Representations on diagrams of categories and applications to parabolic bundles

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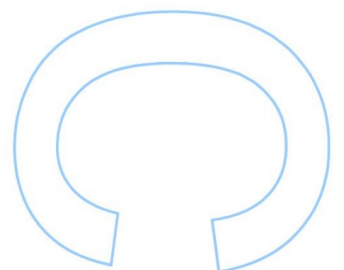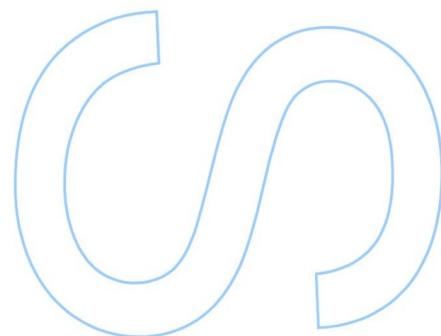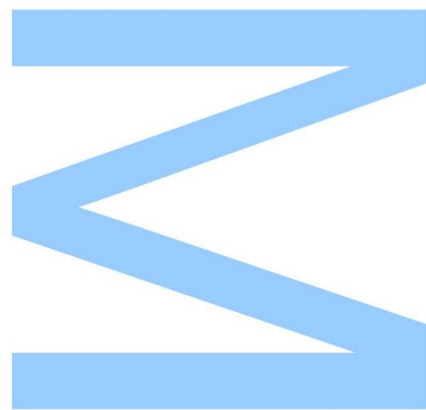Departamento de Matemática
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Resumo

Este trabalho analisa mais de perto as ideias desenvolvidas por [7] e [13], a ideia de representações de *quivers* com torção, colocadas num contexto mais geral. Estas ferramentes são úteis para o estudo de álgebra homologica de certos objectos, entre eles temos os fibrados de *Higgs* e fibrados parabólicos. Sobre esta último, estas ferramentas são utlizadas para estudar a sua teoria de deformação, que é o estudo do espaço vetorial $\mathrm{Ext}(\mathcal{E}_*, \mathcal{E}_*)$ para um fibrado parabólico $\mathcal{E}_*$. As ferramentas aqui desenvolvidas ajudam a dar uma descrição limpa deste objeto.

Palavras-chave: representações de *quivers* com torção, fibrados vetoriais parabólicos, teoria de deformação.

Abstract

This work takes a closer look at the ideas developed by [7] and [13], that of twisted quiver representations, put in a more general setting. These tools are useful to study homological algebra of certain objects, among which we have Higgs bundles and parabolic bundles. In the latter case, we use those tools to study their deformation theory, which is the study of the vector space $\mathrm{Ext}(\mathcal{E}_*, \mathcal{E}_*)$ for a parabolic bundle $\mathcal{E}_*$. The tools developed help give a clean description of this object.

Keywords: twisted quiver representations, parabolic vector bundle, deformation theory.

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Chapter 1

Introduction

Parabolic bundles arose as one of two ways to generalize the Narasimhan-Seshadri theorem ([14], [12]), the other being Higgs bundles ([9], [16]). The moduli space $\mathcal{M} = \mathcal{M}(D, d, \alpha, k)$ of parabolic bundles with divisor D, parabolic degree d, weights α and type of flag k has the structure of a smooth quasiprojective variety, and is an object of active study. The tangent space to $\mathcal{M}$ at a stable point $\mathcal{E}$ is identified with the space of infinitesimal deformations, that is, $\text{Ext}^1(\mathcal{E}, \mathcal{E})$. The study of this space is interesting even in the unstable case, as is done in [2].

In [7], twisted quiver representations are defined and some of its properties are explored, and then some geometric objects are studied under that context, unifying some aspects of seperate theories: quiver representations and Higgs bundles for example. This project was taken further by Mozgovoy in [13], defining categories of representations of diagrams. This author developed some machinery to work with them, and one of the applications was finding a long exact sequence in the category of quasiparabolic bundles with fixed length filtration, involving Hom and Ext^1 groups, hence in particular, encoding the deformation theory of such a quasiparabolic bundle.

Another possible generalization of Gothen and King's work appears in [1], which imposes relations, in the form of taking quotients over ideals of the algebra associated to the twisted representation.

This thesis is mostly based in the work of Mozgovoy in [13]. We will layout the structure of the thesis in the next few paragraphs, but the goal is to understand what are representations of diagrams, and how those tools work, by applying them to parabolic bundles. Thanks to this, we found a way to describe the Ext groups of parabolic bundles as an extension of two

other, simpler, Ext groups.

In chapter 2, we define some objects which will be used on the remainder of the thesis, such as quivers, sheaves, and Grothendieck categories. We review some important ideas and properties of these objects, which we will use to build examples to study with. We also prove an equivalence of categories

$$[\mathcal{C}, \mathcal{T}] \cong A\text{-}\mathbf{Mod}_\mathcal{T}$$

where $\mathcal{C}$ is a category with finitely many objects, $\mathcal{T}$ a tensor category with coproducts, and $A = (A, \mu, \eta)$ an appropriate monoid in $\mathcal{T}$, which is a slight generalization of the well-known equivalence of categories

$$[\mathcal{C}, R\text{-}\mathbf{Mod}] \cong R\mathcal{C}\text{-}\mathbf{Mod},$$

with a similar construction and proof.

Chapter 3 is extensively based on the work of Mozgovoy in [13], in which we cover his idea of representations of diagrams. We restate most of the content, and give different proofs of a few key results, for example:

Theorem 1.1. Let Φ be an abelian, finite quiver diagram with coproducts, and quiver Q. For each $X \in \mathbf{Rep}_*(\Phi)$, the sequence

$$0 \to \bigoplus_{a \in Q_1} \sigma_{ta}(a_* X_{sa}) \xrightarrow{\beta} \bigoplus_{i \in Q_0} \sigma_i(X_i) \xrightarrow{\gamma} X \to 0$$

is exact.

In [13], this was done using the Freyd-Mitchell embedding theorem, and here we give an alternative proof, without using an embedding. With this result, one gets a construction of homological long exact sequences for appropriate representation diagrams. Using the tools of chapter 2, we build a few examples and apply some of the results to them. In particular, we embed quasiparabolic bundles in an abelian category $\mathcal{R} = \mathbf{Rep}_*(\Phi)$, which gives us a way to calculate $\text{Ext}(E, F)$.

In chapter 4, we review the definition of the (abelian) categories of parabolic $\mathcal{O}_X$-modules and parabolic sheaves as done by Yokogawa in [19], where the group $\text{Ext}(\mathcal{F}_*, \mathcal{G}_*)$, is defined, and we show that parabolic bundles are a full subcategory of the category of parabolic sheaves. We also compare the Ext groups of parabolic sheaves with the Ext groups of the category $\mathcal{R}$ in which quasiparabolic bundles are embedded: we find they are isomorphic.

We also find that the Ext group of parabolic bundles is itself an extension of two other Ext groups: the Ext group of coherent sheaves and the Ext group of quiver representations over complex vector spaces. This means that the study of the deformation theory of a parabolic bundle can be done through the deformation theory of the underlying vector bundle, and does not depend on the values of the weights.

Chapter 2

Preliminaries

In this chapter, we introduce a few fundamental concepts, some of which are the tools we will use in the next chapter, and some of which we will use to give illustrative examples.

In what follows, all rings have unit, and unless noted, are commutative. We denote the category of all small categories by **Cat**. Whenever we speak of a category, it will be assumed small unless noted otherwise. For two categories $\mathcal{C}, \mathcal{D}$, we denote the category of functors from $\mathcal{C} \to \mathcal{D}$, with morphisms the natural transformations between them, by $[\mathcal{C}, \mathcal{D}]$.

Two morphisms f, g are said to be composable if and only if $g \circ f$ exists (that is, $\mathrm{dom}(g) = \mathrm{cod}(f)$).

2.1 Some sheaf theory

Let X be a topological space. We can form the category $\mathbf{Op}(X)$, whose objects are the open subsets of X, and the morphisms are the inclusions; in other words, it is the poset of open subsets.

Definition 2.1. Let $\mathcal{C}$ be a category. The category of $\mathcal{C}$**-presheaves** or $\mathcal{C}$**-valued presheaves** over X is the functor category $\mathbf{PSh}_{\mathcal{C}}(X) = [\mathbf{Op}(X)^{\mathsf{op}}, \mathcal{C}]$. Unless there is any ambiguity, we omit the category $\mathcal{C}$.

Let $\mathcal{F}$ be a presheaf. We say $\mathcal{F}$ is a **sheaf** if, for all open subsets U and all open covers $\{U_i\}_{i \in I}$ of U, then the functor $\mathcal{F}|_{\mathcal{J}}$ has a limit and

$$\lim \mathcal{F}|_{\mathcal{J}} \cong \mathcal{F}(U)$$

where $\mathcal{J}$ is the full subcategory of $\mathbf{Op}(X)^{\mathsf{op}}$ consisting of the objects U_i and $U_i \cap U_j$ for each $i, j \in I$.

The full subcategory of $\mathbf{PSh}(X)$ whose objects are sheaves is denoted by $\mathbf{Sh}(X)$, with subscript $\mathcal{C}$ in case of ambiguity.

This general definition may be daunting, but when $\mathcal{C}$ has products, it is equivalent to saying that

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer. The morphisms are the ones induced by the inclusions

$$U_i \cap U_j \subseteq U_k \subseteq U$$

for $k \in \{i, j\}$.

Concretely, a set-valued sheaf is a presheaf such that for every open set U, every open cover $\{U_i\}_{i \in I}$ of U and every family $\{f_i\}_{i \in I}$ such that $f_i \in \mathcal{F}(U_i)$ and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$. An easy way to see this is to consider an arbitrary commutative diagram

$$* \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

and then use the equalizer property.

For nice enough categories $\mathcal{C}$, there exists a functor $\mathbf{PSh}_{\mathcal{C}}(X) \to \mathbf{Sh}_{\mathcal{C}}(X)$ left adjoint to the inclusion functor, a process called **sheafification**. For $\mathcal{F}$ a $\mathcal{C}$-presheaf, we may also call its sheafification $\mathcal{F}^{\mathsf{sh}}$ the sheaf associated to $\mathcal{F}$. In particular, $\mathbf{Set}$, $\mathbf{Ab}$, $\mathbf{CRing}$ all have this property. For more details, one can check [10].

Another helpful observation, if $\mathcal{C}$ has (co)limits of shape $\mathcal{J}$, then $\mathbf{Sh}(C)$ also has pointwise (co)limits of shape $\mathcal{J}$. In particular, products of sheaves are sheaves. It is easy to see this using the equalizer diagram above.

Definition 2.2. Let X, Y be topological spaces, $f \colon X \to Y$ continuous. If $\mathcal{F}$ is a sheaf on X, we define a presheaf $f_* \mathcal{F}$ on Y as

$$V \mapsto \mathcal{F}(f^{-1}(V))$$

which can easily verified to be a sheaf. This is the **direct image** sheaf on Y.

Let X be a topological space, $\mathcal{O}_X$ be a ring-valued sheaf, or sheaf of rings, over X. We call the pair $(X, \mathcal{O}_X)$ a *ringed space*. If $(Y, \mathcal{O}_Y)$ is another ringed space, a ringed space morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f \colon X \to Y$ is continuous and $f^\# \colon \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a sheaf morphism.

Definition 2.3. Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{F}$ be a sheaf of abelian groups over X. We say $\mathcal{F} = (\mathcal{F}, \mu)$ is a $\mathcal{O}_X$-**module** if $\mu \colon \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ is a morphism such that for each U open, $\mu_U \colon \mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is a morphism making $(\mathcal{F}(U), \mu_U)$ into an $\mathcal{O}_X(U)$-module.

Definition 2.4. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is **locally free** if there exists an open covering $\{U_i\}_{i \in I}$ of X such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module for each $i \in I$.

The above definition is useful, since for a scheme X, we have a bijective correspondence between vector bundles and locally free sheaves of finite rank, check [8] for more details. There is a problem, however; the category of locally free sheaves is not abelian, but one may consider the category of *coherent* $\mathcal{O}_X$-modules, which we will not define here. Check [10, p. 206] for the general definition, where $\mathcal{J}$ is the full subcategory of free $\mathcal{O}_X$-modules of finite rank, or [8] for the case when X is a scheme. It is (as seen in the aforementioned references) an abelian category, contains all locally free sheaves of finite rank, and has enough "finiteness" conditions to be easy to work with, particularly sheaf cohomology.

Let E be a vector bundle over a complex scheme X, and $\mathcal{E}$ be the locally free sheaf corresponding to E. We can recover the fiber of E at a point $x \in X$ from $\mathcal{E}$ using the following:

$$E(x) = \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}_x = \mathcal{E}_x / \mathfrak{m}_x \mathcal{E}_x$$

where $\mathbb{C}_x$ is the skyscraper sheaf with value $\mathbb{C}$ in open neighbourhoods of x, and $\mathfrak{m}_x$ is the unique maximal ideal of $\mathcal{O}_{X,x}$. This can be verified locally or check [15].

Commonly, one may study sheaves with some extra structure. One particular case is that of *Higgs sheaves*, on a smooth complex projective curve X.

Definition 2.5. Let X be a smooth complex projective curve, and write K_X for the canonical line bundle, as a locally free sheaf. A **Higgs sheaf** is a pair $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a coherent sheaf and $\phi \colon \mathcal{E} \to \mathcal{E} \otimes K_X$ is a morphism.

One may also define morphisms between Higgs sheaves which preserve its structure, and call it the category of Higgs sheaves.

Higgs bundles correspond to Higgs sheaves $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is locally free, that is, a vector bundle. Check [3] for more details about this subject.

2.2 Quick introduction to representations over a category

Let $\mathcal{C}, \mathcal{D}$ be categories. A **representation** of $\mathcal{C}$ in $\mathcal{D}$, or a $\mathcal{D}$-**representation** of $\mathcal{C}$ is nothing more than a functor $F\colon \mathcal{C} \to \mathcal{D}$. Although deceivingly simple and redundant as it may appear, in certain contexts it gives the correct mindset for a problem.

Example 2.6. Let G be a group, regarded as a one object category whose morphisms are isomophisms, and consider the category $\mathbb{C}$-**Vect**$_{\mathsf{fg}}$ of finite dimensional vector spaces over $\mathbb{C}$. A linear representation of G may be regarded as a functor $\rho\colon G \to \mathbb{C}$-**Vect**$_{\mathsf{fg}}$; to see why, the single object $* \in G$ selects a vector space $V = \rho_*$, and we have a function

$$\rho_{*,*}\colon \mathsf{Hom}(*, *) \to \mathsf{Hom}(V, V)$$

given by ρ. $\mathsf{Hom}(*, *)$ with the composition operation is (isomorphic to) the group G, and this function factors uniquely through the inclusion $\mathsf{Aut}(V) \subseteq \mathsf{Hom}(V, V)$. This unique function is a standard representation $G \to \mathsf{Aut}(V)$; functoriality guarantees that $\rho_1 = \mathsf{id}$ and $\rho_{gh} = \rho_g \circ \rho_h$, so we have a group morphism.

For a richer supply of examples, but mostly for how important it is, we introduce the following notion:

Definition 2.7. A **quiver** or **digraph** is a quadruple $Q = (V, A, s, t)$ such that $s, t\colon A \to V$. We say that

- V is a set of *vertices* or *objects*;
- A is a set of *arrows*;
- s, t are functions which specify the *source* and *target*, respectively, of an arrow.

We can associate to each quiver $Q = (V, A, s, t)$ a category $\mathcal{F}Q$, the *free category generated* by Q. As one can describe the free monoid generated by a set, the construction of $\mathcal{F}Q$ follows a similar train of thought: Let $\mathcal{M}A$ be the free monoid generated by the set A. The category $\mathcal{F}Q$ is defined as follows:

- $\mathsf{Ob}(\mathcal{F}Q) = V$ is the set of objects;

- $\text{Mor}(\mathcal{F}Q) = V \cup \{\, a_1 \ldots a_n \in \mathcal{M}A \mid s(a_k) = t(a_{k+1}) \text{ for } k < n \,\}$ is the set of morphisms;
- $\text{dom}(a_1 \ldots a_n) = s(a_n)$, $\text{dom}(i) = i$;
- $\text{cod}(a_1 \ldots a_n) = t(a_1)$, $\text{cod}(i) = i$;
- id is the inclusion $\text{Ob}(\mathcal{F}Q) \subseteq \text{Mor}(\mathcal{F}Q)$.

We also say two morphisms f, g are composable if $\text{dom}(g) = \text{cod}(f)$, and define their composition to be $g \circ f$, given by concatenation. By virtue of the composable condition, $g \circ f$ is also a morphism. One can easily check that this data satisfies the conditions for being a category. For an appropriate notion of morphism between quivers, it can be shown that given such a morphism ϕ, there is a corresponding functor $\mathcal{F}\phi$. The data of this correspondence can be packed into a functor $\mathcal{F}\colon \textbf{Quiv} \to \textbf{Cat}$, left adjoint to the forgetful functor (see [11, p. 49-50]).

Representations of the form $\mathcal{F}Q \to \mathcal{C}$ for a quiver Q and a category $\mathcal{C}$ are what we call *quiver representations*. For each diagram one can draw down there is a list of examples. We will exhibit a few examples after defining the composition algebra next.

