0,007545775381
$P_{100,16}$	0,006179052996	0,006377232042	0,006576301395	0,006776082664	0,006976397925	0,007177070083	0,007377923215	0,007578782904	0,007779476551	0,007979833679
$P_{100,17}$	0,006603636588	0,006805943477	0,007008752486	0,007211878727	0,007415138576	0,007618350030	0,007821333034	0,008023909803	0,008225905122	0,008427146632
$P_{100,18}$	0,007044300440	0,007250207050	0,007456212532	0,007662127340	0,007867764015	0,008072937517	0,008277465545	0,008481168831	0,008683871425	0,008885400954
$P_{100,19}$	0,007499728292	0,007708611806	0,007917176355	0,008125229627	0,008332582220	0,008539047959	0,008744444187	0,008948592041	0,009151316707	0,009352447654
$P_{100,20}$	0,007968559689	0,008179713765	0,008390117779	0,008599578578	0,008807906741	0,009014916872	0,009220427862	0,009424263133	0,009626250863	0,009826224192
$P_{100,21}$	0,008449356731	0,008662001100	0,008873453067	0,009083520600	0,009292016206	0,009498757191	0,009703565887	0,009906269871	0,010106702146	0,010304701315
$P_{100,22}$	0,008940582104	0,009153871237	0,009365516991	0,009575330431	0,009783127945	0,009988731456	0,010191968623	0,010392673012	0,010590684246	0,010785848139
$P_{100,23}$	0,009440585682	0,009653617028	0,009864548455	0,010073196109	0,010279382196	0,010482935156	0,010683689820	0,010881487539	0,011076176295	0,011267610796
$P_{100,24}$	0,009947597818	0,010159419923	0,010368683468	0,010575211645	0,010778834391	0,010979388517	0,011176717819	0,011370673168	0,011561112575	0,011747901240
$P_{100,25}$	0,010459727924	0,010669348663	0,010875953865	0,011079375693	0,011279453675	0,011476034786	0,011668973512	0,011858131895	0,012043379550	0,012224593677
$P_{100,26}$	0,010974967381	0,011181362431	0,011384290620	0,011583594947	0,011779126315	0,011970743572	0,012158313525	0,012341710934	0,012520818494	0,012695526786
$P_{100,27}$	0,011491196036	0,011693317666	0,011891530909	0,012085691368	0,012275663015	0,012461318172	0,012642537481	0,012819209854	0,012991232402	0,013158510348
$P_{100,28}$	0,012006191761	0,012202977954	0,012395428250	0,012583412519	0,012766809368	0,012945506069	0,013119398481	0,013288390955	0,013452396214	0,013611335229
$P_{100,29}$	0,012517642628	0,012708026516	0,012893665243	0,013074444474	0,013250258873	0,013421011987	0,013586616123	0,013746992196	0,013902069578	0,014051785927
$P_{100,30}$	0,013023161365	0,013206080924	0,013383868478	0,013556426851	0,013723668025	0,013885512982	0,014041891524	0,014192742090	0,014338011556	0,014477655027
$P_{100,31}$	0,013520301781	0,013694709712	0,013863625261	0,014026969596	0,014184673094	0,014336675138	0,014482923897	0,014623376094	0,014757996777	0,014886759070
$P_{100,32}$	0,014006576865	0,014171450571	0,014330501841	0,014483671173	0,014630908217	0,014772171523	0,014907428289	0,015036654089	0,015159832607	0,015276955357
$P_{100,33}$	0,014479478310	0,014633829848	0,014782062829	0,014924137850	0,015060024492	0,015189701029	0,015313154138	0,015430378594	0,015541376975	0,015646159357
$P_{100,34}$	0,014936497157	0,015079383064	0,015215891524	0,015346003775	0,015469709762	0,015587007814	0,015697904313	0,015802413374	0,015900556517	0,015992362334
$P_{100,35}$	0,015375145303	0,015505676163	0,015629610852	0,015746951558	0,015857708787	0,015961901019	0,016059554353	0,016150702157	0,016235384721	0,016313648911
$P_{100,36}$	0,015792977582	0,015910327203	0,016020904629	0,016124733054	0,016221843508	0,016312274491	0,016396071594	0,016473287137	0,016543979801	0,016608214272
$P_{100,37}$	0,016187614111	0,016291028199	0,016387538848	0,016477190067	0,016560033135	0,016636126204	0,016705533918	0,016768327027	0,016824582018	0,016874380744
$P_{100,38}$	0,016556762625	0,016645566826	0,016727382713	0,016802274683	0,016870313747	0,016931577123	0,016986147842	0,017034114360	0,017075570184	0,017110613497
$P_{100,39}$	0,016898240481	0,016971847669	0,017038429121	0,017098068952	0,017150857164	0,017196889230	0,017236265702	0,017269091819	0,017295477133	0,017315535141
