Homotopy Exact Sequence for the Pro-Étale Fundamental Group

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Abstract

We prove the homotopy exact sequence for the pro-étale fundamental group introduced by Bhatt and Scholze. In the proof, we construct an infinite analogue of the Stein factorization. In the case of a general base S, one has to use a certain weaker notion of exactness for the theorem to be true.

Over $S = \mathrm{Spec}(k)$ one gets more: we show that for a geometrically connected scheme of finite type over a field k, the sequence

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

is exact as abstract groups and the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding.

On the way, we prove the general van Kampen theorem and the Künneth formula for the pro-étale fundamental group.

Zusammenfassung

Wir beweisen die exakte Homotopie-Sequenz für die von Bhatt und Scholze eingeführte pro-étale Fundamentalgruppe. Im Beweis konstruieren wir ein unendliches Analogon der Stein-Faktorisierung. Im Falle einer allgemeinen Basis S muss man einen gewissen schwächeren Begriff der Exaktheit verwenden, damit der Satz wahr ist.

Über $S = \mathrm{Spec}(k)$ erhält man mehr: Wir zeigen das für ein geometrisch zusammenhängendes Schema endlichen Typs über einem Körper k, die Sequenz

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

eine exakte Sequenz abstrakter Gruppen und der Morphismus $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ eine topologische Einbettung ist.

Auf dem Weg beweisen wir den allgemeinen Satz von Seifert-van Kampen und die Künnethformel für die pro-étale Fundamentalgruppe.

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Chapter 1

Introduction

In [SGA 1], Grothendieck introduced the étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ of a connected locally noetherian scheme X with a geometric point $\bar{x}$ on it. It is a profinite topological group and parametrizes finite étale covers of the scheme X, in the sense that there is an equivalence of categories between the category of finite étale covers Fét_X and the category $\pi_1^{\text{ét}}(X, \bar{x}) - \text{FSets}$ of finite discrete sets with a continuous action of $\pi_1^{\text{ét}}(X, \bar{x})$. It is done in a following way: one considers a category $\mathcal{C}$ endowed with a functor $F : \mathcal{C} \to \text{FSets}$ satisfying certain axioms. Such $\mathcal{C}$ is called a (finite) Galois category. Then one defines a profinite group $\pi_1(\mathcal{C}, F) = \text{Aut}(F)$. With the axioms imposed, one then shows that $\mathcal{C} \simeq \pi_1(\mathcal{C}, F) - \text{Sets}$. In the context of schemes, one takes $\mathcal{C} = \text{Fét}_X$ and $F = F_{\bar{x}} : \text{Fét}_X \to \text{FSets}$ - the fibre functor obtained using the geometric point $\bar{x}$. Then one defines $\pi_1^{\text{ét}}(X, \bar{x}) = \pi_1(\mathcal{C}, F)$. Different fibre functors lead to isomorphic groups and when considering maps between Galois categories we can either insist on them being compatible or omit them from the notation at the cost of indeterminacy by an inner automorphism of the target fundamental group. For brevity, we will often suppress the geometric points from the notation.

Grothendieck then proceeds in [SGA 1] to develop some crucial properties of $\pi_1^{\text{ét}}(X, \bar{x})$. Among them:

1. The homotopy exact sequence over a general base:

   Theorem 1.1. *([SGA 1, Exp. X, Corollaire 1.4.]) Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\bar{s}$ be a geometric point of S. Then there is an exact sequence*

   $$\pi_1^{\text{ét}}(X_{\bar{s}}) \to \pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(S) \to 1$$

   of fundamental groups.

2. The homotopy exact sequence over a field, i.e. the comparison between the "arithmetic" and "geometric" fundamental groups:

   Theorem 1.2. *([SGA 1, Exp. IX, Théorème 6.1]) Let k be a field with algebraic closure $\bar{k}$. Let X be a quasi-compact and quasi-separated scheme over k. If the base change $X_{\bar{k}}$ is connected, then there is a short exact sequence*

   $$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}) \to \pi_1^{\text{ét}}(X) \to \text{Gal}_k \to 1$$

   of profinite topological groups.

3. "Künneth formula":

   Proposition 1.3. *([SGA 1, Exposé X, Cor. 1.7]) Let X, Y be two connected schemes locally of finite type over an algebraically closed field k and assume that Y is proper. Let us fix $\bar{x}, \bar{y}$ - closed points respectively of X and Y. Then the map induced by the projections is an isomorphism*

   $$\pi_1^{\text{ét}}(X \times_k Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \bar{x}) \times \pi_1^{\text{ét}}(Y, \bar{y})$$

4. Invariance under extensions of algebraically closed fields for proper schemes:

   Proposition 1.4. *([SGA 1, Exp. X, Corollaire 1.8]) Let X be a proper connected scheme over an algebraically closed field k. Let $K \supset k$ be another algebraically closed field. Then $X_K \to X$ induces an isomorphism*
   $$\pi_1^{\text{ét}}(X_K) \xrightarrow{\sim} \pi_1^{\text{ét}}(X).$$

The aim of this article is to extend these (and other) results to a setting of more general covers and a more general fundamental group for schemes. These more general notions were introduced by Bhatt and Scholze in Chapter 7 of [BS]. They considered so-called *infinite* Galois categories $\mathcal{C}$ with a functor $F : \mathcal{C} \to \mathrm{Sets}$ satisfying certain properties analogous to those in the definition of a usual Galois category but adjusted to enable one to work with arbitrary (not necessarily finite) discrete sets and group actions on them. For such a pair, they define a topological group $\pi_1(\mathcal{C}, F)$ by "the same" formula, i.e. $\pi_1(\mathcal{C}, F) = \mathrm{Aut}(F)$ topologized in a suitable way. They call such pair a *tame* infinite Galois category if moreover $\pi_1(\mathcal{C}, F)$ acts transitively on $F(Y)$ for any connected $Y \in \mathcal{C}$. This additional condition is automatically satisfied in (finite) Galois categories and was overlooked in [Noo], where a similar formalism was considered. The authors of [BS] proceed to prove that for a tame infinite Galois category $(\mathcal{C}, F)$ there is an equivalence of categories between $\mathcal{C}$ and the category $\pi_1(\mathcal{C}, F)-\mathrm{Sets}$ of discrete sets with a continuous action of $\pi_1(\mathcal{C}, F)$. The groups $\pi_1(\mathcal{C}, F)$ belong to a class of Noohi groups. This means, by the definition made in [BS], that it is a (Hausdorff) topological group G such that the natural map induces an isomorphism $G \to \mathrm{Aut}(F_G)$ of topological groups, where $F_G : G - \mathrm{Sets} \to \mathrm{Sets}$ denotes the forgetful functor. The authors of [BS] characterize Noohi groups: these are precisely the (Hausdorff) topological groups such that:

1. open subgroups form a basis of open neighbourhoods of $1 \in G$,
2. the group is Raîkov complete.

Using this description, one sees that discrete groups and profinite groups are Noohi. A basic new example of a Noohi group would be $\mathrm{Aut}(S)$ with the compact-open topology, where S is a discrete set.
This new formalism of infinite Galois categories is now applied in the following geometric setting: for a locally topologically noetherian scheme X consider the category of "geometric covers" Cov_X, i.e. schemes Y over X such that $Y \to X$:

1. is étale (not necessarily quasi-compact!)
2. satisfies the valuative criterion of properness.

As Y is not assumed to be of finite type over X, the valuative criterion does not imply that $Y \to X$ is proper (otherwise we would get finite étale morphisms again). A choice of a geometric point $\bar{x}$ on X gives a fibre functor again and one shows that $(\mathrm{Cov}_X, F_{\bar{x}})$ is a tame infinite Galois category and defines the pro-étale fundamental group as
$$\pi_1^{\text{proét}}(X, \bar{x}) = \pi_1(\mathcal{C}, F_{\bar{x}}).$$
The pro-étale fundamental group generalizes the étale fundamental group, as it turns out that $\pi_1^{\text{ét}}(X)$ is a profinite completion of $\pi_1^{\text{proét}}(X)$. It also generalizes the group considered formerly in Chapter X.6 of [SGA 3] (let us denote it by π_1^{SGA3}), namely $\pi_1^{\text{SGA3}}(X)$ is the pro-discrete completion of $\pi_1^{\text{proét}}(X)$. The motivation for the introduction of $\pi_1^{\text{proét}}(X)$ was to be able to detect all $\mathbb{Q}_\ell$-local systems through the finite dimensional continuous $\mathbb{Q}_\ell$-representations. Unless X is geometrically unibranch, this cannot be in general achieved with $\pi_1^{\text{ét}}(X)$ and $\pi_1^{\text{SGA3}}(X)$, but it is possible with $\pi_1^{\text{proét}}(X)$. It is important to keep in mind that in the case of a geometrically unibranch scheme (so e.g. normal) all three groups coincide: $\pi_1^{\text{ét}}(X) \simeq \pi_1^{\text{SGA3}} \simeq \pi_1^{\text{proét}}(X)$, and thus to study $\pi_1^{\text{proét}}(X)$ we work a lot with non-normal schemes.

Our main results are the homotopy exact sequences: over a general (Nagata) base S and over a field (with relaxed assumptions) for $\pi_1^{\text{proét}}$. In the case of a general base S, the sequence of $\pi_1^{\text{proét}}$'s might be not exact on the level of abstract groups (see Remark 5.2), and thus we prove what we call a "weak exactness" (see Definition 2.65). The weak exactness of a sequence of Noohi groups translates to the expected statements about covers via an analogue of the usual dictionary (see Prop. 2.64). Over $S = \mathrm{Spec}(k)$ we are, however, able to obtain actual exactness as abstract groups. Our main results are:

Theorem. (Thm. 5.1) Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume that S is Nagata and connected. Let $\bar{s}$ be a geometric point of S. Then the sequence induced on the pro-étale fundamental groups

$$\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S) \to 1$$

is weakly exact (see Defn. 2.65).

Moreover, the induced map

$$\left(\pi_1^{\text{proét}}(X)/\overline{\overline{\mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{s}}))}}\right)^{\text{Noohi}} \to \pi_1^{\text{proét}}(S)$$

is a homeomorphism. Here, $\overline{\overline{H}}$ denotes the "thick closure" introduced in Defn. 2.58.

The crucial element of the proof is showing the existence of an "infinite Stein factorization" formulated in the following theorem.

Theorem. (Thm. 5.5) Let S be a Nagata scheme. Let $X \to S$ be as above and let $Y \in \mathrm{Cov}_X$ be connected. Then there exists a connected $T \in \mathrm{Cov}_S$ and a morphism $g : Y \to T$ over $X \to S$, such that g has geometrically connected fibres.

Moreover, for any two T_1, T_2 and maps $g_i : Y \to T_i$, $i = 1, 2$, as in the statement, there exists a unique isomorphism $\phi : T_1 \simeq T_2$ in Cov_S making the diagram

$$\begin{array}{ccc} & \nearrow & T_1 \\ Y & & \downarrow \phi \\ & \searrow & T_2 \end{array}$$

commute.

Over a field, we prove the following.

Theorem. (Thm. 4.16) Let X be a geometrically connected scheme of finite type over a field k. Then the sequence

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

is exact as abstract groups.

Moreover, the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding and the map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a quotient map of topological groups.

The most difficult part of the above theorem is showing that $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is injective or, more precisely, a topological embedding. This is Theorem 4.14.

Other interesting results that we prove include:

- the Künneth formula for $\pi_1^{\text{proét}}$ (see Prop. 3.34),
- the invariance of $\pi_1^{\text{proét}}$ under extensions of algebraically closed fields for proper schemes (see Prop. 3.36).

One of the main technical tools is the van Kampen theorem for $\pi_1^{\text{proét}}$ (see Thm. 3.18). It allows us to write the pro-étale fundamental group of a scheme X in terms of the pro-étale fundamental groups of connected components of the normalization X^{ν} of X. As $\pi_1^{\text{proét}}$ and $\pi_1^{\text{ét}}$ coincide for a normal scheme, this method allows us to benefit from the fact, that the theorems in discussion are known for $\pi_1^{\text{ét}}$. This method relies on the results of [Ryd], which show that proper surjective morphisms of finite presentation are morphisms of effective descent for geometric covers. This is why the Nagata assumption appears in many of our theorems: to guarantee that the normalization morphism is finite.

The pro-étale fundamental group was applied in [Lav] to study stratified bundles.

1.1 Conventions and notations

- $H <^\circ G$ will mean that H is an open subgroup of G.
- For groups $H < G$, H^{nc} will denote the normal closure of H in G, i.e. the smallest normal subgroup of G containing H.
- For a field k, we will use $\bar{k}$ to denote its (fixed) algebraic closure and k^{sep} or k^s to denote its separable closure (in $\bar{k}$).
- The topological groups are assumed to be Hausdorff unless specified otherwise or appearing in a context where it is not automatically satisfied (e.g. as a quotient by a subgroup that is not necessarily closed). We will usually comment whenever a non-Hausdorff group appears.
- We assume (almost) every base scheme to be locally topologically noetherian. This does not cause problems when considering geometric covers, as a geometric cover of a locally topologically noetherian scheme is locally topologically noetherian again - this is [BS, Lm. 6.6.10]. Without this assumption the category of geometric covers does not behave in a desired way: see [BS, Example 7.3.12]. On the other hand, in some proofs we base-change to large pro-étale covers (e.g. so-called w-contractible covers) which are usually non-noetherian and some care is needed.
- A "G-set" for a topological group G will mean a discrete set with a continuous action of G unless specified otherwise. We will denote the category of G-sets by $G - \mathrm{Sets}$.
- We will denote the category of sets by Sets.
- For a topological group G, we will denote its Raîkov completion by $\widehat{G}$. See [Dik] or [AT, Chapter 3.6] for an introduction to the Raîkov completion. Keep in mind that they use a different notation for the completion.

Chapter 2

Infinite Galois categories, Noohi groups and $\pi_1^{\text{proét}}$

2.1 Overview of the results in [BS]

Throughout the entire article we use the language and results of [BS], especially of Chapter 7., as this is where the pro-étale fundamental group was defined. We are going to give a quick overview of these results below, but we recommend keeping a copy of [BS] at hand.

Recall our global assumption that topological groups are assumed to be Hausdorff unless specified otherwise or appearing in a context where it is not automatically satisfied (e.g. as a quotient by a subgroup that is not necessarily closed). We will usually comment whenever a non-Hausdorff group appears.

Definition 2.1. ([BS, Defn. 7.1.1.]) Fix a topological group G. Let $G - \mathrm{Sets}$ be the category of discrete sets with a continuous G-action, and let $F_G : G - \mathrm{Sets} \to \mathrm{Sets}$ be the forgetful functor. We say that G is a *Noohi group* if the natural map induces an isomorphism $G \to \mathrm{Aut}(F_G)$ of topological groups. Here, $S \in \mathrm{Sets}$ are considered with the discrete topology, $\mathrm{Aut}(S)$ with the compact-open topology and $\mathrm{Aut}(F_G)$ is topologized using $\mathrm{Aut}(S)$ for $S \in \mathrm{Sets}$.

Remark 2.2. If G is a Noohi group, then its open subgroups form a basis of open neighbourhoods of $1 \in G$. This follows from the definition of a Noohi group and the observation that this condition is satisfied for $\mathrm{Aut}(S)$.

Recall that we denote by $\widehat{G}$ the Raîkov completion of a topological group G, i.e. the completion of G for its two-sided uniformity (see [Dik] or [AT, Chapter 3.6] for an introduction to the Raîkov completion).

Proposition 2.3. *([BS, Prop. 7.1.5.]) Let G be a topological group with a basis of open neighbourhoods of* $1 \in G$ *given by open subgroups. Then there is a natural isomorphism* $\mathrm{Aut}(F_G) \to \widehat{G}$.

In particular, a topological group is Noohi if and only if it satisfies the following conditions:

- *its open subgroups form a basis of open neighbourhoods of* $1 \in G$*,*
- *it is complete.*

Using the above proposition it is easy to give examples of Noohi groups.

Example 2.4. The following classes of topological groups are Noohi:

- discrete groups,
- profinite groups,
- $\mathrm{Aut}(S)$ with the compact-open topology for S a discrete set (see [BS, Lm. 7.1.4.]).

Lemma 2.5. *([BS, Lm. 7.1.8.]) If a topological group G admits an open Noohi subgroup U, then G is itself Noohi.*

Example 2.6. The following groups are Noohi:

- $\mathbb{Q}_\ell$,
- $\overline{\mathbb{Q}_\ell}$ for the colimit topology induced by expressing $\overline{\mathbb{Q}_\ell}$ as a union of finite extensions (in contrast with the situation for the ℓ-adic topology),
- $\mathrm{GL}_n(\mathbb{Q}_\ell)$ for the colimit topology.

Proof. The case of $\mathbb{Q}_\ell$ follows from the completeness of this group. The two other statements are proven in [BS, Example 7.1.7]. □

The notion of a Noohi group is tightly connected to a notion of an infinite Galois category, which we are about to introduce. Here, an object $X \in \mathcal{C}$ is called connected if it is not empty (i.e., initial), and for every subobject $Y \to X$ (i.e., $Y \xrightarrow{\sim} Y \times_X Y$), either Y is empty or $Y = X$.

Definition 2.7. ([BS, Defn. 7.2.1]) An infinite Galois category $\mathcal{C}$ is a pair $(\mathcal{C}, F : \mathcal{C} \to \mathrm{Sets})$ satisfying:

(1) $\mathcal{C}$ is a category admitting colimits and finite limits.

(2) Each $X \in \mathcal{C}$ is a disjoint union of connected (in the sense explained above) objects.

(3) $\mathcal{C}$ is generated under colimits by a set of connected objects.

(4) F is faithful, conservative, and commutes with colimits and finite limits.

The fundamental group of $(\mathcal{C}, F)$ is the topological group $\pi_1(\mathcal{C}, F) := \mathrm{Aut}(F)$, topologized by the compact- open topology on $\mathrm{Aut}(S)$ for any $S \in \mathrm{Sets}$.

Example 2.8. a) If G is a topological group, then $(G - \mathrm{Sets}, F_G)$ is an infinite Galois category.

b) If G is a Noohi group, then for $(\mathcal{C}, F) = (G - \mathrm{Sets}, F_G)$ we have $\pi_1(\mathcal{C}, F) = G$.

The problem with the definition of an infinite Galois category so far is that there exist infinite Galois categories that are not of the form $(G - \mathrm{Sets}, F_G)$, see [BS, Ex. 7.2.3.]. One needs an additional assumption of "tameness" for this to be true. This was overlooked in [Noo], where a similar formalism was considered.

Definition 2.9. ([BS, Defn. 7.2.4.]) An infinite Galois category $(\mathcal{C}, F)$ is tame if for any connected $X \in \mathcal{C}$, $\pi_1(\mathcal{C}, F)$ acts transitively on $F(X)$.

Remark 2.10. The formalism introduced above was also studied in [Lep, Chapter 4.] under the names of "quasiprodiscrete" groups and "pointed classifying categories".

Theorem 2.11. *([BS, Thm. 7.2.5.]) Fix an infinite Galois category $(\mathcal{C}, F)$ and a Noohi group G. Then*

1. *$\pi_1(\mathcal{C}, F)$ is a Noohi group.*
2. *There is a natural identification of $\mathrm{Hom}_{cont}(G, \pi_1(\mathcal{C}, F))$ with the groupoid of functors $\mathcal{C} \to G - \mathrm{Sets}$ that commute with the fibre functors.*
3. *If $(\mathcal{C}, F)$ is tame, then F induces an equivalence $\mathcal{C} \simeq \pi_1(\mathcal{C}, F) - \mathrm{Sets}$.*

We will return later to an abstract study of Noohi groups and infinite Galois categories in subsequent sections on "Noohi completion" (see Section 2.2) and on the dictionary between Noohi groups and $G - \mathrm{Sets}$ (see Section 2.3).

For now, let us return to gathering the results from [BS].

Pro-étale topology and the definition of $\pi_1^{\text{proét}}(X)$

Fix a locally topologically noetherian scheme X.

Definition 2.12. Let $Y \to X$ be a morphism of schemes such that:

1. it is étale (not necessarily quasi-compact!)
2. it satisfies the valuative criterion of properness.

We will call Y a *geometric cover* of X. We will denote the category of geometric covers by Cov_X.

As Y is not assumed to be of finite type over X, the valuative criterion does not imply that $Y \to X$ is proper (otherwise we would simply get a finite étale morphism). See Appendix 2.4 at the end of this chapter to see some useful facts about these notions.

Example 2.13. For an algebraically closed field $\bar{k}$, the category $\mathrm{Cov}_{\mathrm{Spec}(\bar{k})}$ consists of (possibly infinite) disjoint unions of $\mathrm{Spec}(\bar{k})$ and we have $\mathrm{Cov}_{\mathrm{Spec}(\bar{k})} \simeq \mathrm{Sets}$.

More generally, one has:

Lemma 2.14. *([BS, Lm. 7.3.8]) If X is a henselian local scheme, then any $Y \in \mathrm{Cov}_X$ is a disjoint union of finite étale X-schemes.*

Let us choose a geometric point $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ on X. By Example 2.13, this gives a fibre functor $F_{\bar{x}} : \mathrm{Cov}_X \to \mathrm{Sets}$.

Proposition 2.15. *([BS, Lemma 7.4.1]) The pair $(\mathrm{Cov}_X, F_{\bar{x}})$ is a tame infinite Galois category.*

Then one defines

Definition 2.16. The *pro-étale fundamental group* is defined as

$$\pi_1^{\text{proét}}(X, \bar{x}) = \pi_1(\mathrm{Cov}_X, F_{\bar{x}}).$$

In other words, $\pi_1^{\text{proét}}(X, \bar{x}) = \mathrm{Aut}(F_x)$ and this group is topologized using the compact-open topology on $\mathrm{Aut}(S)$ for any $S \in \mathrm{Sets}$.

One can compare the groups $\pi_1^{\text{proét}}(X, \bar{x})$, $\pi_1^{\text{ét}}(X, \bar{x})$ and $\pi_1^{\mathrm{SGA3}}(X, \bar{x})$, where the last group is the group introduced in Chapter X.6 of [SGA 3].

Lemma 2.17. *For a scheme X, the following relations between the fundamental groups hold*

1. *The group $\pi_1^{\text{ét}}(X, \bar{x})$ is the profinite completion of $\pi_1^{\text{proét}}(X)$.*
2. *The group $\pi_1^{\mathrm{SGA3}}(X, \bar{x})$ is the prodiscrete completion of $\pi_1^{\text{proét}}(X, \bar{x})$.*

Proof. This follows from [BS, Lemma 7.4.3] and [BS, Lemma 7.4.6]. □

As shown in [BS, Example 7.4.9], $\pi_1^{\text{proét}}(X, \bar{x})$ is indeed more general than $\pi_1^{\mathrm{SGA3}}(X, \bar{x})$. This can be also seen by combining Example 4.3 with Prop. 4.6 below.

The following lemma is extremely important to keep in mind and will be used many times throughout the paper. Recall that, for example, a normal scheme is geometrically unibranch.

Lemma 2.18. *([BS, Lm. 7.4.10.]) If X is geometrically unibranch, then $\pi_1^{\text{proét}}(X, \bar{x}) \simeq \pi_1^{\text{ét}}(X, \bar{x})$.*

There is another way of looking at the pro-étale fundamental group, which justifies the name "pro-étale". For this, we need to introduce the pro-étale topology for schemes, which is the main topic of [BS].

Definition 2.19. A map $f : Y \to X$ of schemes is called *weakly étale* if f is flat and the diagonal $\Delta_f : Y \to Y \times_X Y$ is flat.

Definition 2.20. The pro-étale site $X_{\text{proét}}$ is the site of weakly étale X-schemes, with covers given by fpqc covers.

These definitions are motivated by the following theorem in the affine case. The map of rings will be called weakly étale if the induced map on spectra is weakly étale.

Theorem 2.21. *Let $f : A \to B$ be a map of rings.*

a) f is étale if and only if f is weakly étale and finitely presented.

b) If f is ind-étale, i.e. B is a filtered colimit of étale A-algebras, then f is weakly étale.

c) ([BS, Theorem 2.3.4.]) If f is weakly étale, then there exists a faithfully flat ind-étale $g : B \to C$ such that $g \circ f$ is ind-étale.

The choice of weakly étale morphisms to define the site is motivated by the fact that weakly étale morphisms are étale local both on the source and the target while being pro-étale is not even Zariski local on the target (see [BS, Example 4.1.12]).

A basic new example of a pro-étale cover is given by the following

Example 2.22. ([BS, Example 4.1.13.] Given a scheme X and closed geometric points $x_1, \dots, x_n$, the map

$$\big(\sqcup_i \operatorname{Spec}(\mathcal{O}^{sh}_{X,x_i})\big) \sqcup \big(X \setminus \{x_1, \dots, x_n\}\big) \to X$$

is a weakly étale cover. However, one cannot take infinitely many points. For example, the map

$$\sqcup_p \operatorname{Spec}(\mathbb{Z}^{sh}_{(p)}) \to \operatorname{Spec}(\mathbb{Z})$$

is not a weakly étale cover as the target is not covered by a quasi-compact open in the source.

However, in the pro-étale topology one can still consider covers that in a sense generalize passing to a strict henselization of a local ring. We will see this later in a notion of being "w-strictly local".

Since an étale map is also weakly étale, we obtain a morphism of topoi:

$$\nu : \operatorname{Shv}(X_{\text{proét}}) \to \operatorname{Shv}(X_{\text{ét}})$$

Lemma 2.23. *([BS, Lm. 5.1.2]) The pullback $\nu^* : \operatorname{Shv}(X_{\text{ét}}) \to \operatorname{Shv}(X_{\text{proét}})$ is fully faithful. Its essential image consists exactly of those sheaves F with $F(U) = \operatorname{colim}_i F(U_i)$ for any $U = \lim_i U_i$, where $i \mapsto U_i$ is a small cofiltered diagram of affine schemes in $X_{\text{ét}}$.*

Definition 2.24. ([BS, Defn. 5.1.3.]) A sheaf $F \in \operatorname{Shv}(X_{\text{proét}})$ is called *classical* if it lies in the essential image of $\nu^* : \operatorname{Shv}(X_{\text{ét}}) \to \operatorname{Shv}(X_{\text{proét}})$.

Lemma 2.25. *([BS, Lm. 5.1.4.]) Let F be a sheaf on $X_{\text{proét}}$. Assume that for some pro-étale cover $\{Y_i \to X\}$, $F|_{Y_i}$ is classical. Then F is classical.*

In the following, we use that for a qcqs scheme Y there exist a map of topoi $\pi : \operatorname{Shv}(Y_{\text{proét}}) \to \operatorname{Shv}(\pi_0(Y)_{\text{proét}})$ and talk about the pullback $\pi^* : \operatorname{Shv}(\pi_0(X)_{\text{proét}}) \to \operatorname{Shv}(X_{\text{proét}})$. It is discussed in [BS, Lemma 4.2.13.] but we will omit discussing it here.

Definition 2.26. ([BS, Defn. 7.3.1.]) Fix $F \in \operatorname{Shv}(X_{\text{proét}})$. We say that F is

(1) *locally constant* if there exists a cover $\{Y_i \to X\}$ in $X_{\text{proét}}$ with $F|_{Y_i}$ constant.

(2) *locally weakly constant* if there exists a cover $\{Y_i \to X\}$ in $X_{\text{proét}}$ with Y_i qcqs such that $F|_{Y_i}$ is classical and is the pullback via π of a sheaf on the profinite set $\pi_0(Y_i)$.

We write Loc_X and $w\mathrm{Loc}_X$ for the corresponding full subcategories of $\mathrm{Shv}(X_{\text{proét}})$.