In the case that $\mathcal{T}$ is a tensor category (that is, an abelian category which is also symmetric monoidal, whose tensor product is additive in both arguments (see [5] for details on these objects, but note that our definition of tensor category conflicts with theirs)), we have a way to "algebrize" a category of representations. Before we jump into that, we have to introduce a few notions:

Definition 2.8. Let $\mathcal{M} = (\mathcal{M}, \otimes, I)$ be a monoidal category. A **monoid** in $\mathcal{M}$ is a triple (M, μ, ϵ), where $M \in \mathcal{M}$, $\mu\colon M \otimes M \to M$ and $\epsilon\colon I \to M$. This data is subject to the following condition: the diagrams

$$
\begin{array}{ccc}
(M \otimes M) \otimes M & \xrightarrow{\ \alpha\ } & M \otimes (M \otimes M) \\
{\scriptstyle \mu \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle 1 \otimes \mu} \\
M \otimes M \xrightarrow{\ \mu\ } & M & \xleftarrow{\ \mu\ } M \otimes M
\end{array}
\qquad
\begin{array}{ccc}
& M \otimes M & \\
{\scriptstyle \epsilon \otimes 1}\nearrow & \big\downarrow{\scriptstyle \mu} & \nwarrow{\scriptstyle 1 \otimes \epsilon} \\
I \otimes M \xrightarrow[\ \lambda\ ]{} & M & \xleftarrow[\ \rho\ ]{} M \otimes I
\end{array}
$$

must commute, where $\alpha\colon (-\otimes-)\otimes- \Rightarrow -\otimes(-\otimes-)$ is the associator and $\lambda_X\colon I \otimes X \to X$, $\rho_X\colon X \otimes I \to X$ are the left and right unitors, respectively. These diagrams are, respectively, the associativity and identity diagrams.

Furthermore, if $\mathcal{M}$ is symmetrical, a **commutative monoid** is a monoid that also makes

the following diagram commute

$$
\begin{array}{ccc}
M \otimes M & \xrightarrow{\;\gamma\;} & M \otimes M \\
& \searrow{\scriptstyle\mu} \quad {\scriptstyle\mu}\swarrow & \\
& M &
\end{array}
$$

where $\gamma_{A,B}\colon A \otimes B \to B \otimes A$ is the braiding.

For two monoids (M, μ, ϵ) and (N, η, ϵ'), a **monoid morphism** is a morphism $f\colon M \to N$ making the following diagrams commute:

$$
\begin{array}{ccc}
M \otimes M & \xstackrel{f \otimes f} & N \otimes N \\
\downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\eta} \\
M & \xrightarrow{\;f\;} & N
\end{array}
\qquad\qquad
\begin{array}{ccc}
& I & \\
{\scriptstyle\epsilon}\swarrow & & \searrow{\scriptstyle\epsilon'} \\
M & \xrightarrow{\;f\;} & N
\end{array}
$$

The monoids with the morphisms above give rise to the category $\mathbf{Mon}_{\mathcal{M}}$ of monoid objects in $\mathcal{M}$ with monoid morphisms. The commutative monoid objects of a symmetrical monoidal category form $\mathbf{CMon}_{\mathcal{M}}$, with the same morphisms. For more details, check [4].

Example 2.9. We list a few examples:

- In $(\mathbf{Set}, \times, *)$, the category of monoid objects is just the category of monoids with monoid morphisms;

- For a ring R, in $(R\text{-}\mathbf{Mod}, \otimes_R, R)$ the category of monoid objects is the category $R\text{-}\mathbf{Alg}$ of R-algebras, with R-algebra morphisms. In the case $R = \mathbb{Z}$ we get the category of noncommutative rings. We get similar results if we consider the category of commutative monoids.

- The category $\mathbb{Z}\text{-}\mathbf{Mod}$ is symmetric monoidal, and its category of commutative monoids is the category of rings.

- For a category $\mathcal{C}$, a monoid object in the monoidal category $([\mathcal{C}, \mathcal{C}], \circ, \mathrm{id}_{\mathcal{C}})$ is a monad.

- Let $\mathcal{M}$ be a monoidal category, $\mathcal{C}$ be any category. Then the functor category $[\mathcal{C}, \mathcal{M}]$ is also (pointwise) monoidal. Its category of monoids is equivalent to the functor category $[\mathcal{C}, \mathbf{Mon}_{\mathcal{M}}]$, and likewise for commutative monoids if $\mathcal{M}$ is symmetrical.

- A consequence of two of the above examples: If $\mathcal{M} = \mathsf{Sh}_{\mathbf{Ab}}(X)$ is the category of abelian group valued sheaves over X, then $\mathbf{CMon}_{\mathcal{M}}$ is the category of ring valued sheaves.

Definition 2.10. Let $\mathcal{M}$ be a monoidal category, and let $R = (R, \mu, \epsilon)$ be a monoid in $\mathcal{M}$. A **left module** over R or **left R-module** is a tuple $M = (M, \eta)$ where $M \in \mathcal{M}$ and a morphism

$\eta \colon R \otimes M \to M$. This morphism makes the following diagrams commute:

$$
\begin{array}{ccc}
(R \otimes R) \otimes M & \xrightarrow{\ \alpha\ } & R \otimes (R \otimes M) \\
\downarrow{\scriptstyle \mu \otimes 1} & & \downarrow{\scriptstyle 1 \otimes \eta} \\
R \otimes M \xrightarrow{\ \eta\ } M & \xleftarrow{\ \eta\ } & R \otimes M
\end{array}
\qquad
\begin{array}{ccc}
 & & R \otimes M \\
 & \nearrow{\scriptstyle \epsilon \otimes 1} & \downarrow{\scriptstyle \eta} \\
I \otimes M & \xrightarrow[\lambda]{} & M
\end{array}
$$

For two R-modules $M = (M, \eta)$ and $M' = (M', \eta')$, a R-module morphism is a morphism $f \colon M \to M'$ making the following diagram commute:

$$
\begin{array}{ccc}
R \otimes M & \xrightarrow{\ 1 \otimes f\ } & R \otimes M' \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\
M & \xrightarrow{\ f\ } & M'
\end{array}
$$

This forms a category of R-**Mod**$_{\mathcal{M}}$, of R-module objects in $\mathcal{M}$ with R-module morphisms.

Example 2.11. We give two common examples:

- Since a ring R is a commutative monoid object in **Ab**, the category R-**Mod** is the category of R-module objects.

- Let $\mathcal{O}_X$ be a sheaf of rings over X. Then $\mathcal{O}_X$-**Mod** is the category of $\mathcal{O}_X$-module objects.

Theorem 2.12. Let $\mathcal{C}$ be a small category with a finite set of objects, and let $\mathcal{T} = (\mathcal{T}, \otimes, I)$ be a tensor category with coproducts, whose tensor product preserves them. Then we have

$$[\mathcal{C}, \mathcal{T}] \cong A\text{-}\mathbf{Mod}_{\mathcal{T}}$$

for $A = (A, \mu, \epsilon)$ an appropriate monoid in $\mathcal{T}$. This is what we call the **category** or **convolution** monoid of $\mathcal{C}$ over $\mathcal{T}$. In the case $\mathcal{T} = R$-**Mod**, replace "monoid" by "R-algebra", and write $A = R\mathcal{C}$ in this case.

This result is a slight generalization of the equivalence of categories

$$[\mathcal{C}, R\text{-}\mathbf{Mod}] \cong R\mathcal{C}\text{-}\mathbf{Mod},$$

and has essentially the same proof, where $R\mathcal{C}$ is the free R-module generated by the morphisms in $\mathcal{C}$, and the product is induced by composition in $\mathcal{C}$.

PROOF. We will imitate the idea for the case $\mathcal{T} = R$-**Mod** as closely as we can. The first step is to define the monoid:

$$A = \bigoplus_{f \in \mathrm{Mor}\,\mathcal{C}} I_f$$

and write $\iota_f\colon I \to A$ for the inclusions. If $f = \mathrm{id}_c$ for some object $c \in \mathcal{C}$, write ι_c instead.

Define $\mu\colon A \otimes A \to A$ to be the unique morphism making the following diagram commute for all morphisms f, g:

$$
\begin{array}{ccc}
I \otimes I & \xrightarrow{\iota_g \otimes \iota_f} & A \otimes A \\
{\scriptstyle \mu_{g,f}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu} \\
I & \xrightarrow{\eta_{g,f}} & A
\end{array}
$$

where we let $\mu_{g,f}$ be the isomorphism $I \otimes I \cong I$ (the left/right unitor) for all f, g, and $\eta_{g,f} = \iota_{g \circ f}$ if f, g are composable, zero otherwise.

Since $\mathcal{C}$ has a finite number of objects, let

$$
\epsilon = \sum_{c \in \mathcal{C}} \iota_c \colon I \to A
$$

One can easily check that (A, μ, ϵ) is a monoid, just verify the associativity diagram commutes for the maps $\mu_{g,f}$, and the result follows by the use of universal properties. The commutativity of the identity diagram follows by a similar idea.

Let $\mathcal{F}, \mathcal{G}\colon \mathcal{C} \to \mathcal{T}$ be functors. Define

$$
M = \bigoplus_{c \in \mathcal{C}} \mathcal{F}c
$$

with inclusions $i_c\colon \mathcal{F}c \to M$ and let $\eta\colon A \otimes M \to M$ be the unique morphism making the following diagram commute

$$
\begin{array}{ccc}
I \otimes \mathcal{F}c & \xrightarrow{\iota_f \otimes i_c} & A \otimes M \\
{\scriptstyle \eta_{f,c,d}}\Big\downarrow & & \Big\downarrow{\scriptstyle \eta} \\
\mathcal{F}d & \xrightarrow{i_d} & M
\end{array}
$$

where $\eta_{f,c,d} = Ff \circ \lambda$ if $f\colon c \to d$ and the zero morphism otherwise. Checking that (M, η) is an A-module is not unlike showing A is a monoid. We can also define the A-module (M', η') given by $\mathcal{G}$.

For a natural transformation $\phi\colon \mathcal{F} \to \mathcal{G}$, we define $\phi^*\colon M \to M'$ to be the unique morphism such that the following diagram

$$
\begin{array}{ccc}
\mathcal{F}c & \xrightarrow{i_c} & M \\
{\scriptstyle \phi_{c,d}^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi^*} \\
\mathcal{G}d & \xrightarrow{i_d'} & M'
\end{array}
$$

commutes, where $\phi_{c,d}^* = \phi_c$ if $c = d$ and the zero morphism otherwise. Verifying ϕ^* is a module morphism is similar to showing that (M, η) is a module: we check the commutativity

of the diagrams at the "generators" (I, $\mathcal{F}c$, $\mathcal{G}c$), then using universal properties to extend to M.

The above construction should make it clear to what are the steps to reverse it; let M be an A-module. If A were an R-algebra, we would decompose M into modules e_cM, where $\{e_c\}_{c \in \mathcal{C}}$ is a complete set of idempotents of M. We can also do this is general, by considering the of the following chain of morphisms:

$$M \xrightarrow{\lambda^{-1}} I \otimes M \xrightarrow{\iota_c \otimes 1} A \otimes M \xrightarrow{\eta} M$$

We suggestively write e_cM for the (co)image of the above triple composite. It is not dificult to show that

$$M \cong \bigoplus_{c \in \mathcal{C}} e_cM$$

which suggests defining $\mathcal{F}c = e_cM$ for each $c \in \mathcal{C}$, just like it is done in the case $\mathcal{C} = R\text{-}\mathbf{Mod}$. Similarly, for each morphism f, and objects c, d, define $\eta_{f,c,d}$ to be the composite

$$I \otimes \mathcal{F}c \xrightarrow{\iota_f \otimes i_c} A \otimes M \xrightarrow{\eta} M \xrightarrow{\pi_d} \mathcal{F}d.$$

Showing $\eta_{f,c,d}$ behaves as we expect amounts to checking that the "restriction" of the action η by $f \colon c' \to d'$ to $\mathcal{F}c$, at $\mathcal{F}d$, is 0 unless $c = c'$ and $d = d'$, in which case we define $Ff = \eta_{f,c,d} \circ \lambda^{-1}$. This is an immediate calculation, just like in $\mathcal{C} = R\text{-}\mathbf{Mod}$. It is also straightforward to check that $\mathcal{F}$ is a functor, and that $\eta_{f,c,d}$ as defined also makes (2.2) commute.

Finally, let $\phi \colon (M, \eta) \to (M', \eta')$ be a module morphism. Let $\mathcal{F}, \mathcal{G}$ be the functors associated to (M, η) and (M', η'), respectively. As we did to define $\mathcal{F}f$ for $f \colon c \to d$, we consider

$$\mathcal{F}c \xrightarrow{i_c} M \xrightarrow{\phi} M' \xrightarrow{\pi'_d} \mathcal{G}d$$

and let $\phi^*_{c,d}$ be this composite, which is zero possibly except when $c = d$, and we recover a natural transformation $\phi^* \colon \mathcal{F} \to \mathcal{G}$ with $\phi^*_c = \phi^*_{c,c}$. $\qquad\qquad\qquad\qquad\square$

Some comments: we can drop the requirement that $\mathcal{C}$ is finite, losing the unit in the process, so we get "semigroup" objects instead. In the particular case $\mathcal{T} = R\text{-}\mathbf{Mod}$, we get R-algebras without unit.

There is a "dual" construction, appropriate for locally finite posets, which recovers the incidence algebra. In this case, we must also require that each morphism f in $\mathcal{C}$ has a finite

number of factorizations. We will not go through the details, but in the case $\mathcal{C}$ finite, this construction and the dual one are the same, check [17] for more details.

In the case $\mathcal{C} = \mathcal{F}Q$ for a quiver Q, $\mathcal{T} = R$-**Mod**, we say A is the *path R-algebra* of the representation.

Example 2.13. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{T}$ be a representation satisfying the hypothesis of the previous result, let A be the composition monoid constructed above.

- If $\mathcal{C} = G$ is a group, $\mathcal{T} = \mathbb{C}$-**Mod**, we have $A = \mathbb{C}[G]$. In fact, we can replace $\mathbb{C}$ by any ring R.

- If $\mathcal{C} = P$ is a finite poset, $\mathcal{T} = R$-**Mod**, then due to a remark above, A is commonly known as the *incidence* algebra. Usually, we have $R = \mathbb{Z}$.

- Let Q be the quiver

$$\bullet \,\circlearrowleft\, x$$

and $\mathcal{T} = R$-**Mod**. Then $A = R[x]$ is the ring of polynomials with coefficients in R.

- Let Q be the quiver

$$\bullet \,\overset{}{\longleftarrow}\, \boxed{x_1}\; x_2\; \cdots\; \Big)\; x_n$$

and $\mathcal{T} = R$-**Mod**. Then $A = R\langle x_1, x_2, \ldots, x_n \rangle$ is the free R-algebra on $\{x_1, x_2, \ldots, x_n\}$, or the ring of polynomials in n noncommuting variables with coefficients in R.

2.3 Grothendieck categories

Definition 2.14. Let $\mathcal{J}$ be a category. If

- $\mathcal{J}$ is nonempty,
- for each pair $i, j \in \mathcal{J}$ of objects, there exists an object $k \in \mathcal{J}$ with arrows $f\colon i \to k$ and $g\colon j \to k$,
- and for each pair of parallel arrows $f, g\colon i \to j$ there exists an object k with an arrow $h\colon j \to k$ such that $hf = hg$,

then we say that $\mathcal{J}$ is a **filtered** category, and a colimit of a functor $\mathcal{F}\colon \mathcal{J} \to \mathcal{C}$ for some category $\mathcal{C}$ is called a **filtered colimit**.

Definition 2.15. Let $\mathcal{C}$ be an abelian category.

- We say $\mathcal{C}$ has **exact** filtered colimits if for all filtered categories $\mathcal{J}$ and functors $F, G\colon \mathcal{J} \to \mathcal{C}$ such that for each $j \in \mathcal{J}$

$$0 \to F_j \to G_j$$

  is exact, then

$$0 \to \operatorname{colim} F \to \operatorname{colim} G$$

  is also exact, whenever both colimits exist.

- We say $g \in \mathcal{C}$ is a **generator** if, given $f\colon c \to d$ we have $f\phi = 0$ for all $\phi\colon g \to c$, then $f = 0$.

- We say $\mathcal{C}$ is a **Grothendieck** category if it has all coproducts, exact filtered colimits and has a generator.

Note that in the definition of exact filtered colimits, we only required left exactness, since all colimits are necessarily right exact (they are left adjoints).

Example 2.16. Many important categories are Grothendieck:

- R-**Mod** for each ring R, and in particular, **Ab**;
- $\mathcal{O}_X$-**Mod** for each ringed space $(X, \mathcal{O}_X)$;
- **Lex**($\mathcal{C}^{\text{op}}$, **Ab**), category of left-exact functors with natural transformations, for each abelian category $\mathcal{C}$. We have a fully faithful and exact embedding of $\mathcal{C}$ into the above category, not unlike Yoneda's embedding ([6]).

Lemma 2.17. If a direct summand is a generator, then the direct sum is a generator.

PROOF. Let $\{G_k\}_{k \in K}$ be a collection of objects in $\mathcal{C}$, and suppose that there exists $i \in K$ such that G_i is a generator. Let $f\colon X \to Y$ be a map such that for all $g\colon \bigoplus_k G_k \to X$ we have $fg = 0$. Let $\phi_i\colon G_i \to X$ be any morphism, and for $k \neq i$, let $\phi_k\colon G_k \to X$ be the zero morphism; then there exists a unique morphism $\phi\colon \bigoplus_k G_k \to X$ factoring ϕ_k through the inclusions. In particular, $\phi\iota_i = \phi_i$, so $f\phi_i = 0$. This implies $f = 0$, hence the sum is a generator. $\qquad\square$

Let $\mathcal{C}$ be a Grothendieck category. $\mathcal{C}$ has several nice properties which make it very interesting to study. Among those:

- $\mathcal{C}$ has an injective cogenerator, and in particular, enough injectives;

- $\mathcal{C}$ has all limits;

- $\mathcal{C}$ is a full subcategory of R-**Mod**, where $R = \mathrm{Hom}(G, G)$, for G a generator of $\mathcal{C}$;

- If a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves (co)limits, then it is a (left) right adjoint;

- If a functor $F\colon \mathcal{C}^{\mathsf{op}} \to$ **Set** preserves limits, then it is representable, that is, $F \cong$ $\mathrm{Hom}(-, C)$ for some $C \in \mathcal{C}$.