$P_{100,40}$	0,017209996035	0,017267912749	0,017318814306	0,017362803643	0,017399988792	0,017430482469	0,017454401682	0,017471867345	0,017483003909	0,017487939008
$P_{100,41}$	0,017490129103	0,017531961025	0,017566836380	0,017594875813	0,017616204223	0,017630950363	0,017639246455	0,017641227812	0,017637032486	0,017626800919

PAGE:	[0-41 / 280-298]

τ =	280	282	284	286	288	290	292	294	296	298
$P_{100,0}$	0,009287948613	0,009780150175	0,010288788194	0,010813958740	0,011355742112	0,011914202774	0,012489389327	0,013081334514	0,013690055255	0,014315552720
$P_{100,1}$	0,007672368530	0,008074321374	0,008489475347	0,008917899439	0,009359649987	0,009814770634	0,010283292328	0,010765233340	0,011260599312	0,011769383326
$P_{100,2}$	0,006577593679	0,006913928763	0,007260914362	0,007618591492	0,007986990978	0,008366133457	0,008756029410	0,009156679202	0,009568073152	0,009990191614
$P_{100,3}$	0,005867286819	0,006156441002	0,006454228263	0,006760662145	0,007075747966	0,007399482873	0,007731855902	0,008072848051	0,008422432376	0,008780574095
$P_{100,4}$	0,005441463440	0,005697184683	0,005959933885	0,006229698490	0,006506459315	0,006790190640	0,007080860305	0,007378429822	0,007682854489	0,007994083524
$P_{100,5}$	0,005226907111	0,005459398007	0,005697628158	0,005941560796	0,006191153845	0,006446360046	0,006707127092	0,006973397765	0,007245110087	0,007522197465
$P_{100,6}$	0,005170067229	0,005386851768	0,005608327561	0,005834435788	0,006065113433	0,006300293449	0,006539904920	0,006783873228	0,007032120228	0,007284564411
$P_{100,7}$	0,005231805515	0,005438384582	0,005648788168	0,005862937671	0,006080751269	0,006302144109	0,006527028495	0,006755314081	0,006986908064	0,007221715368
$P_{100,8}$	0,005383516870	0,005583862908	0,005787301612	0,005993736901	0,006203070341	0,006415201365	0,006630027482	0,006847444487	0,007067346667	0,007289627006
$P_{100,9}$	0,005604270108	0,005801199658	0,006000588975	0,006202326762	0,006406300179	0,006612395072	0,006820496203	0,007030487475	0,007242252147	0,007455673047
$P_{100,10}$	0,005878704200	0,006074157979	0,006271509983	0,006470635914	0,006671410688	0,006873708685	0,007077403985	0,007282370601	0,007488482701	0,007695614824
$P_{100,11}$	0,006195483213	0,006390736401	0,006587376669	0,006785268862	0,006984277773	0,007184268400	0,007385106187	0,007586657255	0,007788788634	0,007991368470
$P_{100,12}$	0,006546163753	0,006741983757	0,006938714402	0,007136211759	0,007334332561	0,007532934464	0,007731876284	0,007931018228	0,008130222120	0,008329351604
$P_{100,13}$	0,006924366482	0,007121131320	0,007318353560	0,007515882521	0,007713568886	0,007911264957	0,008108824888	0,008306104908	0,008502963535	0,008699261772
$P_{100,14}$	0,007325171474	0,007522958747	0,007720765191	0,007918435341	0,008115815790	0,008312755423	0,008509105648	0,008704720604	0,008899457358	0,009093176091
$P_{100,15}$	0,007744678169	0,007943332153	0,008141576518	0,008339252940	0,008536205824	0,008732282532	0,008927333587	0,009121212872	0,009313777806	0,009504889511
$P_{100,16}$	0,008179686216	0,008378868767	0,008577218860	0,008774577192	0,008970787842	0,009165698480	0,009359160556	0,009551029471	0,009741164738	0,009929430124
$P_{100,17}$	0,008627465091	0,008826694630	0,009024672978	0,009221241686	0,009416246317	0,009609536634	0,009800966759	0,009990395328	0,010177685620	0,010362705676
$P_{100,18}$	0,009085588867	0,009284270658	0,009481286077	0,009676479314	0,009869699176	0,010060799238	0,010249637985	0,010436078926	0,010619990708	0,010801247199
$P_{100,19}$	0,009551818849	0,009749268956	0,009944641514	0,010137785094	0,010328553443	0,010516805610	0,010702406049	0,010885224715	0,011065137138	0,011242024479
$P_{100,20}$	0,010024021404	0,010219486092	0,010412467304	0,010602819669	0,010790403508	0,010975084926	0,011156735883	0,011335234258	0,011510463890	0,011682314606
$P_{100,21}$	0,010500111732	0,010692783624	0,010882573208	0,011069342781	0,011252960790	0,011433301896	0,011610247012	0,011783683324	0,011953504309	0,012119609726