Finally, we are ready to state the following important result.

Theorem 2.27. *([BS, Lemma 7.3.9.]) One has* $\mathrm{Loc}_X = w\mathrm{Loc}_X = \mathrm{Cov}_X$ *as subcategories of* $\mathrm{Shv}(X_{\text{proét}})$.

Let us remark that this theorem needs the locally topologically noetherian assumption to hold, as explained in [BS, Ex. 7.3.12]

We conclude by stating a result that was one of the main motivations for introducing the pro-étale fundamental group.

Theorem 2.28. *([BS, Lemma 7.4.7.]) For a local field E, there is an equivalence of categories*

$$\mathrm{Rep}_{E,\mathrm{cont}}(\pi_1^{\text{proét}}(X, \bar{x})) \simeq \mathrm{Loc}_X(E),$$

where $\mathrm{Loc}_X(E)$ denotes the category of of sheaves of E-modules locally free on $X_{\text{proét}}$.

Let us list some notions that play an important role in the pro-étale topology and appear constantly in the proofs of the results stated above. The following are also nicely discussed in [SP, Chapter 0965].

Definition 2.29. ([BS, Defn. 2.1.1.]) A spectral space X is *w-local* if it satisfies:

(1) All open covers split, i.e., for every open cover $\{U_i \hookrightarrow X\}$, the map $\sqcup_i U_i \to X$ has a section.

(2) The subspace $X^c \subset X$ of closed points is closed.

Lemma 2.30. *([BS, Lemma 2.1.4.]) A spectral space X is w-local if and only if $X^c \subset X$ is closed, and every connected component of X has a unique closed point. For such X, the composition $X^c \to X \to \pi_0(X)$ is a homeomorphism.*

Definition 2.31. ([BS, Defn. 2.2.1]) Fix a ring A.

(1) A is *w-local* if $\mathrm{Spec}(A)$ is w-local.

(2) A is *w-strictly local* if A is w-local, and every faithfully flat étale map $A \to B$ has a section.

Lemma 2.32. *([BS, Lm. 2.2.9.]) A w-local ring A is w-strictly local if and only if all local rings of A at closed points are strictly henselian.*

Proposition 2.33. *([BS, Cor. 2.2.14.]) Any ring A admits an ind-étale faithfully flat map $A \to A'$ with A' w-strictly local.*

Let us mention a lemma that fits into the discussion and will be useful later.

Lemma 2.34. *Let S be an affine scheme and $\widetilde{S} \to S$ a pro-étale cover by a w-strictly local affine scheme. Then*

1. *Each connected component of $\widetilde{S}$ is the strict henselization of the local ring at a certain point of S.*
2. *If S is moreover noetherian, connected and normal and η denotes its generic point, then each connected component $c \subset \widetilde{S}$ is noetherian, normal and its generic point is the unique point of c lying over η. The map*

$$\pi_0(\widetilde{S}_\eta) \to \pi_0(\widetilde{S})$$

is a homeomorphism.

Proof. The scheme $\widetilde{S}$ is pro-étale over S and c is a pro-Zariski localization of $\widetilde{S}$ (this is because a connected component of a qcqs scheme can be seen as an inverse limit of clopen subschemes containing it, see Lm. 5.15). In particular, $c \to S$ is weakly étale. By [SP, Lm. 094Z] (it relies on the theorem by Olivier, in our case it is probably an overkill, as the morphism $c \to S$ is actually pro-étale), $c \to S$ induces an isomorphism on the strict henselizations of the local rings. As $\widetilde{S} \to S$ is w-strictly local, c is itself a strictly henselian local ring. It follows that c is a strict henselization of a local ring at some point of S. So, c is noetherian (strict henselization of a noetherian ring is noetherian). Being weakly étale over a normal scheme, c is normal (see [SP, Tag 0950]). The scheme c is local, noetherian and normal, thus integral. By [EGA IV 4, Cor. 18.8.14], the fibre c_η contains only one point - the generic point of c. We are using here that the associated primes of a reduced noetherian ring are precisely the generic points of the irreducible components (see [SP, Lm. 0EME] and [SP, Lm. 05AR]). It follows that $\pi_0(\widetilde{S}_\eta) \to \pi_0(\widetilde{S})$ is a (continuous) bijection. As $\widetilde{S}$ is qcqs, $\pi_0(\widetilde{S})$ is compact (see Cor. 5.16) and so $\pi_0(\widetilde{S}_\eta) \to \pi_0(\widetilde{S})$ is in fact a homeomorphism. □

Definition 2.35. ([BS, Defn. 2.4.1.]) A ring A is w-contractible if every faithfully flat ind-étale map $A \to B$ has a section.

Definition 2.36. A compact Hausdorff space is *extremally disconnected* if the closure of every open is open.

Theorem 2.37. *([Gle]) Extremally disconnected spaces are exactly the projective objects in the category of all compact Hausdorff spaces, i.e., those X for which every continuous surjection $Y \to X$ splits.*

Lemma 2.38. *([BS, Lm. 2.4.8.]) A w-strictly local ring A is w-contractible if and only if $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected.*

Lemma 2.39. *([BS, Lm. 2.4.9.]) For any ring A, there is an ind-étale faithfully flat A-algebra A' with A' w-contractible.*

We recall some useful facts about topologically noetherian schemes.

Lemma 2.40. *([BS, Lemma 6.6.10]) Let T be a topological space.*

(a) If T is noetherian, then T is qcqs and has only finitely many connected components. Moreover, any locally closed subset of T is constructible and noetherian itself.

(b) If T admits a finite stratification with noetherian strata, then T is noetherian.

(c) Assume that X is a topologically noetherian scheme and $Y \to X$ is étale. Then Y is topologically noetherian.

Let us now explain a result that is not stated in [BS], but whose proof is essentially the same as the proof of one of the results there. It is not used in the proof of our main results but fits into the current discussion.

Lemma 2.41. *Let X be a topologically noetherian scheme and let $T \to X$ be a finite morphism. Then T is topologically noetherian.*

Proof. The proof is almost the same as the second part of the proof of [BS, Lm. 6.6.10.(3)]. We want to check the descending chain condition on closed subsets of T. Each closed $Z \subset T$ gives rise to a function $f_Z : X \to \mathbb{N}$, given by $x \mapsto \dim_{\kappa(x)}(\mathcal{O}_Z \otimes_{\mathcal{O}_X} \kappa(x))$. This map is upper semicontinuous, see Lm. 2.42. As X is quasi-compact and $T \to X$ is finite, we see that f_X is bounded (by reducing to the affine case and using the definition of a finite morphism). For each closed $Z \subset T$, $f_Z \leq f_X$ (see Lm. 2.43). Thus, f_Z are bounded independently of Z. Let $\{Z_m\}$ be a descending sequence of closed subsets of Z. The functions f_{Z_m} form a descending sequence. Moreover, by Lm. 2.43, the map $Z \mapsto f_Z$ is injective when restricted to $\{Z_m\}$. Thus, it is enough to check that the sequence f_{Z_m} is eventually constant. As the functions f_Z are bounded independently of Z, let us say by N, it is enough to check that for each $n \in \{0, 1, \ldots, N\}$ the descending chain $X_{m,n} = \{x \in X | f_{Z_m}(x) \geq n\}$ is eventually constant (where n is fixed and m varies). But this follows from the semicontinuity of f_Z and topological noetherianity of X. □

Let us state and prove two lemmas that were used in the above proof.

Lemma 2.42. *Let X be a scheme and $\mathcal{F}$ a coherent sheaf on X. Define $f_{\mathcal{F}} : X \to \mathbb{N}$ by $x \mapsto \dim_{\kappa(x)}(\mathcal{F} \otimes \kappa(x))$. Then $f_{\mathcal{F}}$ is upper semicontinuous.*

Proof. This property is local on X and we can assume X is affine, say $X = \mathrm{Spec}(A)$. Then $\mathcal{F}$ corresponds to a finitely generated A-module $M \simeq A^n/K$. Fix $m \in \mathbb{N}$. We want to show that $U_m = \{x \in X | f_{\mathcal{F}} < m\}$ is open. Let $x \in U_m$. By Nakayama's lemma (see [SP, Lm. 00DV]), $f_{\mathcal{F}}(x)$ equals the cardinality of the minimal set of generators of $M \otimes \mathcal{O}_{X,x}$ and any lift of a basis of $M \otimes \kappa(x)$ to $M \otimes \mathcal{O}_{X,x}$ will form such a set of generators. Denote $j = f_{\mathcal{F}}(x)$. Let $e_1, \ldots, e_n$ be the standard basis of A^n (and by abuse of notation also of $\mathcal{O}^n_{X,x}$). By rearranging, we can assume that (the image of) $e_1, e_2, \ldots, e_j$ forms a minimal set of generators of $M \otimes \mathcal{O}_{X,x}$. Let $M' = \langle e_1, \ldots, e_j \rangle \subset M$. Then M/M' is generated by the images of $e_{j+1}, \ldots, e_n$ and $\big(M/M'\big) \otimes \mathcal{O}_{X,x} = 0$. This implies that it is enough to invert finitely many elements to kill M/M'. So there exist an open affine $x \in V \subset X$ where $e_1, \ldots, e_j$ generate $\mathcal{F}(V)$, and thus contained in U_m, as desired. ☐

Lemma 2.43. *Let $Z \to X$ be a finite morphism of schemes and let $Z' \subsetneq Z$ be a closed subscheme (considered with the reduced induced structure). Let $f_Z = f_{\mathcal{O}_Z}$ be as in Lm. 2.42. Then $f_{Z'} \leqslant f_Z$ and $f_{Z'} \neq f_Z$.*

Proof. Let $x \in X$. We want to show that $f_{Z'}(x) \leqslant f_Z(x)$ and $f_{Z'}(x) < f_Z(x)$ if there exist $z \in Z \setminus Z'$ mapping to x. To show this, we can replace X by $\mathrm{Spec}(A)$, with $A = \mathcal{O}_{X,x}$. Denote $Z = \mathrm{Spec}(B)$ and $Z' = \mathrm{Spec}(B/I)$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Then $f_Z(x) = \dim_{\kappa(x)}(B/\mathfrak{m}B)$ and $f_{Z'}(x) = \dim_{\kappa(x)}(B/(\mathfrak{m}B + I))$. Thus, clearly, $f_{Z'}(x) \leqslant f_Z(x)$. Assume now that there exists $z \in Z \setminus Z'$ such that z maps to x. If $f_{Z'}(x) = f_Z(x)$, then $B/\mathfrak{m}B = B/(\mathfrak{m}B + I)$, i.e. $I \subset \mathfrak{m}B$. If z corresponds to a prime $\mathfrak{n} \subset B$, then $z \notin Z'$ implies that $I \nsubseteq \mathfrak{n}$. But $\mathfrak{m}B \subset \mathfrak{n}$ and $I \subset \mathfrak{m}B$, a contradiction. ☐

Let us continue the discussion of topological noetherianity. Again, the following two results are not used elsewhere in the text, but arose during the work.

Lemma 2.44. *Let $f : Y \to X$ be a morphism of schemes. Assume that X is topologically noetherian and f is flat, separated, locally of finite type and has finite fibers. Then*

1. *Y is quasi-compact.*
2. *Even more, Y is topologically noetherian.*

Proof. Let $\eta \in X$ be the generic point of an irreducible component. As $f^{-1}(\{\eta\})$ is finite and f is locally of finite type, [SP, Lm. 02NW] gives that there exist open affine $U_i \subset Y$, $i = 1, \ldots, n$ and $V \subset X$ with $f(U_i) \subset V$, $\eta \in V$, $f^{-1}(\{\eta\}) \subset \cup_i U_i$ such that each $f_{U_i} : U_i \to V$ is finite. Let $U = \cup_i U_i \subset Y$. It follows that U is quasi-compact. Moreover, by Lm. 2.41, each U_i is topologically noetherian, and thus also U. Shrink V so that V is contained in the irreducible component $\overline{\{\eta\}}$ and replace U_i and U by the intersections with the preimage of the new V. The desired properties of U_i and U still hold. We want to show that now $U = Y_V = f^{-1}(V)$. But each $U_i \to V$ is finite and $Y_V \to V$ is separated, thus the image of each U_i (and so of their finite union U) is closed in Y_V (e.g. by [SP, Lm 01W6]). As $f_V : Y_V \to V$ is flat, the generalizations lift along f_V and, as $V \subset \overline{\{\eta\}}$, every point of Y_V is in the closure of $f^{-1}(\{\eta\})$. But $f^{-1}(\{\eta\}) \subset U$ and U is closed in Y_V and so $U = Y_V$ as desired. Now, $f_1 : Y_1 = Y \setminus U \to X_1 = X \setminus V$ is again flat, separated, locally of finite type and has finite fibers. Repreating the reasoning, there is a point $\eta_1 \in X_1$ and an open $\eta_1 \in V_1 \subset X_1$ such that $f_1^{-1}(V_1)$ is quasi-compact (and in fact topologically noetherian). By topological noetherianity of X, this procedure ends in a finite number of steps. Thus, we get a finite stratification of X on which the statement is true. But both quasi-compactness and topological noetherianity can be checked on a fixed finite stratification (the latter observation is stated in [BS, Lm. 6.6.10(2)]). ☐

Lemma 2.45. *Let $f : X' \to X$ be a morphism of schemes. Assume that X is topologically noetherian, f is of finite presentation, separated and quasi-finite. Then X' is topologically noetherian.*

Proof. By [SP, Lm. 0ASY], there exist closed subschemes $X \supset X_0 \supset X_1 \ldots \supset X_t =$ such that $X_0 \subset X$ is a thickening, and the pullback of $X' \to X$ to each $S_i \setminus S_{i+1}$ is flat. Thus, $f|_{S_i \setminus S_{i+1}}$ satisfies the assumptions of Lm. 2.44. It follows that there is a finite stratification of X' by topologically noetherian spaces. Thus, X' is topologically noetherian (see [BS, Lm. 6.6.10(2)]). □

2.2 Noohi completion

Let $\mathrm{TopGrps}$ denote the category of Hausdorff topological groups (recall that we assume all topological groups to be Hausdorff, unless stated otherwise) and $\mathrm{NoohiGrps}$ to be the full subcategory of Noohi groups. Let G be a topological group. Denote $\mathcal{C}_G = G-\mathrm{Sets}$ and let $F_G : \mathcal{C}_G \to \mathrm{Sets}$ be the forgetful functor. Observe that $(\mathcal{C}_G, F_G)$ is a tame infinite Galois category. To see tameness, consider the natural morphism of topological groups $\alpha_G : G \to \mathrm{Aut}(F_G)$ and observe that already the image $\alpha_G(G)$ acts transitively on any connected object of the category $\mathcal{C}_G$. Thus, the group $\mathrm{Aut}(F_G)$ is a Noohi group. It is easy to see that a morphism $G \to H$ defines an induced morphism of groups $\mathrm{Aut}(F_G) \to \mathrm{Aut}(F_H)$ and check that it is continuous. Let ψ_N : TopGrps $\to$ NoohiGrps be the functor defined by $G \mapsto \mathrm{Aut}(F_G)$. Denote also the inclusion i_N : NoohiGrps $\to$ TopGrps.

Definition 2.46. We call $\psi_N(G)$ the Noohi completion of G and will denote it G^{Noohi}.

Example 2.47. In [BS, Example 7.2.6], it was explained that the category of Noohi groups admits coproducts. Let G_1, G_2 be two Noohi groups and let $G_1 *^N G_2$ denote their coproduct as Noohi groups. Let $G_1 *^{\mathrm{top}} G_2$ be their topological coproduct. It exists and it is a Hausdorff group ([Gra]). Then $G_1 *^N G_2 = (G_1 *^{\mathrm{top}} G_2)^{\mathrm{Noohi}}$.

Proposition 2.48. *For a topological group G, the functor F_G induces an equivalence of categories*

$$G - \mathrm{Sets} \simeq G^{\mathrm{Noohi}} - \mathrm{Sets}$$

Proof. This follows directly from [BS, Theorem 7.2.5]. □

Lemma 2.49. *For any topological group G, the image of $\alpha_G : G \to G^{\mathrm{Noohi}}$ is dense.*

Proof. Let $U \subset G^{\mathrm{Noohi}}$ be open. As G^{Noohi} is Noohi, there exists $q \in G^{\mathrm{Noohi}}$ and an open subgroup $V <^\circ G^{\mathrm{Noohi}}$ such that $qV \subset U$. Consider the G^{Noohi}-set G^{Noohi}/V. It is connected in the category $G^{\mathrm{Noohi}} - \mathrm{Sets}$ and it is isomorphic to a connected object coming from $G - \mathrm{Sets}$ via F_G. Thus, the action of G on G^{Noohi}/V is transitive and so there exists $g \in G$ such that $\alpha_G(g) \cdot [V] = [qV]$, i.e. $\alpha_G(g) \in qV$. Thus, the image of α_G is dense.
□

Observation 2.50. Let $f : H \to G$ be a map of topological groups. Directly from the definitions, one sees that the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ \downarrow{\scriptstyle \alpha_H} & & \downarrow{\scriptstyle \alpha_G} \\ H^{\mathrm{Noohi}} & \xrightarrow{f^{\mathrm{Noohi}}} & G^{\mathrm{Noohi}} \end{array}$$

Lemma 2.51. *(universal property of Noohi completion) Let $f : H \to G$ be a continuous morphism from a topological group to a Noohi group. Then there exists a unique map $f' : H^{\mathrm{Noohi}} \to G$ such that $f' \circ \alpha_H = f$.*

Proof. By the definition of a Noohi gropup, α_G is an isomorphism. Defining $f' := \alpha_G^{-1} \circ f^{\mathrm{Noohi}}$ gives the existence. The uniqueness follows from α_H having dense image. □

Corollary 2.52. *The functor ψ_N is a left adjoint of i_N.*

We now move towards a more explicit description of the Noohi completion.

Lemma 2.53. *Let (G, τ) be a topological group. Denote by $\mathcal{B}$ the collection of sets of the form*

$$x_1\Gamma_1 y_1 \cap x_2\Gamma_1 y_2 \cap \ldots x_m\Gamma_1 y_m$$

where $m \in \mathbb{N}$, $x_i, y_i \in G$ and $\Gamma_i < G$ are open subgroups of G. Then $\mathcal{B}$ is a basis of a group topology τ' on G that is weaker than τ and open subgroups of (G, τ) form a basis of open neighbourhoods of 1_G in (G, τ').

Moreover, the identity map $i' : (G, \tau) \to (G, \tau')$ induces an equivalence of categories $(G, \tau') - \mathrm{Sets} \to (G, \tau) - \mathrm{Sets}$. If $\{1_G\} \subset (G, \tau)$ is thickly closed, i.e. $\cap_{U <^{\circ} G} U = \{1_G\}$ (see Defn. 2.58), then (G, τ') is Hausdorff and $(G, \tau)^{\mathrm{Noohi}} \xrightarrow{\sim} (G, \tau')^{\mathrm{Noohi}}$ is an isomorphism.

Proof. The first statement follows from [Bou, Prop. III.1.1] by taking the filter of subsets of G containing an open subgroup. It is also proven in [Lav, Lm. 1.13] (the proposition is stated there in a particular case, but the proof works for any topological group). The second statement follows from the fact that for a discrete set S, any continuous morphism $(G, \tau) \to \mathrm{Aut}(S)$ factorizes through $i' : (G, \tau) \to (G, \tau')$. ☐

Fact 2.54. Let G be a topological group such that its open subgroups form a basis of open neighbourhoods of 1_G. Then $\widehat{G} \simeq G^{\mathrm{Noohi}}$.

Proof. This is [BS, Prop. 7.1.5]. ☐

Proposition 2.55. *Let (G, τ) be a topological group. Assume that $\{1_G\} \subset (G, \tau)$ is thickly closed (see Defn. 2.58). Then there is a natural isomorphism of groups*

$$G^{\mathrm{Noohi}} \simeq \widehat{(G, \tau')},$$

where τ' denotes the topology described in the previous lemma and $\widehat{\ldots}$ denotes the Raîkov completion.

Proof. We combine Fact 2.54 with the last lemma and get $(G, \tau)^{\mathrm{Noohi}} \simeq (G, \tau')^{\mathrm{Noohi}} \simeq \widehat{(G, \tau')}$. ☐

Observation 2.56. Let G be a topological group and H its normal subgroup. Then the full subcategory of $G - \mathrm{Sets}$ of objects on which H acts trivially is equal to the full subcategory of $G - \mathrm{Sets}$ on which its closure $\bar{H}$ acts trivially and it is equivalent to the category of $G/\bar{H} - \mathrm{Sets}$. So it is an infinite Galois category with the fundamental group equal to $(G/\bar{H})^{\mathrm{Noohi}}$.

Lemma 2.57. *Let X be a connected, locally path-connected, semilocally simply-connected topological space and $x \in X$ a point. Let F_x be the functor taking a covering space $Y \to X$ to the fibre Y_x over the point $x \in X$. Then $(TopCov(X), F_x)$ is a tame infinite Galois category and $\pi_1(TopCov(X), F_x) = \pi_1^{\mathrm{top}}(X, x)$, where we consider $\pi_1^{\mathrm{top}}(X, x)$ with the discrete topology. Here, $TopCov(X)$ denotes denotes the category of covering spaces of X.*

Proof. We first claim that there is an isomorphism: $(\mathrm{TopCov}(X), F_x) \simeq (\pi_1^{\mathrm{top}}(X, x) - \mathrm{Sets}, F_{\pi_1^{\mathrm{top}}(X,x)})$. This is in fact a classical result in algebraic topology, which can be recovered from [Ful, Ch. 13] or [Hat, Ch. 1] and is stated explicitly in [Cak, Cor. 4.1]. This finishes the proof, as discrete groups are Noohi. ☐

2.3 Dictionary between Noohi groups and $G - \mathrm{Sets}$

Definition 2.58. *Let $H \subset G$ be a (not necessarily Noohi) subgroup of a Noohi group G. Then we define a "thick closure" $\overline{\overline{H}}$ of H in G to be the intersection of all open subgroups of G containing H, i.e. $\overline{\overline{H}} := \bigcap_{H \subset U <^{\circ} G} U$. If a subgroup satisfies $H = \overline{\overline{H}}$ we will call it thickly closed in G.*

Observe that in a topological group open subgroups are also closed, so a thickly closed subgroup is also an intersection of closed subgroups, so it is closed in G. Observe that if H is a normal subgroup of G, then H is thickly closed if and only if G/H satisfies the condition that the intersection of its open subgroups is trivial. Observe that if G/H is Noohi then it automatically satisfies this condition as it is Hausdorff and its open subgroups form a basis of opens of the trivial element.

Observe also that an arbitrary intersection of thickly closed subgroups is thickly closed. This justifies, for example, the existence of the smallest normal thickly closed subgroup containing a given group. In fact, we can formulate a more precise observation.

Observation 2.59. Let $H < G$ be a subgroup of a topological group G. Then the smallest normal thickly closed subgroup of G containing H is equal to $\overline{\overline{(H^{nc})}}$, where H^{nc} is the normal closure of H in G.

Let us make an easy observation, that will be useful to keep in mind while reading the proof of the technical proposition below.

Observation 2.60. Let $U < G$ be an open subgroup of a topological group and let $g_0 \in G$. Then the mapping $G/g_0Ug_0^{-1} \to G/U$ given by $[gg_0Ug_0^{-1}] \mapsto [gg_0U]$ is an isomorphism of G-sets.

Given open subgroups $U, V < G$ and some surjective map of G-sets $\phi : G/V \twoheadrightarrow G/U$ we can assume that it is a standard quotient map (i.e. $V \subset U$) up to replacing U by a conjugate open set (more precisely by $g_0Ug_0^{-1}$, where g_0 is such that $\phi([V]) = [g_0U]$).

Remark 2.61. A map $Y' \to Y$ in an infinite Galois category $(\mathcal{C}, F)$ is an epimorphism/monomorphism if and only if the map $F(Y') \to F(Y)$ is surjective/injective. Similarly Y is an initial object if and only if $F(Y) = \emptyset$ and so on. The proofs of those facts are the same as the proofs in [SP, Tag 0BN0]. This justifies using words "injective" or "surjective" when speaking about maps in $(\mathcal{C}, F)$.

Recall the following fact.

Observation 2.62. Let $f : G' \to G$ be a surjective map of topological groups. Then the induced morphism $G'/\ker(f) \to G$ is an isomorphism if and only if f is open. In such case, we say that f is a quotient map. In the language of [Bou, III.2.8] we would call f *strict* and surjective.

Definition 2.63. We will say that an object of a tame infinite Galois category is *completely decomposed* if it is a (possibly infinite) disjoint union of final objects.

Proposition 2.64. *Let $G'' \xrightarrow{h'} G' \xrightarrow{h} G$ be maps between Noohi groups and $\mathcal{C}'' \xleftarrow{H'} \mathcal{C}' \xleftarrow{H} \mathcal{C}$ the corresponding maps of the infinite Galois categories. Then the following hold:*

(1) The map $h' : G'' \to G'$ is a topological embedding if and only if for every connected object X in $\mathcal{C}''$, there exist connected objects $X' \in \mathcal{C}''$ and $Y \in \mathcal{C}'$ and maps $X' \twoheadrightarrow X$ and $X' \hookrightarrow H'(Y)$.

(2) The following are equivalent

(a) The morphism $h : G' \to G$ has dense image.

(b) The functor H maps connected objects to connected objects.

(c) The functor H is fully faithful.

(3) The thick closure of $\mathrm{Im}(h') \subset G'$ is normal if and only if for every connected object Y of $\mathcal{C}'$ s.t. $H'(Y)$ contains a final object of $\mathcal{C}''$, $H'(Y)$ is completely decomposed.

(4) $h'(G'') \subset \mathrm{Ker}(h)$ if and only if the composition $H' \circ H$ maps any object to a completely decomposed object.

(5) Assume that $h'(G'') \subset \mathrm{Ker}(h)$ and that $h : G' \to G$ has dense image. Then the following conditions are equivalent:

(a) the induced map $(G'/\ker(h))^{\mathrm{Noohi}} \to G$ is an isomorphism and the smallest normal thickly closed subgroup containing $\mathrm{Im}(h')$ is equal to $\mathrm{Ker}(h)$,

(b) for any connected $Y \in \mathcal{C}'$ s.t. $H'(Y)$ is completely decomposed, Y is in the essential image of H,

(c) the induced map $(G'/\ker(h))^{\mathrm{Noohi}} \to G$ is an isomorphism and for any connected $Y \in \mathcal{C}'$ s.t. $H'(Y)$ is completely decomposed, there exists $Z \in \mathcal{C}$ and an epimorphism $H(Z) \twoheadrightarrow Y$.