Check [10] for more details on Grothendieck categories.

Chapter 3

Representations of diagrams

For a category $\mathcal{I}$, a **diagram** over $\mathcal{I}$ is a functor $\Phi\colon \mathcal{I} \to \mathbf{Cat}$. Given a morphism a in $\mathcal{I}$, we will normally write a^* or a_* for Φ_a, depending on context.

3.1 Getting our feet wet

Definition 3.1. Let $\mathcal{I}$ be a category, let Φ be a diagram over $\mathcal{I}$, and, for each arrow $p\colon i \to j$, write $p_* = \Phi_a$ whenever confusion won't arise. We define the category $\mathbf{Rep}_*(\Phi)$ as follows. An object X is determined by the following data:

- For each object $i \in \mathcal{I}$, an object $X_i \in \Phi_i$;
- For each arrow $p\colon i \to j$ in $\mathcal{I}$, an arrow $X_p\colon p_* X_i \to X_j$

This data is subject to the following condition: for p, q composable morphisms, we have

$$X_{q \circ p} = X_q \circ q_* X_p \tag{3.1}$$

This condition is motivated by requiring the following diagram to be commutative:

$$
\begin{array}{ccccc}
q_* p_* X_i & \xrightarrow{\;q_* X_p\;} & q_* X_j & \xrightarrow{\;X_q\;} & X_k \\
\big\| & & & & \big\| \\
(q \circ p)_* X_i & & \xrightarrow{\quad X_{q \circ p} \quad} & & X_k
\end{array}
$$

A morphism $f\colon X \to Y$ is, for each $i \in \mathcal{I}$, an arrow $f_i\colon X_i \to Y_i$, and we require these arrows to be such that the following diagram

$$
\begin{array}{ccc}
p_*X_i & \xrightarrow{\ X_p\ } & X_j \\
\big\downarrow{\scriptstyle p_*f_i} & & \big\downarrow{\scriptstyle X_j} \\
p_*Y_i & \xrightarrow{\ Y_p\ } & Y_j
\end{array}
$$

is commutative.

The goal is to take a (nonabelian) category of interest in which we would like to do homological algebra, and embed it in a full subcategory of $\mathbf{Rep}_*(\Phi)$ for an adequate diagram Φ.

We consider a whole class of simple examples. Let $\Phi\colon \mathcal{I} \to \mathbf{Cat}$ be a diagram such that $\Phi_i = \mathcal{C}$ for all $i \in \mathcal{I}$ and $p_* = \mathrm{id}_{\mathcal{C}}$ for all $p\colon i \to j$; the constant diagram to $\mathcal{C}$. Then $\mathbf{Rep}_*(\Phi)$ is equivalent to the functor category $[\mathcal{I}, \mathcal{C}]$, which is just the category of $\mathcal{I}$-representations over $\mathcal{C}$, as defined in the previous chapter.

We can quickly check this: for each $i \in \mathcal{I}$, we have an object $X_i \in \mathcal{C}$, and for each $p\colon i \to j$, a morphism $X_p\colon X_i \to X_j$. These morphisms satisfy

$$
X_{q \circ p} = X_q \circ p_* X_p = X_q \circ X_p
$$

for each pair of composable morphisms p, q, hence $X\colon \mathcal{I} \to \mathcal{C}$ is a functor. A morphism $f\colon X \to Y$ in $\mathbf{Rep}_*(\Phi)$ satisfies

$$
\begin{array}{ccc}
X_i & \xrightarrow{\ X_p\ } & X_j \\
\big\downarrow{\scriptstyle f_i} & & \big\downarrow{\scriptstyle f_j} \\
Y_i & \xrightarrow{\ Y_p\ } & Y_j
\end{array}
$$

which is just a natural transformation, and all natural transformations give morphisms in $\mathbf{Rep}_*(\Phi)$.

Example 3.2. Linear representations of a group, R-module representations of a poset and quiver representations are all special cases of the above, from the previous chapter.

Example 3.3. We have another class of examples; suppose that $\mathcal{I}$ is a discrete category, a category with no non-identity arrows. Then $\mathbf{Rep}_*(\Phi) \cong \prod_{i \in I} \Phi_i$.

To quickly check this, we have for each $i \in \mathcal{I}$ an object Φ_i, and for each morphism $p\colon i \to i$, necessarily an identity, we have a morphism $X_p\colon X_i \to X_i$, an identity as well.

For $X, Y \in \mathbf{Rep}_*(\Phi)$, a morphism is just a map $f_i\colon X_i \to Y_i$ for each $i \in \mathcal{I}$, so the condition (3.1) is automatically satisfied.

Now we consider nontrivial examples: let $\mathcal{M}$ be a monoidal category, and consider a family $\{M_p\}_p$ of objects in $\mathcal{M}$ indexed by the morphisms of $\mathcal{I}$ satisfying

$$M_{q \circ p} = M_q \otimes M_p$$

We define the diagram Φ to be constant on objects, equal to $\mathcal{M}$, and for each arrow $p \colon i \to j$, we define $p_* = M_p \otimes -$. We recover the categories of twisted representations, as defined in [7], letting $\mathcal{M} = R\text{-}\mathbf{Mod}$ for a ring R or $\mathcal{M} = \mathcal{O}_X\text{-}\mathbf{Mod}$ for a sheaf of rings $\mathcal{O}$ over a space X. Furthermore, we observe that if $\mathcal{I} = \mathcal{F}Q$ for a quiver Q, the family $\{M_p\}_p$ is determined by the objects on the arrows of Q. We give a particular case:

Example 3.4. Higgs sheaves over a curve can be described in the following way: Let X be a complex projective curve, let K_X be the canonical line bundle as a locally free $\mathcal{O}_X$-module. We define Q to be the quiver

$$\bullet \,\circlearrowright\, a$$

and define Φ to be a diagram over $\mathcal{F}Q$ such that $\Phi_\bullet = \mathbf{Coh}\,X$ and $a_* = - \otimes K_X$. An object in $\mathbf{Rep}_*(\Phi)^{\mathsf{op}}$ is then an object $E \in \mathbf{Coh}\,X$ and a morphism $\phi \colon E \to E \otimes K_X$. Later we will be allowed to conclude that this category is abelian, because $\mathbf{Coh}\,X$ is abelian and a_* is left exact.

Example 3.5. A final and more involved example which we will study more in-depth in the next chapter: quasiparabolic bundles. Given a smooth complex projective curve X, with reduced divisor $D = p_1 + \ldots + p_n$, a quasiparabolic sheaf $(\mathcal{E}, E_*)$ on X over D is given by the following data:

- A coherent $\mathcal{O}_X$-module $\mathcal{E}$;
- For each point $p \in D$, a positive integer m_p and a filtration

$$\mathcal{E}_p \otimes \mathbb{C}_p = E_{p,0} \supseteq E_{p,1} \supseteq \ldots \supseteq E_{p,m_p} \supseteq E_{p,m_p+1} = 0$$

  We say m_p is the **length** of the parabolic structure at p.

We get a quasiparabolic bundle when we have $\mathcal{E}$ locally free. Given quasiparabolic bundles $(\mathcal{E}, E_*), (\mathcal{F}, F_*)$, a morphism $\phi \colon E \to F$ is defined only when the filtrations are of the same length at every point $p \in D$, and such a morphism satisfies $\phi(E_{p,j}) \subseteq F_{p,j}$ for all p, j. This can be restated by saying that the morphism must preserve filtrations.

Write $\textbf{QPar}_m(X, D)$ for this category of quasiparabolic sheaves of length m. Suppose for simplicity that $D = p$. We proceed to show how to embed this category in $\textbf{Rep}_*(\Phi)$ for an appropriate Φ: let Q be the quiver

$$0 \xrightarrow{a} 1 \to \ldots \to m$$

and define $\Phi_0 = \textbf{Coh}(X)$, $\Phi_k = \textbf{Vect}^{\text{op}}$ for $k > 0$, a_* to be the functor opposite to $\text{Hom}(-, \mathbb{C}_p)$, and $b_* = \text{id}$ for arrows $b \neq a$. This gives us a diagram $\Phi \colon \mathcal{F}Q \to \textbf{Cat}$, and the corresponding category $\textbf{Rep}_*(\Phi)$. Concretely, this is the category with objects a coherent sheaf $\mathcal{E}$ and vector spaces $V_1, \ldots, V_m$, with a chain of linear maps

$$\text{Hom}(\mathcal{E}, \mathbb{C}_p) \cong (\mathcal{E}_p)^* \leftarrow V_1 \leftarrow \ldots \leftarrow V_m$$

Note that the direction of the linear maps is the opposite of the quiver's arrows; this is because the "original" morphisms live in $\textbf{Vect}^{\text{op}}$.

Let $\mathcal{E}$ be a quasiparabolic sheaf of length m. Then we have a filtration

$$E_0 \supset \ldots \supset E_{m+1} = 0$$

where $E_0 = \mathcal{E}_p$. So for each $i \geq 0$, let $V_i = (E_0/E_{m+1-i})^*$. We justify this in a series of steps: We started with

$$E_0 \hookleftarrow \ldots \hookleftarrow E_{m+1}$$

where the maps are inclusions, so by defining $E^k = E_0/E_k$, we get

$$E^0 \twoheadleftarrow \ldots \twoheadleftarrow E^{m+1} \cong E_0$$

where the morphisms are the projections induced by the inclusions above. Next, we dualize:

$$(E^0)^* \twoheadleftarrow \ldots \twoheadleftarrow (E^{m+1})^* \cong (E_0)^*$$

note that this diagram is in $\textbf{Vect}^{\text{op}}$. Finally, reversing the indices ($V_k = (E^{m+1-k})^*$) we get

$$V_{m+1} \to V_m \to \ldots \to V_0 = (E_0)^* \cong \text{Hom}(\mathcal{E}, \mathbb{C}_p)$$
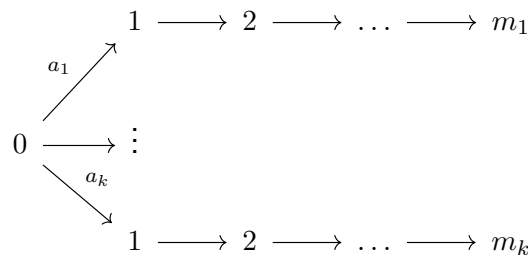
which is a chain of linear maps, as above (that is, a diagram in $\textbf{Vect}$). Removing $V_{m+1} = 0$, we have a chain of length m.

The morphisms $(\mathcal{E}, V_\bullet) \to (\mathcal{F}, W_\bullet)$ in $\textbf{Rep}_*(\Phi)$ are precisely those which make the rectangles

$$
\begin{array}{ccccccc}
E_p^* & \longleftarrow & V_1 & \longleftarrow & \ldots & \longleftarrow & V_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_p^* & \longleftarrow & W_1 & \longleftarrow & \ldots & \longleftarrow & W_m
\end{array}
$$

commute. If the objects are (the image of) a parabolic sheaf, all parabolic sheaf morphisms arise in this way, by preservation of filtrations. This exhibits the parabolic sheaves as a full subcategory of $\mathbf{Rep}_*(\Phi)$, which is abelian.

For general $D = p_1 + \ldots + p_k$, we can use the following quiver

$$
\begin{array}{c}
1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow m_1 \\[2em]
a_1 \nearrow \\
0 \longrightarrow \vdots \\
a_k \searrow \\[2em]
1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow m_k
\end{array}
$$

that is, a tree rooted at 0 with k branches, each with the length of its filtration, and Φ_{a_j} the opposite functor of $\mathrm{Hom}(-, \mathbb{C}_{p_j})$.

What is not clear right now is why we had to go to such length to define Φ as we did, which made the embedding difficult to construct and understand. The reason is simple: We require a_* to be right-exact, so thaat we can guarantee that $\mathbf{Rep}_*(\Phi)$ is abelian. In what follows, we will explore what conditions we need on a diagram Φ to get more good properties on $\mathbf{Rep}_*(\Phi)$.

3.2   Grothendieck construction and the dual version

Let Φ be a diagram over $\mathcal{I}$, and consider the large category $\mathbf{Cat}_{*,\mathrm{clx}}$ of *colax-pointed* categories:

- Objects are given by pairs $(\mathcal{C}, c)$ where $\mathcal{C}$ is a non-empty category and c is an object of $\mathcal{C}$;
- Morphisms are given by pairs $(F, f) \colon (\mathcal{C}, c) \to (\mathcal{D}, d)$ where $F \colon \mathcal{C} \to \mathcal{D}$ is a functor and $f \colon d \to Fc$ is a morphism.
- For $(F, f), (G, g)$ composable arrows, composition is given by

$$(G, g) \circ (F, f) = (G \circ F, Gf \circ g)$$

Write $U\colon \mathbf{Cat}_{*,\mathsf{clx}} \to \mathbf{Cat}$ for the forgetful functor. The **Grothendieck construction** for Φ is the pullback $\Pi\colon \int \Phi \to \mathcal{I}$ of U along Φ, given by the following pullback diagram:

$$
\begin{array}{ccc}
\int \Phi & \longrightarrow & \mathbf{Cat}_{*,\mathsf{clx}} \\
\Big\downarrow{\scriptstyle \Pi} & & \Big\downarrow{\scriptstyle U} \\
\mathcal{I} & \xrightarrow{\ \Phi\ } & \mathbf{Cat}
\end{array}
$$

In other words, $\int \Phi$ has objects (i, c) where $i \in \mathcal{I}$ and $c \in \Phi_i$ and morphisms

$$(p, f)\colon (i, c) \to (j, d)$$

where $p\colon i \to j$ and $f\colon d \to p^*c$, where $p^* = \Phi_p$. So, the Grothendieck construction Π of Φ is the functor projecting to the first coordinate.

Example 3.6. Suppose $\mathcal{I}$ is a set (discrete category) and Φ be a diagram over $\mathcal{I}$. Then $\int \Phi \cong \sum_{i \in I} \Phi_i$.

Note that the category $\sum_{i \in I} \Phi_i$ whose objects are pairs (i, c) for $c \in \Phi_i$, with morphisms given by $(i, f)\colon (i, c) \to (i, d)$ where $f\colon c \to d$ in Φ_i.

Example 3.7. Let $\mathrm{Sh}\colon \mathbf{Top} \to \mathbf{Cat}$ be the functor sending each topological space to its category of sheaves of rings $\mathrm{Sh}(X)$ and each continuous map $f\colon X \to Y$ to the direct image functor $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. The category $\int \mathrm{Sh}$ in the pullback

$$
\begin{array}{ccc}
\int \mathrm{Sh} & \longrightarrow & \mathbf{Cat}_{*,\mathsf{clx}} \\
\Big\downarrow & & \Big\downarrow \\
\mathbf{Top} & \xrightarrow{\ \mathrm{Sh}\ } & \mathbf{Cat}
\end{array}
$$

is the category of ringed spaces, with ringed space morphisms between them: The objects are pairs $(X, \mathcal{O}_X)$ for X topological space and $\mathcal{O}_X$ a sheaf of rings over X, and a morphism $(f, f^\#)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ and a sheaf morphism $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Dually, we can get another construction by replacing $\mathbf{Cat}_{*,\mathsf{clx}}$ by $\mathbf{Cat}_{*,\mathsf{lx}}$, the category of *lax-pointed* categories, with the same objects and similar morphisms; for a morphism $(F, f)\colon (\mathcal{C}, c) \to (\mathcal{D}, d)$ we require $f\colon Fc \to d$ instead, and $(G, g) \circ (F, f) = (G \circ F, g \circ Gf)$ for $(F, f), (G, g)$ composable arrows. The following definition and result will clarify the relationship between both constructions.

Definition 3.8. For a diagram Φ over $\mathcal{I}$, we define the opposite diagram $\Phi^{\mathsf{op}} \colon \mathcal{I} \to \mathbf{Cat}$ such that Φ_i^{op} is the opposite category of Φ_i and $\Phi_p^{\mathsf{op}} \colon \Phi_i^{\mathsf{op}} \to \Phi_j^{\mathsf{op}}$ the opposite functor for $p \colon i \to j$ in $\mathcal{I}$.

Proposition 3.9. Let Φ be a diagram over $\mathcal{I}$, and consider the pullbacks

$$
\begin{array}{ccc}
\int \Phi & \longrightarrow & \mathbf{Cat}_{*,\mathsf{lx}} \\
{\scriptstyle \Pi^*}\downarrow & & \downarrow \\
\mathcal{I} & \xrightarrow{\ \Phi\ } & \mathbf{Cat}
\end{array}
\qquad
\begin{array}{ccc}
\int \Phi^{\mathsf{op}} & \longrightarrow & \mathbf{Cat}_{*,\mathsf{clx}} \\
{\scriptstyle \Pi_*}\downarrow & & \downarrow \\
\mathcal{I} & \xrightarrow{\ \Phi^{\mathsf{op}}\ } & \mathbf{Cat}
\end{array}
$$

Then $\int \Phi$ and $\int \Phi^{\mathsf{op}}$ are isomorphic categories.

PROOF. We only need to find functors $\int \Phi \to \mathbf{Cat}_{*,\mathsf{clx}}$ and $\int \Phi^{\mathsf{op}} \to \mathbf{Cat}_{*,\mathsf{lx}}$. The universal properties of the pullbacks guarantee there are unique morphisms from one category to the other, implying we have an isomorphism.

First, for $p \colon i \to j$ in $\mathcal{I}$, write $p_* = \Phi_p$, $p^* = \Phi_p^{\mathsf{op}}$, and for $f \colon c \to d$ in some category $\mathcal{C}$, write $f^{\mathsf{op}} \colon d \to c$ for the same morphism in $\mathcal{C}^{\mathsf{op}}$. Note that $p_*^{\mathsf{op}} = p^*$.