$P_{100,22}$	0,010978016805	0,011167048749	0,011352808941	0,011535168870	0,011714006579	0,011889206691	0,012060660411	0,012228265517	0,012391926337	0,012551553715
$P_{100,23}$	0,011455652538	0,011640169865	0,011821037995	0,011998139040	0,012171362006	0,012340602777	0,012505764081	0,012666755453	0,012823493169	0,012975900185
$P_{100,24}$	0,011930911578	0,012110023233	0,012285123067	0,012456105141	0,012622870673	0,012785327986	0,012943392444	0,013096986371	0,013246038961	0,013390486180
$P_{100,25}$	0,012401659037	0,012574467927	0,012742920132	0,012906922859	0,013066390662	0,013221245357	0,013371415915	0,013516838351	0,013657455605	0,013793217405
$P_{100,26}$	0,012865734229	0,013031346998	0,013192278942	0,013348451481	0,013499793494	0,013646241191	0,013787737982	0,013924234328	0,014055687593	0,014182061877
$P_{100,27}$	0,013320956933	0,013478493294	0,013631048346	0,013778558637	0,013920968201	0,014058228400	0,014190297757	0,014317141782	0,014438732785	0,014555049695
$P_{100,28}$	0,013765137078	0,013913738788	0,014057085178	0,014195128680	0,014327829158	0,014455153718	0,014577076511	0,014693578530	0,014804647402	0,014910277177
$P_{100,29}$	0,014196087004	0,014334926487	0,014468265768	0,014596073750	0,014718326628	0,014835007677	0,014946107021	0,015051621411	0,015151553991	0,015245914066
$P_{100,30}$	0,014611635620	0,014739924238	0,014862499340	0,014979346706	0,015090459191	0,015195836485	0,015295484863	0,015389416936	0,015477651400	0,015560212786
$P_{100,31}$	0,015009643922	0,015126639855	0,015237742701	0,015342955339	0,015442287433	0,015535755158	0,015623380941	0,015705193185	0,015781226009	0,015851518977
$P_{100,32}$	0,015388021407	0,015493037092	0,015592015733	0,015684977348	0,015771948368	0,015852961352	0,015928054701	0,015997272377	0,016060663626	0,016118282699
$P_{100,33}$	0,015744743005	0,015837152070	0,015923417283	0,016003575651	0,016077670156	0,016145749453	0,016207867576	0,016264083645	0,016314461578	0,016359069808
$P_{100,34}$	0,016077866170	0,016157109794	0,016230141080	0,016297013689	0,016357786753	0,016412524570	0,016461296293	0,016504175641	0,016541240597	0,016572573127
$P_{100,35}$	0,016385547827	0,016451140465	0,016510491387	0,016563670389	0,016610752186	0,016651816090	0,016686945709	0,016716228644	0,016739756193	0,016757623072
$P_{100,36}$	0,016666060887	0,016717595288	0,016762898087	0,016802054527	0,016835154163	0,016862290548	0,016883560921	0,016899065912	0,016908909250	0,016913197485
$P_{100,37}$	0,016917810073	0,016954961531	0,016985930968	0,017010818221	0,017029726796	0,017042763555	0,017050038412	0,017051664042	0,017047755599	0,017038430440
$P_{100,38}$	0,017139346805	0,017161876590	0,017178312967	0,017188769362	0,017193362198	0,017192210585	0,017185436029	0,017173162151	0,017155514412	0,017132619855
$P_{100,39}$	0,017329382934	0,017337140857	0,017338932177	0,017334882765	0,017325120796	0,017309776449	0,017288981632	0,017262869708	0,017231575242	0,017195233751
$P_{100,40}$	0,017486803111	0,017479729196	0,017466852432	0,017448309873	0,017424240169	0,017394783291	0,017360080260	0,017320272899	0,017275503589	0,017225915047
$P_{100,41}$	0,017610675621	0,017588800850	0,017561322312	0,017528386879	0,017490142311	0,017446736999	0,017398319719	0,017345039401	0,017287044903	0,017224484811

PAGE: [0-41 / 300-318]

τ =	300	302	304	306	308	310	312	314	316	318
$P_{100,0}$	0,014957812430	0,015616804382	0,016292483212	0,016984788382	0,017693644387	0,018418960992	0,019160633496	0,019918543006	0,020692556742	0,021482528361
$P_{100,1}$	0,012291566002	0,012827115617	0,013375988242	0,013938127909	0,014513466789	0,015101925393	0,015703412791	0,016317826852	0,016945054485	0,017584971915
$P_{100,2}$	0,010423005088	0,010866474330	0,011320550501	0,011785175309	0,012260281186	0,012745791462	0,013241620565	0,013747674226	0,014263849699	0,014790035985
$P_{100,3}$	0,009147230707	0,009522352121	0,009905880800	0,010297751914	0,010697893505	0,011106226658	0,011522665687	0,011947118319	0,012379485897	0,012819663580