Proof. (1) The proof is virtually the same as for usual Galois categories, but there every injective map is automatically a topological embedding (as profinite groups are compact). Assume that $G'' \to G'$ is a topological embedding. Let $X \in \mathcal{C}''$ be connected and write $X \simeq G''/U$ for an open subgroup $U < G''$. Then there exists an open subset $\widetilde{V} \subset G'$ such that $\widetilde{V} \cap G'' = U$ (as $G'' \to G'$ is a topological embedding) and an open subgroup $V < G'$ such that $V \subset \widetilde{V}$ (as G' is Noohi). Denote $W = V \cap G''$. Then $X' := G''/W \twoheadrightarrow X$ and $X' \hookrightarrow H'(G'/V)$, so we conclude by setting $Y := G'/V$. For the other implication: we want to prove that $G'' \to G'$ is a topological embedding under the assumption from the statement. It is enough to check that the set of preimages ${h'}^{-1}(\mathcal{B})$ of some basis $\mathcal{B}$ of opens of $e_{G'}$ forms a basis of opens of $e_{G''}$. Indeed, assume that this is the case. Firstly, observe that it implies that h' is injective, as both G'' and G' are Hausdorff (and in particular T_0). If U is an open subset of G'', then we can write $U = \bigcup g''_\alpha U_\alpha$ for some $g''_\alpha \in G''$ and $U_\alpha \in {h'}^{-1}(\mathcal{B})$. We can write $U_\alpha = {h'}^{-1}(V_\alpha)$ for some $V_\alpha \in \mathcal{B}$. Then $V = \bigcup h'(g''_\alpha)V_\alpha$ satisfies ${h'}^{-1}(V) = U$ because ${h'}^{-1}(h'(g''_\alpha)V_\alpha) = g''_\alpha U_\alpha$ (by injectivity of h'). So this will prove that the topology on G'' is induced from G' via h'. Let $\mathcal{B} = \{U < G | U \text{ is open}\}$. This is a basis of opens of $e_{G'}$ (as G' is Noohi). We want to check that ${h'}^{-1}(\mathcal{B})$ is a basis of opens of $e_{G''}$. As open subgroups of G'' form a basis of opens of $e_{G''}$ it is enough to show that for any open subgroup $U < G''$ there exists an open subgroup $V < G'$ such that ${h'}^{-1}(V) \subset U$. From the assumption we know that there exist open subgroups $\widetilde{U} < G''$ and $V < G'$ such that $G''/\widetilde{U} \twoheadrightarrow G''/U$ and $G''/\widetilde{U} \hookrightarrow G'/V$. The surjectivity of the first map means that we can assume (up to replacing $\widetilde{U}$ by a conjugate) $\widetilde{U} \subset U$. The injectivity of the second means that we can assume (up to replacing V by a conjugate) that ${h'}^{-1}(V) \subset \widetilde{U}$. Indeed, the injectivity implies that if $h'(g'')V = V$, then $g''\widetilde{U} = \widetilde{U}$ which translates immediately to ${h'}^{-1}(V) \subset \widetilde{U}$. So we have also ${h'}^{-1}(V) \subset U$, which is what we wanted to prove.

(2) The equivalence between (a) and (b) follows from the observation that a map between Noohi groups $G' \to G$ has a dense image if and only if for any open subgroup U of G, the induced map on sets $G' \to G/U$ is surjective. Here, we only use that open subgroups form a basis of open neighbourhoods of $1_G \in G$.

Now, the functor H is automatically faithful and conservative (because $F_{G'} \circ H = F_G$ is faithful and conservative). Assume that (b) holds. Let $S, T \in G-\mathrm{Sets}$ and let $g \in \mathrm{Hom}_{G'-\mathrm{Sets}}(H(S), H(T))$. We have to show that g comes from $g_0 \in \mathrm{Hom}_{G-\mathrm{Sets}}(S,T)$. We can and do assume S, T connected for that. Let $\Gamma_g \subset H(S) \times H(T)$ be the graph of g. It is a connected subobject. As $H(S) \times H(T) = H(S \times T)$, the assumption (b) implies that each connected component of $H(S) \times H(T)$ is the pullback of a connected component Γ_0 of $S \times T$. Thus, Γ_g is the pullback of some $\Gamma_0 \subset S \times T$. By conservativity of H, the projection $p_{\Gamma_0} : \Gamma_0 \to S$ is an isomorphism, as this is true for $p_{\Gamma_g} : \Gamma_g \to H(S)$. The composition $q_{\Gamma_0} \circ p_{\Gamma_0}^{-1} : S \to T$ maps via H to g.

Conversely, assume (c) holds. Let $S \in G - \mathrm{Sets}$ be connected. We want to show that $H(S)$ is connected. Suppose on the contrary that $H(S) = A \sqcup B$ with $A, B \in G' - \mathrm{Sets}$. Let $T = \bullet \sqcup \bullet \in G - \mathrm{Sets}$ be a two-element set with a trivial G-action. Then $\mathrm{Hom}_{G-\mathrm{Sets}}(S,T)$ has precisely two elements, while $\mathrm{Hom}_{G'-\mathrm{Sets}}(H(S), H(T)) = \mathrm{Hom}_{G'-\mathrm{Sets}}(A \sqcup B, \bullet \sqcup \bullet)$ has at least four.

(3) Assume first that the thick closure of $\mathrm{im}(h')$ is normal. Let $Y = G'/U$ be an element of $\mathcal{C}'$ whose pullback to $G'' - \mathrm{Sets}$ contains the final object. This means that G'' fixes one of the classes, let's say $[g'U]$. This is equivalent to ${g'}^{-1}h'(G'')g'$ fixing $[U]$, i.e. ${g'}^{-1}h'(G'')g' \subset U$. But this implies immediately that $\overline{\overline{({g'}^{-1}h'(G'')g')}} \subset U$. Let $\tilde{g} \in G'$ be any element. We have $\overline{\overline{({g'}^{-1}h'(G'')g')}} = {g'}^{-1}\overline{\overline{h'(G'')}}g' = \overline{\overline{h'(G'')}} = \tilde{g}^{-1}\overline{\overline{h'(G'')}}\tilde{g}$ from the assumption that $\overline{\overline{h'(G'')}}$ is normal. So $\tilde{g}^{-1}h'(G'')\tilde{g} \subset \tilde{g}^{-1}\overline{\overline{h'(G'')}}\tilde{g} \subset U$ and we conclude that $h'(G'')$ fixes an arbitrary class $[\tilde{g}U]$. This shows that G'/U pulls back to a completely decomposed object.

The other way round: assume that for every connected object Y of $\mathcal{C}'$ s.t. $H'(Y)$ contains a final object, $H'(Y)$ is completely decomposed. Let U be an open subgroup of G' containing $h'(G'')$. Then G'' fixes $[U] \in G/U$ and so, by assumption, fixes every $[g'U] \in G/U$. This implies that for any $g' \in G'$ ${g'}^{-1}h'(G'')g' \subset U$ which easily implies that also $h'(G'')^{nc} \subset U$. As this is true for any U containing

$h'(G'')$ we get that $\overline{\overline{h'(G'')}} = \overline{\overline{(h'(G'')^{nc})}}$ and the last group is the smallest normal thickly closed subgroup of G' containing $h'(G'')$ (Observation 2.59).

(4) The same as for usual Galois categories, we use that $\cap_{U<^\circ G} U = 1_G$.

(5)
- (b) $\Rightarrow$ (c): Assume (b). We only need to show, that $(G'/\ker(h))^{\mathrm{Noohi}} \to G$ is an isomorphism. This is equivalent to showing that H induces an equivalence $G'/\ker(h) - \mathrm{Sets} \simeq G - \mathrm{Sets}$. As $G'/\ker(h) - \mathrm{Sets} \simeq \{S \in G' - \mathrm{Sets} | \ker(h) \text{ acts trivially on } S\} \subset \{S \in G - \mathrm{Sets} | G'' \text{ acts trivially on } S\}$, the assumption of (b) implies that the functor $G - \mathrm{Sets} \to G'/\ker(h) - \mathrm{Sets}$ is essentially surjective. By the global assumption that $G' \to G$ has dense image, it is fully faithful (see (2)).
- (c) $\Rightarrow$ (b): Assume (c). Let $Y \in \mathcal{C}'$ be connected and such that $H'(Y)$ is completely decomposed. We have $Z \in \mathcal{C}$ and an epimorphism $H(Z) \twoheadrightarrow Y$. As $\ker(h)$ acts trivially on $H(Z)$, we conclude that it also acts trivially on Y. Thus, by abuse of notation, $Y \in G'/\ker(h) - \mathrm{Sets}$. But $G'/\ker(h) - \mathrm{Sets} \simeq (G'/\ker(h))^{\mathrm{Noohi}} - \mathrm{Sets} \simeq G - \mathrm{Sets}$ from the assumption. Thus, we see that Y is in the essential image of H.
- (b) $\Rightarrow$ (a): Assume (b). We give two proofs of this fact.
  Elementary proof: We have proven above that (b) $\Rightarrow (G'/\ker(h))^{\mathrm{Noohi}} \simeq G$. Let N be the smallest normal thickly closed subgroup of G' containing $h'(G'')$. Observe that $N \subset \ker h$ (as $\ker(h)$ is thickly closed). Let U be an open subgroup containing N. We want to show that U contains $\ker h$. This will finish the proof as both N and $\ker h$ are thickly closed. Write $Y = G'/U$. Observe that G'/U pulls back to a completely decomposed G''-set if and only if for any $g' \in G'$ there is $g'h'(G''){g'}^{-1} \subset U$. Indeed, $h'(G'')$ fixes $[g'U] \in G'/U$ if and only if $g'h'(G''){g'}^{-1}$ fixes $[U]$. So $N \subset U$ implies that Y pulls back to a completely decomposed G''-set and, by assumption, Y is isomorphic to a pull-back of some G-set and so $\ker(h)$ acts trivially on Y. This implies that $\ker h \subset U$, which finishes the proof.
  Alternative proof: We already know that (b) $\Rightarrow (G'/\ker(h))^{\mathrm{Noohi}} \simeq G$. Let $N \subset \ker(h)$ be as in the elementary proof above. Consider the map $G/N \twoheadrightarrow G/\ker(h)$. The assumption (b) and full faithfulness of H (by the global assumption and using (2)) imply that $(G'/N)^{\mathrm{Noohi}} \to G$ is an isomorphism. Thus, $(G'/N)^{\mathrm{Noohi}} \simeq (G'/\ker(h))^{\mathrm{Noohi}}$. Using Prop. 2.55, we check that the canonical maps $G'/N \to (G'/N)^{\mathrm{Noohi}}$ and $G'/\ker(h) \to (G'/\ker(h))^{\mathrm{Noohi}}$ are injective. Thus, $G'/N \twoheadrightarrow G'/\ker(h)$ is injective and so $N = \ker(h)$.
- (a) $\Rightarrow$ (b): Assume (a). Let $Y = G'/U$ be a connected G'-set that pulls back via h' to a completely decomposed object. As we have seen while proving "(b) $\Rightarrow$ (a)", this implies that for any $g' \in G'$ $g'h'(G''){g'}^{-1} \subset U$, so $H^{nc} \subset U$ and so also $\overline{\overline{(H^{nc})}} \subset U$. But, by Observation 2.59, there is $N = \overline{\overline{(H^{nc})}}$. By assumption, we have $N = \ker h$ and so we conclude that $\ker h \subset U$. But then, by assumption $(G'/\ker(h))^{\mathrm{Noohi}} \simeq G$, Y is in the essential image of H.

$\square$

To distinguish between exactness in the usual sense (i.e. on the level of abstract groups) and notions of exactness appearing in Prop. 2.64, we introduce a new notion. It will be mainly used in the context of Noohi groups.

Definition 2.65. Let $G'' \xrightarrow{h'} G' \xrightarrow{h} G \to 1$ be a sequence of topological groups such that $\mathrm{im}(h') \subset \ker(h)$. Then we will say that the sequence is

(1) *weakly exact on the right* if h has dense image,

(2) *weakly exact in the middle* if $\overline{\overline{\mathrm{im}(h')}} = \ker(h)$, i.e. the thick closure of the image of h' in G' is equal to the kernel of h,

(3) *weakly exact* if it is both weakly exact on the right and weakly exact in the middle.

Observation 2.66. Let G be a topological group such that the open subgroups form a local base at 1_G. Let $W \subset G$ be a subset. Then the topological closure of W can be written as $\overline{W} = \cap_{V<^\circ G} WV$.

The following lemma can be found on p.79 of [Lep].

Lemma 2.67. *Let G be a topological group such that that the open subgroups for a basis of topology. Let $H \lhd G$ be a normal subgroup. Then*

$$\overline{H} = \overline{\overline{H}}$$

i.e. the usual topological closure and the thick closure coincide.

Proof. We compute that $\overline{H} = \cap_{V <^{\circ} G} HV \overset{(*)}{\supset} \cap_{H < U <^{\circ} G} U = \overline{\overline{H}} \supset \overline{H}$. The inclusion $(*)$ follows from the fact that HV is an (open) subgroup of G as H is normal. □

2.4 Appendix on valuative criteria, normalization and pro-étale descent

We recall the precise statements of the valuative criteria.

Lemma 2.68. *([SP, Lemma 01KE] Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) Specializations lift along any base change of f.

(2) The morphism f satisfies the existence part of the valuative criterion.

Lemma 2.69. *([SP, Section 01KY] Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) f is separated.

(2) f is quasi-separated and satisfies the uniqueness part of the valuative criterion.

Lemma 2.70. *([SP, Lemma 01KC]) Let $f : X \to S$ be a morphism of schemes.*

(a) If f is universally closed then specializations lift along any base change of f.

(b) If f is quasi-compact and specializations lift along any base change of f, then f is universally closed.

We will sometimes shorten "the valuative criterion of properness" to "VCoP".

Lemma 2.71. *Let $g : T \to S$ be a map of schemes. The properties:*

(a) g is étale

(b) g is separated

(c) g satisfies the existence part of VCoP

can be checked fpqc-locally on S.

Proof. The cases of étale and separated morphisms are proven in [SP, Section 02YJ]. For the last part: satisfying the existence part of VCoP is equivalent to specializations lifting along any base-change of g. It is easy to see that this property can be checked Zariski locally. Thus, if $S' \to S$ is an fpqc cover such that the base-change $g' : X' \to S'$ satisfies specialization lifting for any base-change, we can assume that S, S' are affine with $S' \to S$ faithfully flat. Let $T \to S$ be any morphism. Consider the diagram:

$$\begin{array}{ccccc} X' & \longleftarrow & S' \times_S X \times_S T & \longrightarrow & T \times_S X \\ \downarrow & & \downarrow & & \downarrow \\ S' & \longleftarrow & S' \times_S T & \longrightarrow & T \end{array}$$

Let $\xi' \in T \times_S X$, let ξ be its image in T and let $t \in T$ be such that $\xi \rightsquigarrow t$. We need to find $t' \in T \times_S X$ over t such that $\xi' \rightsquigarrow t'$. Let $Z = \overline{\{\xi'\}} \subset T \times_S X$ be the closure of $\{\xi'\}$. We need to show that the set-theoretic image $W \subset T$ of Z in T contains t. It is enough to show, that W is stable under specialization or, equivalently, that $T \setminus W$ is stable under generalization. But, from flatness ([SP, Lemma 03HV]), generalizations lift along $S' \times_S T \to T$. Thus, it is enough to show that the preimage of $T \setminus W$ in $S' \times_S T$ is stable under generalizations or, equivalently (using the surjectivity of $S' \times_S T \to T$), that the preimage of W in $S' \times_S T$ is closed under specializations. But an easy diagram chasing (using the fact that the right square of the diagram above is cartesian) shows that the preimage of W in $S' \times_S T$ is the image of a closed subset of $S' \times_S X \times T$. We conclude, because specializations lift along $S' \times_S X \times_S T \to S' \times_S T$ by assumption. □

Lemma 2.72. *Let $f : Y \to X$ be a geometric cover of a locally topologically noetherian scheme. Then f is separated.*

Proof. As X is locally topologically noetherian, Y is also locally topologically noetherian. As intersection of two noetherian opens is noetherian again and in particular quasi-compact, we easily get that f is quasi-separated. This is also stated in [BS, Remark 7.3.3]. A quasi-separated morphism satisfying VCoP is separated (see [SP, Tag 01KY]). □

Normalization revisited

The normalization of a scheme plays an important role in this text. We work with the definition and the level of generality as in Stacks Project, i.e. the normalization is defined for schemes such that every quasi-compact open has finitely many irreducible components. See [SP, 035E] for the definition and some further discussion.

The following two results give some additional insight on normalization at the level of generality we are working, but do not play an essential role later on.

Lemma 2.73. *(Universal property of normalization) Let X, X' be schemes such that every quasi-compact open has finitely many irreducible components. Assume that X' is normal and let $\pi : X' \to X$ be a dominant morphism of schemes such that the generic points of the irreducible components of X' map to the generic points of the irreducible components of X. The following assertions are equivalent.*

(1) $\pi : X' \to X$ is the normalization of X.

(2) For every integral normal scheme Y and every morphism $f : Y \to X$ such that the generic point of Y maps to the generic point of one of the irreducible components of X, there exists a unique morphism $f' : Y \to X'$ such that $\pi \circ f' = f$.

Proof. If X and X' are integral, this is [GW, Prop. 12.44]. We reduce the general case to this one as follows. Let us show (2) ⇒ (1) first. By [SP, Lm. 0357]), X' is a disjoint union of integral normal schemes. The restriction of π to a connected component of X' factorizes through some irreducible component of X and the obtained map is dominant. Fix an irreducible component Z of X considered with the induced reduced structure. The scheme Z is reduced and irreducible, and thus integral. Let Z^ν be its normalization. By the lifting property in (2) applied to $Z^\nu \to Z \to X$, there is a connected component $Z' \subset X'$ such that $Z^\nu \to X$ factorizes as $Z^\nu \to Z' \to X$. We claim that $Z' \simeq Z^\nu$. Let $T \to Z$ be a map from an integral normal scheme. By the known special case of the lemma ([GW, Prop. 12.44]), $T \to Z$ factorizes through Z^ν, and thus (using the map $Z^\nu \to Z'$) through Z'. By ([GW, Prop. 12.44]) again, we only need to check uniqueness of the lift $T \to Z'$ to show $Z' \simeq Z^\nu$. But two different lifts $T \to Z'$ would contradict the uniqueness of the lift $T \to X'$, guaranteed by (2). We have proven $Z' \simeq Z^\nu$. All we need to check to show that $X' \simeq X^\nu$ is that Z' is the unique connected component of X' dominating Z. Assume that there is another such component $Z'' \subset X'$. As Z'' is integral and normal, by the universal property in the integral case, $Z'' \to Z$ factorizes through Z^ν. But $Z' \simeq Z^\nu$. Thus the map $Z'' \to X$ admits two different lifts $Z'' \to X'$, i.e. the structural embedding $Z'' \subset X'$ and $Z'' \to Z^\nu \simeq Z' \subset X'$. This contradicts the uniqueness part of the assumption in (2). This finishes the proof of (2) ⇒ (1). The implication (1) ⇒ (2) is proven in [SP, Lm. 035Q]. Let us include a short proof for convenience. Let $Y \to X$ be a dominant map from an integral normal scheme. It factorizes through an irreducible component Z of X. As $X' = X^\nu$, there is a unique connected component Z' of X' dominating Z and in fact equal (uniquely isomorphic) to Z^ν. As $Y \to Z$ is dominant, any lift $Y \to X'$ has to factorize through Z'. By [GW, Prop. 12.44], there is such a lift and it is unique. □

Corollary 2.74. *Let X, Y be schemes such that every quasi-compact open has finitely many irreducible components. Let $Y \to X$ be a morphism such that the generic points of the irreducible components of Y map to the generic points of the irreducible components of X (e.g. flat). Let $X^\nu \to X$ be the normalization. Assume that the base-change $Y' = X^\nu \times_X Y$ is normal, every quasi-compact open of Y' has finitely many irreducible components and the map $Y' \to Y$ sends the generic points of the irreducible components of Y' to the generic points of the irreducible components of Y. Then Y' is (uniquely isomorphic to) the normalization Y^ν of Y.*

Proof. One easily checks that $Y' \to Y$ is dominant. Let us check that $Y' \to Y$ satisfies the universal property and apply Lm. 2.73. Let $f : T \to Y$ be a morphism from an integral normal scheme T such that the generic point of T maps to the generic point of an irreducible component of Y. Then the composition $T \to Y \to X$ satisfies the same property and the universal property of $X^\nu \to X$ says that the map $T \to X$ factorizes uniquely through X^ν. Thus $T \to Y$ factorizes uniquely through Y', as desired. Indeed, if $f \mapsto g$ via $\mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, X)$, then $\mathrm{Hom}_Y(T, Y')$ can be identified with the subset $\mathrm{Hom}(T, X^\nu) \times_{\{g\}} \{f\} \subset \mathrm{Hom}(T, X^\nu) \times_{\mathrm{Hom}(T,X)} \mathrm{Hom}(T, Y) = \mathrm{Hom}_X(T, Y')$. But $\mathrm{Hom}(T, X^\nu) \times_{\{g\}} \{f\}$ is a singleton by the universal property of $X^\nu \to X$. □

Compare the last result with [SP, Lm. 0CBM], where the special case of strict henselization is treated.

A remark on pro-étale descent

The following results arose as an attempt of the author to understand why at some point of the proof of [BS, Lemma 7.3.9.] the fpqc sheaf obtained via descent is automatically an algebraic space. The main question being of the existence of a cover by étale schemes. As explained to me via e-mail by Prof. Scholze, this is a consequence of the fact that the objects of Loc_X are classical (see Defn. 2.24 and Lm. 2.23).

Below we present another approach that, however, requires an additional assumption that the maps in the pro-étale presentation of the considered cover are surjective. By following the construction (see e.g. [SP, Section 097Q]) one sees that this can be guaranteed when constructing w-strictly local covers. This is for example sufficient for our needs in the proof of Theorem 5.5. However, for w-contractible covers this condition will usually not be satisfied. In the Prop. 2.75 and Lm. 2.76 below, the results are Zariski local on the base S and thus in the proofs we will assume it to be affine.

Proposition 2.75. *Let S be a qcqs scheme and let $\widetilde{S} \to S$ be a pro-étale cover with a presentation as a limit over a directed inverse system $\widetilde{S} = \varprojlim_\lambda S_\lambda$, where $S_\mu \to S_\lambda$ are all affine étale. Assume that all the maps $S_\mu \to S_\lambda$ in the inverse system are surjective. Let $\widetilde{T} \to \widetilde{S}$ be a scheme with a fpqc-descent datum $\phi : p^*\widetilde{T} \xrightarrow{\sim} q^*\widetilde{T}$ with respect to $\widetilde{S} \to S$. For any λ, this gives a descent datum ϕ_λ for $\widetilde{T}$ with respect to $\widetilde{S} \to S_\lambda$ as well. Let T (and T_λ) denote the fpqc-sheaf on S (S_λ respectively) obtained by the fpqc descent. Then for any affine open $\widetilde{U} \subset \widetilde{T}$ there exists λ_0 such that ϕ_{λ_0} induces a descent datum on $\widetilde{U} \to \widetilde{T}$ with respect to $\widetilde{S} \to S_{\lambda_0}$, i.e. ϕ_{λ_0} restricts to an isomorphism $p^*_{\lambda_0}\widetilde{U} \xrightarrow{\sim} q^*_{\lambda_0}\widetilde{U}$. Moreover, in the obtained morphism of fpqc sheaves $U_{\lambda_0} \to T_{\lambda_0}$, U_{λ_0} is representable by a scheme.*

Proof. Observe that there is a canonical isomorphism (coming from the diagonal morphism at each level) $\widetilde{S} \simeq \varprojlim_\lambda \widetilde{S} \times_{S_\lambda} \widetilde{S}$. The isomorphisms ϕ_λ and the projections p_λ, q_λ form inverse systems respectively and their limits are: $\varprojlim p_\lambda = \mathrm{id}_{\widetilde{S}}$, $\varprojlim q_\lambda = \mathrm{id}_{\widetilde{S}}$, $\varprojlim \phi_\lambda = \mathrm{id}_{\widetilde{T}}$. Thus, in the limit, the ϕ_λ's do restrict to an isomorphism from $p^*_\lambda \widetilde{U}$ to $q^*_\lambda \widetilde{U}$. We need to show that this holds already for some λ_0. The schemes $V_\lambda = q^*_\lambda \widetilde{U}$ and $V'_\lambda = \phi_\lambda(p^*_\lambda \widetilde{U})$ are affine open subschemes of $q^*_\lambda \widetilde{T}$ that become equal in the limit. The symmetric differences $\Delta_\lambda = (V_\lambda \setminus V'_\lambda) \cup (V'_\lambda \setminus V_\lambda)$ are affine schemes and they form an inverse system. Indeed, if $\mu > \lambda$, then V_μ is the preimage of V_λ via $q^*_\mu \widetilde{T} \to q^*_\lambda \widetilde{T}$ and analogously for V'_μ. So $V'_\mu \setminus V_\mu$ maps to $V'_\lambda \setminus V_\lambda$ and similarly for $V_\mu \setminus V'_\mu$. Thus, Δ_λ form a directed inverse system of affine schemes with an empty limit. By [SP, Tag 01Z2], there exists λ_0 such that Δ_{λ_0} is empty, which is precisely what we wanted to show. The last part follows from [SP, Tag 0247], because $\widetilde{U} \subset \widetilde{T}$ is an affine open, $\widetilde{T}$ is quasi-separated and so $\widetilde{U} \to \widetilde{T}$ is quasi-affine. □

In what follows, let us work with the definition of an algebraic space as in the Stacks Project, i.e. [SP, Definition 025Y].

Lemma 2.76. *Let S be a topologically noetherian scheme and let $T \in \mathrm{Loc}_S$ (notation as in [BS]). Let $\widetilde{S} \to S$ be a pro-étale cover satisfying the assumptions of Prop. 2.75. Assume that the restriction $\widetilde{T} = T_{|\widetilde{S}}$ is a constant sheaf (i.e. can be written as $\sqcup_\alpha \widetilde{S}$). Then T is represented by an algebraic space.*

Proof. T is an fpqc sheaf on S and in turn an fppf sheaf on S. We need to check that: a) the diagonal morphism $T \to T \times T$ is representable, b) there is an étale surjective morphism $U \to T$ with U a scheme. Let us start with showing b): let us cover $\widetilde{T}$ with open affine subschemes $\widetilde{U}_i$. As $\widetilde{T}$ comes from S, it is equipped with a descent datum with respect to $\widetilde{S} \to S$. By Prop. 2.75, for each U_i there exist λ_i such that $\widetilde{U}_i \to \widetilde{T}$ descends to a morphism $U_i \to T_{\lambda_i}$ with U_i a scheme. Let f_i be the composition of $U_i \to T_\lambda$ and $T_\lambda \to T$. We claim that this is an étale morphism of fpqc sheaves. Indeed, each morphism of the composition is representable, because each of them becomes a quasi-affine morphism of schemes after an fpqc base-change and thus we can apply [SP, Tag 0247]. Similarly, both $U_i \to T_\lambda$ and $T_\lambda \to T$ are étale, because each of them becomes étale morphism of schemes after a suitable fpqc base-change. To show a) we again observe that $\Delta : T \to T \times T$ becomes $\widetilde{T} \to \widetilde{T} \times_{\widetilde{S}} \widetilde{T}$ after an fpqc base-change and thus a quasi-affine morphism of schemes (using that being quasi-affine can be checked on a chosen affine cover of the target we translate the problem to checking that the intersection of two affine opens in $\widetilde{T}$ is quasi-compact, but this follows from quasi-separatedness of $\widetilde{T}$ over $\widetilde{S}$) and we apply [SP, Tag 0247] to see that Δ is representable. □

Chapter 3

Seifert-van Kampen theorem for $\pi_1^{\text{proét}}$ and its applications

3.1 Abstract Seifert–van Kampen theorem for infinite Galois categories

We aim at recovering a general version of van Kampen theorem, as presented in [Sti], in the case of the pro-étale fundamental group. Most of the definitions and proofs are virtually the same as in [Sti], after replacing "Galois category" with "(tame) infinite Galois category" and "profinite" with "Noohi", but still some additional technical difficulties appear here and there. We make the necessary changes in the definitions and deal with those difficulties below.

Denote by $\Delta_{\leq 2}$ a category whose objects are $[0] = \{0\}, [1] = \{0,1\}, [2] = \{0,1,2\}$ and has strictly increasing maps as morphisms. There are face maps $\partial_i : [n-1] \to [n]$ for $n = 1, 2$ and $0 \leq i \leq n$ which omit the value i and vertices $v_i : [0] \to [2]$ with image i.