Let $G \colon \int \Phi \to \mathbf{Cat}_{*,\mathsf{clx}}$ be such that

$$
G(i,c) = (\Phi_i^{\mathsf{op}}, c) \quad \text{and} \quad G(p,f) = (p^*, f^{\mathsf{op}})
$$

To check this is well defined, note that if $f \colon p_* c \to d$ is a morphism in Φ_j, then $f^{\mathsf{op}} \colon d \to p^* c$ is a morphism in Φ_j^{op}. We can define the other functor in almost exactly the same way. $\square$

In this next proposition, we will show how to use the Grothendieck construction for a diagram Φ to give an easier definition of $\mathbf{Rep}_*(\Phi)$:

Proposition 3.10. Let Φ be a diagram over $\mathcal{I}$, consider its Grothendieck construction Π with lax-pointed categories. The category $\mathbf{Rep}_*(\Phi)$ is equivalent to the full subcategory of sections of Π of the functor category $[\mathcal{I}, \int \Phi]$.

PROOF. Let $X \colon \mathcal{I} \to \int \Phi$ be a section of Π. For $p \colon i \to j$ in $\mathcal{I}$, we can write, if we abuse notation, $X_i = (i, X_i)$ and $X_p = (p, X_p)$, it becomes clear that X is a representation, since $X_i \in \Phi_i$ and $X_p \colon p_* X_i \to X_j$, and the condition 3.1 is satisfied. That any representation gives a section of Π is immediate as well.

What remains is to check that the morphisms correspond to natural transformations. For X, Y sections, consider a natural transformation $f \colon X \to Y$ between them. For $i \in \mathcal{I}$, we

may (continue to abuse notation and) write $f_i = (\mathrm{id}_i, f_i)$. For a morphism $p\colon i \to j$, we have the following commutative diagram:

$$
\begin{array}{ccc}
X_i & \xrightarrow{\ X_p\ } & X_j \\
\downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_j} \\
Y_i & \xrightarrow{\ Y_p\ } & Y_j
\end{array}
$$

Here, the abuse of notation becomes deceiving, because of the composition of morphisms: remember that $Y_p \circ f_i = Y_p \circ p_* f_i$ (on the LHS, abuse of notation, on the RHS, actual composition) and $X_p\colon p_* X_i \to X_j$. So diagram above translates to

$$
\begin{array}{ccc}
p_* X_i & \xrightarrow{\ X_p\ } & X_j \\
\downarrow{\scriptstyle p_* f_i} & & \downarrow{\scriptstyle f_j} \\
p_* Y_i & \xrightarrow{\ Y_p\ } & Y_j
\end{array}
$$

which is just the diagram expressing the representation morphism condition. Equivalently, it is clear that a representation morphism gives a natural transformation between sections. $\square$

Example 3.11. For Φ a diagram over a discrete category, as in Example 3.3, we verify $\mathbf{Rep}_*(\Phi) \cong \prod_{i \in I} \Phi_i$; a way to define $\prod_i \Phi_i$ in terms of $\sum_i \Phi_i$ is exactly the category of functors $\mathcal{I} \to \sum_i \Phi_i$ which are sections of the "projection" $\sum_i \Phi_i \to \mathcal{I}$.

Next, we consider the dual concept: given such a diagram Φ, we can define the category $\mathbf{Rep}^*(\Phi)$:

Definition 3.12. For a category $\mathcal{I}$, let $\Phi\colon \mathcal{I} \to \mathbf{Cat}$ be a diagram, and write $p^* = \Phi_p$. The objects of $\mathbf{Rep}^*(\Phi)$ are given by data

- For each object $i \in \mathcal{I}$, an object $X^i \in \Phi_i$;
- For each morphism $p\colon i \to j$ in $\mathcal{I}$, a morphism $X^p\colon X^j \to p^* X^i$

This data is subject to the following condition: for p, q composable morphisms, we have

$$
X^{q \circ p} = p^* X^q \circ X^p
$$

A morphism $f\colon X \to Y$ is determined by the following data: for each $i \in \mathcal{I}$, an arrow $f^i\colon X^i \to Y^i$, and for each morphism $p\colon i \to j$, the diagram

$$
\begin{array}{ccc}
X^i & \xrightarrow{\ X^p\ } & p^* X^j \\
\downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle p^* f^j} \\
Y^i & \xrightarrow{\ Y^p\ } & a^* Y^j
\end{array}
$$

commutes.

Much like $\mathbf{Rep}_*(\Phi)$, one can quickly check that $\mathbf{Rep}^*(\Phi)$ is equivalent to the category of sections of Π, where Π is the Grothendieck construction for Φ using the colax-pointed categories.

Proposition 3.13. Let Φ be a diagram. We have

$$\mathbf{Rep}^*(\Phi^{\mathsf{op}}) \cong \mathbf{Rep}_*(\Phi)^{\mathsf{op}}$$

PROOF. Consider the pullbacks

$$
\begin{array}{ccc}
\int \Phi & \longrightarrow & \mathbf{Cat}_{*,\mathsf{lx}} \\
\downarrow{\scriptstyle \Pi_*} & & \downarrow{\scriptstyle U} \\
\mathcal{I} & \xrightarrow{\ \Phi\ } & \mathbf{Cat}
\end{array}
\qquad
\begin{array}{ccc}
\int \Phi^{\mathsf{op}} & \longrightarrow & \mathbf{Cat}_{*,\mathsf{clx}} \\
\downarrow{\scriptstyle \Pi^*} & & \downarrow{\scriptstyle U} \\
\mathcal{I} & \xrightarrow{\ \Phi^{\mathsf{op}}\ } & \mathbf{Cat}
\end{array}
$$

We have seen, in Proposition 3.8, that $\int \Phi$ and $\int \Phi^{\mathsf{op}}$ are isomorphic, hence the categories $[I, \int \Phi]$ and $[I, \int \Phi^{\mathsf{op}}]$ are isomorphic; if X is a section of Π_*, write X^{op} for the corresponding section of Π^*, and the same backwards.

Let $f\colon X \to Y$ be a natural transformation between sections of Π_*. We have, for a morphism $p\colon i \to j$

$$
\begin{array}{ccc}
p_* X_i & \xrightarrow{\ X_p\ } & X_j \\
\downarrow{\scriptstyle p_* f_i} & & \downarrow{\scriptstyle f_j} \\
p_* Y_i & \xrightarrow{\ Y_p\ } & Y_j.
\end{array}
$$

So define $f^{\mathsf{op}}\colon Y^{\mathsf{op}} \to X^{\mathsf{op}}$ such that $(f^{\mathsf{op}})^i = (f_i)^{\mathsf{op}}$. We get a diagram similar to the one above, but with the arrows flipped and indices raised, so $f^{\mathsf{op}}\colon Y^{\mathsf{op}} \to X^{\mathsf{op}}$ is a natural transformation. The reverse direction is equivalent, and hence the categories $\mathbf{Rep}^*(\Phi)^{\mathsf{op}}$ and $\mathbf{Rep}_*(\Phi^{\mathsf{op}})$ are isomorphic. $\square$

For every result we prove about $\mathbf{Rep}_*(\Phi)$, there is a corresponding result for $\mathbf{Rep}^*(\Phi)$, and this result allows us to take the liberty to prove just one of the cases, since the dual one will implicitly follow.

Definition 3.14. Let Φ, Ψ be diagrams over $\mathcal{I}$ and $\mathcal{I}^{\mathsf{op}}$ respectively. We say Φ, Ψ are a pair of **adjoint** diagrams if for all $i \in \mathcal{I}$ we have $\Phi_i = \Psi_i$, and for each $p\colon i \to j$ in $\mathcal{I}$, $p_* = \Phi_p$ and $p^* = \Psi_p$ are a pair of adjoint functors.

Proposition 3.15. Let Φ, Ψ be a pair of adjoint diagrams. Then

$$\mathbf{Rep}_*(\Phi) \cong \mathbf{Rep}^*(\Psi)$$

PROOF. For $p\colon i \to j$, let $\phi^p\colon \mathrm{Hom}_{\Phi_j}(p_*(-),-) \to \mathrm{Hom}_{\Psi_i}(-,p^*(-))$ be the adjunction's natural isomorphism.

Define a functor $\mathcal{F}\colon \mathbf{Rep}_*(\Phi) \to \mathbf{Rep}^*(\Psi)$ as follows: for $f\colon X \to Y$ a morphism in $\mathbf{Rep}_*(\Phi)$ and $p\colon i \to j$ a morphism in $\mathcal{I}$ we have

- $\mathcal{F}(X)^i = X_i$;
- $\mathcal{F}(X)^p = \phi^p_{X_i,X_j}(X_p)\colon X_i \to p^*X_j$
- $\mathcal{F}(f)^i = f_i$.

So $\mathcal{F}(X)$ is an object in $\mathbf{Rep}_*(\Phi)$, and we must check that $\mathcal{F}(f)$ is a morphism. This amounts to proving that the following rectangle

$$
\begin{array}{ccccc}
X^i & \xrightarrow{h_{X_i}} & p^*p_*X^i & \xrightarrow{p^*X^p} & p^*X^j \\
\downarrow{\scriptstyle p_*f^i} & & \downarrow{\scriptstyle p^*p_*f^i} & & \downarrow{\scriptstyle p^*f^j} \\
Y^i & \xrightarrow{h_{Y_i}} & p^*p_*Y^i & \xrightarrow{p^*Y^p} & p^*Y^j
\end{array}
$$

is a commutative diagram, where $h_Z = \phi^p_{Z,p_*Z}(\mathrm{id})$. By applying p^* to the diagram expressing that f is a representation morphism shows that the right square commutes, and since ϕ^p is a natural transformation, the left square commutes.

In a similar fashion, one can define a functor $\mathcal{G}\colon \mathbf{Rep}^*(\Psi) \to \mathbf{Rep}_*(\Phi)$ such that for a morphism $f\colon X \to Y$ in $\mathbf{Rep}^*(\Psi)$ and a morphism $p\colon i \to j$ we have

- $\mathcal{G}(X)_i = X^i$;
- $\mathcal{G}(X)_p = (\phi^p_{X_i,X_j})^{-1}(X^p)$;
- $\mathcal{G}f_i = f^i$.

That $\mathcal{F}, \mathcal{G}$ are actually functors can be checked easily.

It is clear, by inspection, that $\mathcal{G}\mathcal{F}$ and $\mathcal{F}\mathcal{G}$ are exactly the identities of $\mathbf{Rep}_*(\Phi)$ and $\mathbf{Rep}^*(\Psi)$, respectively. Hence the categories are isomorphic. $\square$

3.3 Properties of Rep$_*(\Phi)$ and Rep$^*(\Phi)$

This section is devoted to listing the several properties of $\mathbf{Rep}_*(\Phi)$ under some suitable conditions on Φ. First we define some abbreviations:

Definition 3.16. Let Φ be a diagram over $\mathcal{I}$. We say that

- Φ is a **finite** diagram if $\mathcal{I}$ has finitely many objects.

- Φ is a **quiver** diagram if $\mathcal{I}$ is generated by a quiver Q, and is a **finite** quiver diagram if Q is finite.

- Φ is an **abelian** diagram if for all $i \in \mathcal{I}$, Φ_i is an abelian category, and for all morphisms p, Φ_p is additive;

- Φ has **(co)products** if for all $i \in \mathcal{I}$, Φ_i has coproducts and for each morphism p, Φ_p preserves them;

- Φ is a **(left, right) exact** diagram if Φ is abelian and for each morphism p, Φ_p is (left, right) exact;

- Φ has **exact filtered colimits/cofiltered limits** if Φ is abelian, has coproducts/products and for each $i \in \mathcal{I}$, Φ_i has exact filtered colimits/cofiltered limits;

- Φ has **(injective, projective) (co)generators** if for each $i \in \mathcal{I}$, Φ_i has (injective, projective) (co)generators;

- Φ is a **(co-)Grothendieck** diagram if Φ has exact filtered colimits (cofiltered limits) and (co)generators;

- Φ has **enough injectives/projectives** if for each $i \in \mathcal{I}$, Φ_i has enough injectives/projectives.

This will come in handy when stating hypothesis for the next few results.

Definition 3.17. Let Φ be a diagram over $\mathcal{I}$. For each $i \in \mathcal{I}$, define a functor $\rho_i \colon \mathbf{Rep}_*(\Phi) \to \Phi_i$ such that $\rho_i(X) = X_i$ and $\rho_i(f) = f_i$ for all $f \colon X \to Y$. Equivalently, we can define functors $\rho^i \colon \mathbf{Rep}^*(\Phi) \to \Phi_i$ for each $i \in \mathcal{I}$.

The next result is trivial in the sense that, for each $i \in \mathcal{I}$, we can embed Φ_i into $\mathbf{Rep}_*(\Phi)$, but will come in handy.

Proposition 3.18. Let Φ be a diagram. ρ_i has a right adjoint; dually, ρ^i has a left adjoint. These adjoints are fully faithful.

PROOF. Define $\tau_i \colon \Phi_i \to \mathbf{Rep}_*(\Phi)$ to be such that, for each $M \in \Phi_i$:

- $\tau(M)_j = 0$ for $i \neq j$, and $\tau(M)_i = M$;
- $\tau(M)_p \colon p_* \tau(M)_j \to \tau(M)_k$ is the zero morphism for $a \neq \mathrm{id}$, and the identity if $p = \mathrm{id}$.

Given $f \colon M \to N$ in Φ_i, we define $\tau(f)_j = 0$ for $i \neq j$, and $\tau(f)_i = f$. This trivially satisfies the conditions for being a representation morphism.

We define $\phi\colon \mathsf{Hom}(X_i, M) \to \mathsf{Hom}(X, \tau(M))$ such that for $f\colon X_i \to M$, define $\phi(f)_i = f$ and $\phi(f)_j = 0$ otherwise, which is clearly a natural isomorphism. Full faithfulness is the observation that $\rho_i(\tau_i(M)) = M$ and then applying the adjunction isomorphism. $\qquad\square$

Proposition 3.19. Let Φ be an abelian, right exact diagram over $\mathcal{I}$. Then **Rep**$_*(\Phi)$ is abelian. Furthermore, $X \to Y \to Z$ is exact in **Rep**$_*(\Phi)$ iff for all $i \in \mathcal{I}$, $X_i \to Y_i \to Z_i$ is exact. In particular, ρ_i is an exact functor for each $i \in \mathcal{I}$.

PROOF. Let $\mathcal{R} = $ **Rep**$_*(\Phi)$. First, we begin to show that for each pair $X, Y \in \mathcal{R}$, $\mathsf{Hom}_\mathcal{R}(X, Y)$ has the structure of an abelian group, by showing it is a subgroup of

$$\prod_{i \in I} \mathsf{Hom}_{\Phi_i}(X_i, Y_i).$$

The zero morphism $0\colon X \to Y$ satisfies $Y_p \circ p_* 0_i = 0_j \circ X_p$ for all $p\colon i \to j$, so it belongs in $\mathsf{Hom}_\mathcal{R}(X, Y)$. For $f, g\colon X \to Y$, we can verify that $f - g$ is in $\mathsf{Hom}_\mathcal{R}(X, Y)$, since p_* is in particular additive for all $p\colon i \to j$:

$$Y_a \circ o_*(f_i - g_i) = Y_a \circ p_* f_i - Y_p \circ p_* g_i = f_i \circ X_p - g_i \circ X_p = (f_i - g_i) \circ X_p$$

Thus $\mathsf{Hom}_\mathcal{R}(X, Y)$ is an abelian group.

For X_1, X_2 objects, we define $X_1 \oplus X_2$ such that $(X_1 \oplus X_2)_i = X_{1,i} \oplus X_{2,i}$ for each $i \in \mathcal{I}$ and

$$(X_1 \oplus X_2)_p\colon p_*(X_{1,i} \oplus X_{2,i}) \to X_{1,j} \oplus X_{2,j}$$

to be the unique morphism making the following diagram commute

$$
\begin{array}{ccccc}
p_* X_{1,i} & \longrightarrow & p_* X_{1,i} \oplus p_* X_{2,i} & \longleftarrow & p_* X_{2,i} \\
\downarrow{\scriptstyle X_{1,p}} & & \downarrow & & \downarrow{\scriptstyle X_{2,p}} \\
X_{1,j} & \longrightarrow & X_{1,j} \oplus X_{2,j} & \longleftarrow & X^{2,j}
\end{array}
$$

for each $p\colon i \to j$, remarking that $p_*(X_{1,i} \oplus X_{2,i}) = p_* X_{1,i} \oplus p_* X_{2,i}$ by right exactness. This implies, in particular, that the inclusions $\iota_k\colon X_k \to X_1 \oplus X_2$ are representation morphisms, and likewise for the projections.

For $k = 1, 2$, consider morphisms $g_k\colon X_k \to Z$. Then for each $i \in \mathcal{I}$, there exist unique $h_i\colon X_{1,i} \oplus X_{2,i} \to Z_i$ which factor $g_{k,i}$ through the inclusions $\iota_{k,i}\colon X_{k,i} \to X_{1,i} \oplus X_{2,i}$. To check h is a representation morphism, for each k we have a maps $p_* X_{k,i} \to Z_j$, given by

$g_{k,j} \circ X_{k,p} = Z_p \circ p_* g_{k,i}$, since g is a representation morphism. Factoring g, we get

$$h_j \circ \iota_{k,j} \circ X_{k,p} = h_j \circ (X_1 \oplus X_2)_p \circ p_* \iota_{k,i}$$
$$= Z_p \circ p_* h_i \circ p_* \iota_{k,i}$$

where the first equality is consequence of ι_k being a representation morphism. The given maps $p_* X_{k,i} \to Z_j$ factor uniquely through $p_* \iota_{k,i}$ for $k = 1, 2$, implying $h_j \circ (X_1 \oplus X_2)_p = Z_p \circ p_* h_i$. This is enough to show that $X \oplus Y$ is a biproduct.