$P_{100,4}$	0,008312060197	0,008636721978	0,008968000689	0,009305822663	0,009650108908	0,010000775280	0,010357732659	0,010720887128	0,011090140159	0,011465388800
$P_{100,5}$	0,007804588850	0,008092208899	0,008384978135	0,008682813117	0,008985626606	0,009293327744	0,009605822220	0,009923012449	0,010244797746	0,010571074505
$P_{100,6}$	0,007541121086	0,007801702549	0,008066218261	0,008334575021	0,008606677141	0,008882426618	0,009161723309	0,009444465102	0,009730548081	0,010019866698
$P_{100,7}$	0,007459638838	0,007700579421	0,007944436353	0,008191107338	0,008440488726	0,008692475689	0,008946962389	0,009203842148	0,009463007608	0,009724350890
$P_{100,8}$	0,007514177382	0,007740888762	0,007969651390	0,008200354974	0,008432888863	0,008667142219	0,008903004189	0,009140364061	0,009379111423	0,009619136310
$P_{100,9}$	0,007670632776	0,007887013908	0,008104699181	0,008323571683	0,008543515023	0,008764413509	0,008986152305	0,009208617584	0,009431696678	0,009655278214
$P_{100,10}$	0,007903642088	0,008112440387	0,008321886582	0,008531858683	0,008742236016	0,008952899392	0,009163731256	0,009374615833	0,009585439264	0,009796089730
$P_{100,11}$	0,008194266233	0,008397352910	0,008600501190	0,008803585638	0,009006482855	0,009209071633	0,009411233097	0,009612850839	0,009813811036	0,010014002565
$P_{100,12}$	0,008528272342	0,008726852202	0,008924961430	0,009122472813	0,009319261830	0,009515206791	0,009710188967	0,009904092704	0,010096805531	0,010288218259
$P_{100,13}$	0,008894863293	0,009089634617	0,009283445265	0,009476167914	0,009667678524	0,009857856471	0,010046584648	0,010233749574	0,010419241476	0,010602954372
$P_{100,14}$	0,009285740267	0,009477016786	0,009666876134	0,009855192506	0,010041843926	0,010226712354	0,010409683772	0,010590648268	0,010769500105	0,010946137774
$P_{100,15}$	0,009694412963	0,009882217129	0,010068175084	0,010252164130	0,010434065881	0,010613766354	0,010791156036	0,010966129943	0,011138587665	0,011308433405
$P_{100,16}$	0,010115693776	0,010299828336	0,010481711042	0,010661223811	0,010838253313	0,011012691031	0,011184433303	0,011353381365	0,011519441369	0,011682524396
$P_{100,17}$	0,010545328401	0,010725431654	0,010902898320	0,011077616369	0,011249478906	0,011418384204	0,011584235726	0,011746942135	0,011906417294	0,012062580255
$P_{100,18}$	0,010979727567	0,011155316337	0,011327903443	0,011497384254	0,011663659600	0,011826635777	0,011986224543	0,012142343105	0,012294914095	0,012443865529
$P_{100,19}$	0,011415773582	0,011586276999	0,011753433013	0,011917145639	0,012077324617	0,012233885396	0,012386749100	0,012535842491	0,012681097916	0,012822453248
$P_{100,20}$	0,011850682239	0,012015468624	0,012176581589	0,012333934931	0,012487448379	0,012637047550	0,012782663890	0,012924234610	0,013061702610	0,013195016397
$P_{100,21}$	0,012281905600	0,012440304193	0,012594723962	0,012745089505	0,012891331498	0,013033386622	0,013171197481	0,013304712509	0,013433885872	0,013558677365
$P_{100,22}$	0,012707064958	0,012858383778	0,013005440218	0,013148170574	0,013286517301	0,013420428914	0,013549859882	0,013674770509	0,013795126815	0,013910900410
$P_{100,23}$	0,013123906053	0,013267446828	0,013406464971	0,013540909240	0,013670734568	0,013795901942	0,013916378269	0,014032136239	0,014143154181	0,014249415918
$P_{100,24}$	0,013530270649	0,013665341528	0,013795654388	0,013921171074	0,014041859560	0,014157693804	0,014268653591	0,014374724375	0,014475897117	0,014572168117
$P_{100,25}$	0,013924080130	0,014050006662	0,014170966230	0,014286934251	0,014397892161	0,014503827247	0,014604732471	0,014700606294	0,014791452491	0,014877279973
$P_{100,26}$	0,014303327855	0,014419462598	0,014530449396	0,014636277576	0,014736942311	0,014832444433	0,014922790241	0,015007991302	0,015088064258	0,015163030629
$P_{100,27}$	0,014666077864	0,014771808870	0,014872240316	0,014967375627	0,015057223844	0,015141799418	0,015221121996	0,015295216220	0,015364111512	0,015427841870
$P_{100,28}$	0,015010468113	0,015105226455	0,015194564218	0,015278498968	0,015357053595	0,015430256094	0,015498139344	0,015560740887	0,015618102710	0,015670271024