The category of 2-*complexes in a category* $\mathscr{C}$ is the category of contravariant functors $T_\bullet : \Delta_{\leq 2} \to \mathscr{C}$. We denote $T_n = T_\bullet([n])$ and call it *the n-simplices of $T_\bullet$*. $T(\partial_i)$ is called the *i-th boundary map*.

By a 2-*complex E* we mean a 2-complex in the category of sets. We often think of E as a category: its objects are the elements of E_n for $n = 0, 1, 2$ and its morphisms are obtained by defining $\partial : s \to t$ where $s \in E_n$ and $t = E(\partial)(s)$. Let $\Delta_n = \{\sum_{i=0}^{n} \lambda_i e_i \in \mathbb{R}_{\geq 0}^{n+1} | \sum_i \lambda_i = 1\}$ denote the topological n-simplex. Then we define $|E| = \bigsqcup E_n \times \Delta_n / \sim$, where $\sim$ identifies $(s, d(x))$ with $(E(\partial)(s), x)$ for all $\partial : [m] \to [n]$ and its corresponding linear map $d : \Delta_m \to \Delta_n$ sending e_i to $e_{\partial(i)}$, and $s \in E_n$ and $x \in \Delta_m$. We call E *connected* if $|E|$ is a connected topological space.

Definition 3.1. *Noohi group data* $(\mathscr{G}, \alpha)$ on a 2-complex E consists of the following:

1. A mapping (not necessarily a functor!) $\mathscr{G}$ from the category E to the category of Noohi groups: to a complex $s \in E_n$ is attributed a Noohi group $\mathscr{G}(s)$ and to a map $\partial : s \to t$ is attached a continuous morphism $\mathscr{G}(\partial) : \mathscr{G}(s) \to \mathscr{G}(t)$.

2. For every triple $v \in E_0$, $e \in E_1$, $f \in E_2$ and boundary maps ∂', ∂ such that $\partial'(f) = e$, $\partial(e) = v$, an element $\alpha_{vef} \in \mathscr{G}(v)$ (its existence is a part of the definition) such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{G}(f) & \xrightarrow{\mathscr{G}(\partial')} & \mathscr{G}(e) \\ \downarrow{\scriptstyle \mathscr{G}(\partial\partial')} & & \downarrow{\scriptstyle \mathscr{G}(\partial)} \\ \mathscr{G}(v) & \xrightarrow{\alpha_{vef}(\cdot)\alpha_{vef}^{-1}} & \mathscr{G}(v) \end{array}$$

Definition 3.2. Let $(\mathscr{G},\alpha)$ be Noohi group data on the 2-complex E. A $(\mathscr{G},\alpha)$-*system* M on E consists of the following:

(1) For every simplex $s\in E$ a $\mathscr{G}(s)$-set M_s.

(2) For every boundary map $\partial: s\to t$ a map of $\mathscr{G}(s)$-sets $m_\partial : M_s \to \mathscr{G}(\partial)^*(M_t)$, such that:

(3) for every triple $v\in E_0$, $e\in E_1$, $f\in E_2$ and boundary maps ∂',∂ such that $\partial'(f)=e$, $\partial(e)=v$ the following diagram commutes

$$\begin{array}{ccc} M_f & \xrightarrow{m_{\partial'}} & M_e \\ {\scriptstyle m_{\partial\partial'}}\downarrow & & \downarrow{\scriptstyle m_\partial} \\ M_v & \xrightarrow{\alpha_{vef}\cdot} & M_v \end{array}$$

Definition 3.3. A $(\mathscr{G},\alpha)$-'system is called *locally constant* if all the maps m_∂ are bijections.

Observe that $\alpha\cdot : m\mapsto \alpha m$ is a $\mathscr{G}(v)$-equivariant map $M_v\to(\alpha()\alpha^{-1})^*M_v$. Observe that there is an obvious notion of a morphism of $(\mathscr{G},\alpha)$-systems: a collection of $\mathscr{G}(s)$-equivariant maps that commute with the m's. Let us denote by $\mathrm{lcs}(E,(\mathscr{G},\alpha))$ the category of locally constant $(\mathscr{G},\alpha)$-systems.

Let $M\in\mathrm{lcs}(\mathscr{G},\alpha)$ for Noohi group data $(\mathscr{G},\alpha)$ on some 2-complex E. We define oriented graphs $E_{\le 1}$ and $M_{\le 1}$ (which will be an oriented graph *over* $E_{\le 1}$) as in [Sti], but our graphs $M_{\le 1}$ are possibly infinite. For $E_{\le 1}$ the vertices are E_0 and edges E_1 such that ∂_0 (resp. ∂_1) map an edge to its target (resp. origin). For $M_{\le 1}$ the vertices are $\bigsqcup_{v\in E_0} M_v$ and edges are $\bigsqcup_{e\in E_1} M_e$ serves as the set of edges. The target/origin maps are induced by the m_∂ and the map $M_{\le 1}\to E_{\le 1}$ is the obvious one.

There is an obvious topological realization functor for graphs $|\cdot|$. By applying this functor to the above construction we get a topological covering (because M is locally constant) $|M_{\le 1}|\to|E_{\le 1}|$. This gives a functor

$$|\cdot_{\le 1}| : \mathrm{lcs}(E,(\mathscr{G},\alpha))\to \mathrm{TopCov}(|E_{\le 1}).$$

Choosing a maximal subtree T of $|E_{\le 1}|$ gives a fibre functor $F_T : \mathrm{TopCov}(|E_{\le 1}|)\to$Sets by $(p: Y\to|E_{\le 1}|)\mapsto \pi_0(p^{-1}(|T|))$. The pair $(\mathrm{TopCov}(|E_{\le 1}|),F_T)$ is an infinite Galois category and the resulting fundamental group $\pi_1(\mathrm{Cov}(|E_{\le 1}|),F_T))$ is isomorphic to $\pi_1^{\mathrm{top}}(|E_{\le 1}|)$ (see Lemma 2.57) which is in turn isomorphic to $\mathrm{Fr}(E_1)/\langle\langle\{\vec{e}|e\in T\}^{\mathrm{Fr}(E_1)}\rangle\rangle = \mathrm{Fr}(\vec{e}|e\in E_1\setminus T)$, where $\mathrm{Fr}(\dots)$ denotes a free group on the given set of generators and $\langle\langle\{\vec{e}|e\in T\}^{Fr(E_1)}\rangle\rangle$ denotes the normal closure in $\mathrm{Fr}(E_1)$ of the subgroup generated by $\{\vec{e}\in T\}$. Here, $\vec{e}$ acts on $F_T(M)$ via

$$\pi_0(p^{-1}(|T|))\cong\pi_0(p^{-1}(\partial_0(e)))\cong\pi_0(p^{-1}(|e|))\cong\pi_0(p^{-1}(\partial_1(e))\cong\pi_0(p^{-1}(|T|))$$

As in [Sti], for every $s\in E_0$ and $M\in\mathrm{lcs}(E,(\mathscr{G},\alpha))$ we have that $F_T(M)$ can be seen canonically as a $\mathscr{G}(s)$-module by $M_s=\pi_0(p^{-1}(s))\cong\pi_0(p^{-1}(T))$. Denote $\pi_1(E_{\le 1},T)=\mathrm{Fr}(E_1)/\langle\langle\{\vec{e}|e\in T\}^{\mathrm{Fr}(E_1)}\rangle\rangle$. Putting the above together we get a functor

$$Q : \mathrm{lcs}(E,(\mathscr{G},\alpha))\to(*^N_{v\in E_0}\mathscr{G}(v) *^N \pi_1(E_{\le 1},T)) - \mathrm{sets}$$

Remark 3.4. In the setting of usual ("finite") Galois categories, it is usually enough to say that a particular morphism between two Galois categories is exact, because of the following fact ([SP, Tag 0BMV]): Let G be a topological group. Let $F:\mathrm{Finite}{-}G{-}\mathrm{Sets}\to\mathrm{Sets}$ be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the forgetful functor.

As we do not know if an analogous fact is true for infinite Galois categories, given two infinite Galois categories $(\mathcal{C},F)$, $(\mathcal{C}',F')$ and a morphism $\phi:\mathcal{C}\to\mathcal{C}'$, we are usually more interested in checking whether $F\simeq F'\circ\phi$. If ϕ satisfies this condition it also commutes with finite limits and arbitrary colimits. Indeed, we have a map $\mathrm{colim}\phi(X_i)\to\phi(\mathrm{colim}X_i)$ that becomes an isomorphism after applying F' (as F' and $F=F'\circ\phi$ commute with colimits) and we conclude by conservativity of F'. Similarly for finite limits.

Proposition 3.5. *Let $(E, (\mathscr{G}, \alpha))$ be a connected 2-complex with Noohi group data. Define a functor $F : \mathrm{lcs}(E, (\mathscr{G}, \alpha)) \to$ Sets in the following way: pick any simplex s and define F by $M \mapsto M_s$. Then $(\mathrm{lcs}(E, (\mathscr{G}, \alpha)), F)$ is a tame infinite Galois category.*

Moreover, the obtained functor

$$Q : lcs(E, (\mathscr{G}, \alpha)) \to (*^N_{v \in E_0} \mathscr{G}(v) *^N \pi_1(E_{\leq 1}, T)) - sets$$

satisfies $F \simeq F_{\mathrm{forget}} \circ Q$ and maps connected objects to connected objects.

Proof. We first check conditions (1), (2) and (4) of [BS, Def. 7.2.1]. Then we show that Q maps connected objects to connected objects and we use the proof of this last fact to show the condition (3).

Colimits and finite limits: they exist simplexwise and taking limits and colimits is functorial so we get a system as candidate for a colimit/finite limit. This will be a locally constant system, as the colimit/finite limit of bijections between some G-sets is a bijection.

Each M is a disjoint union of connected objects: let us call $N \in \mathrm{lcs}(\mathscr{G}, \alpha)$ a *subsystem* of M if there exists a morphism $N \to M$ such that for any simplex s the map $N_s \to M_s$ is injective (we then identify, for any simplex s, N_s with a subset of M_s). We can intersect such subsystems in an obvious way and observe that it gives another subsystem. So for any element $a \in M_v$ there exists the smallest subsystem N of M such that $a \in N_v$. We see readily that for any vertices v, v' and $a \in M_v, a' \in M_{v'}$ the smallest subsystems N and N' containing one of them are either equal or disjoint (in the sense that, for each simplex s, N_s and N'_s are disjoint as subsets of M_s). It is easy to see that in this way we have obtained a decomposition of M into a disjoint union of connected objects.

F is faithful, conservative and commutes with colimits and finite limits: observe that $\phi_s : \mathrm{lcs}(E, (\mathscr{G}, \alpha)) \ni M \mapsto M_s \in \mathscr{G}(s) - \mathrm{Sets}$ is faithful, conservative and commutes with colimits and finite limits and $F = F_s \circ \phi_s$, where F_s is the usual forgetful functor on $\mathscr{G}(s)-$Sets.

It is obvious that $F \simeq F_{\mathrm{forget}} \circ Q$. We are now going to show that Q preserves connected objects. Take a connected object $M \in \mathrm{lcs}(E, (\mathscr{G}, \alpha))$ and suppose that N is a non-empty subset of $F_T(M)$ stable under the action of $\pi_1(E_{\leq 1}, T)$ and $\mathscr{G}(v)$ for $v \in E_0$. Stability under the action of $\pi_1(E_{\leq 1}, T)$ shows that N can be extended to a subgraph $N_{\leq 1} \subset M_{\leq 1}$: for an edge e of $M_{\leq 1}$ we declare it to be an edge of $N_{\leq 1}$ if one of its ends touches a connected component of $p^{-1}(|T|)$ corresponding to an element of N. This is well defined, as in this case both ends touch such a component - this is because the action of $m_{\partial_1} m_{\partial_0}^{-1}$ equals the action of $\vec{e} \in \pi_1(E_{\leq 1}, T)$.

Now we want to show that it extends to 2-simplexes. This is a local question and we can restrict to simplices in the boundary of a given face $f \in E_2$. Define N_f as a preimage of N_s via any ∂ s.t. $\partial(f) = s$. We see that if the choice is independent of s, then we have extended N to a locally constant system. To see the independence it is enough to prove that if (vef) is a barycentric subdivision (i.e. we have ∂ and ∂' such that $\partial'(f) = e$ and $\partial(e) = v$), then $m_{\partial\partial'}^{-1}(N_v) = m_{\partial'}^{-1}(N_e)$. But from the $\mathscr{G}(v)$-invariance we have $N_v = \alpha_{vef}^{-1}(N_v)$ and so

$$m_{\partial\partial'}^{-1}(N_v) = m_{\partial\partial'}^{-1}(\alpha_{vef}^{-1}(N_v)) = m_{\partial'}^{-1} m_{\partial}^{-1}(N_v) = m_{\partial'}(N_e)$$

and thus N can be seen as an element of $\mathrm{lcs}(E, (\mathscr{G}, \alpha))$ which is a subobject of M, which contradicts connectedness of M.

To see that $\mathrm{lcs}(E, (\mathscr{G}, \alpha))$ is generated under colimits by a set of connected objects, observe that in the above proof of the fact that Q preserves connected objects, we have in fact shown the following statement.

Fact 3.6. Let $M \in \mathrm{lcs}(\mathscr{G}, \alpha)$ and let Z be a connected component of $Q(M)$. Then there exists a subsystem $W \subset M$ such that $Q(W) = Z$.

We want to show that there exists a set of connected objects in $\mathrm{lcs}(\mathscr{G}, \alpha)$ such that any connected object of $\mathrm{lcs}(\mathscr{G}, \alpha)$ is isomorphic to an element in that set. As an analogous fact is true in $(*^N_{v \in E_0} \mathscr{G}(v) *^N \pi_1(E_{\leq 1}, T))-$sets, it is easy to see that it is enough to check that, for any X, Y, if $QX \simeq QY$, then $X \simeq Y$. Let $X, Y \in \mathrm{lcs}(\mathscr{G}, \alpha)$ be connected. Assume that $QX \simeq QY$. Looking at the graph of this isomorphism, we find a connected subobject $Z \subset QX \times QY$ that maps isomorphically on QX and QY via the respective projections. By the above fact, we know that there exists $W \subset X \times Y$ such that $QW = Z$. Because $F \simeq F_{\mathrm{forget}} \circ Q$ and F is conservative, we see that the projections $W \to X$ and $W \to Y$ must be isomorphisms. This shows $X \simeq Y$ as desired.

The only claim left is that $\mathrm{lcs}(E(\mathscr{G},\alpha))$ is tame, but this follows from tameness of $(*^N_{v\in E_0}\mathscr{G}(v)*^N\pi_1(E_{\leq 1},T))-$Sets, the equality $F \simeq F_{\mathrm{forget}} \circ Q$ and the fact that Q maps connected objects to connected objects. □

Let us denote by $\pi_1(E,\mathscr{G},s)$ the fundamental group of the infinite Galois category $(\mathrm{lcs}(E,\mathscr{G}),F_s)$. The proposition above tells us that there is a continuous map of Noohi groups with dense image $*^N_{v\in E_0}\mathscr{G}(v) *^N \pi_1(E_{\leq 1},T)) \to \pi_1(E,\mathscr{G},s)$. We now proceed to describe the kernel.

Recall that $\pi_1(E_{\leq 1},T) = \mathrm{Fr}(E_1)/\langle\langle\{\vec{e}|e\in T\}^{\mathrm{Fr}(E_1)}\rangle\rangle$.

Theorem 3.7. *(abstract Seifert-Van Kampen theorem for infinite Galois categories) Let E be a connected 2-complex with group data $(\mathscr{G},\alpha)$. With notations as above, the functor Q induces an isomorphism of Noohi groups*

$$(*^N_{v\in E_0}\mathscr{G}(v) *^N \pi_1(E_{\leq 1},T)/\bar{H})^{\mathrm{Noohi}} \to \pi_1(E,\mathscr{G},s)$$

where $\bar{H}$ is the closure of the group

$$H = \left\langle\left\langle \begin{array}{c|c} \mathscr{G}(\partial_1)(g)\vec{e} = \vec{e}\mathscr{G}(\partial_0)(g) & e\in E_1\,,\, g\in\mathscr{G}(e) \\ \overrightarrow{(\partial_2 f)}\alpha^{(f)}_{102}(\alpha^{(f)}_{120})^{-1}\overrightarrow{(\partial_0 f)}\alpha^{(f)}_{210}(\alpha^{(f)}_{201})^{-1}\left(\overrightarrow{(\partial_1 f)}\right)^{-1}\alpha^{(f)}_{021}(\alpha^{(f)}_{012})^{-1} & f\in E_2 \end{array}\right\rangle\right\rangle$$

where $\langle\langle\cdot\rangle\rangle$ denotes the normal closure of the subgroup generated by the indicated elements and α's come from the definition of a $(\mathscr{G},\alpha)$-system for each given f.

Proof. The same proof as the proof of [Sti, Thm. 3.2 (2)] shows that Q induces an equivalence of categories between the infinite Galois categories $(\mathrm{lcs}(E,\mathscr{G}),F_s)$ and the full subcategory of objects of $*^N_{v\in E_0}\mathscr{G}(v)*^N\pi_1(E_{\leq 1},T)-$Sets on which H acts trivially. We conclude by Observation 2.56. □

Remark 3.8. It is important to note that we can replace free Noohi products by free topological products in the statement above, as we take the Noohi completion of the quotient anyway. More precisely, the canonical map

$$(*^{\mathrm{top}}_{v\in E_0}\mathscr{G}(v) *^{\mathrm{top}} \pi_1(E_{\leq 1},T)/\bar{H}_1)^{\mathrm{Noohi}} \to (*^N_{v\in E_0}\mathscr{G}(v) *^N \pi_1(E_{\leq 1},T)/\bar{H})^{\mathrm{Noohi}}$$

is an isomorphism, where H_1 is the normal closure in $*^{\mathrm{top}}_{v\in E_0}\mathscr{G}(v) *^{\mathrm{top}} \pi_1(E_{\leq 1},T)$ of a group having the same generators as H. This is because the categories of $G-\mathrm{Sets}$ are the same for those two Noohi groups.

Fact 3.9. A topological free product $*^{\mathrm{top}}_i G_i$ of topological groups has as an underlying space the free product of abstract groups $*^{\mathrm{top}}_i G_i$. This follows from the original construction of Graev [Gra].

3.2 Application to the pro-étale fundamental group

Descent data

Let $T_\bullet$ be a 2-complex in a category $\mathscr{C}$ and let $\mathscr{F}\to\mathscr{C}$ be a category fibred over $\mathscr{C}$, with $\mathscr{F}(S)$ as a category of sections above the object S.

Definition 3.10. The category $\mathrm{DD}(T_\bullet,\mathscr{F})$ of *descent data* for $\mathscr{F}/\mathscr{C}$ relative $T_\bullet$ has as objects pairs (X',ϕ) where $X'\in\mathscr{F}(T_0)$ and ϕ is an isomorphism $\partial_0^*X' \xrightarrow{\sim} \partial_1^*X'$ in $\mathscr{F}(T_1)$ such that the *cocycle condition* holds, i.e., the following commutes in $\mathscr{F}(T_2)$:

$$\begin{array}{ccc} v_2^*X' & \xrightarrow{\partial_0^*\phi} & v_1^*X' \\ & {\scriptstyle\partial_1^*\phi}\searrow \quad \swarrow{\scriptstyle\partial_2^*\phi} & \\ & v_0^*X' & \end{array}$$

Morphisms $F : (X', \phi) \to (Y', \psi)$ in $\mathrm{DD}(T_\bullet, \mathscr{F})$ are morphisms $F : X' \to Y'$ in $\mathscr{F}(T_0)$ such that its two pullbacks $\partial_0^* f$ and $\partial_1^* f$ are compatible with ϕ, ψ, i.e., $\partial_1^* f \circ \phi = \psi \circ \partial_0^* f$.

Let $h : S' \to S$ be a map of schemes. There is an associated 2-complex of schemes

$$S_\bullet(h) : S' \leftleftarrows S' \times_S S' \overset{\textstyle\leftarrow}{\leftleftarrows} S' \times_S S' \times_S S'$$

The value of ∂_i is the projection under omission of the i^{th} component. We abbreviate $\mathrm{DD}(S_\bullet(h), \mathscr{F})$ by $\mathrm{DD}(h, \mathscr{F})$. Observe that h^* gives a functor $h^* : \mathscr{F}(S) \to \mathrm{DD}(h, \mathscr{F})$.

Definition 3.11. In the above context $h : S' \to S$ is called an *effective descent* morphism for $\mathscr{F}$ if h^* is an equivalence of categories.

Proposition 3.12. *Let $g : S' \to S$ be a proper, surjective morphism of finite presentation, then g is a morphism of effective descent for geometric covers.*

Proof. This is [Lav, Prop. 1.16] and relies on the results of [Ryd]. More precisely, this follows from [Ryd, Prop. 5.4], [Ryd, Thm. 5.19], then checking that the obtained algebraic space is a scheme (using étaleness and separatedness, see [SP, Tag 0417]) and that it still satisfies the valuative criterion (see the proof of Prop. 3.32 and Appendix 2.4). □

Discretisation of descent data

We would like to apply the procedure described in [Sti, §4.3] but to the pro-étale fundamental group. However, in the classical setting of Galois categories, given a category $\mathcal{C}$ and functors $F, F' : \mathcal{C} \to \mathrm{Sets}$ such that $(\mathcal{C}, F)$ and $(\mathcal{C}, F')$ are Galois categories (i.e. F, F' are fibre functors), there exists an isomorphism (not unique) between F and F'. Choosing such an isomorphism is called "choosing a path" between F and F'. However, it is not clear whether an analogous statement is true for tame infinite Galois categories as the proof does not carry over to this case (see the proof of [SP, Lemma 0BN5] or in [SGA 1] - these proofs are essentially the same and rely on the pro-representability result of Grothendieck [Gro60, Prop. A.3.3.1]).

Question 3.13. Let $\mathcal{C}$ be a category and $F, F' : \mathcal{C} \to \mathrm{Sets}$ be two functors such that $(\mathcal{C}, F)$ and $(\mathcal{C}, F')$ are tame infinite Galois categories. Is it true that F and F' are isomorphic?

As we do not know the answer to this question, we have to make an additional assumption when trying to discretise the descent data. Fortunately, it will always be satisfied in the geometric setting, which is our main case of interest.

Definition 3.14. Let $(\mathcal{C}, F)$, $(\mathcal{C}', F')$ be two infinite Galois categories and let $\phi : \mathcal{C} \to \mathcal{C}'$ be a functor. We say that ϕ is *compatible* if there exists an isomorphism of functors $F \simeq F' \circ \phi$.

Let $\mathscr{F} \to \mathscr{C}$ be fibred in tame infinite Galois categories. More precisely, we have a notion of connected objects in $\mathscr{C}$ and any $T \in \mathscr{C}$ is a coproduct of connected components. Over connected objects $\mathscr{F}$ takes values in tame infinite Galois categories (i.e. over a connected $Y \in \mathscr{C}$ there exists a functor $F_Y : \mathscr{F}(Y) \to \mathrm{Sets}$ such that $(\mathscr{F}(Y), F_Y)$ is a tame infinite Galois category but we do not fix the functor).

Definition 3.15. Let $T_\bullet$ be a 2-complex in $\mathscr{C}$. Let $E = \pi_0(T_\bullet)$ be its 2-complex of connected components: the 2-complex in Sets built by degree-wise application of the connected component functor. We will say that $T_\bullet$ is a *compatible* 2-complex if one can fix fibre functors F_s of $\mathscr{F}(s)$ for each simplex $s \in E$ such that $(\mathscr{F}(s), F_s)$ is tame and for any boundary map $\partial : s \to s'$ there exists an isomorphism of fibre functors $F_s \circ T(\partial)^* \xrightarrow{\sim} F_{s'}$.

The 2-complexes that will appear in the (geometric) applications below will always be compatible. From now on, we will assume all 2-complexes to be compatible, even if not stated explicitly. Let $T_\bullet$ be a compatible 2-complex in $\mathscr{C}$. Fix fibre functors F_s and isomorphisms between them as in the definition of a compatible 2-complex. For any ∂, denote the fixed isomorphism by $\vec{\partial}$. For a 2-simplex (vef) of the barycentric subdivision with $\partial' : f \to e$ and $\partial : e \to v$ we define

$$\alpha_{vef} = \vec{\partial}'\vec{\partial}\left(\overrightarrow{\partial\partial'}\right)^{-1}$$

or, more precisely,

$$\alpha_{vef} = T(\partial)(\vec{\partial}')\vec{\partial}(\overrightarrow{\partial\partial'})^{-1} \in \mathrm{Aut}(F_v) = \pi_1(\mathscr{F}(s), F_s).$$

We define Noohi group data $(\mathscr{G}, \alpha)$ on E in the following way: $\mathscr{G}(s) = \pi_1(\mathscr{F}(s), F_s)$ for any simplex $s \in E$ and to $\partial : s \to s'$ is associated $\mathscr{G}(\partial) : \pi_1(\mathscr{F}(s), F_s) \overset{T(\partial)^*}{\to} \pi_1(\mathscr{F}(s'), F_s \circ T(\partial)^*) \overset{\vec{\partial}()\vec{\partial}^{-1}}{\longrightarrow} \pi_1(\mathscr{F}(s'), F_{s'})$. We define elements α as described above and we easily check that this gives Noohi group data.

Proposition 3.16. *The choice of functors F_s and the choice of $\vec{\partial}$ as above fix a functor*

$$\mathrm{discr} : \mathrm{DD}(T_\bullet, \mathscr{F}) \to \mathrm{lcs}(E, (\mathscr{G}, \alpha))$$

which is an equivalence of categories.

Proof. Given a descent datum (X', ϕ) relative $T_\bullet$ we have to attach a locally constant $(\mathscr{G}, \alpha)$-system on E in a functorial way. For $v \in E_0, e \in E_1$ and $f \in E_2$, the definition of suitable $\mathscr{G}(v)$ (or $\mathscr{G}(e)$ or $\mathscr{G}(f)$) sets and maps m_∂ between them can be given by the same formulas as in [Sti, Prop. 4.4.] and also the same computations as in [Sti, Prop. 4.4.] show that we obtain an element of $\mathrm{lcs}(E, (\mathscr{G}, \alpha))$. Again, the reasoning of [Sti, Prop. 4.4.] gives a functor in the opposite direction: given $M \in \mathrm{lcs}(E, (\mathscr{G}, \alpha))$ we define $X' \in \mathscr{F}(T_0) = \prod_{v \in E_0} \mathscr{F}(v)$ as $X'_{|v}$ corresponding to M_v for all $v \in E_0$. Maps from edges to vertices define a map $\phi : T(\partial_0)^* X' \to T(\partial_1)^* X'$ and to check the cocycle condition one reverses the argument of the proof that discr gives a locally constant system. □

To apply the last proposition we need to know that the compatibility condition holds in the setting we are interested in, i.e. in the geometry of pro-étale covers.