Next, consider a morphism $f \colon X \to Y$. Then for each $i \in I$ let $k_i \colon K_i \to X_i$ and $c_i \colon Y_i \to C_i$. be the kernel and cokernel of f_i, respectively. Given $p \colon i \to j$, we have the diagram

$$
\begin{array}{ccccccc}
p_* K_i & \xrightarrow{p_* k_i} & p_* X_i & \xrightarrow{p_* f_i} & p_* Y_i & \xrightarrow{p_* c_i} & p_* C_i \\
\Big\downarrow{\scriptstyle K_p} & & \Big\downarrow{\scriptstyle X_p} & & \Big\downarrow{\scriptstyle Y_p} & & \Big\downarrow{\scriptstyle C_p} \\
K_j & \xrightarrow{k_j} & X_j & \xrightarrow{f_j} & Y_j & \xrightarrow{c_j} & C_j
\end{array}
$$

where K_p, C_p are the unique morphisms making the diagram commute by the universal property of kernel and cokernel, respectively: note that right exactness of p_* guarantees that $p_* c_i$ is the cokernel of $p_* f_i$. As a consequence, K, C are representations, and $k \colon K \to X$, $c \colon Y \to C$ are representation morphisms. To check they are the kernel, cokernel respectively, consider maps $g \colon Z \to X$ and $h \colon Y \to W$ such that fg and hf are zero morphisms. Then for each $i \in \mathcal{I}$, there exist unique $\hat{g}_i \colon Z_i \to K_i$ and $\hat{h}_i \colon C_i \to W_i$ such that

$$k_i \hat{g}_i = g_i \quad \text{and} \quad \hat{h}_i c_i = h_i.$$

so for each $a \colon i \to j$, we have

$$k_j \circ \hat{g}_j \circ Z_p = X_p \circ p_* k_i \circ p_* \hat{g}_i = k_j \circ K_p \circ p_* \hat{g}_i$$

$$W_a \circ p_* \hat{h}_i \circ p_* c_i = \hat{h}_j \circ c_j \circ Y_p = \hat{h}_j \circ C_p \circ p_* c_i$$

and since $k_j, p_* c_i$ are a monomorphism and an epimorphism respectively, we conclude that $\hat{g}$ and $\hat{h}$ are representation morphisms. Everything up until now also allows us to observe that for each $i \in \mathcal{I}$, ρ_i is an exact functor.

Finally, let $f \colon X \to Y$ be a morphism. Now it must be shown that the morphism $\hat{f}$

$$X \to \mathrm{coker}(\ker f) \xrightarrow{\hat{f}} \ker(\mathrm{coker}\, f) \to Y$$

is an isomorphism, that is to say, we must have $\hat{f}_i$ an isomorphism for each $i \in \mathcal{I}$. So we apply ρ_i to the diagram above, getting (the indices go inside because ρ_i is exact)

$$X_i \to \mathsf{coker}(\mathsf{ker}\, f_i) \xrightarrow{\hat{f}_i} \mathsf{ker}(\mathsf{coker}\, f_i) \to Y_i$$

and because Φ_i is abelian, we have just verified $\hat{f}_i$ is an isomorphism.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact in $\mathcal{R}$, then $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ for each $i \in \mathcal{I}$ by exactness of ρ_i.

Conversely, suppose that we have morphisms $f \colon X \to Y$ and $g \colon Y \to Z$ such that for each $i \in \mathcal{I}$, $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ is exact. Then for each $i \in \mathcal{I}$

$$\rho_i(\mathsf{Ker}\, g) = \mathsf{Ker}\, g_i = \mathsf{Im}\, f_i = \rho_i(\mathsf{Im}\, f)$$

now consider $a \colon i \to j$. The maps $\mathsf{Ker}\, g_i \to Y_i$ and $\mathsf{Im}\, f_i \to Y_i$ are the same for all $i \in \mathcal{I}$, so the maps $(\mathsf{Ker}\, g)_p$ and $(\mathsf{Im}\, f)_p$ coincide. Hence $\mathsf{Ker}\, g = \mathsf{Im}\, f$, proving exactness in $\mathcal{R}$. $\square$

For convenience, we state the dual result:

Corollary 3.20. Let Ψ be a left exact diagram over $\mathcal{I}$. Then **Rep**$^*(\Psi)$ is abelian, moreover $X \to Y \to Z$ is exact iff $X^i \to Y^i \to Z^i$ is exact.

With stronger conditions on Φ, we can extract even more information about **Rep**$_*(\Phi)$. The main goal is finding conditions that guarantee enough projectives. Before getting to that point, we have the following result, whose proof is omitted since it uses almost the same argument as for binary (co)products.

Proposition 3.21. Let Φ be a diagram with (co)products. Then **Rep**$_*(\Phi)$ has all (co)products, and for each $i \in \mathcal{I}$, ρ_i preserves them.

The next proposition is arguably one of the most important; under good enough conditions, it allows us to transport properties from Φ to **Rep**$_*(\Phi)$ more easily. [13].

Proposition 3.22. Let Φ be a diagram with coproducts. For each $i \in \mathcal{I}$, define a functor, $\sigma_i \colon \Phi_i \to$ **Rep**$_*(\Phi)$ such that for each $j \in \mathcal{I}$, $q \colon j \to k$ in $\mathcal{I}$, $M \in \Phi_i$ and $f \colon M \to N$ in Φ_i we have

- $\sigma_i(M)_j = \bigoplus\limits_{p \colon i \to j} p_* M$;
- $\sigma_i(M)_q \colon q_* \sigma_i(M)_j \to \sigma_i(M)_k$ the (inclusion) morphism induced by the inclusions:

$$q_* \sigma_i(M)_j = \bigoplus_{p \colon i \to j} q_* p_* M \to \bigoplus_{r \colon i \to k} r_* M = \sigma_i(M)_k$$

- $\sigma_i(f)_j = \bigoplus_{p\colon i\to j} p_* f.$

We have σ_i left adjoint to ρ_i. Moreover, if Φ is left exact, and has exact coproducts, then σ_i is left exact.

PROOF. Write $\mathcal{R} = \mathbf{Rep}_*(\Phi)$, and $\iota_p\colon p_* M \to \sigma_i(M)_j$ for the inclusion morphism, with $p\colon i \to j$. For each $X \in \mathcal{R}$, and $M \in \Phi_i$, we want to define isomorphisms

$$\phi_{M,X}\colon \mathsf{Hom}_{\mathcal{R}}(\sigma_i(M), X) \to \mathsf{Hom}_{\Phi_i}(M, X_i)$$

$$\psi_{M,X}\colon \mathsf{Hom}_{\Phi_i}(M, X_i) \to \mathsf{Hom}_{\mathcal{R}}(\sigma_i(M), X)$$

Omitting indices, for $f\colon \sigma_i(M) \to X$ we define $\phi(f)$ to be the composite

$$M \xrightarrow{\iota_{\mathsf{id}}} \bigoplus_{p\colon i\to i} p_* M = \sigma_i(M)_i \xrightarrow{f_i} X_i$$

and for $g\colon M \to X_i$ define $\psi(g)_j$ to be the unique morphism such that

$$\psi(g)_j \circ \iota_p = X_p \circ p_* g$$

for all $p\colon i \to j$. So for $g\colon M \to X_i$

$$\phi(\psi(g)) = \psi(g)_i \circ \iota_{\mathsf{id}} = g$$

and since the following diagram commutes for all $p\colon i \to j$

for $f\colon \sigma_i(M) \to X$, we get

$$\psi(\phi(f))_j \circ \iota_p = X_p \circ p_*\phi(f) = X_p \circ p_* f_i \circ p_*\iota_{\mathsf{id}} = f_j \circ \sigma_i(M)_i \circ p_*\iota_{\mathsf{id}} = f_j \circ \iota_p$$

for each $p\colon i \to j$, hence we conclude that $\psi(\phi(f))_j = f_j$, so $\psi\phi$ and $\phi\psi$ are identities, hence ϕ and ψ are isomorphisms.

Naturality boils down to the fact that $\sigma_i(g)_i\iota_{\mathsf{id}_M} = \iota_{\mathsf{id}_N}g$ for a morphism $g\colon M \to N$ in Φ_i, given by definition of $\sigma_i(g)$. We have

$$f_i \circ \phi(h) \circ g = f_i \circ h_i \circ \iota_{\mathsf{id}_N} \circ g = f_i \circ h_i \circ \sigma_i(g)_i \circ \iota_{\mathsf{id}_M} = \phi(f \circ h \circ \sigma_i(g))$$

for $f\colon X \to Y$, $g\colon M \to N$ and $h\colon \sigma_i(N) \to X$.

Left exactness is a simple consequence, all we need to show is preservation of kernels. Since coproducts and p_* are assumed left exact for all $p\colon i \to j$, this is guaranteed. $\qquad\square$

A quick sidenote: The hypothesis can be slightly weakened, while keeping essentially the same proof. Let $\kappa = \max_{i,j}(\operatorname{card}\operatorname{Hom}_{\mathcal{I}}(i,j))$. We only need Φ_i to have coproducts of size $\leq \kappa$, and for p_* to preserve them. This still allows for σ_i to be well-defined for each $i \in \mathcal{I}$, and the adjointness follows. Similarly requiring left exactness of products and p_* of size $\leq \kappa$ is also enough to have σ_i exact. This can be sharpened even further, if we look at each σ_i individually.

We will sometimes use the above fact to our advantage, particularly when $\mathcal{I}$ has finite hom-sets, and Φ is an right exact diagram. We may conclude that the adjoints already exist in this setting, and if we take Φ to be exact, we have σ_i exact. As a more concrete example, let $\mathcal{I}$ be generated by an acyclic quiver, and choose Φ_i to be categories of finite dimensional vector spaces, with right exact functors between them.

Since this result is so useful, we state the dual result, which can be weakened as we remarked.

Corollary 3.23. Let Ψ be a diagram with products. For each $i \in \mathcal{I}$, define a functor, $\sigma^i \colon \Psi_i \to$ **Rep**$^*(\Psi)$ such that for each $j \in \mathcal{I}$, $M \in \Psi_i$

$$\sigma^i(M)^j = \prod_{p \colon i \to j} p^* M$$

and for $q \colon j \to k$, define

$$\sigma^i(M)^q \colon \sigma^i(M)^j \to q^*\sigma^i(M)^k$$

to be the morphism induced by the projections

$$\prod_{r \colon i \to k} r^* M = \sigma^i(M)^k \to (qp)^* M = q^* p^* M$$

For $f \colon M \to N$ in Ψ_i, let

$$\sigma^i(f)^j = \prod_{a \colon i \to j} a^* f$$

We have σ^i right adjoint to ρ^i. Moreover, σ^i is right exact if Ψ is right exact and has exact products.

From this set of adjunctions, there is a lot we can say about **Rep**$_*(\Phi)$:

Corollary 3.24. Let Φ be a diagram with coproducts, and let $i \in \mathcal{I}$. If $M \in \Phi_i$ is projective, then so is $\sigma_i(M)$.

PROOF. Let $\mathcal{R} = \mathbf{Rep}_*(\Phi)$. We have an isomorphism

$$\mathrm{Hom}_{\mathcal{R}}(\sigma_i(M), X) \cong \mathrm{Hom}_{\Phi_i}(M, X_i)$$

Since the functor $\mathrm{Hom}_{\Phi_i}(M, \rho_i(-))$ is exact, being the composition of two exact functors, we conclude $\sigma_i(M)$ is projective. □

With no conditions of Φ, and almost the same proof, if $M \in \Phi_i$ is injective, then so is $\tau_i(M)$.

Corollary 3.25. Let Φ be a Grothendieck diagram. Then $\mathbf{Rep}_*(\Phi)$ and $\mathbf{Rep}^*(\Phi)$ are Grothendieck categories.

This result follows immediately by the following two lemmas:

Lemma 3.26. Suppose Φ has exact filtered colimits. Then filtered colimits in $\mathbf{Rep}_*(\Phi)$ and $\mathbf{Rep}^*(\Phi)$ are exact.

PROOF. Let $\mathcal{J}$ be a filtered category, and suppose that for $X, Y \colon \mathcal{J} \to \mathbf{Rep}_*(\Phi)$ we have

$$0 \to X_j \to Y_j$$

exact for each $j \in \mathcal{J}$ Then for all $i \in I$, we have

$$0 \to \rho_i X_j \to \rho_i Y_j$$

exact in Φ_i. Since this category has exact filtered colimits,

$$0 \to \mathrm{colim}\, \rho_i X \to \mathrm{colim}\, \rho_i Y$$

and since ρ^i preserves colimits,

$$0 \to \rho_i \, \mathrm{colim}\, X \to \rho_i \, \mathrm{colim}\, Y$$

is exact for all $i \in I$, hence we conclude

$$0 \to \mathrm{colim}\, X \to \mathrm{colim}\, Y$$

is exact, showing that $\mathbf{Rep}_*(\Phi)$ also has exact filtered colimits. The proof is the same for $\mathbf{Rep}^*(\Phi)$. □

Lemma 3.27. Let Φ be a diagram with coproducts and generators. For each $i \in \mathcal{I}$, let $G_i \in \Phi_i$ be a generator. Then

$$\hat{G} = \bigoplus_{i \in I} \sigma_i(G_i) \qquad \tilde{G} = \bigoplus_{i \in I} \tau_i(G_i)$$

are generators in $\mathcal{R}_* = \textbf{Rep}_*(\Phi)$ and $\mathcal{R}^* = \textbf{Rep}^*(\Phi)$, respectively.

PROOF. We begin by fixing $f \colon X \to Y$ in. Suppose that, for all $\phi \colon \hat{G} \to X$ we have $f\phi = 0$. Now fix $i \in \mathcal{I}$, and let $g \colon G_i \to X_i$ be any morphism. Since

$$\textsf{Hom}_{\mathcal{R}_*}(\sigma_i(G_i), X) \cong \textsf{Hom}_{\Phi_i}(G_i, X_i),$$

let $\hat{g}$ be the corresponding morphism in $\mathcal{R}_*$. We have $f\hat{g} = 0$ by hypothesis, and so $f_i\hat{g}_i = 0$. Writing $\iota G_i \to \sigma_i(G_i)_i$ for the inclusion reveals that $f_i\hat{g}_i\iota = f_i g = 0$. Since G_i is a generator, this implies $f_i = 0$. This is true for all $i \in \mathcal{I}$, hence $f = 0$.

The proof for $\mathcal{R}^*$ is similar but easier, because $\rho^i(\tau^i(G_i)) = G_i$. $\qquad\qquad\square$

Corollary 3.28. Some applications of this result to "trivial" cases, which are known without resorting to these methods:

- Let $\mathcal{C}$ be a Grothendieck category. Then $[\mathcal{I}, \mathcal{C}]$ is a Grothendieck category for any $\mathcal{I}$. Proof: $[\mathcal{I}, \mathcal{C}]$ is equivalent to $\textbf{Rep}_*(\Phi)$ for Φ the constant diagram to $\mathcal{C}$. It is a diagram of Grothendieck categories, and the identity functor preserves colimits.
- Products of Grothendieck categories are Grothendieck. Proof: Let $\mathcal{I}$ be a set/discrete category, and $\{\Phi_i\}_{i \in \mathcal{I}}$ be a family of Grothendieck categories. We have $\prod_i \Phi_i$ equivalent to $\textbf{Rep}_*(\Phi)$, and identity functors preserve colimits.

Corollary 3.29. Let Φ be a diagram with coproducts and projective generators. If $P_i \in \Phi_i$ is a projective generator for each $i \in \mathcal{I}$, then

$$\hat{P} = \bigoplus_{i \in \mathcal{I}} \sigma_i(P_i)$$

is a projective generator.

PROOF. Let $P_i, \hat{P}$ be as above. Then by the previous lemma, $\hat{P}$ is a generator. By a previous corollary, $\sigma_i(P_i)$ is projective for each $i \in \mathcal{I}$. $\hat{P}$ is therefore a coproduct of projectives, hence it must be projective itself. $\qquad\qquad\square$

The next few results give us short exact sequences which we will use to prove and make use of the existence of enough projectives:

Proposition 3.30. Let Φ be a diagram, either finite or with coproducts. For each $X \in$ **Rep**$_*(\Phi)$, we have an epimorphism

$$\gamma \colon \bigoplus_{i \in \mathcal{I}} \sigma_i(X_i) \to X$$

where, for each $j \in \mathcal{I}$, γ_j is induced by morphisms $X_p \colon p_* X_i \to X_j$ for each $p \colon i \to j$.

PROOF. For each $j \in \mathcal{I}$, the identity $X_j \to X_j$ is one of the maps inducing γ_j, hence it is a right factor of the identity, implying it is an epimorphism. □

Theorem 3.31. Let Φ be a diagram, either finite or with coproducts, and enough projectives. Then $\mathcal{R} = $ **Rep**$_*(\Phi)$ has enough projectives.

PROOF. Let $X \in \mathcal{R}$. Then for each $i \in \mathcal{I}$, there exist P_i projective and an epimorphism $\eta_i \colon P_i \to X_i$. Since σ_i is right exact, $\sigma_i(\eta_i)$ is an epimorphism, and hence

$$\hat{\eta} \colon \hat{P} \to \bigoplus_{i \in \mathcal{I}} \sigma_i(X_i)$$

where $\hat{P}$ is the generator constructed in (3.27). Using the previous proposition, we have $\gamma \hat{\eta} \colon \hat{P} \to X$ epimorphism with $\hat{P}$ projective. □

One can give an explicit expression for the kernel of γ, if Φ is given by a finite quiver. This result, which is one of the central results of this work, relies crucially on the following lemma:

Lemma 3.32. Let $\mathcal{C}$ be an abelian category with coproducts (in particular we have colimits), $\{Z_n\}_{n \in \mathbb{N}}$ a set of objects and a set of morphisms $f_n \colon Z_n \to Z_n$ for $n \geq 0$, satisfying $\iota_n f_n = f_{n+1} \iota_n$, $f_n^{n+1} = 0$, and let $f_\infty \colon Z_\infty \to Z_\infty$ their filtered colimit. Then id $- f_\infty$ is invertible.