$P_{100,29}$	0,015334716866	0,015417983312	0,015495739779	0,015568017861	0,015634854136	0,015696289934	0,015752371103	0,015803147784	0,015848674180	0,015889008337
$P_{100,30}$	0,015637131206	0,015708442107	0,015774186021	0,015834408315	0,015889158952	0,015938492245	0,015982466622	0,016021144390	0,016054591502	0,016082877338
$P_{100,31}$	0,015916116838	0,015975069264	0,016028430590	0,016076259562	0,016118619085	0,016155575977	0,016187200725	0,016213567252	0,016234752679	0,016250837105
$P_{100,32}$	0,016170188578	0,016216444711	0,016257118746	0,016292282274	0,016322010570	0,016346382351	0,016365479529	0,016379386975	0,016388192291	0,016391985580
$P_{100,33}$	0,016397981006	0,016431271805	0,016459022539	0,016481316979	0,016498242081	0,016509887737	0,016516346538	0,016517713536	0,016514086023	0,016505563308
$P_{100,34}$	0,016598258902	0,016618387021	0,016633049749	0,016642342256	0,016646362372	0,016645210338	0,016638988577	0,016627801461	0,016611755098	0,016590957114
$P_{100,35}$	0,016769927132	0,016776769092	0,016778252279	0,016774482376	0,016765567177	0,016751616350	0,016732741215	0,016709054522	0,016680670243	0,016647703369
$P_{100,36}$	0,016912039714	0,016905547322	0,016893833726	0,016877014132	0,016855205303	0,016828525329	0,016797093419	0,016761029685	0,016720454951	0,016675490561
$P_{100,37}$	0,017023807869	0,017004008880	0,016979155917	0,016949372642	0,016914783712	0,016875514567	0,016831691228	0,016783440104	0,016730887807	0,016674160976
$P_{100,38}$	0,017104606855	0,017071604880	0,017033744260	0,016991155974	0,016943971439	0,016892322315	0,016836340315	0,016776157029	0,016711903759	0,016643711354
$P_{100,39}$	0,017153981472	0,017107955139	0,017057291771	0,017002128470	0,016942602230	0,016878849758	0,016811007306	0,016739210505	0,016663594220	0,016584292406
$P_{100,40}$	0,017171650102	0,017112851497	0,017049661689	0,016982222674	0,016910675808	0,016835161651	0,016755819813	0,016672788811	0,016586205943	0,016496207158
$P_{100,41}$	0,017157507233	0,017086259625	0,017010888609	0,016931539818	0,016848357739	0,016761485575	0,016671065114	0,016577236602	0,016480138638	0,016379908066

PAGE: [0-41 / 320-338]

τ =	320	322	324	326	328	330	332	334	336	338
$P_{100,0}$	0,022288298287	0,023109694073	0,023946530765	0,024798611289	0,025665726847	0,026547657317	0,027444171678	0,028355028429	0,029279976020	0,030218753298
$P_{100,1}$	0,018237444956	0,018902329302	0,019579470834	0,020268705931	0,020969861789	0,021682756757	0,022407200671	0,023142995198	0,023889934186	0,024647804016
$P_{100,2}$	0,015326114076	0,015871957199	0,016427431070	0,016992394157	0,017566697948	0,018150187223	0,018742700336	0,019344069489	0,019954121026	0,020572675715
$P_{100,3}$	0,013267540558	0,013723000262	0,014185920588	0,014656174117	0,015133628348	0,015618145924	0,016109584865	0,016607798803	0,017112637216	0,017623945667
$P_{100,4}$	0,011846525863	0,012233440121	0,012626016499	0,013024136274	0,013427677264	0,013836514033	0,014250518083	0,014669558053	0,015093499912	0,015522207156
$P_{100,5}$	0,010901736369	0,011236674411	0,011575777308	0,011918931511	0,012266021420	0,012616929555	0,012971536723	0,013329722185	0,013691363817	0,014056338276
$P_{100,6}$	0,010312313931	0,010607781452	0,010906159781	0,011207338443	0,011511206120	0,011817650799	0,012126559915	0,012437820495	0,012751319290	0,013066942914
$P_{100,7}$	0,009987763747	0,010253137713	0,010520364250	0,010789334881	0,011059941329	0,011332075642	0,011605630320	0,011880498432	0,012156573725	0,012433750739
$P_{100,8}$	0,009860329349	0,010102581893	0,010345786151	0,010589835313	0,010834623665	0,011080046701	0,011326001227	0,011572385455	0,011819099103	0,012066043470
$P_{100,9}$	0,009879252248	0,010103510381	0,010327945884	0,010552453798	0,010776931037	0,011001276481	0,011225391061	0,011449177836	0,011672542065	0,011895391274
$P_{100,10}$	0,010006457574	0,010216435403	0,010425918195	0,010634803385	0,010842990948	0,011050383480	0,011256886255	0,011462407293	0,011666857405	0,011870150243