Lemma 3.17. *Let $f : X' \to X$ be a morphism of locally topologically noetherian connected schemes. Let $\bar{x}'$ be a geometric point of X' and $\bar{x}$ a geometric point of X. Then the functor $f^* : \mathrm{Cov}_X \to \mathrm{Cov}_{X'}$ is a compatible functor between infinite Galois categories $(\mathrm{Cov}_X, \mathrm{ev}_{\bar{x}})$ and $(\mathrm{Cov}_{X'}, \mathrm{ev}_{\bar{x}'})$, i.e. the functors $\mathrm{ev}_{\bar{x}}$ and $\mathrm{ev}_{\bar{x}'} \circ f^*$ are isomorphic.*

Proof. Looking at the image of $\bar{x}'$ (as a geometric point) on X, we reduce to the case when both $\bar{x}'$ and $\bar{x}$ lie on the same scheme X. In that case we proceed as in the proof of [BS, Lm. 7.4.1], i.e. we choose a finite (it is possible as the scheme is locally topologically noetherian and connected) sequence of points $x = x_0, x_1, \ldots, x_n = x'$ (where x means the image of $\bar{x}$ in X and similarly for x'), such that, for each $i < n$, x_{i+1} is either a specialization or a generization of x_i. Then for each i we choose a geometric point $\bar{x}_i$ and a valuation ring R_i with a map $\mathrm{Spec} R_i \to X$ such that the special and the generic point are $\bar{x}_i$ and $\bar{x}_{i+1}$ (not necessarily in this order). This induces an isomorphism of the fibre functors $\mathrm{ev}_{\bar{x}_i} \simeq \mathrm{ev}_{\bar{x}_{i+1}}$. Indeed, it is enough to show that any connected geometric cover of $\mathrm{Spec}(R_i)$ is trivial, as then the pullback via $\mathrm{Spec}(R_i) \to X$ of any geometric cover will give a functorial bijection between the two geometric fibres. But either by [BS, Lm. 7.4.10] and using that R_i is normal, or by checking that such R_i is strictly henselian and using [BS, Lm. 7.3.8], we see that any connected geometric cover of $\mathrm{Spec}(R_i)$ is finite étale. By using the valuative criterion and the fact that the residue field of R_i is algebraically closed, one sees that any finite étale cover of $\mathrm{Spec}(R_i)$ has a section and thus any connected geometric cover is trivial. □

The above results combine to recover the analogue of [Sti, Cor. 5.3] in the pro-étale setting.

Corollary 3.18. *Let $h : S' \to S$ be an effective descent morphism for geometric covers. Assume that S is connected and $S, S', S' \times_S S', S' \times_S S' \times_S S'$ are locally topologically noetherian. Let $S' = \bigsqcup_v S'_v$ be the decomposition into connected components. Let $\bar{s}$ be a geometric point of S, let $\bar{s}(t)$ be a geometric point of the simplex $t \in \pi_0(S_\bullet(h))$, and let T be a maximal tree in the graph $\Gamma = \pi_0(S_\bullet(h))_{\leq 1}$. For every boundary map $\partial : t \to t'$ let $\gamma_{t',t} : \bar{s}(t') \to S_\bullet(h)(\partial)\bar{s}(t)$ be a fixed path (i.e. an isomorphism of fibre functors as in Lm. 3.17). Then canonically with respect to all these choices*

$$\pi_1^{\text{proét}}(S, \bar{s}) \cong \Big(\big(*^N_{v \in E_0} \pi_1^{\text{proét}}(S'_v, \bar{s}(v)) *^N \pi_1(\Gamma, T) \big) / \overline{H} \Big)^{\text{Noohi}}$$

where H is the normal subgroup generated by the cocycle and edge relations

$$\pi_1^{\text{proét}}(\partial_1)(g)\vec{e} = \vec{e}\pi_1^{\text{proét}}(\partial_0)(g) \tag{3.1}$$

$$\overrightarrow{(\partial_2 f)}\alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1}\overrightarrow{(\partial_0 f)}\alpha_{210}^{(f)}(\alpha_{201}^{(f)})^{-1}\left(\overrightarrow{(\partial_1 f)}\right)^{-1}\alpha_{021}^{(f)}(\alpha_{012}^{(f)})^{-1} = 1 \tag{3.2}$$

for all parameter values $e \in S_1(h)$, $g \in \pi_1^{\text{proét}}(e, \bar{s}(e))$, and $f \in S_2(h)$. The map $\pi_1^{\text{proét}}(\partial_i)$ uses the fixed path $\gamma_{\partial_i(e),e}$ and $\alpha_{ijk}^{(f)}$ is defined using $v \in S_0(h)$ and $e \in S_1(h)$ determined by $v_i(f) = v$, $\{\partial_0(e), \partial_1(e)\} = \{v_i(f), v_j(f)\}$ as

$$\alpha_{ijk}^{(f)} = \gamma_{v,e}\gamma_{e,f}\gamma_{v,f}^{-1} \in \pi_1^{\text{proét}}(v, \bar{s}(v)).$$

Remark 3.19. Similarly as in Rmk. 3.8, we could replace $*^N$ by $*^{\text{top}}$ in the above, as we take the Noohi completion of the whole quotient anyway.

Remark 3.20. We will often use Cor. 3.18 for h - the normalization map (or similar situations), where the connected components S'_v are normal. In this case $\pi_1^{\text{proét}}(S'_v, \bar{s}(v)) = \pi_1^{\text{ét}}(S'_v, \bar{s}_v)$. This implies that $\pi_1^{\text{proét}}(\partial_1)$ factorizes through the profinite completion of $\pi_1^{\text{proét}}(e, \bar{s}(e))$, which can be identified with $\pi_1^{\text{ét}}(e, \bar{s}(e))$. Moreover, the map $\pi_1^{\text{proét}}(e, \bar{s}(e)) \to \pi_1^{\text{ét}}(e, \bar{s}(e))$ has dense image and, in the end, we take the closure $\overline{H}$ of H. The upshot of this discussion is that in the definition of generators of H we might consider $g \in \pi_1^{\text{ét}}(e, \bar{s}(e))$ instead of $g \in \pi_1^{\text{proét}}(e, \bar{s}(e))$ and $\pi_1^{\text{ét}}(\partial_i)$ instead of $\pi_1^{\text{proét}}(\partial_i)$, $i \in \{0, 1\}$, i.e.

$$\pi_1^{\text{proét}}(S, \bar{s}) \cong \left(\left(*_{v \in E_0}^{\text{top}} \pi_1^{\text{ét}}(S'_v, \bar{s}(v)) *^{\text{top}} \pi_1(\Gamma, T)\right)/\overline{H}\right)^{\text{Noohi}}$$

where H is the normal subgroup generated by

$$\pi_1^{\text{ét}}(\partial_1)(g)\vec{e}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{e}^{-1} \text{ for all } e \in S_1(h), g \in \pi_1^{\text{ét}}(e, \bar{s}(e))$$

and

$$\overrightarrow{(\partial_2 f)}\alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1}\overrightarrow{(\partial_0 f)}\alpha_{210}^{(f)}(\alpha_{201}^{(f)})^{-1}\left(\overrightarrow{(\partial_1 f)}\right)^{-1}\alpha_{021}^{(f)}(\alpha_{012}^{(f)})^{-1} \text{ for all } f \in S_2(h).$$

Let us move on to some applications.

Ordered descent data

Let $\mathscr{F}$ be a category fibred over $\mathscr{C}$ with a fixed splitting cleavage (i.e. the associated pseudo-functor is a functor). Assume that $\mathscr{C}$ is some subcategory of the category of locally topologically noetherian schemes with the property that finite fibre products in $\mathscr{C}$ are the same as the finite fibre products as schemes. Let $h = \bigsqcup_{i \in I} h_i : S' = \bigsqcup_i S'_{i \in I} \to S$ be a morphism of schemes and let $<$ be a total order on the set of indices I. Let $S_\bullet^<(h) \subset S_\bullet(h)$ be the open and closed sub-2-complex of schemes in $\mathscr{C}$ of ordered partial products

$$S_0^<(h) = S' \quad , \quad S_1^<(h) = \bigsqcup_{i<j} S'_i \times_S S'_j \quad , \quad S_2^<(h) = \bigsqcup_{i<j<k} S'_i \times_S S'_j \times_S S'_k$$

Proposition 3.21. *Let $h = \bigsqcup_{i \in I} h_i : S' = \bigsqcup_i S'_{i \in I} \to S$ be a morphism of schemes such that, for every $i, j \in I$, the maps induced by the diagonal morphisms $\Delta_i^* : \mathscr{F}(S'_i \times_S S'_i) \to \mathscr{F}(S'_i)$ and $(\Delta_i \times \mathrm{id}_{S'_j})^* : \mathscr{F}(S'_i \times_S S'_j \times_S S'_i) \to \mathscr{F}(S'_i \times S'_j)$ are fully faithful. Then the natural open and closed immersion $S_\bullet^<(h) \hookrightarrow S_\bullet(h)$ induces an equivalence of categories*

$$\mathrm{DD}(h, \mathscr{F}) \overset{\cong}{\to} \mathrm{DD}(S_\bullet^<(h), \mathscr{F})$$

Proof. Let $Y \in \mathscr{F}(S'_i)$ and consider $\partial_0^* Y, \partial_1^* Y \in \mathscr{F}(S'_i \times_S S'_i)$ obtained via maps induced by the projections $\mathscr{F}(S'_i) \to \mathscr{F}(S'_i \times_S S'_i)$. We first claim that there is exactly one isomorphism $\partial_0|^*_{S_i \times_S S_i} Y \to \partial_1|^*_{S_i \times_S S_i} Y$ as in the definition of descent data. Observe that $\Delta_i^* \partial_0^* Y = Y, \Delta_i^* \partial_1^* Y = Y$ and from the assumption any isomorphism $\phi : \partial_0^* Y|_{S_i} \to \partial_1^* Y|_{S_i}$ corresponds to some isomorphism in $\psi \in \mathrm{Hom}_{S_i}(Y|_{S_i}, Y|_{S_i})$. Pulling back the cocycle condition via the diagonal ${\Delta_{2,i}}^* : \mathscr{F}(S'_i \times_S S'_i \times_S S'_i) \to \mathscr{F}(S'_i)$ we get $\psi = \mathrm{id}_{Y|_{S_i}}$, so there is at most one map ϕ as above (we use here and below that we work with a splitting cleavage and so, by definition, the pullback functors preserve compositions of maps). Moreover, our assumptions imply that ${\Delta_{2,i}}^*$ is fully faithful as well, which shows that $\phi : \partial_0^* Y|_{S_i} \to \partial_1^* Y|_{S_i}$ corresponding to $\mathrm{id}_{Y|_{S_i}}$ will satisfy the condition. A similar reasoning shows that if we have ϕ_{ij} specified for $i < j$, then ϕ_{ji} is uniquely determined and the if ϕ_{ij}'s satisfy the cocycle condition on S_{ijk} for $i < j < k$, then ϕ_{ij}'s together with ϕ_{ji}'s obtained will satisfy the cocycle condition on any $S_{\alpha\beta\gamma}, \alpha, \beta, \gamma \in \{i, j, k\}$ ☐

Observation 3.22. If the map of schemes $S'_i \to S$ is injective, i.e. if the diagonal map $S'_i \to S'_i \times_S S'_i$ is an isomorphism, then (still assuming splitting of the cleavage) the assumptions of the proposition are satisfied.

Two examples

Example 3.23. Let k be a field and C be $\mathbb{P}^1_k$ with two k-rational closed points p_0 and p_1 glued (see [Sch05] for results on gluing schemes). Denote by p the node (i.e. the image of p_i's in C). We want to compute $\pi_1^{\text{proét}}(C)$. By the definition of C, we have a map $h : \widetilde{C} = \mathbb{P}^1 \to C$ (which is also the normalization). It is finite, so it is an effective descent map for pro-étale covers. Thus, we can use the van Kampen theorem. We claim that $\widetilde{C} \times_C \widetilde{C} \simeq \widetilde{C} \sqcup p_{01} \sqcup p_{10}$ as schemes over C, where $p_{\alpha\beta}$ are equal to $\mathrm{Spec}(k)$ and map to the node of C via the structural map. This can be computed directly with an equation for the nodal cubic, but we choose a different way. It is enough to check that $\mathrm{Hom}_C(Y, \widetilde{C} \sqcup p_{01} \sqcup p_{10}) \simeq \mathrm{Hom}_C(Y, \widetilde{C}) \times \mathrm{Hom}_C(Y, \widetilde{C})$ for any C-scheme $s : Y \to C$. Observe that

$$\mathrm{Hom}_C(Y, \widetilde{C} \sqcup p_{01} \sqcup p_{10}) \simeq$$
$$\overset{(1)}{\simeq} \bigsqcup_{\{(U,V) | U,V \text{ - disjoint clopen subschemes of } Y \text{ contained in } s^{-1}(p)\}} \mathrm{Hom}_C(Y \setminus (U \sqcup V), \widetilde{C}) \simeq$$
$$\overset{(2)}{\simeq} \mathrm{Hom}_C(Y, \widetilde{C}) \times \mathrm{Hom}_C(Y, \widetilde{C})$$

Where the identification (1) is via $f \mapsto f|_{Y \setminus (f^{-1}(p_{01}) \sqcup f^{-1}(p_{10}))}$ and the identification (2) is via $f|_{\{f=g\} = Y \setminus ((f^{-1}(p_0) \cap g^{-1}(p_1)) \sqcup (f^{-1}(p_1) \cap g^{-1}(p_0)))} \hookleftarrow (f, g)$. This last map makes sense as $\{f = g\}$ is closed (as $\widetilde{C}$ is separated) and $Y \setminus \{f = g\} = (f^{-1}(p_0) \cap g^{-1}(p_1)) \sqcup (f^{-1}(p_1) \cap g^{-1}(p_0))$ (as $\widetilde{C} \to C$ is an isomorphism outside $\{p_0, p_1\}$) and so the sets $f^{-1}(p_0) \cap g^{-1}(p_1)$ and $f^{-1}(p_1) \cap g^{-1}(p_0)$ are clopen. From the above identifications one can also conclude that, via $\widetilde{C} \times_C \widetilde{C} \simeq \widetilde{C} \sqcup p_{01} \sqcup p_{10}$, points p_{01}, p_{10} map to p_0, p_1 respectively via the projection on the first factor $\widetilde{C} \times_C \widetilde{C} \to \widetilde{C}$ and to p_1, p_0 respectively via the projection on the second factor while both projections restricted to $\widetilde{C} \subset \widetilde{C} \sqcup p_{01} \sqcup p_{10}$ become just the identity $\widetilde{C} \to \widetilde{C}$.

In a similar way one can identify $\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C} \simeq \widetilde{C} \sqcup p_{001} \sqcup p_{010} \sqcup p_{011} \sqcup p_{100} \sqcup p_{101} \sqcup p_{110}$. One can also think of the points p_0 and p_1 in the copy of $\widetilde{C}$ on the right hand side of this identification as points denoted p_{000} and p_{111}, and similarly for the two points lying in the copy of $\widetilde{C}$ in the description of $\widetilde{C} \times_C \widetilde{C}$. This way it is easy to keep track where the points $p_{\alpha\beta\gamma}$ and $p_{\alpha\beta}$ map via projections, namely the projection $\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C} \to \widetilde{C} \times_C \widetilde{C}$ omitting the first factor maps p_{abc} to p_{bc} and so on.

We fix a geometric point $\bar{b} = \mathrm{Spec}(\bar{k})$ over the base scheme $\mathrm{Spec}(k)$ and fix geometric points $\bar{p}_0$ and $\bar{p}_1$ over p_0 and p_1 that map to $\bar{b}$. Then we fix geometric points on $\widetilde{C}, p_{01}, p_{10} \subset \widetilde{C} \sqcup p_{01} \sqcup p_{10} \simeq \widetilde{C} \times_C \widetilde{C}$ in a compatible way and similarly for connected components of $\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C}$ (i.e. let us say that $\bar{p}_{\alpha\beta\gamma} \mapsto \bar{p}_\alpha$ via v_0 and $\bar{p}_{\alpha\beta} \mapsto \bar{p}_\alpha$). We fix a path γ from $\bar{p}_0$ to $\bar{p}_1$ that becomes trivial on $\mathrm{Spec}(k)$ via the structural map (this can be done by viewing $\bar{p}_0$ and $\bar{p}_1$ as geometric points on $\widetilde{C}_{\bar{k}}$, choosing the path on $\widetilde{C}_{\bar{k}}$ first and defining γ to be its image). Let $\bar{p}$ be the fixed geometric point on C given by the image of $\bar{p}_0$ (or, equivalently, $\bar{p}_1$). We use Cor. 3.18 to compute $\pi_1^{\text{proét}}(C, \bar{p})$. We choose $\bar{p}_0$ as the base point $\bar{s}(\widetilde{C})$ for $\widetilde{C} \in \pi_0(S_0(h))$, $\widetilde{C} \in \pi_0(S_1(h))$ and $\widetilde{C} \in$

$\pi_0(S_2(h))$. Then for any $t, t' \in \pi_0(S_\bullet(h))$ and the boundary map $\partial : t \to t'$, we use either identity or γ to define $\gamma_{t',t} : \bar{s}(t') \to S_\bullet(h)(\partial)\bar{s}(t)$ as all the points $\bar{p}_{abc}$ map ultimately either to $\bar{p}_0$ or $\bar{p}_1$. Then the $\alpha_{ijk}^{(f)}$'s (defined as in Cor. 4.18) are trivial for any f and so the relation (2) in this corollary reads $\overrightarrow{(\partial_2 f)}\overrightarrow{(\partial_0 f)}\left(\overrightarrow{(\partial_1 f)}\right)^{-1} = 1$. Applying this to different faces $f \in \pi_0(\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C})$ gives that the image of $\pi_1(\Gamma, T) \simeq \mathbb{Z}^{*3}$ in $\pi_1^{\text{proét}}(C, \bar{p})$ is generated by a single edge (in our case only one maximal tree can be chosen - consisting of just a single vertex). The choice of paths made guarantees $\pi_1^{\text{proét}}(\partial_0)(g) = \pi_1^{\text{proét}}(\partial_1)(g)$ in $\pi_1^{\text{proét}}(\widetilde{C}, \bar{p}_0)$ for any $g \in \pi_1^{\text{proét}}(p_{ab}, \bar{p}_{ab}) = \text{Gal}(k)$. So relation (1) in Cor. 3.18 implies that the image of $\pi_1^{\text{proét}}(\widetilde{C}, \bar{p}_0) \simeq \text{Gal}(k)$ in $\pi_1^{\text{proét}}(C, \bar{p}_0)$ commutes with the elements of the image of $\pi_1(\Gamma, T)$. Putting this together we get

$$\pi_1^{\text{proét}}(C, \bar{p}) \simeq$$
$$\simeq \left((\pi_1^{\text{proét}}(\widetilde{C}, \bar{p}_0) *^N \pi_1(\Gamma, T)) / \langle \pi_1^{\text{proét}}(\partial_1)(g)\vec{e} = \vec{e}\pi_1^{\text{proét}}(\partial_2)(g), \overrightarrow{(\partial_2 f)}\overrightarrow{(\partial_0 f)}\left(\overrightarrow{(\partial_1 f)}\right)^{-1} = 1 \rangle \right)^{\text{Noohi}} \simeq$$
$$\simeq \left(\text{Gal}_k \times \mathbb{Z}\right)^{\text{Noohi}} = \text{Gal}_k \times \mathbb{Z}$$

Example 3.24. Let $X_1, \ldots, X_m$ be geometrically connected normal curves over a field k and let $Y_{m+1}, \ldots, Y_n$ be nodal curves over k as in Ex. 3.23. Let $x_i : \text{Spec}(k) \to X_i$ be rational points and let y_j denote the node of Y_j. Let $X := \cup_\bullet X_i \cup_\bullet Y_j$ be a scheme over k obtained via gluing of X_i's and Y_j's along the rational points x_i and y_j (in the sense of [Sch05]). The notation $\cup_\bullet$ denotes gluing along the obvious points. More precisely, we can glue iteratively, for example: $X_1 \cup_\bullet X_2 \cup_\bullet X_3 = (X_1 \cup_\bullet X_2) \cup_\bullet X_3$ and show that it does not depend on the order of gluing: $(X_1 \cup_\bullet X_2) \cup_\bullet X_3 = X_1 \cup_\bullet (X_2 \cup_\bullet X_3)$ by checking that both schemes satisfy the obvious universal property (of triple gluing). The point of gluing gives a rational point $x : \text{Spec}(k) \to X$. We choose a geometric point $\bar{b}$ over the base $\text{Spec}(k)$ and choose a geometric point $\bar{x}$ over x such that it maps to $\bar{b}$. The maps $X_i \to X$ and $Y_j \to X$ are closed immersions (this is basically [Sch05, Lm. 3.8]). We also get geometric points $\bar{x}_i$ and $\bar{y}_j$ over x_i and y_j that map to $\bar{b}$ as well. Let h be the map $\widetilde{X} := \sqcup_{1 \leq i \leq m} X_i \sqcup_{m+1 \leq i \leq n} Y_i \to X$. Again, one can easily understand the fibre products $\widetilde{X} \times_X \widetilde{X}$ and $\widetilde{X} \times_X \widetilde{X} \times_X$. For example, we have $X_i \times_X X_j \simeq x$ as schemes over X for $i \neq j$. This can be checked by an explicit computation of the gluing in an affine neighbourhood of the gluing point, but we would like to avoid that. Instead, one can use the universal property to check that $(X_i \times_X X_j)_{\text{red}} \simeq x$ (which is enough for us by topological invariance, see Prop. 3.28) by showing the following: for a pair of points $a_i \in X_i$, $a_j \in X_j$ such that at least one of the points a_i or a_j is not the gluing point, there is a scheme T and maps $\alpha_1 : X_1 \to T, \alpha_2 : X_2 \to T, \ldots, \alpha_{m+1} : Y_{m+1} \to T, \ldots$ that agree on the gluing points but the images of a_i and a_j (under the maps α_i, α_j) are different. This can be achieved by choosing $T = X_1 \times_k X_2 \times_k ... \times_k X_m \times_k Y_{m+1} \times_k \ldots \times_k Y_n$ and the maps $\alpha_r : X_r \to T$ to be $\alpha_r = (x_1, x_2, ..., \text{id}_{X_r}, ..., y_n)$, where $x_1 : \text{Spec}(k) \to X_1, x_2 : \text{Spec}(k) \to X_2, \ldots$ are the gluing points. This implies that $X_i \times_X X_j$ is a closed subscheme of (let's say) X_i with one point as an underlying set and there is a map $\text{Spec}(k) \to X_i \times_X X_j$. Thus, $(X_i \times_X X_j)_{\text{red}} = \text{Spec}(k)$. In the same way we see that other double fibre products over X of the considered curves are either equal to x or to one of the curves and similarly for the triple products (at least up to reduction). We now apply Cor. 3.18 to the morphism h to compute $\pi_1^{\text{proét}}(X, \bar{x})$. We choose $\bar{x}_i, \bar{y}_j$ as the base points for connected components of $\widetilde{X}$ and $\bar{x}_i$ or $\bar{y}_j$ or $\bar{x}$ for connected components of $\widetilde{X} \times_X \widetilde{X}$ and $\widetilde{X} \times_X \widetilde{X} \times_X \widetilde{X}$. We can then choose identity as a path $\gamma_{t',t}$ for any $t', t \in \pi_0(S_\bullet(h)$ with boundary map $\partial : t \to t'$. This way all the $\alpha_{ijk}^{(f)} = 1$. So the relation (2) in van Kampen reads $\overrightarrow{(\partial_2 f)}\overrightarrow{(\partial_0 f)}\left(\overrightarrow{(\partial_1 f)}\right)^{-1} = 1$. We can use Prop. 3.21 to simplify the situation, i.e. we can consider intersections $X_i \times_X X_{i'}$, $Y_j \times_X Y_{j'}$ for $i < i', j < j'$ and $X_i \times_X Y_j$ for i, j any. Similarly for triple intersections. Then the graph $\Gamma = \pi_0(S_\bullet(h))_{\leq 1}$ is easy to understand and the the relations $\overrightarrow{(\partial_2 f)}\overrightarrow{(\partial_0 f)}\left(\overrightarrow{(\partial_1 f)}\right)^{-1} = 1$ imply that the image of $\pi_0(\Gamma)$ in $\pi_1^{\text{proét}}(X)$ is trivial. Using the rational points x_i and y_i we can write $\pi_1^{\text{proét}}(X_i, \bar{x}_i) = \pi_1^{\text{ét}}(X_i, \bar{x}_i) = \pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) \rtimes \text{Gal}_k$ and similarly for Y_j's. Here and later in this proof, $\overline{X}_i$ (and similarly for other schemes) denotes the base-change of X_i to $\bar{k}$. The relation (1) of Cor. 3.18 reads $\iota_i(\sigma) = \iota_{i'}(\sigma)$, where $\sigma \in \text{Gal}_k,\ i, i' = 1, \ldots$ n, where $\iota_i : \text{Gal}_k \to \text{Gal}_{k,i}$ identifies $\text{Gal}_{k,i}$ with some fixed copy of Gal_k. Using Ex. 3.23 and the fact that X_i are normal, we get

$$\pi_1^{\text{proét}}(X,\bar{x}) \simeq \left(*_{1\le i\le m}^N(\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)\rtimes\mathrm{Gal}_{k,i})*_{m+1\le j\le n}^N(\mathbb{Z}\times\mathrm{Gal}_{k,j})/\langle\iota_i(\sigma)=\iota_{i'}(\sigma)|\sigma\in\mathrm{Gal}_k,\ i,i'=1,\dots,n\rangle\right)^{\text{Noohi}}.$$

We want to write the fundamental group in a different form and we look at the category of group-sets to do so:

$$\left(*_{1\le i\le m}^N(\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)\rtimes\mathrm{Gal}_{k,i})*_{m+1\le j\le n}^N(\mathbb{Z}\times\mathrm{Gal}_{k,j})/\langle\iota_i(\sigma)=\iota_{i'}(\sigma)|\sigma\in\mathrm{Gal}_k,\ i,i'=1,\dots,n\rangle\right)^{\text{Noohi}}-\mathrm{Sets}\simeq$$

$$\simeq\left\{S\in\left(*_{1\le i\le m}^N(\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)\rtimes\mathrm{Gal}_{k,i})*_{m+1\le j\le n}^N(\mathbb{Z}\times\mathrm{Gal}_{k,j})\right)^{\text{Noohi}}-\mathrm{Sets}\middle|\forall_{i,i',\sigma}\forall_{s\in S}\iota_i(\sigma)\cdot s=\iota_{i'}(\sigma)\cdot s\right\}\simeq$$

$$\simeq\left\{S\in\left(*_{1\le i\le m}^N\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)*^N\mathbb{Z}^{*n-m}*\mathrm{Gal}_k\right)^{\text{Noohi}}-\mathrm{Sets}\right|$$

$$\left|\forall_{\sigma\in\mathrm{Gal}_k}\forall_i\forall_{\substack{\gamma\in\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)\\ w\in\mathbb{Z}^{*n-m}}}\forall_{s\in S}(\sigma\cdot(\gamma\cdot s)={}^{\sigma}\gamma\cdot(\sigma\cdot s)\text{ and }\sigma\cdot(w\cdot s)=w\cdot(\sigma\cdot s)\right\}$$

We have used Obs. 2.56 and Lm. 3.25 below.

Lemma 3.25. *Let K and Q be topological groups and assume we have a continuous action $K\times Q\to K$ respecting multiplication in K. Then $K\rtimes Q$ with the product topology (on $K\times Q$) is a topological group and there is an isomorphism*

$$K*_{\text{top}}Q/\langle\langle qkq^{-1}={}^qk\rangle\rangle\to K\rtimes Q$$

Proof. That $K\rtimes Q$ becomes a topological group is easy from the continuity assumption of the action. The isomorphism is obtained as follows: from the universal property we have a continuous homomorphism $K*_{\text{top}}Q\to K\rtimes Q$ and the kernel of this map is the smallest normal subgroup containing the elements $qkq^{-1}({}^qk)^{-1}$ (this follows from the fact that the underlying abstract group of $K*_{\text{top}}Q$ is the abstract free product of the underlying abstract groups, similarly for $K\rtimes Q$ and that we know the kernel in this case). So we have a continuous map that is an isomorphism of abstract groups. We have to check that the inverse map $K\rtimes Q\ni kq\mapsto kq\in K*_{\text{top}}Q/\langle\langle qkq^{-1}={}^qk\rangle\rangle$ is continuous. It is enough to check that the map $K\times Q\ni(k,q)\mapsto kq\in K*_{\text{top}}Q$ (of topological spaces) is continuous, but this follows from the fact that the maps $K\to K*_{\text{top}}Q$ and $Q\to K*_{\text{top}}Q$ are continuous and that the multiplication map $(K*_{\text{top}}Q)\times(K*_{\text{top}}Q)\to K*_{\text{top}}Q$ is continuous. ☐

Some remarks on free products

The following observation is easy to check and not used in the article, but lets one to get rid of some false intuitions about the topology on free products.