PROOF. The trick to this proof is to note that, for a noncommutative ring R and $x \in R$ we have

$$(1 - x)(1 + x + x^2 + \ldots) = (1 + x + x^2 + \ldots)(1 - x) = 1$$

whenever there is an appropriate notion of "$1 + x + x^2 + x^3 + \ldots$". For instance, this is true when x is nilpotent, since the sum is finite. In this case, we have $R = \mathrm{Hom}(Z_\infty, Z_\infty)$, $x = f_\infty$, and must find another way around the infinite sum.

Let $\iota_{\leq n}\colon Z_n \to Z_\infty$ be the induced inclusion. Define

$$g_n = \sum_{j=0}^{n} f_n^j \colon Z_n \to Z_n$$

and note that $\iota_n g_n = g_{n+1}\iota_n$, since $f_n^{n+1} = 0$ and $\iota_n f_n = f_{n+1}\iota_n$, so we may write g_∞ for the colimit of the g_n with the morphisms ι_n. calculating directly, we note that $(\text{id} - f_n)g_n = g_n(\text{id} - f_n) = \text{id}$. The maps $(\text{id} - f_\infty)g_\infty$, $g_\infty(\text{id} - f_\infty)$, $\text{id}\colon Z_\infty \to Z_\infty$ are induced by $(\text{id} - f_n)g_n$, $g_n(\text{id} - f_n)$, $\text{id}\colon Z_n \to Z_n$ respectively, and these three are all equal, either by the above remark or by calculating explicitly. Hence $\text{id} - f_\infty$ is an isomorphism, as we wanted. $\qquad\square$

A classical example: let $C \in \mathcal{C}$. If $Z_n = C$ for all $n \in \mathbb{N}$, and $f_n\colon C \to C$ is the identity for $n > 0$, this is the left-shift operator. The above lemma says that (identity + left-shift) is an invertible operator.

Theorem 3.33. Let Φ be an abelian, finite quiver diagram with coproducts and quiver Q. For each $X \in$ **Rep**$_*(\Phi)$, the sequence

$$0 \to \bigoplus_{a \in Q_1} \sigma_{ta}(a_* X_{sa}) \xrightarrow{\beta} \bigoplus_{i \in Q_0} \sigma_i(X_i) \xrightarrow{\gamma} X \to 0$$

is exact, where γ_j is induced by $X_p\colon p_* X_i \to X_j$ for each $p\colon i \to j$, as defined above, and for each $j \in Q$, β_j is induced by maps

$$(\text{id}, -q_* X_a)\colon q_* a_* X_{sa} \to q_* a_* X_{sa} \oplus q_* X_{ta}.$$

This result has a proof in [13], using embeddings of abelian categories in categories of modules over a ring. Here, we give an alternative proof, independent of such embeddings.

PROOF. Fix $j \in Q_0$. For this proof, it is useful to define, for each $k \geq 0$

$$Z_k = \bigoplus_{\substack{\text{cod } p = j \\ l(p) = k}} p_* X_{\text{dom } p} \qquad \hat{Z}'_n = \bigoplus_{k=1}^{n} Z_n, \qquad \hat{Z}_n = \hat{Z}'_n \oplus X_j,$$

$$Z'_\infty = \bigoplus_{n>0} Z_n, \quad Z_\infty = Z'_\infty \oplus X_j.$$

where $l(p)$ is the length of the path $p = p_{l(p)}\ldots p_1$. It is not difficult to check that if we let $\iota_n\colon \hat{Z}_n \to \hat{Z}_{n+1}$ be the inclusions, then Z_∞ is the filtered colimit of the $\hat{Z}_n$ with those inclusions, and likewise for Z'_∞ and $\hat{Z}'_n$ with similar inclusions ι'_n.

We must check exactness of

$$0 \to Z_{>0} \xrightarrow{\beta_j} Z_{\geq 0} \xrightarrow{\gamma_j} X_j \to 0$$

at $Z_{>0}$ and $Z_{\geq 0}$; we already know exactness at X_j by (3.30).

We define $\beta'_{j,n} \colon \hat{Z}_n \to \hat{Z}_n$ to be 0 for $n = 0$ and for $n > 0$, the map induced by $q_* X_a \colon (qa)_* X_{sa} \to q_* X_{ta}$, for each path q of length $\leq n-1$ and each arrow a, such that a, q are composable. To apply the previous lemma, we need to show that $\iota_n \beta'_{j,n} = \beta'_{j,n+1} \iota_n$, and, for $n > 0$, $\beta'^{n+1}_{j,n} = 0$. The first relation follows by a look at the definition of $\beta'_{j,n}$. To prove the second, note that for $p \colon i \to j$ of length $k > 0$, one can prove by induction that, at $p_* X_i$, $\beta^k_{j,n}$ is induced by X_p, and $\beta^{k+1}_{j,n}$ is induced by zero. Hence $\beta^{n+1}_{j,n}$ is induced by zero maps at all $p_* X_i$ with p of length $\leq n$, and is therefore zero. The previous lemma guarantees that $\mathrm{id} - \beta'_{j,\infty}$ is invertible, where $\beta'_{j,\infty}$ is the colimit of $\beta'_{j,n}$ with ι_n, and we note that β_j can be equivalently defined to be $(\mathrm{id} - \beta'_{j,\infty})\iota'$, where $\iota' \colon Z'_\infty \to Z_\infty$ is the inclusion. This makes it clear that β_j is a monomorphism.

We're left with showing that $\ker \gamma_j = \mathrm{im}\, \beta_j$. We will make an attempt to simplify the problem: since γ_j can be seen as a map $Z_{>0} \oplus Z_0 \to Z_0$, by decomposing the morphism into maps $\gamma'_j \colon Z_{>0} \to Z_0$ and $\mathrm{id} \colon Z_0 \to Z_0$, the kernel is determined as the pullback of this pair. Let $\gamma^\vee_j \colon Z_{>0} \to Z_{\geq 0}$ be induced by $\mathrm{id} \colon Z_{>0} \to Z_{>0}$ and $-\gamma'_j \colon Z_{>0} \to Z_0$. This is the kernel map, and the problem then becomes finding an isomorphism $Z_{>0} \to Z_{>0}$ such that $\beta_j = \gamma^\vee_j \phi$.

We define, for $n \geq 1$, $\phi'_n \colon \hat{Z}'_n \to \hat{Z}'_n$ to be $\hat{\pi}_n \circ \beta'_{j,n} \circ \hat{\iota}_n$, where

$$\hat{\iota}_n \colon \hat{Z}'_n \to \hat{Z}_n \qquad \hat{\pi}_n \colon \hat{Z}_n \to \hat{Z}'_n$$

are the inclusion and projection, respectively. The proof that $\phi'^n_n = 0$, and $\iota'_n \phi'_n = \phi'_{n+1} \iota'_n$ resembles the analogous one for $\beta'_{j,n}$, since the definitions are nearly equivalent. By letting ϕ'_∞ be the colimit, the previous lemma says that $\phi = \mathrm{id} - \phi'_\infty$ is invertible.

It is not difficult to confirm that $\phi = \pi'\beta_j$, so to check that $\beta_j = \gamma^\vee_j \phi$, we only need to check that $\pi_0 \beta_j = \pi_0 \gamma^\vee_j \phi$. We have

$$\pi_0 \gamma^\vee_j \phi = -\gamma'_j \circ (\mathrm{id} - \phi') = \gamma'_j \phi - \gamma'_j.$$

These are maps $Z'_\infty \to Z_0$, so we're aiming to understand their behaviour at each Z_n, $n \geq 1$.

- γ'_j is induced by X_p at $p_* X_i$ for each $p \colon i \to j$, for p of length $n \geq 1$;

- ϕ is induced by the zero map at Z_1, and for paths p of length $n \geq 1$ and arrows a with a, p composable, it is induced by $p_* X_a$;

- Combining the two previous ones, $\gamma'_j \phi$ is induced by the zero map at Z_1, and by $X_p \circ p_* X_a = X_{p \circ a}$ for p of length ≥ 1 and arrow a with a, p composable.

This means that $\gamma'_j \phi - \gamma'_j$ is induced by zero maps $Z_n \to Z_0$ for $n > 1$, and induced by maps $-X_a$ at Z_1. β_j is induced by maps $(pa)_* X_{sa} \to (pa)_* X_{sa} \oplus p_* X_{ta}$ for each path p and arrow a with a, p composable, and the only nonzero ones remaining after projecting to Z_0 are the ones given by $-X_a$ for each arrow a. This proves $\beta_j = \gamma_j^\vee \phi$. $\qquad \square$

We remark that if we assume that the quiver above is acyclic, then we don't need arbitrary coproducts, nor the previous lemma; finite coproducts suffice, and the objects Z_n are zero for n greater than the maximum length for a path. For reference, we state the dual result:

Proposition 3.34. Let Φ be an abelian, finite quiver diagram with products and quiver Q. For each $X \in$ **Rep**$^*(\Phi)$, the sequence

$$0 \to X \xrightarrow{\gamma} \bigoplus_{i \in Q_0} \sigma_i(X_i) \xrightarrow{\beta} \bigoplus_{a \in Q_1} \sigma_{ta}(a_* X_{sa}) \to 0$$

is exact, where for each $j \in Q$, γ_j is defined to be the map induced by the X^p for each $p \colon i \to j$ in $\mathcal{I}$, and β_j is induced by maps

$$(\mathsf{id}, -q^* X^a) \colon q^* a^* X^{sa} \oplus q^* X^{sa} \to q^* a^* X^{sa}.$$

Before we construct the long exact sequences, we will need the following result, given in [13]:

Lemma 3.35. Let $\mathcal{A}, \mathcal{B}$ be abelian categories such that either $\mathcal{A}$ has enough projectives or $\mathcal{B}$ has enough injectives, and $L \colon \mathcal{A} \to \mathcal{B}$, $R \colon \mathcal{B} \to \mathcal{A}$ be a pair of adjoint functors, with L exact if we have enough projectives, R exact if we have enough injectives. Then we have the following isomorphism

$$\mathsf{Ext}^n_{\mathcal{B}}(L(-), -) \cong \mathsf{Ext}^n_{\mathcal{A}}(-, R(-))$$

for all $n \geq 0$.

PROOF. Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$. If $\mathcal{A}$ has enough projectives, let

$$P^\bullet \to X \to 0$$

be a projective resolution of X. Since L is exact, we have

$$L(P^\bullet) \to L(X) \to 0$$

exact. Applying functors $\mathsf{Hom}(-, R(Y))$, $\mathsf{Hom}(-, Y)$ to each of the sequences above, we get

$$0 \to \mathsf{Hom}(X, R(Y)) \to \mathsf{Hom}(P^\bullet, R(Y))$$

$$0 \to \mathsf{Hom}(L(X), Y) \to \mathsf{Hom}(L(P^\bullet), Y)$$

and for each $n \in \mathbb{N}$, we have

$$\mathsf{Ext}^n_{\mathcal{A}}(X, R(Y)) \cong \mathsf{Hom}_{\mathcal{A}}(P^n, R(Y)) \cong \mathsf{Hom}_{\mathcal{B}}(L(P^n), Y) \cong \mathsf{Ext}^n_{\mathcal{B}}(L(X), Y)$$

If we have enough injectives in $\mathcal{B}$ instead, then we assume R is exact, and the proof is dual: use an injective resolution of Y and consider functors $\mathsf{Hom}(L(X), -)$ and $\mathsf{Hom}(X, -)$ instead. $\qquad\square$

Theorem 3.36. Let Φ be an exact finite quiver diagram with enough projectives, exact coproducts and quiver Q, and let $\mathcal{R} = \mathbf{Rep}_*(\Phi)$. Then we have the following long exact sequence

$$0 \to \mathsf{Hom}_{\mathcal{R}}(X, Y) \to \bigoplus_{i \in Q_0} \mathsf{Hom}_{\Phi_i}(X_i, Y_i) \to \bigoplus_{a \in Q_1} \mathsf{Hom}_{\Phi_{sa}}(a_* X_{sa}, Y_{ta})$$

$$\to \mathsf{Ext}^1_{\mathcal{R}}(X, Y) \to \bigoplus_{i \in Q_0} \mathsf{Ext}^1_{\Phi_i}(X_i, Y_i) \to \bigoplus_{a \in Q_1} \mathsf{Ext}^1_{\Phi_{sa}}(a_* X_{sa}, Y_{ta})$$

$$\to \mathsf{Ext}^2_{\mathcal{R}}(X, Y) \to \dots$$

for each $X, Y \in \mathcal{R}$.

PROOF. By the previous theorem, given $X \in \mathcal{R}$, we have the following short exact sequence

$$0 \to \bigoplus_{a \in Q_1} \sigma_{ta}(a_* X_{sa}) \to \bigoplus_{i \in Q_0} \sigma_i(X_i) \to X \to 0.$$

For $Y \in \mathcal{R}$, the functor $\mathsf{Hom}(-, Y)$ is left exact, so we get

$$0 \to \mathsf{Hom}_{\mathcal{R}}(X, Y) \to \mathsf{Hom}_{\mathcal{R}}\left(\bigoplus_{i \in Q_0} \sigma_i(X_i), Y\right) \to \mathsf{Hom}_{\mathcal{R}}\left(\bigoplus_{a \in Q_1} \sigma_{sa}(a_* X_{sa}), Y\right).$$

There are isomorphisms

$$\mathsf{Hom}_{\mathcal{R}}\left(\bigoplus_{i \in Q_0} \sigma_i(X_i), Y\right) \cong \bigoplus_{i \in Q_0} \mathsf{Hom}_{\Phi_i}(X_i, Y_i)$$

$$\mathsf{Hom}_{\mathcal{R}}\left(\bigoplus_{a\in Q_1}\sigma_{ta}(a_*X_{sa}),Y\right)\cong\bigoplus_{a\in Q_1}\mathsf{Hom}_{\Phi_i}(a_*X_{sa},Y_{ta})$$

and together with the fact that σ_i is an exact functor for each $i\in Q_0$ and $\mathcal{R}$ has enough projectives, we can use the previous lemma, and conclude that the long exact sequence we get from the right derived functors of $\mathsf{Hom}(-,Y)$ is isomorphic to the one above. $\qquad\square$

The next result gives the same sequence under different conditions, which allows us to exploit the adjunction $\mathsf{Hom}(a_*(-),-)\cong\mathsf{Hom}(-,a^*(-))$ whenever we have a pair of adjoint diagrams.

Theorem 3.37. Let Φ,Ψ be an adjoint pair of exact, finite quiver diagrams, with quivers Q, Q^{op} respectively. Suppose that we also have either exact products and enough injectives, or exact coproducts and enough projectives. Then, for each $X,Y\in\mathcal{R}=\textbf{Rep}_*(\Phi)\cong\textbf{Rep}^*(\Psi)$, we have we following long exact sequence

$$0\to\mathsf{Hom}_{\mathcal{R}}(X,Y)\to\bigoplus_{i\in Q_0}\mathsf{Hom}_{\Phi_i}(X_i,Y_i)\to\bigoplus_{a\in Q_1}\mathsf{Hom}_{\Phi_{sa}}(a_*X_{sa},Y_{ta})$$

$$\to\mathsf{Ext}^1_{\mathcal{R}}(X,Y)\to\bigoplus_{i\in Q_0}\mathsf{Ext}^1_{\Phi_i}(X_i,Y_i)\to\bigoplus_{a\in Q_1}\mathsf{Ext}^1_{\Phi_{sa}}(a_*X_{sa},Y_{ta})$$

$$\to\mathsf{Ext}^2_{\mathcal{R}}(X,Y)\to\dots$$

PROOF. Since Φ,Ψ is an adjoint pair of diagrams, we have the adjunction $\mathsf{Hom}(a_*X_{sa},Y_{ta})\cong\mathsf{Hom}(X_{sa},a^*Y_{ta})$ for each arrow a. Then we can apply either the previous theorem or its dual, and use the adjunction to adjust the sequence if necessary. $\qquad\square$

Now, we test out these results on a few examples:

Example 3.38. Let Q be a finite quiver, and Φ be a diagram such that $\Phi_i=\textbf{Vect}$ for all $i\in Q_0$ and $\Phi_a=\mathsf{id}$ for all $a\in Q_1$. Here we have **Vect** a Grothendieck category with enough projectives, which has exact coproducts, and the identity functor is clearly exact and coproduct preserving, so we can apply Theorems 3.33 and 3.36.

Write $m_{i,j}=\#\mathsf{Hom}_{\mathcal{F}Q}(i,j)$ for each $i,j\in Q_0$. Fixing $V\in\textbf{Rep}_*(\Phi)=[\mathcal{F}Q,\textbf{Vect}]=\mathcal{R}$, we have the following short exact sequence for each $j\in Q_0$

$$0\to\bigoplus_{a\in Q_1}V_{sa}^{m_{ta,j}}\to\bigoplus_{i\in Q_0}V_i^{m_{i,j}}\to V_j\to0.$$

Fixing $W \in \mathcal{R}$, this gives the following exact sequence,

$$0 \to \mathsf{Hom}_{\mathcal{R}}(V,W) \to \bigoplus_{i \in Q_0} \mathsf{Hom}(V_i, W_i) \to \bigoplus_{a \in Q_1} \mathsf{Hom}(V_{sa}, W_{ta}) \to \mathsf{Ext}_{\mathcal{R}}(V,W) \to 0,$$

since every short exact sequence of vector spaces splits, that is, $\mathsf{Ext}(M,N) = 0$ for any vector spaces M, N.