$P_{100,11}$	0,010213317105	0,010411649227	0,010608896482	0,010804959470	0,010999741908	0,011193150685	0,011385095913	0,011575490961	0,011764252492	0,011951300486
$P_{100,12}$	0,010478225058	0,010666723542	0,010853614826	0,011038803586	0,011222198104	0,011403710307	0,011583255790	0,011760753847	0,011936127472	0,012109303372
$P_{100,13}$	0,010784786129	0,010964638529	0,011142417307	0,011318032191	0,011491396929	0,011662429305	0,011831051149	0,011997188339	0,012160770790	0,012321732447
$P_{100,14}$	0,011120464046	0,011292386002	0,011461815062	0,011628667000	0,011792861950	0,011954324404	0,012112983203	0,012268771514	0,012421626809	0,012571490828
$P_{100,15}$	0,011475575999	0,011639928935	0,011801410353	0,011959943043	0,012115454429	0,012267876546	0,012417146013	0,012563203989	0,012705996132	0,012845472544
$P_{100,16}$	0,011842546460	0,011999428498	0,012153096354	0,012303480752	0,012450517260	0,012594146248	0,012734312837	0,012870966840	0,013004062702	0,013133559423
$P_{100,17}$	0,012215355237	0,012364671597	0,012510463789	0,012652671314	0,012791238669	0,012926115280	0,013057255432	0,013184618193	0,013308167333	0,013427871232
$P_{100,18}$	0,012589130769	0,012730648466	0,012868362498	0,013002221902	0,013132180799	0,013258198305	0,013380238450	0,013498270077	0,013612266742	0,013722206615
$P_{100,19}$	0,012959851822	0,013093242353	0,013222578858	0,013347820562	0,013468931804	0,013585881934	0,013698645206	0,013807200666	0,013911532034	0,014011627586
$P_{100,20}$	0,013324129992	0,013449002836	0,013569599681	0,013685890487	0,013797850299	0,013905459131	0,014008701843	0,014107568009	0,014202051787	0,014292151789
$P_{100,21}$	0,013679052295	0,013794981362	0,013906440543	0,014013410952	0,014115878718	0,014213834838	0,014307275047	0,014396199668	0,014480613469	0,014560525519
$P_{100,22}$	0,014022068355	0,014128613031	0,014230521992	0,014327787819	0,014420407975	0,014508384648	0,014591724604	0,014670439022	0,014744543346	0,014814057122
$P_{100,23}$	0,014350910610	0,014447632601	0,014539581260	0,014626760820	0,014709180212	0,014786852904	0,014859796731	0,014928033731	0,014991589977	0,015050495411
$P_{100,24}$	0,014663538844	0,014750015766	0,014831610178	0,014908338022	0,014980219716	0,015047279973	0,015109547631	0,015167055473	0,015219840050	0,015267941513
$P_{100,25}$	0,014958102600	0,015033938995	0,015104812359	0,015170750285	0,015231784572	0,015287951037	0,015339289335	0,015385842772	0,015427658127	0,015464785471
$P_{100,26}$	0,015232916615	0,015297752898	0,015357574448	0,015412420329	0,015462333497	0,015507360619	0,015547551875	0,015582960771	0,015613643958	0,015639661046
$P_{100,27}$	0,015486445655	0,015539965391	0,015588447557	0,015631942388	0,015670503672	0,015704188556	0,015733057352	0,015757173344	0,015776602605	0,015791413813
$P_{100,28}$	0,015717296055	0,015759231827	0,015796135956	0,015828069438	0,015855096456	0,015877284172	0,015894702537	0,015907424098	0,015915523814	0,015919078870
$P_{100,29}$	0,015924211921	0,015954350006	0,015979490858	0,015999705728	0,016015068651	0,016025656245	0,016031547518	0,016032823678	0,016029567952	0,016021865405
$P_{100,30}$	0,016106074472	0,016124258467	0,016137507653	0,016145902928	0,016149527549	0,016148466941	0,016142808506	0,016132641436	0,016118056534	0,016099146043
$P_{100,31}$	0,016261903380	0,016268036893	0,016269325362	0,016265858630	0,016257728466	0,016245028376	0,016227853418	0,016206300020	0,016180465809	0,016150449447
$P_{100,32}$	0,016390859236	0,016384907727	0,016374227392	0,016358916244	0,016339073776	0,016314800781	0,016286199169	0,016253371803	0,016216422325	0,016175455007
$P_{100,33}$	0,016492246505	0,016474238333	0,016451642913	0,016424565583	0,016393112710	0,016357391518	0,016317509919	0,016273576352	0,016225699627	0,016173988778
$P_{100,34}$	0,016565516455	0,016535543189	0,016501148319	0,016462443601	0,016419541373	0,016372554393	0,016321595673	0,016266778340	0,016208215484	0,016146020026