Observation 3.26. Let $n\ge 2$, let $G_1,\dots,G_n$ be non-trivial (Hausdorff) topological groups such that not all of them have discrete topology - let's say that G_n is not discrete. Let $U_i\lneq G_i$ be an open proper subgroup for some $i\in\{1,\dots,n-1\}$. Then the subgroup $G_1*\dots*G_{i-1}*U_i*G_{i+1}*\dots*G_n$ is not open in $G_1*_{\text{top}}G_2*_{\text{top}}\dots*_{\text{top}}G_n$ (the statement makes sense as the underlying abstract group of a topological free product is just the abstract free product of underlying abstract groups).

A topological group is called *topologically residually finite* if the intersection of all its open subgroups of finite index is trivial. Observe that we distinguish between residually finite (as an abstract group) and topologically residually finite (i.e. we consider only open subgroups).

Proposition 3.27. *Let G be the free topological product $*_i^{\text{top}}G_i$, where all G_i are either profinite and topologically finitely generated or discrete and residually finite. Then G is topologically residually finite.*

Proof. It is true that an abstract product $*_i^{\text{top}}G_i$ is residually finite, as in general free products of residually finite groups are residually finite (see [Gru]). So the proof will be finished if we prove that every homomorphism (as abstract groups) $G\to H$, where H is finite, is continuous. But from the universal property it is enough to check the analogous property for each G_i. For discrete G_i's this is obvious. For profinite and topologically finitely generated G_i's this follows from a difficult result of Nikolov and Segal in [NS], which says that a profinite topologically finitely generated group is *strongly complete*, i.e. each subgroup of finite index of it is open. ☐

3.3 Künneth formula and topological invariance

Topological invariance of the pro-étale fundamental group

We are going to prove that universal homeomorphisms of schemes induce equivalence on categories of geometric covers. This result could be also obtained by combining [BS, Lemma 5.4.2] with the equivalence $\mathrm{Cov}_X \simeq \mathrm{Loc}_X$ of [BS, Lemma 7.3.9.], but we are going to give a slightly different argument.

Theorem 3.28. *Let $h : X' \to X$ be a universal homeomorphism of locally topologically noetherian schemes. Then*

$$Cov_X \to Cov_{X'}, V \mapsto V \times_X X'$$

is an equivalence. Thus, if X is connected, then h induces an isomorphism $\pi_1^{\text{proét}}(X') \to \pi_1^{\text{proét}}(X)$.

We start first with a weaker lemma.

Lemma 3.29. *Let X be a locally topologically noetherian connected scheme, X_{red} its associated reduced subscheme and let $\bar{x}$ be a geometric point of X then $\pi_1^{\text{proét}}(X_{red}, \bar{x}) \simeq \pi_1^{\text{proét}}(X, \bar{x})$*

Proof. See [Lav, Lm. 1.15] or argue as follows: by [SGA 1, Exp. I, Thm 8.3] the category of schemes that are étale over X is equivalent to the category of schemes that are étale over X_{red} and in our case (for $X_{red} \to X$) it is easy to see that covers correspond to covers and maps satisfying valuative criterion of properness correspond exactly to maps satisfying valuative criterion of properness. □

Recall

Definition 3.30. *We say a scheme X is a* thickening *of a scheme X_0 if X_0 is a closed subscheme of X and the underlying topological spaces are equal.*

We say a scheme X is a first order thickening *of a scheme X_0 if X_0 is a closed subscheme of X and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ defining X_0 has square zero.*

Corollary 3.31. *If $X_0 \to X$ is a thickening, the induced map on pro-étale fundamental groups is an isomorphism.*

Proof. In this situation we can naturally identify $X_{0,red} = X_{red}$ and so we conclude by the previous lemma. □

Proposition 3.32. *Let $h : X' \to X$ be a universal homeomorphism. Then f is a morphism of effective descent for geometric covers.*

Proof. Theorem 5.19 of [Ryd] asserts that universally open and surjective morphisms are morphisms of effective $\mathbf{E}$-descent, where $\mathbf{E}$ denotes the fibred category over Sch consisting of étale morphisms $Y \to X$ where Y is an algebraic space. We have to check that if $f' : Y' \to X'$ with $Y' \in \mathrm{Cov}_{X'}$ descends to an algebraic space Y, then Y is a scheme and belongs to Cov_X. By Lm. 2.72, f' is separated and so f is also separated: we have to check that $d : X \to X \times_S X$ is closed, but we know that $d' : X' \to X' \times_{S'} X'$ is closed and d' is a base change of d via the universal homeomorphism $X' \times_{S'} X' \to X \times_S X$. As f is étale, f is also locally quasi-finite. But an algebraic space that is separated and locally quasi-finite over a scheme is a scheme as well (see [SP, Tag 03XX]). As f is separated, it satisfies the uniqueness part of VCoP (see [SP, Tag 01KY]). On the other hand, satisfying the existence part of VCoP is equivalent to lifting of specializations via any base-change ([SP, Tag 01KE]), so a purely topological condition and we easily see that it is satisfied by f' if and only if it is satisfied by f. This finishes the proof. □

Proof. (of Thm. 3.28) We can assume that X is connected. A universal homeomorphism is the same thing as an integral, universally injective, surjective morphism (see [SP, Tag 04DF]). In particular, the diagonal $\Delta : X' \to X' \times_X X'$ is surjective (by [SP, Tag 01S4]) and a surjective immersion is a thickening. Thus, by the above corollary, we see that $\mathrm{Cov}_{X' \times_X X'} \to \mathrm{Cov}_{X'}$ is an equivalence. So Prop. 3.21 applies. As X' is connected, $\mathrm{DD}(S_\bullet^<(h), \mathrm{Cov}) = \mathrm{Cov}_{X'}$, so we get $\mathrm{Cov}_X = \mathrm{Cov}_{X'}$ as desired.

□

Künneth formula for $\pi_1^{\text{proét}}$

Let X, Y be two connected schemes locally of finite type over an algebraically closed field k and assume that Y is proper. Let us fix closed points $\bar{x}$ and $\bar{y}$ of X and Y respectively. With these assumptions, the classical statement says that the "Künneth formula" for $\pi_1^{\text{ét}}$ holds, i.e.

Fact 3.33. ([SGA 1, Exposé X, Cor. 1.7]) With the above assumptions, the map induced by the projections is an isomorphism

$$\pi_1^{\text{ét}}(X \times_k Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \bar{x}) \times \pi_1^{\text{ét}}(Y, \bar{y})$$

We want to establish analogous statement for $\pi_1^{\text{proét}}$.

Proposition 3.34. *Let X, Y be two connected schemes locally of finite type over an algebraically closed field k and assume that Y is proper. Let us fix $\bar{x}, \bar{y}$ - closed points respectively of X and Y. Then the map induced by the projections is an isomorphism*

$$\pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$$

Before we start the proof let us state and prove the surjectivity of the above map as a lemma. Properness is not needed for this.

Lemma 3.35. *Let X, Y be two connected schemes over an algebraically closed field k with k-points on them: $\bar{x}$ on X and $\bar{y}$ on Y. Then the map induced by the projections*

$$\pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$$

is surjective.

Proof. Consider the map $(\mathrm{id}_X, \bar{y}) : X = X \times_k \bar{y} \to X \times_k Y$. It is easy to check that the map induced on fundamental groups $\pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$ is given by $(\mathrm{id}_{\pi_1^{\text{proét}}(X,\bar{x})}, 1_{\pi_1^{\text{proét}}(Y,\bar{y})}) : \pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$. Analogous fact holds if we consider $(\bar{x}, \mathrm{id}_Y) : Y \to X \times_k Y$. As a result, the image $\mathrm{im}(\pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y}))$ contains the set $(\pi_1^{\text{proét}}(X, \bar{x}) \times \{1_{\pi_1^{\text{proét}}(Y,\bar{y})}\}) \cup (\{1_{\pi_1^{\text{proét}}(X,\bar{x})}\} \times \pi_1^{\text{proét}}(Y, \bar{y}))$. This finishes the proof, as this set generates $\pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$. $\square$

Proof. (of Prop. 3.34) As X, Y are locally of finite type over a field, the normalization maps are finite and we can apply Prop. 3.12. Let $\widetilde{X} \to X$ be the normalization of X and let $\widetilde{X} = \sqcup_v \widetilde{X}_v$ be its decomposition into connected components and let us fix a closed point $x_v \in \widetilde{X}_v$ for each v.

We first deal with a particular case.

Claim: the statement of Prop. 3.34 holds under the additional assumption that, for any v, the projections induce isomorphisms

$$\pi_1^{\text{proét}}(\widetilde{X}_v \times_k Y, (x_v, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y}).$$

Let us prove the claim. Apply Cor. 3.18 to $h : \widetilde{X} \to X$. We choose $\bar{x}$ and x_v's as geometric points $\bar{s}(t)$ of the corresponding simplexes $t \in \pi_0(S_\bullet(h))_0$ and choose $\bar{s}(t)$ to be arbitrary closed points (of suitable double and triple fibre products) for $t \in \pi_0(S_\bullet(h))_2$. We fix a maximal tree T in $\Gamma = \pi_0(S_\bullet(h))_{\leq 1}$ and fix paths $\gamma_{t',t} : \bar{s}(t') \to S_\bullet(h)(\partial)\bar{s}(t)$. Thus, we get $\pi_1^{\text{proét}}(X, \bar{x}) \cong \Big(\big(*_v^N \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) *^N \pi_1(\Gamma, T) \big) / \overline{H} \Big)^{\text{Noohi}}$ where H is defined as in Cor. 3.18.

Observe now that $\widetilde{X}_v \times_k Y$ are connected (as k is algebraically closed) and that $h \times \mathrm{id}_Y : \widetilde{X} \times Y \to X \times Y$ is an effective descent morphism for pro-étale covers. So we might use Cor. 3.18 in this setting. As $(\widetilde{X}_v \times Y) \times_{X \times Y} (\widetilde{X}_w \times Y) = (X_v \times_X X_w) \times_k Y$, and similarly for triple products, we can identify in a natural way $i^{-1} : \pi_0(S_\bullet(h \times \mathrm{id}_Y)) \xrightarrow{\sim} \pi_0(S_\bullet(h))$. In particular we can identify the graph $\Gamma_Y = \pi_0(S_\bullet(h \times \mathrm{id}_Y))_{\leq 1}$ with Γ and we choose the maximal tree T_Y of Γ_Y as the image of T via this identification. For $t \in \pi_0(S_\bullet(h))$ choose

$(\bar{s}(t), \bar{y})$ as the closed base points for $i(t) \in \pi_0(S_\bullet(h \times \mathrm{id}_Y))$. Denote by α_{ijk} elements of various $\pi_1^{\text{proét}}(\widetilde{X}_v)$ defined as in Cor. 3.18 and by $\vec{e}$ elements of $\pi_1(\Gamma, T)$. By the choices and identifications above we can identify $\pi_1(\Gamma_Y, T_Y)$ with $\pi_1(\Gamma, T)$. Using van Kampen and the assumption, we write

$$\begin{aligned}\pi_1^{\text{proét}}(X \times Y, (\bar{x}, \bar{y})) &\cong \Big(\big(*_v^N \, \pi_1^{\text{proét}}(\widetilde{X}_v \times Y, (x_v, \bar{y})) *^N \pi_1(\Gamma_Y, T_Y)\big)/\overline{H_Y}\Big)^{\text{Noohi}} \\ &\cong \Big(\big(*_v^N \, (\pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y})_v) *^N \pi_1(\Gamma, T)\big)/\overline{H_Y}\Big)^{\text{Noohi}}.\end{aligned}$$

Here $\pi_1^{\text{proét}}(Y, \bar{y})_v$ denotes a "copy" of $\pi_1^{\text{proét}}(Y, \bar{y})$ for each v. By Lm. 3.35, for $T \in \pi_0(S_\bullet(h))$ the natural map $\pi_1^{\text{proét}}(T \times Y, (\bar{s}(T), \bar{y})) \to \pi_1^{\text{proét}}(T, \bar{s}(T)) \times \pi_1^{\text{proét}}(Y, \bar{y})$ is surjective. It follows that the relations defining H_Y (as in Cor. 3.18) can be written as

$$\begin{aligned}&\pi_1^{\text{proét}}(\partial_1)(g) h_{y,1} \vec{e} = \vec{e} \pi_1^{\text{proét}}(\partial_0)(g) h_{y,0},\ e \in e(\Gamma),\ g \in \pi_1^{\text{proét}}(e, \bar{s}(e)),\ e \in S_1(h),\ h_y \in \pi_1^{\text{proét}}(Y, \bar{y}) \\ &\overrightarrow{(\partial_2 f)} \alpha_{102} \alpha_{120}^{-1} \overrightarrow{(\partial_0 f)} \alpha_{210} \alpha_{201}^{-1} \Big(\overrightarrow{(\partial_1 f)}\Big)^{-1} \alpha_{021} \alpha_{012}^{-1} = 1,\ f \in S_2(h)\end{aligned}$$

where α's in the second relation are elements of suitable $\pi_1^{\text{proét}}(\widetilde{X}_v)$'s and are the same as in the corresponding generators of H. The $h_{y,i}$ denotes a copy of element $h_y \in \pi_1^{\text{proét}}(Y, \bar{y})$ in a suitable $\pi_1^{\text{proét}}(Y, \bar{y})_v$. Varying e and h_y while choosing $g = 1 \in \pi_1^{\text{proét}}(e, \bar{s}(e))$ for every e, gives that $h_{y,1}\vec{e} = \vec{e} h_{y,0}$. For $e \in T$ we have $\vec{e} = 1$ and so the first relation reads $h_{y,1} = h_{y,0}$, i.e. it identifies $\pi_1^{\text{proét}}(Y, \bar{y})_v$ with $\pi_1^{\text{proét}}(Y, \bar{y})_w$ for v, w - ends of the edge e. As T is a maximal tree in Γ, it contains all the vertices, so the first relation identifies $\pi_1^{\text{proét}}(Y, \bar{y})_v = \pi_1^{\text{proét}}(Y, \bar{y})_w$ for any two vertices v, w and we will denote this subgroup (of the quotient) by $\pi_1^{\text{proét}}(Y, \bar{y})$. This way $h_{y,1}\vec{e} = \vec{e} h_{y,0}$ reads simply $h_y \vec{e} = \vec{e} h_y$, so elements of $\pi_1^{\text{proét}}(Y, \bar{y})$ commute with elements of $\pi_1(\Gamma, T)$. Moreover, elements of $\pi_1^{\text{proét}}(Y, \bar{y})$ commute with elements of each $\pi_1^{\text{proét}}(\widetilde{X}_v, x_v)$, as this was true for $\pi_1^{\text{proét}}(Y, \bar{y})_v$. On the other hand, choosing $h_y = 1$ in the first relation and looking at the second relation, we see that H_Y contains all the relations of H. Using notations from the above discussion, we can sum it up by writing

$$H_Y = \langle\langle \text{relations generating } H, h_{y,0} = h_{y,1}, h_y \vec{e} = \vec{e} h_y, h_y g = g h_y \ (g \in \pi_1^{\text{proét}}(\widetilde{X}_v, x_v))\rangle\rangle.$$

Putting this together, we get equivalences of categories

$$\begin{aligned}&\Big(\big(*_v^N \, (\pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y})_v) *^N \pi_1(\Gamma, T)\big)/\overline{H_Y}\Big) - \text{Sets} \\ \cong &\{S \in \big(*_v^N \, (\pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y})_v) *^N \pi_1(\Gamma, T)\big) - \text{Sets} | H_Y \text{ acts trivially on } S\} \\ \overset{\spadesuit}{\cong} &\{S \in \big((*_v^N \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) *^N \pi_1(\Gamma, T)) \times \pi_1^{\text{proét}}(Y, \bar{y})\big) - \text{Sets} | H \text{ acts trivially on } S\} \\ \cong &\Big(\big(\big(*_v^N \, \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) *^N \pi_1(\Gamma, T)\big)/\overline{H}\big) \times \pi_1^{\text{proét}}(Y, \bar{y})\Big) - \text{Sets} \\ \cong &\Big(\pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})\Big) - \text{Sets},\end{aligned}$$

where equality $\spadesuit$ follows from the fact that for topological groups G_1, G_2 there is equivalence $(G_1 \times G_2) - \text{Sets} \cong \{S \in G_1 *^N G_2 - \text{Sets} | \forall_{g_1 \in G_1, g_2 \in G_2} \forall_{s \in S} g_1 g_2 s = g_2 g_1 s\}$ (see Lm. 3.25).
This finishes the proof of the claim, i.e. of a particular case of the proposition.

The general case follows from the claim proven above in the following way: let $\sqcup_v \widetilde{X}_v = \widetilde{X} \to X$ and $\sqcup_u \widetilde{Y}_u = \widetilde{Y} \to Y$ be decompositions into connected components of the normalizations of X and Y. Fix v and note that $\pi_1^{\text{proét}}(X_v \times_k Y) = \pi_1^{\text{proét}}(X_v) \times \pi_1^{\text{proét}}(Y)$ by applying the claim to Y and $\widetilde{X}_v$. This is possible, as $\widetilde{Y}_u$'s, $\widetilde{X}_v$ and the products $\widetilde{Y}_u \times_k \widetilde{X}_v$ (for all u) are normal varieties and so their pro-étale fundamental groups are equal to the usual étale fundamental groups (by Lm. 2.18) for which the equality $\pi_1^{\text{ét}}(\widetilde{Y}_u \times_k \widetilde{X}_v) = \pi_1^{\text{ét}}(\widetilde{Y}_u) \times \pi_1^{\text{ét}}(\widetilde{X}_v)$ is known (see Fact 3.33). Thus, for any v, we have that $\pi_1^{\text{proét}}(X_v \times_k Y) = \pi_1^{\text{proét}}(X_v) \times \pi_1^{\text{proét}}(Y)$. We can now apply the claim to X and Y and finish the proof in the general case. ☐

3.4 Invariance of $\pi_1^{\text{proét}}$ of a proper scheme under a base-change $K \supset k$ of algebraically closed fields

Proposition 3.36. *Let X be a proper scheme over an algebraically closed field k. Let $K \supset k$ be another algebraically closed field. Then the pullback induces an equivalence of categories*

$$F : \mathrm{Cov}_X \to \mathrm{Cov}_{X_K}$$

In particular, if X is connected, $X_K \to X$ induces an isomorphism

$$\pi_1^{\text{proét}}(X_K) \xrightarrow{\sim} \pi_1^{\text{proét}}(X).$$

Proof. Let $X^\nu \to X$ be the normalization. It is finite and thus a morphism of effective descent for geometric covers. Let us show that the functor F is essentially surjective. Let $Y' \in \mathrm{Cov}_{X_K}$. As k is algebraically closed and X^ν is normal, we conclude that X^ν is geometrically normal and thus the base change $(X^\nu)_K$ is normal as well (see [SP, Tag 038O]). Pulling Y' back to $(X^\nu)_K$ we get a disjoint union of schemes finite étale over $(X^\nu)_K$ with a descent datum. It is a classical result ([SGA 1, Exp. X, Cor. 1.8]) that the pullback induces an equivalence $\mathrm{Fét}_{X^\nu} \to \mathrm{Fét}_{X^\nu_K}$ of finite étale covers and similarly for the double and triple products $X_2^\nu = X^\nu \times_X X^\nu$, $X_3^\nu = X^\nu \times_X X^\nu \times_X X^\nu$. These equivalences obviously extend to categories whose objects are (possibly infinite) disjoint unions of finite étale schemes (over X^ν, X_2^ν, X_3^ν respectively) with étale morphisms as arrows. These categories can be seen as subcategories of Cov_{X^ν} and so on. These subcategories are moreover stable under pullbacks between $\mathrm{Cov}_{X_i^\nu}$. Putting this together we see, that $Y'' = Y' \times_{X_K} (X^\nu)_K$ with its descent datum is isomorphic to a pullback of a descent datum from X^ν. Thus, we conclude that there exists $Y \in \mathrm{Cov}_X$ such that $Y' \simeq Y_K$. Full faithfulness of F is shown in the same way. If X is connected, it can be also proven more directly, as F being fully faithful is equivalent to preserving connectedness of geometric covers, but any connected $Y \in \mathrm{Cov}_X$ is geometrically connected and thus Y_K remains connected by Lm. 2.64 (2). □

Chapter 4

Homotopy exact sequence over a field

4.1 Statement of the results and examples

Observation 4.1. By Thm. 3.28, the category of geometric covers is invariant under universal homeomorphisms. In particular, for a connected X over a field and k'/k purely inseparable, there is $\pi_1^{\text{proét}}(X_{k'}) = \pi_1^{\text{proét}}(X)$. Similarly, we can replace X by X_{red} and so assume X to be reduced when convenient. In this case base change to separable closure X_{k^s} is reduced as well. We will often use this observation without an explicit reference.

Theorem 4.2. *Let k be a field and fix an algebraic closure $\bar{k}$. Let X be a geometrically connected scheme of finite type over k. Let $\bar{x} : \mathrm{Spec}(\bar{k}) \to X_{\bar{k}}$ be a geometric point on $X_{\bar{k}}$. Then the induced sequence*

$$\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \to \mathrm{Gal}_k \to 1$$

of topological groups is weakly exact in the middle (i.e. the smallest thickly closed subgroup of the image of the first map is equal to the kernel of the second map) and $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a topological quotient map.

Proof.

1. The map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is surjective and and open: let $U < \pi_1^{\text{proét}}(X)$ be an open subgroup. We want show that the image of U in Gal_k is open. U corresponds to a geometric cover Y of X and $U = \pi_1^{\text{proét}}(Y)$. Via this identification the morphism $U \to \mathrm{Gal}_k$ corresponds to the morphism $\pi_1^{\text{proét}}(Y) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$. So it is enough to check that the image of $\pi_1^{\text{proét}}(Y) \to \mathrm{Gal}_k$ is open. Observe that Y is locally of finite type over k. So it has a closed point with residue field $\mathrm{Spec}(l)$ for l a finite extension of k. Thus, we get $\mathrm{Gal}_l \to \pi_1^{\text{proét}}(Y) \to \mathrm{Gal}_k$ and see that the image $\pi_1^{\text{proét}}(Y) \to \mathrm{Gal}_k$ contains an open subgroup, so is open. We have shown that $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is an open morphism. In particular the image of $\pi_1^{\text{proét}}(X)$ in Gal_k is open and so also closed. On the other hand, this image is dense as we have the following diagram

$$\begin{array}{ccc} & \pi_1^{\text{proét}}(X) \longrightarrow & \pi_1^{\text{proét}}(\mathrm{Spec}(k)) \\ & \downarrow & \| \\ \widehat{\pi_1^{\text{proét}}(X)}^{\text{prof}} & = \pi_1^{\text{ét}}(X) \longrightarrow\!\!\!\!\rightarrow & \pi_1^{\text{ét}}(\mathrm{Spec}(k)) \end{array}$$

   where $\widehat{\ldots}^{\text{prof}}$ means the profinite completion. In the diagram, the left vertical map has dense image and the lower horizontal is surjective. This shows that $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is surjective.

2. The composition $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \to \mathrm{Gal}_k$ is trivial: this is clear thanks to Prop. 2.64 and the fact that the map $X_{\bar{k}} \to \mathrm{Spec}(k)$ factorizes through $\mathrm{Spec}(\bar{k})$.

3. Normality of the thick closure of the image of the first map: As remarked above, $\pi_1^{\text{proét}}(X_{\bar{k}}) = \pi_1^{\text{proét}}(X_{k^s})$, where k^s denotes the separable closure. Thus, we are allowed to replace $\bar{k}$ with k^s in the proof of this point. Moreover, by the same remark, we can and do assume X to be reduced. Let $Y \to X$ be a connected geometric cover s.t. there exists a section $s : X_{k^s} \to Y \times_X X_{k^s} = Y_{k^s}$ over X_{k^s}. Observe that any such section is a clopen immersion: this follows immediately from the equivalence of categories of $\pi_1^{\text{proét}}(X_{k^s})-$Sets and geometric covers. Define $\bar{T} := \bigcup_{\sigma \in Gal(k)} {}^{\sigma}s(X_{k^s}) \subset Y_{k^s}$. At this step, we could simplify the rest of the proof by using Lm. 5.19 to conclude that $\pi_0(Y_{k^s})$ is finite, but this is not necessary. Observe that two images of sections in the sum either coincide or are disjoint as X_{k^s} is connected and they are clopen. Now, $\bar{T}$ is obviously open, but we claim that it is also a closed subset. This can be checked locally, i.e. we need to check that $T \cap U_\lambda$ is closed in U_λ for an open cover $\{U_\lambda\}$ of Y_{k^s}. Observe that X_{k^s} is noetherian and as $Y_{k^s} \to X_{k^s}$ is étale, Y_{k^s} itself is locally noetherian. But a topologically noetherian neighbourhood can only intersect a finite number of disjoint clopen sets, so our claim follows. Now, by [SP, Tag 038B] $\bar{T}$ descends to a closed subset $T \subset Y$. It is also open as T is the image of $\bar{T}$ via projection $Y_{k^s} \to Y$ which is surjective and open map. Indeed, surjectivity is clear and openness is easy as well and is a particular case of a general fact, that any map from a scheme to a field is universally open ([SP, Tag 0383]). By connectedness of Y we see that $T = Y$. So $Y_{k^s} = \bar{T}$. But this last one is a disjoint union of copies of X_{k^s}, which is what we wanted to show by Prop. 2.64.

4. The smallest normal thickly closed subgroup of $\pi_1^{\text{proét}}(X)$ containing the image of $\pi_1^{\text{proét}}(X_{\bar{k}})$ is equal to $\ker(\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k)$: As we already know that this image is contained in the kernel and that the map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a quotient map of topological groups, we can apply Prop. 2.64. Let Y be a geometric cover of X such that $Y_{\bar{k}}$ - its pull-back to $X_{\bar{k}}$, is isomorphic as a geometric cover to a disjoint union of $X_{\bar{k}}$, let's say $Y_{\bar{k}} = \bigsqcup_\alpha X_{\bar{k},\alpha}$, where by $X_{\bar{k},\alpha}$ we label different copies of $X_{\bar{k}}$. As before, we could now use Lm. 5.19 and Lm. 5.20 to simplify the rest of the proof but it is not really needed. Denote by i_α the inclusion $X_{\bar{k},\alpha} \to Y_{\bar{k}}$ over $X_{\bar{k}}$. Pick one $X_{\bar{k},\alpha_0}$. As X is quasi-compact, by [EGA IV 3, Thm. 8.8.2], we see that there exists a finite separable field extension l/k and an X_l-morphism $X_l \to Y_l$ whose base-change to $\bar{k}$ is equal to i_{α_0}. Denote this l by l_{α_0} and do the same reasoning for any $X_{\bar{k},\alpha}$ to obtain a finite separable field extension l_α/k and a morphism $X_{l_\alpha} \to Y_{l_\alpha}$ whose base-change is equal to i_α. By composing with projections $Y_{l_\alpha} \to Y$ we get also morphisms $j_\alpha : X_{l_\alpha} \to Y$ of geometric covers of X. Consider now the geometric cover $Z = \bigsqcup_\alpha \mathrm{Spec}(l_\alpha) \to \mathrm{Spec}(k)$. Observe that $Z \times_{\mathrm{Spec}(k)} X = \bigsqcup_\alpha X_{l_\alpha}$ has a morphism to Y over X given by j_α on the appropriate components. As the pull-back of $Z \to Y$ to $X_{\bar{k}}$ is surjective, $Z \to Y$ itself must be surjective (for example by an analogous statement for group-sets).