We take a look at a particular case: let Q be the quiver

$$0 \to 1 \to \ldots \to n.$$

In this case, $m_{i,j} \leq 1$ since this is a poset, and the short exact sequence simplifies to

$$0 \to \bigoplus_{i=0}^{j-1} V_i \to \bigoplus_{i=0}^{j} V_i \to V_j \to 0.$$

Although these simplifications of the short exact sequence are not necessary to determine the long exact sequence, nor help with simplifying it, we're able to reason about it more easily, as we will see in the next chapter.

Example 3.39. Let Q be a finite quiver, $\Phi_i = R\text{-}\mathbf{Mod}$ for a ring R, and let $\{M_a\}_{a \in Q_1}$ be a family of projective R-modules. Define $a_* = M_a \otimes_R -$. This functor has a right adjoint, which we will write $a^* = \mathsf{Hom}(M_a, -)$, and this is also exact. $R\text{-}\mathbf{Mod}$ is a Grothendieck category, with exact products and a_* is an exact, coproduct preserving functor, meaning we are under the conditions of Theorem 3.35. For $N, P \in \mathbf{Rep}_*(\Phi)$, we have the long exact sequence

$$0 \to \mathsf{Hom}_{\mathcal{R}}(N,P) \to \bigoplus_{i \in Q_0} \mathsf{Hom}_R(N_i, P_i) \to \bigoplus_{a \in Q_1} \mathsf{Hom}_R(M_a \otimes_R N_{sa}, P_{ta})$$

$$\to \mathsf{Ext}^1_{\mathcal{R}}(N,P) \to \bigoplus_{i \in Q_0} \mathsf{Ext}^1_R(N_i, P_i) \to \bigoplus_{a \in Q_1} \mathsf{Ext}^1_R(M_a \otimes_R N_{sa}, P_{ta})$$

$$\to \mathsf{Ext}^2_{\mathcal{R}}(N,P) \to \ldots$$

which corresponds to Theorem 4.1 in [7] (one can replace $R\text{-}\mathbf{Mod}$ by $\mathcal{O}_X\text{-}\mathbf{Mod}$ with small changes).

Example 3.40. Let X be a smooth complex projective curve, divisor $D = p$, and let $(\mathcal{E}, E_*), (\mathcal{F}, F_*)$ be quasiparabolic bundles of length n, and a quiver Q given by

$$0 \xrightarrow{a} 1 \to \ldots \to n$$

and $\Phi_0 = \mathbf{QCoh}(X) = \mathcal{Q}$, $\Phi_j = \mathbf{Vect}^{\mathsf{op}}$ for $j > 0$, Φ_a the opposite functor of $\mathsf{Hom}(-, \mathbb{C}_p)$, and $\Phi_b = \mathsf{id}$ for arrows $b \neq a$, similar to what was given in the final example of the first

section. Φ_a is an exact functor, since $\mathbb{C}_p$ is an injective object. Also, both **QCoh**(X) and **Vect** are Grothendieck categories, and the latter also has exact products and enough injectives, meaning we are under the conditions of Theorem 3.36. For $j = 0$, we have the following exact sequence:

$$0 \to 0 \to \mathcal{E} \to \mathcal{E} \to 0$$

and the remaining short exact sequences for $j > 0$ are given by

$$0 \to \bigoplus_{i=0}^{j-1} V_i \to \bigoplus_{i=0}^{j} V_i \to V_j \to 0$$

for $V_k = (E_0/E_{n+1-k})^*$. Note that this sequence lives in **Vect**$^{\mathsf{op}}$. If we also write $W_k = (F_0/F_{n+1-k})^*$, and write $\hat{\mathcal{E}} = (\mathcal{E}, V_\bullet)$, $\hat{\mathcal{F}} = (\mathcal{F}, W_\bullet)$, the long exact sequence is given by

$$0 \to \mathsf{Hom}_{\mathcal{R}}(\hat{\mathcal{E}}, \hat{\mathcal{F}}) \to \mathsf{Hom}_{\mathcal{Q}}(\mathcal{E}, \mathcal{F}) \oplus \bigoplus_{i=1}^{n} \mathsf{Hom}(E_i, F_i) \to \bigoplus_{i=0}^{n-1} \mathsf{Hom}(E_i, F_{i+1})$$

$$\to \mathsf{Ext}^1_{\mathcal{R}}(\hat{\mathcal{E}}, \hat{\mathcal{F}}) \to \mathsf{Ext}^1_{\mathcal{Q}}(\mathcal{E}, \mathcal{F}) \to 0 \to \mathsf{Ext}^2_{\mathcal{R}}(\hat{\mathcal{E}}, \hat{\mathcal{F}}) \to \mathsf{Ext}^2_{\mathcal{Q}}(\mathcal{E}, \mathcal{F}) \to 0 \to \dots$$

In particular, we note that for $m > 1$

$$\mathsf{Ext}^m_{\mathcal{R}}(\hat{\mathcal{E}}, \hat{\mathcal{F}}) \cong \mathsf{Ext}^m_{\mathcal{Q}}(\mathcal{E}, \mathcal{F}).$$

Chapter 4

Parabolic Sheaves

We begin this section by defining the tools used in [19] to define parabolic sheaves. Although the theory of the next section applies to any scheme X and any effective Cartier divisor D, we will restrict ourselves to the case that X is a complex projective curve and D is reduced, that is, a finite set of points in X. With the tools developed in the last chapter, we give simple proofs of some parabolic sheaf properties.

4.1   Parabolic $\mathcal{O}_X$-modules

Definition 4.1. Let $\mathcal{O}_X$-**Mod** be the category of $\mathcal{O}_X$-modules, and consider $\mathbb{R}$ as a category: objects are real numbers, morphisms induced by the usual order, and if $\alpha \leq \beta$ we denote the morphism by $i^{\alpha,\beta}$. An $\mathbb{R}$-**filtered** $\mathcal{O}_X$-module is a presheaf $\mathbb{R}^{\mathsf{op}} \to \mathcal{O}_X$-**Mod**. For such a presheaf $\mathcal{E}_*$, we write

- $\mathcal{E}_\alpha$ for $\mathcal{E}_*(\alpha)$,
- $i_\mathcal{E}^{\alpha,\beta}$ for $\mathcal{E}_*(i^{\alpha,\beta})$

for real numbers $\alpha \leq \beta$.

Given two $\mathbb{R}$-filtered $\mathcal{O}_X$-modules $\mathcal{E}_*$ and $\mathcal{F}_*$, a morphism $f_* \colon \mathcal{E}_* \to \mathcal{F}_*$ of filtered $\mathcal{O}_X$-modules is a presheaf morphism, and we write $f_*(\alpha) = f_\alpha$.

Write $\mathcal{O}_X$-**FilMod** for the presheaf category above.

To give the next definition, we need two extra pieces: Firsly, let $\mathcal{E}_*$ be a filtered $\mathcal{O}_X$-module. We have a bifunctor

$$\mathcal{O}_X\text{-}\mathbf{FilMod} \times \mathbb{R}^{\mathsf{op}} \to \mathcal{O}_X\text{-}\mathbf{FilMod}$$

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sending $(\mathcal{E}_*, \alpha)$ to the filtered $\mathcal{O}_X$-module $\mathcal{E}[\alpha]_*$, defined to be

$$\mathcal{E}[\alpha]_\beta = \mathcal{E}_{\alpha+\beta}$$

$$i_\mathcal{E}^{\beta,\gamma}[\alpha] = i_\mathcal{E}^{\alpha+\beta,\alpha+\gamma}$$

If we have a morphism $f_* \colon \mathcal{E}_* \to \mathcal{F}_*$, we have a morphism $f[\alpha]_* \colon \mathcal{E}[\alpha]_* \to \mathcal{F}[\alpha]_*$ such that $f[\alpha]_\beta = f_{\alpha+\beta}$. Moreover, if $\alpha \leq \beta$, then we have a morphism

$$i_\mathcal{E}^{[\alpha,\beta]} \colon \mathcal{E}[\beta]_* \to \mathcal{E}[\alpha]_*$$

such that $(i_\mathcal{E}^{[\alpha,\beta]})_\gamma = i_\mathcal{E}^{\alpha+\gamma,\beta+\gamma}$. Finally, we define $f_*^{[\alpha,\beta]} \colon \mathcal{E}[\beta]_* \to \mathcal{F}[\alpha]_*$ to be the diagonal of the following commutative square

$$
\begin{array}{ccc}
\mathcal{E}[\beta]_* & \xrightarrow{\ i_\mathcal{E}^{[\alpha,\beta]}\ } & \mathcal{E}[\alpha]_* \\
{\scriptstyle f[\beta]_*}\downarrow & & \downarrow{\scriptstyle f[\alpha]_*} \\
\mathcal{F}[\beta]_* & \xrightarrow[\ i_\mathcal{F}^{[\alpha,\beta]}\ ]{} & \mathcal{F}[\alpha]_*.
\end{array}
$$

Secondly, given an $\mathcal{O}_X$-module $\mathcal{F}$, and an $\mathbb{R}$-filtered $\mathcal{O}_X$-module $\mathcal{E}_*$, we can define an $\mathbb{R}$-filtered $\mathcal{O}_X$-module $\mathcal{E}_* \otimes \mathcal{F}$ as follows:

$$(\mathcal{E}_* \otimes \mathcal{F})_\alpha = \mathcal{E}_\alpha \otimes \mathcal{F}$$

$$i_{\mathcal{E}_* \otimes \mathcal{F}}^{\alpha,\beta} = i_\mathcal{E}^{\alpha,\beta} \otimes \mathrm{id}_\mathcal{F}$$

Definition 4.2. A **parabolic $\mathcal{O}_X$-module** w.r.t D is a pair $\mathcal{E}_* = (\mathcal{E}_*, j_\mathcal{E})$ where

- $\mathcal{E}_*$ is an $\mathbb{R}$-filtered $\mathcal{O}_X$-module,
- $j_\mathcal{E} \colon \mathcal{E}_* \otimes \mathcal{O}_X(-D) \to \mathcal{E}[1]_*$ is an isomorphism,

such that

$$i_\mathcal{E}^{[0,1]} \circ j_\mathcal{E} = \mathrm{id}_\mathcal{E} \otimes \iota_D$$

where $\iota_D \colon \mathcal{O}_X(-D) \to \mathcal{O}_X$ is the induced map.

For $(\mathcal{E}_*, j_\mathcal{E})$ and $(\mathcal{F}_*, j_\mathcal{F})$ parabolic $\mathcal{O}_X$-modules, a morphism $f_* \colon \mathcal{E}_* \to \mathcal{F}_*$ is parabolic iff the diagram

$$
\begin{array}{ccc}
\mathcal{E}_* \otimes \mathcal{O}_X(-D) & \xrightarrow{\ f_* \otimes \mathrm{id}\ } & \mathcal{F}_* \otimes \mathcal{O}_X(-D) \\
{\scriptstyle j_\mathcal{E}}\downarrow & & \downarrow{\scriptstyle j_\mathcal{F}} \\
\mathcal{E}[1]_* & \xrightarrow[\ f[1]_*\ ]{} & \mathcal{F}[1]_*
\end{array}
$$

is commutative. We denote this category by **Par**(X, D).

As seen in [19, p.129], the category above is Grothendieck and an abelian subcategory of $[\mathbb{R}^{\mathrm{op}}, \mathcal{O}_X\textbf{-Mod}]$ (this last category is also Grothendieck by Corollary 3.28).

Definition 4.3. Let $\mathcal{E}_*$ be a parabolic $\mathcal{O}_X$-module. $\mathcal{E}_*$ is a **parabolic sheaf** if there exist $n \geq 1$ and a sequence $\alpha_0, \ldots, \alpha_{n+1} \in [0,1]$ with $\alpha_0 = 0$ and $\alpha_{n+1} = 1$ such that $i_{\mathcal{E}}^{x,\alpha_{k+1}}$ is an isomorphism for each $k \leq n$ and each x with $\alpha_k < x < \alpha_{k+1}$, and $\mathcal{E}_\alpha$ is a coherent sheaf for all $\alpha \in \mathbb{R}$.

Whenever we have a parabolic sheaf $\mathcal{E}_*$ with sequence $\alpha_0, \ldots, \alpha_n$, we say the sequence is a **system of weights** whenever $i^{\alpha_k, \alpha_{k+1}}$ is *not* an isomorphism for all $k < n$. The following result is useful:

Proposition 4.4. Let $\mathcal{A}$ be the full subcategory of $\textbf{Par}(X, D)$ of parabolic sheaves. Then $\mathcal{A}$ is an abelian subcategory.

PROOF. By [18], we only need to check that $0 \in \mathcal{A}$ and $\mathcal{A}$ is closed under products, kernels and cokernels.

The first one is obvious, for any sequence $\alpha_0, \ldots, \alpha_{n+1} \in [0,1]$, $i_0^{x,\alpha_{k+1}}$ is the zero morphism on the zero $\mathcal{O}_X$-module, which is an isomorphism for all x.

Let $\mathcal{E}_*, \mathcal{F}_*$ be parabolic sheaves, with sequences

$$\alpha_0^{\mathcal{E}}, \ldots \alpha_{m+1}^{\mathcal{E}}, \;\; \alpha_0^{\mathcal{F}}, \ldots \alpha_{n+1}^{\mathcal{F}} \in [0,1]$$

for some $m, n \geq 1$, with $\alpha_0^{\mathcal{E}} = \alpha_0^{\mathcal{F}} = 0, \alpha_{m+1}^{\mathcal{E}} = \alpha_{n+1}^{\mathcal{F}} = 1$, and let

$$S = \left\{ \alpha_i^{\mathcal{E}}, \alpha_j^{\mathcal{F}} \;\middle|\; 0 \leq i \leq m+1, 0 \leq j \leq n+1 \right\},$$

$M = \#S - 2$ and β be the unique increasing sequence $\{0, \ldots, M+1\} \to S$.

Consider $\mathcal{E}_* \oplus \mathcal{F}_*$. For each $0 \leq k \leq M+1$, $\beta_k < x < \beta_{k+1}$, we have $i_{\mathcal{E} \oplus \mathcal{F}}^{x,\beta_{k+1}} = i_{\mathcal{E}}^{x,\beta_{k+1}} \oplus i_{\mathcal{F}}^{x,\beta_{k+1}}$. One may check that both morphisms on the right are isomorphisms, and since products of coherent sheaves are coherent, $\mathcal{E}_* \oplus \mathcal{F}_*$ is a parabolic sheaf.

Now suppose we have a parabolic morphism $f_* \colon \mathcal{E}_* \to \mathcal{F}_*$. Since $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are defined pointwise just like $\oplus$, checking both are parabolic sheaves is done in almost the same way. $\square$

Note that parabolic sheaves aren't preserved under arbitrary coproducts, even if we drop the coherent sheaf condition: consider the parabolic sheaf $\mathcal{E}_*^n$ defined on $[0,1]$ such that $\mathcal{E}_\alpha^n = \mathcal{O}_X$ for $n < \alpha^{-1}$ and 0 otherwise. There is no finite sequence in $[0,1]$ satisfying the above property.

4.2 Embedding

In [12], parabolic bundles are defined for reduced divisors $D = p_1 + \ldots + p_n$: a parabolic bundle is a pair (E, α) where E a vector bundle and for each $p \in D$ we have a positive integer n_p and a filtration of inclusions

$$E_p = E_{p,0} \supseteq E_{p,1} \supseteq \ldots \supseteq E_{p,n_p} \supseteq E_{p,n_p+1} = 0$$

and for each p an increasing sequence $\alpha_*(p)$ in $[0,1]$ with $\alpha_0(p) = 0$ and $\alpha_{n_p+1}(p) = 1$. We say α is a **system of weights**, a definition for which the above is analogous.

For two parabolic bundles $E_* = (E, \alpha)$, $F_* = (F, \beta)$, a parabolic bundle morphism $f \colon E_* \to F_*$ is a vector bundle morphism $f \colon E \to F$ such that for each $p \in D$ we have

$$\alpha_j(p) > \beta_k(p) \text{ implies } f(E_{p,j}) \subseteq F_{p,k+1}$$

for all j, k. This condition can be more easily visualized, in blue, in the following diagram:
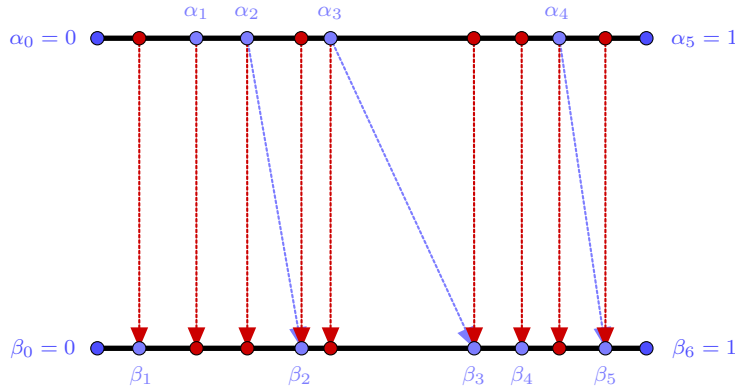

Figure 4.1: Map between filtrations

The lines in red suggest how this condition is equivalent to preservation of filtrations, by looking at this picture as if it were a commutative diagram, where the top and bottom horizontal lines are left-directed inclusions. We will make this precise in the following result:

Theorem 4.5. Let $\mathcal{C}$ be the category of parabolic bundles on X with divisor D, as defined above. Then $\mathcal{C}$ is equivalent to the full subcategory of the category of parabolic sheaves on X over D whose objects $\mathcal{E}_*$ are such that $\mathcal{E}_0$ is locally free and $i_{\mathcal{E}}^{\alpha,\beta}$ is a monomorphism for all $\alpha \leq \beta$.