$P_{100,35}$	0,016610269719	0,016568485755	0,016522468406	0,016472334897	0,016418202593	0,016360188843	0,016298410835	0,016232985463	0,016164029190	0,016091657931
$P_{100,36}$	0,016626258200	0,016572879721	0,016515476984	0,016454171699	0,016389085280	0,016320338708	0,016248052395	0,016172346066	0,016093338639	0,016011148117
$P_{100,37}$	0,016613386117	0,016548689438	0,016480196708	0,016408033112	0,016332323122	0,016253190371	0,016170757537	0,016085146236	0,015996476918	0,015904868774
$P_{100,38}$	0,016571710064	0,016496029401	0,016416798001	0,016334143506	0,016248192445	0,016159070124	0,016066900529	0,015971806231	0,015873908298	0,015773326219
$P_{100,39}$	0,016501437976	0,016415162675	0,016325596970	0,016232869938	0,016137109169	0,016038440673	0,015936988797	0,015832876145	0,015726223510	0,015617149808
$P_{100,40}$	0,016402926945	0,016306498227	0,016207052265	0,016104718565	0,015999624799	0,015891896728	0,015781658136	0,015669030765	0,015554134265	0,015437086141
$P_{100,41}$	0,016276679878	0,016170587132	0,016061760868	0,015950330039	0,015836421445	0,015720159676	0,015601667057	0,015481063611	0,015358467011	0,015233992551

τ =	340	342	344	346	348	350	352	354	356	358
$P_{100,0}$	0,031171089941	0,032136706910	0,033115316899	0,034106624786	0,035110328084	0,036126117397	0,037153676869	0,038192684633	0,039242813262	0,040303730207
$P_{100,1}$	0,025416383961	0,026195446543	0,026984757896	0,027784078127	0,028593161680	0,029411757693	0,030239610366	0,031076459311	0,031922039918	0,032776083697
$P_{100,2}$	0,021199549037	0,021834551479	0,022477488823	0,023128162437	0,023786369565	0,024451903615	0,025124554448	0,025804108660	0,026490349865	0,027183058975
$P_{100,3}$	0,018141566033	0,018665336748	0,019195093033	0,019730667131	0,020271888536	0,020818584226	0,021370578888	0,021927695143	0,022489753765	0,023056573903
$P_{100,4}$	0,015955541002	0,016393360573	0,016835523095	0,017281884079	0,017732297504	0,018186616002	0,018644691031	0,019106373055	0,019571511708	0,020039955968
$P_{100,5}$	0,014424521153	0,014795787129	0,015170010129	0,015547063468	0,015926819992	0,016309152227	0,016693932508	0,017081033114	0,017470326400	0,017861684917
$P_{100,6}$	0,013384577964	0,013704111152	0,014025429420	0,014348420056	0,014672970806	0,014998969980	0,015326306553	0,015654870266	0,015984551713	0,016315242437
$P_{100,7}$	0,012711924906	0,012990992643	0,013270851454	0,013551400006	0,013832538219	0,014114167340	0,014396190015	0,014678510356	0,014961034003	0,015243668184
$P_{100,8}$	0,012313121523	0,012560237968	0,012807299315	0,013054213943	0,013300892156	0,013547246231	0,013793190467	0,014038641223	0,014283516956	0,014527738248
$P_{100,9}$	0,012117635309	0,012339186389	0,012559959153	0,012779870696	0,012998840603	0,013216790977	0,013433646459	0,013649334246	0,013863784103	0,014076928367
$P_{100,10}$	0,012072202331	0,012272933099	0,012472264906	0,012670123058	0,012866435818	0,013061134412	0,013254153030	0,013445428821	0,013634901882	0,013822515242
$P_{100,11}$	0,012136558255	0,012319952458	0,012501413100	0,012680873530	0,012858270436	0,013033543824	0,013206637004	0,013377496564	0,013546072336	0,013712317367
$P_{100,12}$	0,012280211960	0,012448787350	0,012614967338	0,012778693379	0,012939910564	0,013098567582	0,013254616684	0,013408013637	0,013558717680	0,013706691469
$P_{100,13}$	0,012480011256	0,012635549139	0,012788291962	0,012938189491	0,013085195345	0,013229266951	0,013370365481	0,013508455797	0,013643506384	0,013775489280
$P_{100,14}$	0,012718309540	0,012862033095	0,013002615774	0,013140015931	0,013274195925	0,013405122060	0,013532764507	0,013657097229	0,013778097901	0,013895747831
$P_{100,15}$	0,012981587715	0,013114300457	0,013243573835	0,013369375092	0,013491675568	0,013610450620	0,013725679533	0,013837345430	0,013945435175	0,014049939280
$P_{100,16}$	0,013259420488	0,013381613780	0,013500111500	0,013614890073	0,013725930052	0,013833216024	0,013936736506	0,014036483842	0,014132454097	0,014224646948
$P_{100,17}$	0,013543702790	0,013655639330	0,013763662497	0,013867758147	0,013967916246	0,014064130756	0,014156399520	0,014244724146	0,014329109891	0,014409565542