□

The main result of this chapter is the following theorem and its corollary that will be proved in a later section.

Theorem. (see Thoerem 4.14 below) Let k be a field and fix its algebraic closure $\bar{k}$. Let X be a scheme of finite type over k such that the base change $X_{\bar{k}}$ is connected. Then the induced map

$$\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$$

is a topological embedding.

By Prop. 2.64, it translates to the following statement in terms of covers: every geometric cover of $X_{\bar{k}}$ can be dominated by a cover that embeds into a base-change to $\bar{k}$ of a geometric cover of X (i.e. defined over k). In practice, we prove that every connected geometric cover of $X_{\bar{k}}$ can be dominated by a cover of X_l for l/k finite.

For finite covers, the analogous statement is very easy to prove simply by finiteness condition. But for general geometric covers this is non-trivial and maybe even slightly surprising as we show by counterexamples (Ex. 4.3 and Ex. 4.4) that it is not always true that a connected geometric cover of $X_{\bar{k}}$ is isomorphic to a base-change of a cover of X_l for some finite extension l/k. This last statement is, however, stronger than what we need to prove and thus does not contradict our theorem. Observe, that the stronger statement is true for finite covers and, even more generally, whenever $\pi_1^{\text{proét}}(X_{\bar{k}})$ is pro-discrete, as proven in Prop. 4.6.

Theorems 4.2 and 4.14 with an additional input from the van Kampen theorem combine to an elegant statement below.

Theorem. (see Theorem 4.16 below) With the assumptions as in Thm. 4.2, the sequence of abstract groups

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

is exact.

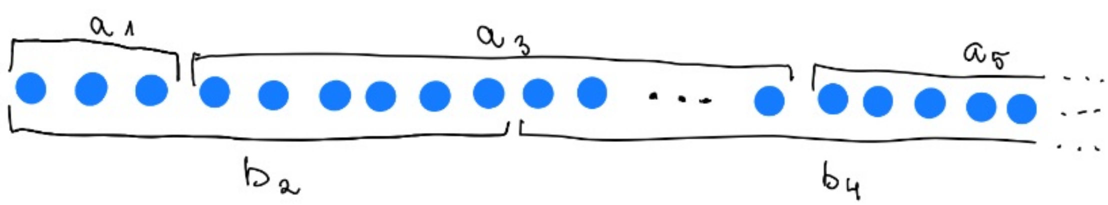
Moreover, the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding and the map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a quotient map of topological groups.

As promised above, we give examples of geometric covers of $X_{\bar{k}}$ that cannot be defined over any finite field extension l/k.

Example 4.3. Let $X_i = \mathbb{G}_{m,\mathbb{Q}}$, $i = 1, 2$. Define X to be the gluing $X = \cup_{\bullet} X_i$ of these schemes at the rational points $1_i : \mathrm{Spec}(\mathbb{Q}) \to X_i$ corresponding to 1. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and so fix a geometric point $\bar{b}$ over the base $\mathrm{Spec}(\mathbb{Q})$. This gives geometric points $\bar{x}_i$ on $\overline{X}_i = X_{i,\overline{\mathbb{Q}}}$ and X_i lying over 1_i, which we choose as base points for the fundamental groups involved. Similarly, we get a geometric point $\bar{x}$ over the point of gluing x that maps to $\bar{b}$. Then Example 3.24 gives us a description of the fundamental group $\pi_1^{\text{proét}}(X, \bar{x}) \simeq \Big(*_{i=1,2} \, (\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) \rtimes \mathrm{Gal}_{\mathbb{Q},i}) / \langle\langle \iota_1(\sigma) = \iota_2(\sigma) | \sigma \in \mathrm{Gal}_{\mathbb{Q}} \rangle\rangle \Big)^{\text{Noohi}}$ and of its category of sets:

$$\pi_1^{\text{proét}}(X, \bar{x}) - \text{Sets} \simeq$$
$$\Big\{ S \in \Big(*_{\text{top}} \, \pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1) *_{\text{top}} \pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2) *_{\text{top}} \mathrm{Gal}_{\mathbb{Q}} \Big) - \text{Sets} \Big|$$
$$\Big| \forall_{\sigma \in \mathrm{Gal}_{\mathbb{Q}}} \forall_i \forall_{\gamma \in \pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i)} \forall_{s \in S} \sigma \cdot (\gamma \cdot s) = {}^{\sigma}\gamma \cdot (\sigma \cdot s) \Big\}$$

For the base change $\overline{X}$ to $\overline{\mathbb{Q}}$ we have: $\pi_1^{\text{proét}}(\overline{X}, \bar{x}) \simeq \pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1) *^N \pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2)$. Recall that the groups $\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i)$ are isomorphic to $\widehat{\mathbb{Z}}(1)$ as the $\mathrm{Gal}_{\mathbb{Q}}$-modules. Let $S = \mathbb{N}_{>0}$. Let us define a $\pi_1^{\text{proét}}(\overline{X}, \bar{x})$-action on S, which means giving actions by $\pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1)$ and $\pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2)$ (no compatibilities of the actions required). Let ℓ be a fixed prime number (e.g. $\ell = 3$). We will give two different actions of $\mathbb{Z}_\ell$ on S which will define actions of $\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i)$ by projections on $\mathbb{Z}_\ell$. We start by dividing S into consecutive intervals labelled $a_1, a_3, a_5, \ldots$ of cardinality $\ell^1, \ell^3, \ell^5, \ldots$ respectively. These will be the orbits under the action of $\pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1)$. Similarly, we divide S into consecutive intervals $b_2, b_4, b_6 \ldots$ of cardinality $\ell^2, \ell^4, \ldots$.



We still have to define the action on each a_m and b_m. For b_m's we choose the identifications $b_m \simeq \mathbb{Z}/\ell^m$ (as an $\mathbb{Z}_\ell$-module) arbitrarily. For a_m's we choose the identifications arbitrarily with one caveat: we demand that for any even number m, the intersection $b_m \cap a_{m+1}$ contains the elements $\bar{0}, \bar{1} \in \mathbb{Z}/\ell^{m+1}$ via the identification $a_{m+1} \simeq \mathbb{Z}/\ell^{m+1}$. As $|b_m \cap a_{m+1}| \equiv \ell \mod \ell^2$, $b_m \cap a_{m+1}$ contains at least two elements and we see that choosing such labelling is always possible.

Assume that S corresponds to a cover that can be defined over a finite Galois extension $K/\mathbb{Q}$. Fix $s_0 \in a_1 \cap b_2$. By increasing K, we might and do assume that Gal_K fixes s_0. Let p be a prime number $\neq \ell$ that splits completely in K and $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying above p. Let $\phi_{\mathfrak{p}} \in \mathrm{Gal}_K$ be a Frobenius element (which depends on the choice of the decomposition group and the coset of the inertia subgroup). It acts on $\mathbb{Z}_\ell(1)$ via multiplication by p (and this action is independent of the choice of $\phi_{\mathfrak{p}}$). Choose $N > 0$ such that $p^N \equiv 1 \mod \ell^2$ and let m be the biggest number such that $p^N \equiv 1 \mod \ell^m$. If m is odd, we look at $p^{\ell N}$ instead. In this case $m+1$ is the biggest number such that $p^{\ell N} \equiv 1 \mod \ell^{m+1}$ and so, by changing N if necessary, we can assume that m is even, > 1. The whole point of the construction is the following: if $s \in a_i \cap b_j$ with $i, j < m$ is fixed by $\phi_{\mathfrak{p}}^N$, then so are $g \cdot s$ and $h \cdot s$ (for $h \in \pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1)$ and $g \in \pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2)$). Then moving such s with g's and h's to $b_m \cap a_{m+1}$ leads to a contradiction. Indeed, let $s_1 \in b_m \cap a_{m+1} \subset S$ correspond to $\bar{0} \in \mathbb{Z}/\ell^{m+1}$ via $a_{m+1} \simeq \mathbb{Z}/\ell^{m+1}$ (it is possible by the choices made in the construction of S). Write $s_1 = g_m h_{m-1} \dots h_3 g_2 h_1 \cdot s_0$ with $h_i \in \pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1)$ and $g_j \in \pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2)$ (this form is not unique, of course). This is possible thanks to the fact that the sets a_i, b_j form consecutive intervals separately, such that b_j intersects non-trivially a_{j-1} and a_{j+1}. By the construction of S again, there is an element $s_2 \in b_m \cap a_{m+1}$ corresponding to $\bar{1} \in \mathbb{Z}/\ell^{m+1}$ via $a_{m+1} \simeq \mathbb{Z}/\ell^{m+1}$. We can now write s_2 in two ways:

$$s_2 = \zeta_{\ell^{m+1}} \cdot s_1 = g \cdot s_1,$$

where $g \in \pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2)$ and $\zeta_{\ell^{m+1}}$ gives an action of $1 \in \hat{\mathbb{Z}} \simeq \pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1)$ on $\mathbb{Z}/\ell^{m+1}$ when viewed multiplicatively $\mathbb{Z}/\ell^{m+1} \simeq \mu_{\ell^{m+1}}$. By the choices made, the action of $\phi_{\mathfrak{p}}^N$ fixes the elements s_1 and $g \cdot s_1$, while it moves $\zeta_{\ell^{m+1}} \cdot s_1$. Indeed, $\phi_{\mathfrak{p}}^N \cdot (\zeta_{\ell^{m+1}} \cdot s_1) = (\phi_{\mathfrak{p}}^N \zeta_{\ell^{m+1}} \phi_{\mathfrak{p}}^{-N}) \cdot (\phi_{\mathfrak{p}}^N \cdot s_1) = \zeta_{\ell^{m+1}}^{p^N} \cdot (\phi_{\mathfrak{p}}^N \cdot s_1) = \zeta_{\ell^{m+1}}^{p^N} \cdot s_1 = \overline{p^N} \neq \bar{1} \in \mathbb{Z}/\ell^{m+1} \simeq a_{m+1}$ - a contradiction.

Example 4.4. Let $X_i = \mathbb{G}_{m,\mathbb{Q}}, i = 1, 2, 3$ and let Y_4, Y_5 be the nodal curves obtained from gluing 1 and -1 on $\mathbb{P}^1_{\mathbb{Q}}$ (see Ex. 3.23). Define X to be the gluing $X = \cup_\bullet X_i \cup_\bullet Y_j$ of all these schemes at the rational points corresponding to 1 (or the image of 1 in the case of Y_j). We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and so fix a geometric point $\bar{b}$ over the base $\operatorname{Spec}(\mathbb{Q})$. As in Example 4.3, we get base points $\bar{x}_i, \bar{y}_i, \bar{x}$ on $\overline{X}_i, \overline{Y}_i$ and $\overline{X}$ lying over 1. Then Example 3.24 gives us a description of the fundamental group $\pi_1^{\text{proét}}(X, \bar{x}) \simeq \Big(*^N_{i=1,2,3} \, (\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) \rtimes \mathrm{Gal}_{\mathbb{Q},i}) *^N_{j=4,5} (\mathbb{Z} \times \mathrm{Gal}_{\mathbb{Q},j}) / \langle\langle \iota_i(\sigma) = \iota_{i'}(\sigma) | \sigma \in \mathrm{Gal}_{\mathbb{Q}}, \ i, i' = 1, \dots, 5 \rangle\rangle \Big)^{\text{Noohi}}$ and of its category of sets:

$$\pi_1^{\text{proét}}(X, \bar{x}) - \text{Sets} \simeq \Big\{ S \in \Big(*^N_{1 \le i \le 3} \, \pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) *^N \mathbb{Z}^{*2} *^N \mathrm{Gal}_{\mathbb{Q}} \Big) - \text{Sets} \Big| \, \Big| \forall_{\sigma \in \mathrm{Gal}_{\mathbb{Q}}} \forall_i \forall_{\substack{\gamma \in \pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) \\ w \in \mathbb{Z}^{*2}}} \forall_{s \in S} \, \sigma \cdot (\gamma \cdot s) = ^{\sigma} \gamma \cdot (\sigma \cdot s) \text{ and } \sigma \cdot (w \cdot s) = w \cdot (\sigma \cdot s) \Big\}$$

For the base change $\overline{X}$ to $\overline{\mathbb{Q}}$ we have: $\pi_1^{\text{proét}}(\overline{X}, \bar{x}) \simeq *^N_{i=1,2,3} \pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) *^N \mathbb{Z} *^N \mathbb{Z}$. The groups $\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i)$ are isomorphic to $\widehat{\mathbb{Z}}(1)$ as the $\mathrm{Gal}_{\mathbb{Q}}$-modules. Let $G = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Q}_\ell)$ be the subgroup of upper triangular matrices. Recall that $\mathbb{Z}_\ell^\times \simeq \mu_{\ell-1} \times (1 + \ell \mathbb{Z}_\ell) \simeq \mathbb{Z}/(\ell - 1) \times \mathbb{Z}_\ell$ and so it possesses a single topological generator u_1. Define a continuous homomorphism $\psi : H := *^{\text{top}}_{i=1,2,3} \pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) *^{\text{top}} \mathbb{Z} *^{\text{top}} \mathbb{Z} \to G$ by:

$$\pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1) \simeq \widehat{\mathbb{Z}} \ni 1 \mapsto \begin{pmatrix} u_1 & 0 \\ & 1 \end{pmatrix}, \ \pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2) \simeq \widehat{\mathbb{Z}} \ni 1 \mapsto \begin{pmatrix} 1 & 0 \\ & u_1 \end{pmatrix},$$

$$\pi_1^{\text{ét}}(\overline{X}_3, \bar{x}_3) \simeq \widehat{\mathbb{Z}} \ni 1 \mapsto \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix},$$

$$\pi_1^{\text{ét}}(\overline{Y}_4, \bar{y}_4) \simeq \mathbb{Z} \ni 1 \mapsto \begin{pmatrix} \ell & 0 \\ & 1 \end{pmatrix}, \ \pi_1^{\text{ét}}(\overline{Y}_5, \bar{y}_5) \simeq \mathbb{Z} \ni 1 \mapsto \begin{pmatrix} 1 & 0 \\ & \ell \end{pmatrix}.$$

It is easy to see that ψ is surjective by writing for any matrix in G

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} = \begin{pmatrix} a'\ell^k & b'\ell^m \\ & d'\ell^n \end{pmatrix} = \begin{pmatrix} a'\ell^k & \\ & d'\ell^{(k-m)+n} \end{pmatrix} \begin{pmatrix} 1 & b'/a' \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \ell^{m-k} \end{pmatrix}$$

where $a', b', c' \in \mathbb{Z}_\ell^\times$. Let $U \subset G$ be the subgroup of matrices with elements in $\mathbb{Z}_\ell$, i.e. $U = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Z}_\ell)$. It is an open subgroup of G. Thus, using ψ and the fact that $H^{\text{Noohi}} = \pi_1^{\text{proét}}(\overline{X}, \bar{x})$, we get that

$S := G/U$ defines a $\pi_1^{\text{proét}}(\overline{X}, \bar{x})$-set. It is connected and so corresponds to a connected geometric cover of $\overline{X}$. Assume that it can be defined over a finite extension l of k. We can assume l/k is Galois. By the description above, it means that there is a compatible action of groups $\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i)$, $\mathbb{Z}^{*2}$ and Gal_l on S. By increasing l, we can assume moreover that Gal_l fixes $[U]$. Write $H = \widehat{\mathbb{Z}}_A *^{\text{top}} \widehat{\mathbb{Z}}_B *^{\text{top}} \widehat{\mathbb{Z}}_D *^{\text{top}} \mathbb{Z}_E *^{\text{top}} \mathbb{Z}_F$ as before, where the subscripts allow us to distinguish between copies of $\widehat{\mathbb{Z}}$. With this notation,

$$\psi(1_A) = \begin{pmatrix} u_1 & \\ & 1 \end{pmatrix},\ \psi(1_D) = \begin{pmatrix} 1 & \\ & u_1 \end{pmatrix},\ \psi(1_B) = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix},\ \psi(1_E) = \begin{pmatrix} \ell & \\ & 1 \end{pmatrix},\ \psi(1_F) = \begin{pmatrix} 1 & \\ & \ell \end{pmatrix}.$$

Let $\phi_\mathfrak{p}$ represent the conjugacy class of p-Frobenius in Gal_k. Recall that $\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) \simeq \widehat{\mathbb{Z}}(1)$ as a Gal_k-module and that the action of Gal_l on S commutes with the action of $\mathbb{Z}_E$ and $\mathbb{Z}_F$. Choose $p \neq \ell$ that splits completely in Gal_l, fix a prime $\mathfrak{p}$ of l dividing p and let $\phi_\mathfrak{p} \in \mathrm{Gal}_l$ denote a fixed Frobenius element. Let $n \gg 0$. An easy calculation shows that $\psi(n_E^{-1} 1_A 1_B 1_A^{-1} (-u_1)_B n_E) = 1_{\mathrm{GL}_2(\mathbb{Q}_\ell)} \in U$. Then $\phi_\mathfrak{p} \cdot [U] = \phi_\mathfrak{p} \cdot (n_E^{-1} 1_A 1_B 1_A^{-1} (-u_1)_B n_E [U]) = {}^{\phi_\mathfrak{p}}(n_E^{-1} 1_A 1_B 1_A^{-1} (-u_1)_B n_E) \cdot (\phi_\mathfrak{p} \cdot [U]) = n_E^{-1} p_A p_B p_A^{-1} (-pu)_B n_E \cdot [U]$. But

$$\begin{aligned} &\psi(n_E^{-1} p_A 1_B p_A^{-1} (-pu_1)_B n_E) = \\ &= \begin{pmatrix} \ell^{-n} & \\ & 1 \end{pmatrix} \begin{pmatrix} u_1^p & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & p \\ & 1 \end{pmatrix} \begin{pmatrix} u_1^{-p} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -pu_1 \\ & 1 \end{pmatrix} \begin{pmatrix} \ell^n & \\ & 1 \end{pmatrix} = \\ &= \begin{pmatrix} \ell^{-n} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & u_1^p p \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -pu_1 \\ & 1 \end{pmatrix} \begin{pmatrix} \ell^n & \\ & 1 \end{pmatrix} = \\ &= \begin{pmatrix} \ell^{-n} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & p(u_1^p - u_1) \\ & 1 \end{pmatrix} \begin{pmatrix} \ell^n & \\ & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & \ell^{-n} p(u_1^p - u_1) \\ & 1 \end{pmatrix} \notin K. \end{aligned}$$

So $\phi_\mathfrak{p} \cdot [U] = n_E^{-1} p_A 1_B p_A^{-1} (-pu_1)_B n_E \cdot [U] \neq [U]$ - a contradiction.

It is important to note, that the above (counter-)examples are possible only when considering the geometric covers that are not trivialized by an étale cover (but one really needs to use the pro-étale cover to trivialize them). In [BS], the category of geometric covers trivialized by an étale cover on X is denoted by $\mathrm{Loc}_{X_{\text{ét}}}$ and the authors prove the following

Fact 4.5. ([BS, Lemma 7.4.5]) Under $\mathrm{Loc}_X \simeq \pi_1^{\text{proét}}(X) - \mathrm{Sets}$, the full subcategory $\mathrm{Loc}_{X_{\text{ét}}} \subset \mathrm{Loc}_X$ corresponds to the full subcategory of those $\pi_1^{\text{proét}}(X) - \mathrm{Sets}$ where an open subgroup acts trivially.

We are now going to prove:

Proposition 4.6. *Let X be a geometrically connected separated scheme of finite type over a field k. Let $Y \in \mathrm{Cov}_{X_{\bar{k}}}$ be such that $Y \in \mathrm{Loc}_{(X_{\bar{k}})_{\text{ét}}}$. Then there exists a finite extension l/k such and $Y_0 \in \mathrm{Cov}_{X_l}$ such that $Y \simeq Y_0 \times_{X_l} X_{\bar{k}}$.*

Proof. By topological invariance (Thm. 3.28), we can replace $\bar{k}$ by k^{sep} if desired. By the assumption $Y \in \mathrm{Loc}_{(X_{\bar{k}})_{\text{ét}}}$, there exists an étale cover of finite type that trivializes Y. Being of finite type, it is a base-change $X'_{\bar{k}} = X' \times_{\mathrm{Spec}(l)} \mathrm{Spec}(\bar{k}) \to X_{\bar{k}}$ of an étale cover $X' \to X_l$ for some finite extension l/k. Thus, $Y_{|X'_{\bar{k}}}$ is constant (i.e. $\simeq \sqcup_{s \in S} X' = \underline{S}$) and the isomorphism between the pull-backs of $Y_{|X'_{\bar{k}}}$ via the two projections $X'_{\bar{k}} \times_{X_{\bar{k}}} X'_{\bar{k}} \rightrightarrows X'_{\bar{k}}$ is expressed by an element of a constant sheaf $\underline{\mathrm{Aut}(S)}(X'_{\bar{k}} \times_{X_{\bar{k}}} X'_{\bar{k}}) = \mathrm{Aut}(\underline{S})(X'_{\bar{k}} \times_{X_{\bar{k}}} X'_{\bar{k}})$ (we use the fact that $X'_{\bar{k}}$ is étale over $X_{\bar{k}}$ and thus $\pi_0(X'_{\bar{k}} \times_{X_{\bar{k}}} X'_{\bar{k}})$ is discrete, in this case even finite). By enlarging l, we can assume that the connected components of the schemes involved: X', $X' \times_{X_l} X'$ etc. are geometrically connected over l. Define $Y'_0 = \sqcup_{s \in S} X'$. The discussion above shows that the descent datum on $Y_{|X'_{\bar{k}}}$ with respect to $X'_{\bar{k}} \to X_{\bar{k}}$ is in fact the pull-back of a descent datum on Y'_0 with respect to $X' \to X_l$. As étale covers are morphisms of effective descent for geometric covers (this follows from the fpqc descent for fpqc sheaves and the equivalence $\mathrm{Cov}_{X_l} \simeq \mathrm{Loc}_{X_l}$ of [BS, Lemma 7.3.9.]), the proof is finished. ☐

Remark 4.7. Over a scheme with a non-discrete set of connected components, $\underline{\mathrm{Aut}(S)}$ might not be equal to $\mathrm{Aut}(\underline{S})$.

4.2 Preparation for the proof of Theorem 4.14

We are going to use the following proposition.

Proposition 4.8. *Let X be a scheme of finite type over a field k with a k-rational point x_0 and assume that $X_{\bar{k}}$ is connected. Let $Y_1, \dots, Y_N$ be a set of connected finite étale covers of $X_{\bar{k}}$. Then there exists a finite étale cover Y of X, such that $Y_{\bar{k}}$ is a Galois cover of $X_{\bar{k}}$ and for all $1 \le i \le N$ there exists a surjective map $Y_{\bar{k}} \twoheadrightarrow Y_i$ of covers of $X_{\bar{k}}$.*

Proof. We can first choose one finite Galois étale cover of $X_{\bar{k}}$ that dominates all $Y_1, \dots, Y_N$ and thus assume $N = 1$ and $Y_1/X_{\bar{k}}$ Galois. Let us choose a geometric point $\bar{x}_0$ over x_0. Existence of a k-rational point gives a splitting $s : \mathrm{Gal}_k \to \pi_1^{\text{ét}}(X)$, allowing us to write $\pi_1^{\text{ét}}(X) \simeq \pi_1^{\text{ét}}(X_{\bar{k}}) \rtimes \mathrm{Gal}_k$ and so an action of Gal_k on $\pi_1^{\text{ét}}(X_{\bar{k}})$. The main point is to prove the following lemma.

Lemma 4.9. *The set of conjugates of the open subgroup $\pi_1^{\text{ét}}(Y_1)$ in $\pi_1^{\text{ét}}(X_{\bar{k}})$ under the action of Gal_k is finite.*

Let us prove the lemma. There is a finite Galois extension l/k and a finite étale cover $Y \to X_l$, such that $Y_1 \to X_{\bar{k}}$ is a base change of it and that Y has an l-rational point y lying over x_0 through the composition $Y \to X_l \to X$ (we have used the fact, that a base-change by a purely inseparable extension of fields induces an equivalence of categories of finite étale covers, so we could assume l/k separable). We can choose the geometric point $\bar{y}$ over y such that it maps to $\bar{x}$. This provides a splitting $s_l : \mathrm{Gal}_l \to \pi_1^{\text{ét}}(Y)$ that gives an action of Gal_l on $\pi_1^{\text{ét}}(Y_1)$. Thanks to the compatibilities of the choices, the restriction of the action $\mathrm{Gal}_k \to \mathrm{Aut}(\pi_1^{\text{ét}}(X_{\bar{k}}))$ to Gal_l preserves $\pi_1^{\text{ét}}(Y_1)$ and is equal to the action coming from s_l. From this, we see that the number of different conjugates of $\pi_1^{\text{ét}}(Y_1)$ under Gal_k is at most $[l : k]$. This finishes the proof of the lemma.

Let us finish the proof of the proposition. Define an intersection $V = \cap_{\sigma \in \mathrm{Gal}_k} {}^{\sigma}\pi_1^{\text{ét}}(Y_1)$. Thanks to the lemma above, V is open in $\pi_1^{\text{ét}}(X_{\bar{k}})$. The subgroup V is obviously fixed by the action of Gal_k. Thus, $W = V \rtimes \mathrm{Gal}_k = V\mathrm{Gal}_k < \pi_1^{\text{ét}}(X_{\bar{k}}) \rtimes \mathrm{Gal}_k$ is a subgroup and it is actually open (as the topology on $\pi_1^{\text{ét}}(X_{\bar{k}}) \rtimes \mathrm{Gal}_k$ is the product topology). Moreover, $\pi_1^{\text{ét}}(X)/W = \pi_1^{\text{ét}}(X_{\bar{k}})/V$, which shows that the cover corresponding to $\pi_1^{\text{ét}}(X_{\bar{k}})/V$ is isomorphic to the one defined over k. As $\pi_1^{\text{ét}}(Y_1)$ was normal in $\pi_1^{\text{ét}}(X_{\bar{k}})$, it is easy to see that the same will be true for V as well. ☐

Remark 4.10. The proof of Lemma 4.9. would be easier, if we knew that the group $\pi_1^{\text{ét}}(X_{\bar{k}})$ is small, i.e. that for each number n it posses only a finite number of subgroups of index n. This is for example true for proper schemes over a field, as their fundamental groups are topologically finitely generated ([SGA 1, Exp. X, Théorème 2.9]). Smallness was also studied in [HH].

We note the following easy fact on topological groups.

Lemma 4.11. *Let G be a topological group (by our convention it is Hausdorff). Let V be an open subgroup of G, let N be a normal subgroup of G and $\bar{N}$ its closure. Denote by U and $\underline{U}$ the image of V in G/N and in $G/\bar{N}$ respectively. Then the map*

$$(G/N)/U \ni \bar{g}U \mapsto \bar{g}\underline{U} \in (G/\bar{N})/\underline{U}$$

is a bijection of G-sets.

Proof. This follows from the fact that the obvious maps induce $(G/N)/U \simeq G/NV$ and $(G/\bar{N})/\underline{U} \simeq G/\bar{N}V$ plus we have that $NV = \bar{N}V$ as NV is open and so closed and contains N. ☐

Before we start, we also note the following lemma.