PROOF. Let (E, α) be a parabolic bundle in the sense of [12], denote by $\mathcal{E}$ the locally free sheaf associated to the vector bundle E, and for each $p \in D$, $j \in \{0, \dots, n_p + 1\}$, write $E_p^j = E_p/E_{p,j}$.

For $x \in [0, 1]$, define

$$\mathcal{E}_x = \mathsf{Ker}\left(\mathcal{E} \to \bigoplus_{p \in D} E_p^{j_p(x)}\right)$$

where we define, for each $p \in D$

$$j_p(x) = \min\left\{\, j \in \{0, \dots, n_p + 1\} \mid x \le \alpha_j(p) \,\right\}$$

Now, define $S = \{\, a_j(p) \mid p \in D \text{ and } 0 \le j \le n_p + 1 \,\}$, and let $N = \#S - 2$. Let $\hat{\alpha} \colon \{0, \dots, N + 1\} \to S$ be the unique increasing function. In particular, $\hat{\alpha}_0 = 0$ and $\hat{\alpha}_{N+1} = 1$. For $k \le N$, we define $i_{\mathcal{E}}^{\hat{\alpha}_k, \hat{\alpha}_{k+1}}$ to be the unique morphism making the diagram commute:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_{\hat{\alpha}_{k+1}} & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_{p \in D} E_p^{j_p(\hat{\alpha}_{k+1})} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}_{\hat{\alpha}_k} & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_{p \in D} E_p^{j_p(\hat{\alpha}_k)} & \longrightarrow & 0
\end{array}
$$

where the middle vertical morphism is the identity, and the right vertical morphism is induced by either $E_p^{j_p(\alpha_k)} = E_p^{j_p(\alpha_{k+1})}$ or a projection $E_p^{j_p(\alpha_{k+1})} \to E_p^{j_p(\alpha_k)}$.

For $x < \hat{\alpha}_{k+1}$, let $i_{\mathcal{E}}^{\hat{\alpha}_k, x}$ be the identity. Then extend $\mathcal{E}_*$ to $\mathbb{R}$ by tensoring with $\mathcal{O}_X(-D)$, defining $\mathcal{E}_x = \mathcal{E}_{x-n} \otimes \mathcal{O}(-nD)$, where $n = \lfloor x \rfloor$, and likewise define $i_{\mathcal{E}}^{a,b}$ for all $a \le b$.

First, note that we can identify each vector space E_p^j with a skyscraper sheaf supported at p, which is coherent, and finite coproducts of coherent sheaves are coherent. Since $\mathcal{E}$ is locally free, it is also coherent. From this, we conclude that $\mathcal{E}_x$ is coherent for all $x \in [0, 1]$, since it is the kernel of a coherent sheaf morphism. We may take the sequence $\hat{\alpha}$, which by definition satisfies the required property. Hence $\mathcal{E}_*$ is a coherent parabolic sheaf.

For (E, α), (F, β) parabolic bundles in the sense of [12], write $\mathcal{E}_*$, $\mathcal{F}_*$ respectively for the corresponding coherent parabolic sheaves as defined above. Let $f \colon (E, \alpha) \to (F, \beta)$ be a parabolic bundle morphism. For $x \in [0, 1]$, we define $\hat{f}_x \colon \mathcal{E}_x \to \mathcal{F}_x$ to be unique morphism

making the diagram commute

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_{p \in D} E_p^{j_p(x)} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\hat{f}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_x & \longrightarrow & \mathcal{F} & \longrightarrow & \bigoplus_{p \in D} F_p^{k_p(x)} & \longrightarrow & 0
\end{array}
$$

where the vertical right one is induced by $f(E_{p,j(x)}) \subseteq F_{p,k(x)}$. We need to make sure this inclusion does occur: let k' be the largest such that $\alpha_{j(x)}(p) > \beta_{k'}(p)$. Then $x \leq \alpha_j(p) \leq \beta_{k'+1}(p)$, we have a chain of inclusions $f(E_p, j) \subseteq F_{p,k'+1}$, and $k(x) = k' + 1$ by definition.

Now given a map $\hat{f}_* \colon \mathcal{E}_* \to \mathcal{F}_*$, we want to reconstruct f. Let $f \colon E \to F$ be the vector bundle morphism corresponding to $\hat{f}_0$, and consider the unique morphism making the diagram commute, given by the cokernel:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_{p \in D} E_p^{j_p(x)} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\hat{f}_x} & & \downarrow{\scriptstyle\hat{f}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_x & \longrightarrow & \mathcal{F} & \longrightarrow & \bigoplus_{p \in D} F_p^{k_p(x)} & \longrightarrow & 0.
\end{array}
$$

Note that $\operatorname{Coker}(\mathcal{E}_x \to \mathcal{E}) \cong \bigoplus_{p \in D} E_p^{j_p(x)}$, since it is the kernel of an epimorphism, so the dashed morphism must be the same as the one induced by the inclusions $f(E_{p,j(x)}) \subseteq F_{p,k(x)}$, which implies $f(E_{p,j}) \subseteq F_{p,k+1}$ whenever $\alpha_j(p) > \beta_k(p)$. To see this, suppose without losing generality that k is the largest satisfying the hypothesis, and take $x = \alpha_j(p)$. Then we have $j_p(x) = j$, $k_p(x) = k + 1$, as wanted to see.

Wrapping things up, this proves we have a fully faithful embedding from the parabolic bundles in the sense of Mehta and Seshadri into the category of coherent parabolic sheaves.

To show equivalence, let $\mathcal{E}_*$ be a coherent parabolic sheaf such that $\mathcal{E}_0$ is locally free, and for $0 \leq \alpha \leq \beta \leq 1$, $i_{\mathcal{E}}^{\alpha,\beta}$ is a monomorphism. To construct (E, α), let E be the vector bundle associated to $\mathcal{E}_0$, and for each $p \in D$, define $\hat{\alpha}_k(p) = \alpha_k$. Write $E^j = \operatorname{Coker}(i_{\mathcal{E}}^{0,\alpha_j})$, so that we have the following exact sequence

$$
0 \to \mathcal{E}_{\alpha_j} \to \mathcal{E} \to E^j \to 0
$$

and since looking at stalks is an exact functor, we may define $E_{p,j}$ to be the vector space at the stalk at p of $\mathcal{E}_{\alpha_j}$, where E_p^j plays the role of $E_p/E_{p,j}$, as was done in the other direction.

It is easy to see that we have $E_{p,j+1} \subseteq E_{p,j}$ for $j \leq n$, so we have a filtration. After removing unnecessary weights, we get a parabolic bundle, and a bunch of calculations which were suggested throughout shows that $\mathcal{E}_* \cong \hat{\mathcal{E}}_*$ where $\hat{\mathcal{E}}_*$ is the coherent parabolic sheaf associated to the object we just defined. $\qquad\square$

In [13], Theorem 5.1, the author embeds quasiparabolic bundles over X wrt to $D = p$ of length n into an abelian category $\mathcal{R}_n$ as was done in the end of the first section of the previous chapter. As suggested in the above reference, we will show that the Ext groups of parabolic bundles can be calculated in $\mathcal{R}_n$. In particular, we conclude that the Ext group does not depend on the values of weights themselves, but on their ordering. The next result will make this precise.

Proposition 4.6. Suppose $D = p$. Let (E, α), (F, β) be parabolic bundles in the sense of [12], write $\mathcal{E}_*$, $\mathcal{F}_*$ respectively for the corresponding coherent parabolic sheaves, and write $\hat{E}, \hat{F}$ denote the objects associated to the quasiparabolic bundles from E, F in $\mathcal{R} = \mathbf{Rep}_*(\Phi)$ for an appropriate Φ. We have $\mathrm{Ext}_\mathcal{P}(\mathcal{E}, \mathcal{F}) \cong \mathrm{Ext}_\mathcal{R}(\hat{E}, \hat{F})$.

PROOF. Let $\gamma \colon \{0, \ldots, N+1\} \to [0,1]$ be the unique increasing map with values in the image of α and β, and define the quiver

$$Q = (0 \xrightarrow{a} 1 \ldots \to N + 1).$$

Define $\Phi \colon \mathcal{F}Q \to \mathbf{Cat}$ as was done in the first section of last chapter, and define $E_k = E_{\gamma(k)}$, $F_k = F_{\gamma(k)}$. Then $\hat{E}, \hat{F} \in \mathcal{R}$ are given by

- $\hat{E}_0 = \mathcal{E}$, $\hat{E}_i = V_i$;
- $\hat{F}_0 = \mathcal{F}$, $\hat{F}_i = W_i$.

and for all $p \colon i \to j$, $\hat{E}_p$ and $\hat{F}_p$ are induced by the inclusions. For $X = E, F$, given an arrow $k \to k+1$, we denote these inclusions by $i_X^{k,k+1}$, which are epimorphisms in $\mathbf{Vect}^{\mathrm{op}}$.

Let $\hat{G}$ be an extension of $\hat{F}$ by $\hat{E}$. For each $i \in Q_0$, we have

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{E}_k & \longrightarrow & \hat{G}_k & \longrightarrow & \hat{F}_k & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle i_E^{k,k+1}} & & \downarrow{\scriptstyle i_G^{k,k+1}} & & \downarrow{\scriptstyle i_F^{k,k+1}} & & \\
0 & \longrightarrow & \hat{E}_{k+1} & \longrightarrow & \hat{G}_{k+1} & \longrightarrow & \hat{F}_{k+1} & \longrightarrow & 0
\end{array}
$$

By the short 5-lemma, $i_G^{k,k+1}$ is also an epimorphism, hence there exists a parabolic bundle (G, γ) inducing $\hat{G}$, and the morphisms $\hat{E} \to \hat{G}$ and $\hat{G} \to \hat{F}$ are also given by parabolic

bundle morphisms. Embedding (G, γ) in $\mathcal{P}$ as $\mathcal{G}_*$, together with those two morphisms, we get a short exact sequence of parabolic sheaves

$$0 \to \mathcal{E}_* \to \mathcal{G}_* \to \mathcal{F}_* \to 0,$$

so we have determined an extension of $\mathcal{F}_*$ by $\mathcal{E}_*$.

Conversely, suppose we start out with the above exact sequence of parabolic sheaves. The previous proposition guarantees that

$$0 \to \mathcal{E}_0 \to \mathcal{G}_0 \to \mathcal{F}_0 \to 0$$

is exact and extensions of locally free sheaves are locally free. Also, we have, for each k,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_{\alpha_k} & \longrightarrow & \mathcal{G}_{\alpha_k} & \longrightarrow & \mathcal{F}_{\alpha_k} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_{\mathcal{E}}^{k,k+1}} & & \downarrow{\scriptstyle i_{\mathcal{G}}^{k,k+1}} & & \downarrow{\scriptstyle i_{\mathcal{F}}^{k,k+1}} & & \\
0 & \longrightarrow & \mathcal{E}_{\alpha k+1} & \longrightarrow & \mathcal{G}_{\alpha k+1} & \longrightarrow & \mathcal{F}_{\alpha k+1} & \longrightarrow & 0
\end{array}
$$

so again by the short 5-lemma, $i_{\mathcal{G}}^{k,k+1}$ is a monomorphism. Proving that $i_{\mathcal{G}}^{x,\alpha_{k+1}}$ is an isomorphism for each $x \in (\alpha_k, \alpha_{k+1})$ also follows by the short 5-lemma. Hence $\mathcal{G}_*$ is a parabolic sheaf, so there exists a parabolic bundle (G, γ) and parabolic bundle morphisms $(E, \alpha) \to (G, \gamma)$, $(G, \gamma) \to (F, \beta)$, and we get corresponding morphisms in $\mathbf{Rep}_*(\Phi)$, hence we have determined an extension of $\hat{F}$ by $\hat{E}$.

To make sure these transitions do in fact preserve/reflect the monomorphisms and epimorphisms present, note that a parabolic bundle morphism is a (mono,epi)morphism if and only if the underlying vector bundle morphism is. It is also not difficult to see that the parabolic sheaf determined by the extension in $\mathbf{Rep}_*(\Phi)$ determines an object isomorphic to the original. $\square$

Two of the long exact sequences at the end of the previous chapter allow us to prove the following result, not found in the literature, giving some insight on how to work with the machinery of diagram representations.

Proposition 4.7. Let $(E, \alpha), (F, \beta)$ be parabolic bundles over X with respect to $D = p$, write γ for the join of the sequences α and β, and choose n so that the length of γ is $n + 1$. Let $\mathcal{R} = \mathbf{Rep}_*(\Phi)$ be the category embedding quasiparabolic bundles of length n, as done previously, where $\Phi \colon \mathcal{F}Q \to \mathbf{Cat}$, with $Q = (0 \to 1 \to \ldots \to n)$. We have the following short exact sequence

$$0 \to \mathrm{Ext}_{\mathcal{Q}}(V_*, W_*) \to \mathrm{Ext}_{\mathcal{R}}(\hat{E}, \hat{F}) \to \mathrm{Ext}_{\mathbf{Coh}X}(\mathcal{E}, \mathcal{F}) \to 0$$

where $\mathcal{Q} = [Q, \mathbf{Vect}]$, $\hat{E}, \hat{F}$ the quasiparabolic structure from the parabolic bundles (E, α), (F, β), $\mathcal{E}, \mathcal{F}$ are the locally free sheaves associated to E, F respectively, $V_k = (E_0/E_{n+1-k})^*$ and $W_k = (F_0/F_{n+1-k})^*$.

PROOF. Consider the diagram

$$V_0 \leftarrow V_1 \leftarrow \ldots V_n,$$

induced by the filtration of E, which is the object V_* we are considering, and similarly for W_*.

The projections of the short exact sequences to each Φ_k for $\hat{E}$ are given by

$$0 \rightarrow 0 \rightarrow E \xrightarrow{\mathrm{id}} E \rightarrow 0$$

in $\mathbf{QCoh}(X)$ for $k = 0$, and

$$0 \rightarrow \bigoplus_{j=0}^{k-1} V_j \xrightarrow{\beta_k^0} \bigoplus_{j=0}^{k} V_j \xrightarrow{\gamma_k^0} V_k \rightarrow 0$$

for $k > 0$, in $\mathbf{Vect}^{\mathrm{op}}$. For V_*, the short exact sequences are given by

$$0 \rightarrow V_0 \xrightarrow{\mathrm{id}} V_0 \rightarrow 0 \rightarrow 0$$

for $k = 0$, and

$$0 \rightarrow V_k \xrightarrow{\gamma_k^1} \bigoplus_{j=0}^{k} V_j \xrightarrow{\beta_k^1} \bigoplus_{j=0}^{k-1} V_j \rightarrow 0$$

for $k > 0$, both in $\mathbf{Vect}$.

From Examples 3.40 and 3.38, we get the two following exact sequences, now with some named arrows:

$$0 \rightarrow \mathrm{Hom}_{\mathcal{R}}(E, F) \rightarrow \mathrm{Hom}_{\mathbf{QCoh}X}(E, F) \oplus \bigoplus_{i=1}^{n} \mathrm{Hom}(E^i, F^i) \xrightarrow{d_0} \bigoplus_{i=1}^{n} \mathrm{Hom}(E^{i+1}, F^i)$$

$$\xrightarrow{f} \mathrm{Ext}_{\mathcal{R}}(E, F) \rightarrow \mathrm{Ext}_{\mathbf{QCoh}X}(E, F) \rightarrow 0$$

$$0 \rightarrow \mathrm{Hom}_{\mathcal{Q}}(E, F) \rightarrow \bigoplus_{i=1}^{n+1} \mathrm{Hom}(E^i, F^i) \xrightarrow{d_1} \bigoplus_{i=1}^{n} \mathrm{Hom}(E^{i+1}, F^i) \xrightarrow{\pi} \mathrm{Ext}_{\mathcal{Q}}(E, F) \rightarrow 0$$

We note that $\mathrm{Im}(d_0) \cong \mathrm{Im}(d_1)$. To see why, from the adjunctions and left exactness of $\mathrm{Hom}(-, F)$, we have

$$\mathrm{Im}(d_0) \cong \mathrm{Coim}(\mathrm{Hom}_{\mathcal{R}}(\beta^0, F)) \cong \mathrm{Coker}(\mathrm{Hom}_{\mathcal{R}}(\gamma^0, F)),$$

and $\mathrm{Im}(d_1) \cong \mathrm{Coker}(\mathrm{Hom}_{\mathcal{Q}}(\gamma^1, F))$. The short exact sequences for the vertices $k > 0$ in $\mathcal{R}$ and $\mathcal{Q}$ coincide, and in particular $\gamma_k^1 = \gamma_k^0$. In the vertex $k = 0$, γ_0^i is the identity in both cases, so the cokernel is 0.

Hence we have $\mathrm{Ker}\, f = \mathrm{Ker}\, \pi$, and since π is an epimorphism, there exists a unique ϕ with $\mathrm{Ker}\, \phi = 0$ making the following diagram commute

$$
\bigoplus_{i=1}^{n} \mathsf{Hom}(E^{i+1}, F^i) \xrightarrow{\;\;\;\pi\;\;\;} \mathsf{Ext}_{\mathcal{Q}}(E, F)
$$

$$
f \searrow \qquad \swarrow \phi
$$

$$
\mathsf{Ext}_{\mathbf{QCoh}(X)}(E, F)
$$

and satisfying $\mathrm{Im}\, \phi = \mathrm{Im}\, f$. This gives us the desired exact sequence

$$
0 \to \mathsf{Ext}_{\mathcal{Q}}(E, F) \xrightarrow{\phi} \mathsf{Ext}_{\mathcal{R}}(E, F) \to \mathsf{Ext}_{\mathbf{QCoh}(X)}(E, F) \to 0,
$$

concluding the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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