$P_{100,18}$	0,013828072364	0,013929851050	0,014027534003	0,014121116711	0,014210598694	0,014295983382	0,014377277990	0,014454493394	0,014527643999	0,014596747613
$P_{100,19}$	0,014107480031	0,014199086382	0,014286447830	0,014369569610	0,014448460874	0,014523134550	0,014593607214	0,014659898950	0,014722033217	0,014780036711
$P_{100,20}$	0,014377870936	0,014459216328	0,014536199096	0,014608834268	0,014677140621	0,014741140540	0,014800859877	0,014856327804	0,014907576676	0,014954641883
$P_{100,21}$	0,014635949033	0,014706901232	0,014773403184	0,014835479658	0,014893158974	0,014946472851	0,014995456257	0,015040147262	0,015080586890	0,015116818972
$P_{100,22}$	0,014879003841	0,014939410780	0,014995308846	0,015046732414	0,015093719174	0,015136309975	0,015174548666	0,015208481948	0,015238159223	0,015263632440
$P_{100,23}$	0,015104783675	0,015154491948	0,015199660778	0,015240333925	0,015276558193	0,015308383271	0,015335861579	0,015359048111	0,015378000280	0,015392777771
$P_{100,24}$	0,015311403435	0,015350272642	0,015384599043	0,015414435465	0,015439837487	0,015460863276	0,015477573434	0,015490030835	0,015498300476	0,015502449327
$P_{100,25}$	0,015497277992	0,015525191816	0,015548585841	0,015567521566	0,015582062923	0,015592276120	0,015598229477	0,015599993274	0,015597639597	0,015591242193
$P_{100,26}$	0,015661074427	0,015677949098	0,015690352488	0,015698354291	0,015702026303	0,015701442256	0,015696677666	0,015687809676	0,015674916912	0,015658079337
$P_{100,27}$	0,015801678068	0,015807468722	0,015808861205	0,015805932860	0,015798762777	0,015787431643	0,015772021579	0,015752615999	0,015729299464	0,015702157540
$P_{100,28}$	0,015918168506	0,015912873838	0,015903277693	0,015889464447	0,015871519866	0,015849530952	0,015823585797	0,015793773438	0,015760183722	0,015722907168
$P_{100,29}$	0,016009802764	0,015993468257	0,015972951444	0,015948343059	0,015919734865	0,015887219500	0,015850890337	0,015810841353	0,015767166992	0,015719962042
$P_{100,30}$	0,016076003474	0,016048723448	0,016017401538	0,015982134119	0,015943018222	0,015900151397	0,015853631580	0,015803556962	0,015750025870	0,015693136649
$P_{100,31}$	0,016116350466	0,016078269116	0,016036306221	0,015990563028	0,015941141080	0,015888142081	0,015831667775	0,015771819825	0,015708699700	0,015642408573
$P_{100,32}$	0,016130574596	0,016081886168	0,016029494992	0,015973506399	0,015914025650	0,015851157825	0,015785007701	0,015715679651	0,015643277536	0,015567904612
$P_{100,33}$	0,016118552924	0,016059501131	0,015996942291	0,015930984994	0,015861737417	0,015789307216	0,015713801421	0,015635326340	0,015553987468	0,015469889403
$P_{100,34}$	0,016080304590	0,016011181380	0,015938762061	0,015863157656	0,015784478439	0,015702833839	0,015618332351	0,015531081447	0,015441187502	0,015348755717
$P_{100,35}$	0,016015986932	0,015937130666	0,015855202724	0,015770315725	0,015682581221	0,015592109616	0,015499010085	0,015403390504	0,015305357381	0,015205015795
$P_{100,36}$	0,015925891486	0,015837684616	0,015746642178	0,015652877555	0,015556502769	0,015457628408	0,015356363561	0,015252815756	0,015147090911	0,015039293277
$P_{100,37}$	0,015810439645	0,015713305945	0,015613582582	0,015511382890	0,015406818566	0,015299999613	0,015191034284	0,015080029038	0,014967088499	0,014852315411
$P_{100,38}$	0,015670177826	0,015564579232	0,015456644764	0,015346486916	0,015234216290	0,015119941559	0,015003769423	0,014885804577	0,014766149678	0,014644905323
$P_{100,39}$	0,015505772021	0,015392205143	0,015276562135	0,015158953881	0,015039489155	0,014918274591	0,014795414649	0,014671011602	0,014545165512	0,014417974218
$P_{100,40}$	0,015318001713	0,015196994074	0,015074174062	0,014949650232	0,014823528829	0,014695913775	0,014566906652	0,014436606692	0,014305110772	0,014172513413
$P_{100,41}$	0,015107753118	0,014979859164	0,014850418694	0,014719537247	0,014587317891	0,014453861211	0,014319265312	0,014183625819	0,014047035883	0,013909586187