Lemma 4.12. *Let X be a scheme of finite type over k and let l/k be a separable (possibly infinite) extension of fields. If $\widetilde{X} \to X$ is the normalization of X, then $\left(\widetilde{X}\right)_l \to X_l$ is the normalization of X_l.*

Proof. We first show that $\left(\widetilde{X}\right)_l$ is normal. This follows immediately from the fact, that $X_l \to X$ is weakly étale (as a base-change of a weakly étale morphism) and [SP, Tag 0950]. It can be also proven more directly: if l/k is finite, this follows from the fact, that a smooth base-change commutes with normalization and in general we can localize and use the fact, that a colimit of normal rings is normal.
Let now W be the normalization of X_l. We have $\left(\widetilde{X}\right)_l \to W$, because $\left(\widetilde{X}\right)_l$ is normal. This descends to $\left(\widetilde{X}\right)_{l'} \to W_0$ over $l' \subset l$ that is finite over k. But then W_0 is normal (as an image of a normal scheme via faithfully flat map) and $\left(\widetilde{X}\right)_{l'}$ is the normalization of $X_{l'}$ and thus $\left(\widetilde{X}\right)_{l'} \to W_0$ is an isomorphism, which finishes the proof. □

Let us state a lemma concerning the "functoriality" of the van Kampen theorem. It is important that the diagram formed by the schemes $X_1, X_2, \widetilde{X}, \widetilde{X}_1$ in the statement is cartesian.

Lemma 4.13. *Let $f : X_1 \to X_2$ be a morphism of connected schemes and $h : \widetilde{X} \to X_2$ be a morphism of schemes. Denote by $h_1 : \widetilde{X}_1 \to X_1$ the base-change of h via f. Assume that h and h_1 are effective descent morphisms for geometric covers and that local topological noetherianity assumptions are satisfied for the schemes involved as in the statement of Cor. 3.18. Assume that for any connected component $W \in \pi_0(S_\bullet(h))$, the base-change W_1 of W via f is connected. Choose the geometric points on $W_1 \in \pi_0(S_\bullet(h_1))$ and paths between the obtained fibre functors as in Cor. 3.18 and choose the geometric points and paths on $W \in \pi_0(S_\bullet(h))$ as the images of those chosen for $\widetilde{X}_1$. Identify the graphs $\Gamma = \pi_0(S_\bullet(h))_{\leqslant 1}$ and $\Gamma_1 = \pi_0(S_\bullet(h_1))_{\leqslant 1}$ (it is possible thanks to the assumption made) and choose a maximal tree T in Γ. Using the above choices, use Cor. 3.18. to write the fundamental groups $\pi_1^{\text{proét}}(X_1) \simeq \left((*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W_1)) *^{\text{top}} \pi_1(\Gamma_1, T)/\langle R'\rangle\right)^{\text{Noohi}}$ and $\pi_1^{\text{proét}}(X_2) \simeq \left((*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W)) *^{\text{top}} \pi_1(\Gamma, T)/\langle R\rangle\right)^{\text{Noohi}}$.*

Then the map of fundamental groups $\pi_1^{\text{proét}}(f) : \pi_1^{\text{proét}}(X_1) \to \pi_1^{\text{proét}}(X_2)$ is the Noohi completion of the map

$$\Big((*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W_1)) *^{\text{top}} \pi_1(\Gamma_1, T)\Big)/\langle R'\rangle \to \Big(*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W)) *^{\text{top}} \pi_1(\Gamma, T)\Big)/\langle R\rangle,$$

which is induced by the maps $\pi_1^{\text{proét}}(W_1) \to \pi_1^{\text{proét}}(W)$ and identity on $\pi_1(\Gamma_1, T)$ (which makes sense after identification of Γ_1 with Γ).

Proof. It is clear that on (the image of) $\pi_1^{\text{proét}}(W_1)$ (in $\pi_1^{\text{proét}}(X_1)$) the map is the one induced from $f_W : W_1 \to W$. The part about $\pi_1(\Gamma_1, T)$ follows from the fact that $\pi_1(\Gamma_1, T) < \pi_1^{\text{proét}}(X_1)$ acts in the same way as $\pi_1(\Gamma, T) < \pi_1^{\text{proét}}(X_2)$ on any geometric cover of X_2. This follows from the choice of points and paths on $W \in \pi_0(S_\bullet(h))$ as the images of the points and paths on the corresponding connected components $W_1 \in \pi_0(S_\bullet(h_1))$. The maps as in the statement give a morphism $\phi : (*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W_1)) *^{\text{top}} \pi_1(\Gamma_1, T) \to (*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W)) *^{\text{top}} \pi_1(\Gamma, T)$ and it is easy to check that $\phi(R') \subset R$, which finishes the proof. □

4.3 Proof that $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding

Theorem 4.14. *Let k be a field and fix its algebraic closure $\bar{k}$. Let X be a scheme of finite type over k such that the base-change $X_{\bar{k}}$ is connected. Then the induced map*

$$\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$$

is a topological embedding.

Proof. For any finite extension l/k, the map $\pi_1^{\text{proét}}(X_l) \to \pi_1^{\text{proét}}(X)$ is an embedding of an open subgroup and we have a factorization $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X_l) \to \pi_1^{\text{proét}}(X)$. Indeed, we write l/k as a tower $l/l'/k$, where l'/k is separable and l/l' is purely inseparable. Then, the assertion is true for l'/k and the map $X_l \to X_{l'}$ induces isomorphism on the pro-étale fundamental groups (thanks to the topological invariance). Thus, we can replace k by a finite extension l before starting the proof, if convenient. Let $h : \widetilde{X} \to X$ be the normalization map. We choose l/k so that any connected component of $(\widetilde{X})_l$ and any connected component of the double and triple fibre product of $(\widetilde{X})_l$ over X_l is geometrically connected and has an l-rational point. Write $l/l'/k$, where l' is the maximal separable extension of k in l. Then $(\widetilde{X})_{l'}$ is the normalization of $X_{l'}$ (because of Lm. 4.12). Let $(\widetilde{X})_{l'} = X_1 \sqcup \ldots \sqcup X_n$ be the decomposition into connected components. As X_v are normal and as l/l' is purely inseparable, we get that $\pi_1^{\text{proét}}((X_v)_l) = \pi_1^{\text{ét}}((X_v)_l)$. Thus, we can base-change whole problem to l and assume that $l = k$ from now on at the cost that $h : \widetilde{X} = \sqcup_v X_v \to X$ is not necessarily the normalization, but but a finite surjective morphism such with X_v geometrically connected and such that $\pi_1^{\text{proét}}((X_v)_l) = \pi_1^{\text{ét}}((X_v)_l)$ for any algebraic field extension l/k and X_v's and their double and triple fibre products over X have k-rational points. Let us choose the base points. Denote $\widetilde{X}_{\bar{k}} = (\widetilde{X})_{\bar{k}}$. For each $t \in \pi_0(\widetilde{X}) \cup \pi_0(\widetilde{X} \times_X \widetilde{X}) \cup \pi_0(\widetilde{X} \times_X \widetilde{X} \times_X \widetilde{X})$), we fix a k-rational point $x(t)$ on t. Then we fix a $\bar{k}$-point $\bar{x}(\bar{t})$ on $\bar{t} = t_{\bar{k}} \in \pi_0(\widetilde{X}_{\bar{k}}) \cup \pi_0(\widetilde{X}_{\bar{k}} \times_{X_{\bar{k}}} \widetilde{X}_{\bar{k}}) \cup \pi_0(\widetilde{X}_{\bar{k}} \times_{X_{\bar{k}}} \widetilde{X}_{\bar{k}} \times_{X_{\bar{k}}} \widetilde{X}_{\bar{k}}))$ such that $\bar{x}(\bar{t})$ lies over $x(t)$. This gives also a geometric point $\bar{x}(t)$ of t lying over $x(t)$. Let us say that the image of $\bar{x}(\overline{X}_1)$ in $\overline{X}$ will be the fixed geometric point $\bar{x}$ of $\overline{X}$ and its image in X the fixed geometric point of X. For any $W_{\bar{k}}, W'_{\bar{k}} \in \pi_0(S_\bullet(\bar{h}))$ and every boundary map $\bar{\partial} : W_{\bar{k}} \to W'_{\bar{k}}$, we fix paths $\gamma_{W'_{\bar{k}}, W_{\bar{k}}}$ (notation as in Cor. 3.18) between the chosen geometric points by choosing a sequence of points (and valuation rings) on $W'_{\bar{k}}$ as in the proof of Cor. 3.17. We define $\gamma_{W',W}$ to be the image of this path. This choice of $\gamma_{W',W}$ guarantees an important property for us. As both $\bar{x}(W)$ and $\bar{x}(W')$ map to the same geometric point on the base scheme $\mathrm{Spec}(k)$, the path $\gamma_{W',W}$ maps to an automorphism of the fibre functor of $\mathrm{Cov}(\mathrm{Spec}(k))$ (given by fixing the algebraic closure $\bar{k}/k$ from the beginning). The fact that $\gamma_{W',W}$ is the image of the path $\gamma_{W'_{\bar{k}}, W_{\bar{k}}}$ on $W'_{\bar{k}}$ assures that this automorphism is in fact identity. This can be seen in the following way: a Galois cover $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$ becomes a disjoint union $\sqcup_{\sigma \in \mathrm{Gal}(l/k)} W'_{\bar{k},\sigma} \to W'_{\bar{k}}$ after a base-change to $W'_{\bar{k}}$ and the above fibre functor takes a copy of the point $\bar{x}(W')$ lying on some $W'_{\bar{k},\sigma_0}$ to the (copy of) the point $\bar{x}(W)$ lying on the same $W'_{\bar{k},\sigma_0}$, which is exactly what we wanted to show. This implies that the following diagram commutes:

$$\begin{array}{ccc} \pi_1^{\text{ét}}(W, \bar{x}(W)) & & \\ \downarrow & \searrow & \\ & & \mathrm{Gal}_k \\ & \nearrow & \\ \pi_1^{\text{ét}}(W', \bar{x}(W')) & & \end{array}$$

Similarly, we also get commutativity of the following diagram:

$$\begin{array}{ccc} \pi_1^{\text{ét}}(W_{\bar{k}}, \bar{x}(W)) & \longrightarrow & \pi_1^{\text{ét}}(W, \bar{x}(W)) \\ \downarrow & & \downarrow \\ \pi_1^{\text{ét}}(W'_{\bar{k}}, \bar{x}(W')) & \longrightarrow & \pi_1^{\text{ét}}(W', \bar{x}(W')) \end{array}$$

Denote by $\bar{h} : \widetilde{X}_{\bar{k}} \to X_{\bar{k}}$, the base-change of h. We choose a maximal tree T in the graph $\Gamma = \pi_0(S_\bullet(h))_{\leqslant 1}$. Thanks to the assumption that all $W \in \pi_0(S_\bullet(h))$ are geometrically connected we can identify (via bijection given by $W \mapsto W_{\bar{k}}$) $\pi_0(S_\bullet(h))$ with $\pi_0(S_\bullet(\bar{h}))$ and in particular the graph Γ with $\bar{\Gamma} = \pi_0(S_\bullet(\bar{h}))_{\leqslant 1}$. After making these choices, we can apply Cor. 3.18. to write the fundamental groups of $(X, \bar{x})$ and $(X_{\bar{k}}, \bar{x})$ (the chosen paths $\gamma_{W',W}$ are the images of the paths $\gamma_{W'_{\bar{k}}, W_{\bar{k}}}$ and we use the latter to apply Cor. 3.18. to $X_{\bar{k}}$). This allows us to write

$$\pi_1^{\text{proét}}(X) \simeq \left((*_v^{\text{top}} \pi_1^{\text{ét}}(X_v)) *^{\text{top}} \pi_1(\Gamma, T)) / \overline{\langle R_1, R_2 \rangle^{nc}} \right)^{\text{Noohi}}$$

where $\overline{(\ldots)}$ denotes the topological closure, $\langle R \rangle^{nc}$ denotes the normal subgroup generated by the set R, R_1 denotes the set of relations of the form $\pi_1^{\text{ét}}(\partial_1)(g)\vec{e} = \vec{e}\pi_1^{\text{ét}}(\partial_0)(g)$ and R_2 denotes the set of relations of the form $\overrightarrow{(\partial_2 f)} \alpha_{102}^{(f)} (\alpha_{120}^{(f)})^{-1} \overrightarrow{(\partial_0 f)} \alpha_{210}^{(f)} (\alpha_{201}^{(f)})^{-1} \left(\overrightarrow{(\partial_1 f)}\right)^{-1} \alpha_{021}^{(f)} (\alpha_{012}^{(f)})^{-1} = 1$. Observe that although a priori the elements

g in the definition of R_1 run through $\pi_1^{\text{proét}}(W)$ (where $W \in \pi_0(X_v \times_X X_w)$ for some v, w), we can in fact assume that g run through $\pi_1^{\text{ét}}(W)$, as any $\pi_1^{\text{proét}}(W) \to \pi_1^{\text{ét}}(X_v)$ will factorize through the profinite completion $\pi_1^{\text{ét}}(W)$ of $\pi_1^{\text{proét}}(W)$ and we take closures at the end (see Rmk. 3.20). Using the rational points $x(X_v)$, we write $\pi_1^{\text{ét}}(X_v) \simeq \pi_1^{\text{ét}}(\overline{X}_v) \rtimes \mathrm{Gal}_k$ and can rewrite the above as

$$\pi_1^{\text{proét}}(X) \simeq \left((*_v^{\text{top}}(\pi_1^{\text{ét}}(\overline{X}_v) \rtimes \mathrm{Gal}_{k,v}))*^{\text{top}}\pi_1(\Gamma, T))/\overline{\langle R_1, R_2\rangle^{nc}}\right)^{\text{Noohi}}$$

where we write $\mathrm{Gal}_{k,v}$ to distinguish between different copies of Gal_k in the free product. Similarly, we write

$$\pi_1^{\text{proét}}(X_{\bar{k}}) \simeq \left((*_v^{\text{top}}\pi_1^{\text{ét}}(\overline{X}_v))*^{\text{top}}\pi_1(\Gamma, T))/\overline{\langle \bar{R}_1, \bar{R}_2\rangle^{nc}}\right)^{\text{Noohi}}.$$

The relations $\bar{R}_1$ and $\bar{R}_2$ are analogous to R_1 and R_2, but for Cor. 3.18. applied to $\bar{h}$. Observe that from the fact explained above, that the path $\gamma_{W',W}$ is an image of $\gamma_{W'_{\bar{k}},W_{\bar{k}}}$ and map to the identity in Gal_k, we see in fact that $\alpha_{abc}^{(f)}$'s appearing in R_1 (and so a priori elements of $\pi_1^{\text{ét}}(X_v)$'s) are in fact in $\pi_1^{\text{ét}}(\overline{X}_v)$ and are exactly the same as the analogous $\alpha_{abc}^{(f)}$'s appearing in $\bar{R}_2$. This ultimately gives that $R_2 = \bar{R}_2$. Similarly, we can see that $\bar{R}_1 \subset R_1$ by choosing $g \in \pi_1^{\text{ét}}(W_{\bar{k}}) \subset \pi_1^{\text{ét}}(W)$ in the definition of R_1.
Moreover, the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ can be seen via the above identifications as the Noohi completion of the map

$$(*_v^{\text{top}}\pi_1^{\text{ét}}(\overline{X}_v))*^{\text{top}}\pi_1(\Gamma, T))/\overline{\langle \bar{R}_1, \bar{R}_2\rangle^{nc}} \to (*_v^{\text{top}}(\pi_1^{\text{ét}}(\overline{X}_v) \rtimes \mathrm{Gal}_{k,v}))*^{\text{top}}\pi_1(\Gamma, T))/\overline{\langle R_1, R_2\rangle^{nc}}$$

induced by the maps $\pi_1^{\text{ét}}(\overline{X}_v) \to \pi_1^{\text{ét}}(X)$ and the identity on $\pi_1(\Gamma, T)$ - see Lm. 4.13. We will often abuse the notation and write ∂_i intstead of $\bar{\partial}_i$. This should not lead to confusion as for $W \in \pi_0(S_\bullet(h))$, the map $\pi_1^{\text{ét}}(\partial_i)_{|\pi_1^{\text{ét}}(W_{\bar{k}})}$ is equal (after restricting the range) to $\pi_1^{\text{ét}}(\bar{\partial}_i)$.

By Prop. 2.56, to prove the theorem, it is enough to prove the following statement in terms of the covers: any connected geometric cover Y of $X_{\bar{k}}$ can be dominated by a cover defined over a finite separable extension l/k. Let $S = Y_{\bar{x}}$ be a $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$-set corresponding to Y and fix some point $s_0 \in S$. We divide the proof into steps:

1) To ease the notation, let us denote $G_v = \pi_1^{\text{ét}}(\overline{X}_v)$ and $D = \pi_1(\Gamma, T)$. Recall that $D = \pi_1(\Gamma, T) \simeq \mathrm{Fr}(\vec{e}|e \in \Gamma \setminus T)$. For $e \in \Gamma \setminus T$, denote by F_e the free subgroup generated by $\vec{e}$ in D. It is abstractly ismorphic to $\mathbb{Z}$ and $D \simeq *_e F_e$. At this step we want to define an open subgroup U of $(*_v^{\text{top}}\bar{G}_v *^{\text{top}} D)/\langle \bar{R}_1, \bar{R}_2\rangle^{nc}$ such that the quotient space by U will turn out to be the sought set - this makes sense thanks to Lm. 4.11. We start by defining a subgroup V of $*_v^{\text{top}}\bar{G}_v *^{\text{top}} D$. Let O_v be the set of orbits under G_v contained in S and for $e \in \Gamma \setminus T$, let O_e be the set of orbits under the action of F_e.

Definition 4.15. Let $\omega =$ "$g_1 \ldots g_r$" be an abstract (not necessarily reduced) word in G_v's and $\mathrm{Fr}(\vec{e}|e \in \Gamma \setminus T)$, i.e. each g_i is an element of some G_v or of $\langle \vec{e}\rangle$ for $e \in \Gamma \setminus T$. To be clear: as we consider abstract words, also the trivial elements are allowed to appear. Let us define the *plain* form of ω as the (abstract) word obtained as follows: for all $e \in \Gamma \setminus T$, remove all the trivial elements 1_{F_e} appearing in ω and expand all $\vec{e}^a$ appearing in ω, i.e. replace all $\vec{e}^a$ in ω by $\vec{e}^{\,\mathrm{sgn}(a)}\vec{e}^{\,\mathrm{sgn}(a)} \ldots \vec{e}^{\,\mathrm{sgn}(a)}$ ($|a|$-times). For example, for some $e_1, e_2, e_3 \in \Gamma \setminus T$ (not necessarily different) and $g_v \in G_v, g_w \in G_w$, the plain form of the word "$g_v e_1^2 e_2^{-3} g_w 1_{F_{e_3}}$" would be $g_v\vec{e_1}\vec{e_1}\vec{e_2}^{-1}\vec{e_2}^{-1}\vec{e_2}^{-1}g_w$.
We define the *plain length* of a word ω, denoted by $l_{\mathrm{pl}}(\omega)$, to be the number of letters in the plain form of ω. For example $l_{\mathrm{pl}}($"$g_v\vec{e}_1^2\vec{e}_2^{-3}g_w 1_{F_{e_3}}$"$) = 7$.

Every abstract word ω gives an element $\mathrm{red}(\omega) \in *_v G_v * D$ in an obvious way. We will often abuse the notation and still write ω instead of $\mathrm{red}(\omega)$, especially when talking about the action of $\mathrm{red}(\omega)$ on some $*_v G_v * D$-set. For $N \in \mathbb{N}$, let us define what can be called the "set of G_v-orbits reachable in at most N steps" as

$$O_v^N := \{c \in O_v | \exists_{\text{ abstract word } \omega}(\omega \cdot s_0 \in c) \wedge (l_{\mathrm{pl}}(\omega) \leq N)\}.$$

Define O_e^N in the same way. An easy but crucial observation is that the sets O_v^N, O_e^N are finite. For all $N \in \mathbb{N}$ and each $v \in \{1, \ldots, n\}$, we fix $c_v^N \in G_v \rtimes \mathrm{Gal}_k -$ Sets that is Galois as a G_v-set and for any $c \in O_v^N$ there exists a G_v-surjection $p_{c,v}^N : c_v^N \twoheadrightarrow c$. We can choose such c_v^N thanks to Prop. 4.8. In the case $e \in \Gamma \setminus T$, we let $c_e^N = F_e$. By choosing c_v^N's inductively, we can and do assume moreover that $\forall_N c_v^{N+1} \twoheadrightarrow c_v^N$. For any abstract word $\omega = "g_k \dots g_1"$, denote the index v_j by declaring $g_j \in G_{v_j}$ or $g_j \in F_{v_j}$. Thus, for brevity, we also allow $v_j \in \Gamma \setminus T$. Let us first define an auxiliary notion (on the set of abstract words) of being *looplike*, denoted $\omega \sim \Omega$. Namely, for an abstract word $\omega = "g_M \dots g_1"$,

$$w \sim \Omega \Leftrightarrow \begin{cases} \omega \text{ is in a plain form}, \\ M = 2m+1 \text{ for some number } m, \\ \forall_{1 \le j \le m} v_{m+1+j} = v_{m+1-j}, \\ g_{m+1} \text{ acts trivially on } c_{v_{m+1}}^{N(\omega)}, \\ \forall_{1 \le j \le m} g_{m+1+j} g_{m+1-j} \text{ acts trivially on } c_{v_{m+1-j}}^{N(\omega)}, \end{cases}$$

where $N(\omega)$ is defined as follows. For two vertices $v, w \in \Gamma$, define $\operatorname{dist}(v, w)$ to be the number of edges in the unique path (we neglect that Γ is directed) in T that starts in v, ends in w and does not go two times through any vertex. For an edge $\vec{e}$, let $\operatorname{vert}_-(\vec{e})$ and $\operatorname{vert}_+(\vec{e})$ denote the initial and the terminal vertex of $\vec{e}$ accordingly. For any $g_v \in G_v$, $g_w \in G_w$, $e_1, e_2 \in \Gamma \setminus T$ (we allow $e_1 = e_2$) and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, let

$$\begin{aligned} \operatorname{dist}(g_v, g_w) &= \operatorname{dist}(v, w), \\ \operatorname{dist}(g_v, \vec{e}_1^{\pm}) &= \operatorname{dist}(v, \operatorname{vert}_{\mp}(\vec{e}_1)), \\ \operatorname{dist}(\vec{e}_1^{\pm}, g_v) &= \operatorname{dist}(\operatorname{vert}_{\pm}(\vec{e}_1), v) + 1, \\ \operatorname{dist}(\vec{e}_1^{\epsilon_1}, \vec{e}_2^{\epsilon_2}) &= \operatorname{dist}(\operatorname{vert}_{\operatorname{sgn}(\epsilon_1)}(\vec{e}_1), \operatorname{vert}_{-\operatorname{sgn}(\epsilon_2)}(\vec{e}_2)) + 1. \end{aligned}$$

Observe that this new $\operatorname{dist}(\cdot, \cdot)$ is not symmetric in general. Let now

$$N(\omega) = \sum_{1 \le j \le 2m} \operatorname{dist}(g_{j+1}, g_j) + 1.$$

We will say that an element $g \in *_v G_V * D$ is *looplike*, denoted $g \sim \Omega$, if there exists an abstract looplike word $\omega \sim \Omega$ such that $g = \operatorname{red}(\omega)$. We will call any such ω a *looplike presentation* of g. Now, we define V to be a subgroup of $*_v G_v * D$ generated by $g \in *_v G_v * D$ that are looplike, i.e.

$$V = \langle \{ g \in *_v G_v * D | g \sim \Omega \} \rangle.$$

We want to show that $V < *_v^{\text{top}} G_v *^{\text{top}} D$ is open or, equivalently, that the action given by the multiplication on the left on $S_0' = ((*_v G_v * D)/V, \text{discrete top.})$ is continuous. By the universal property of the free topological product, we can check it separately on each G_v and F_e. In the case of F_e's the continuity is obvious, thus we focus on G_v's. For each $[gV]$, we have to find an open subgroup of G_v inside of $\operatorname{Stab}_{G_v}([gV])$. But $h \in G_v$ satisfies $h \in \operatorname{Stab}_{G_v}([gV]) \Leftrightarrow [hgV] = [gV] \Leftrightarrow [g^{-1}hgV] = [V] \Leftrightarrow g^{-1}hg \in V$. If we write $g = \operatorname{red}(\omega)$ where $\omega = "g_m \dots g_2 g_1"$ is in the plain form, then $g^{-1}hg$ is an image of word whose plain form is $"g_1^{-1} \dots g_m^{-1} h g_m \dots g_2 g_1"$. Thus, we see that $\ker(G_v \to \operatorname{Aut}(c_v^{2mn+1}))$ can be taken as the desired open subgroup of G_v.

Observe now, that $V \subset \operatorname{Stab}_{*_v^{\text{top}} G_v *^{\text{top}} D}(s_0)$, where $s_0 \in S$ was fixed above. Indeed, it follows from the fact that if $\omega \sim \Omega$ and $\omega = "g_{2m+1} g_{2m} \dots g_1"$ in its plain form, then $\operatorname{red}(\omega) \cdot s_0 = g_{2m+1} \dots g_1 \cdot s_0 = g_{2m+1} \dots g_{m+3} g_{m+2} (g_{m+1} \cdot (g_m g_{m-1} \dots g_1 \cdot s_0)) = g_{2m+1} \dots g_{m+3} (g_{m+2} g_m) \cdot (g_{m-1} \dots g_1 \cdot s_0) = g_{2m+1} \dots g_{m+4} (g_{m+3} g_{m-1}) \cdot (g_{m-2} \dots g_1 \cdot s_0) = \dots = s_0$. The last calculation follows from the definition of c_v^M's (i.e. that they dominate suitable G_v-orbits of S), the fact that for $N_1 \ge N_2$ there is $c_v^{N_1} \twoheadrightarrow c_v^{N_2}$ and an important observation that $N(w) \ge l_{\text{pl}}(w)$.

Define $U = \operatorname{im}(V) < (*_v^{\text{top}} G_v *^{\text{top}} D)/\langle \bar{R}_1, \bar{R}_2 \rangle^{nc}$. U is open as an image of an open subgroup via quotient map. Thus, $S' = ((*_v^{\text{top}} G_v *^{\text{top}} D)/\langle \bar{R}_1, \bar{R}_2 \rangle^{nc})/U$ is discrete. The calculation above shows that $U \subset \operatorname{Stab}_{(*_v^{\text{top}} G_v *^{\text{top}} D)/\langle \bar{R}_1, \bar{R}_2 \rangle^{nc}}(s_o)$. Thus, $S' \twoheadrightarrow S$.

2) We want to show that S can be dominated by a connected $\pi_1^{\text{proét}}(X_{\bar{k}})$-set which in turn embeds into $\pi_1^{\text{proét}}(X)$-set. This will follow, if we show that S' is in fact a $\pi_1^{\text{proét}}(X_l)$-set for some finite field extension

l/k. For this we need some additional notation. Let $\vec{E} \in \pi_0(S_1(h))$ be an edge in the graph Γ with initial vertex X_v and terminal vertex $X_w \in \pi_0(S_0(h))$, i.e. we have $\partial_1 : E \to X_v$ and $\partial_0 : E \to X_w$. For any $\sigma \in \mathrm{Gal}_k$, we define $\delta_{\sigma,E}$ as an element of G_v such that σ maps to $\delta_{\sigma,E}\sigma$ via composition $\mathrm{Gal}_k = \pi_1^{\text{ét}}(\mathrm{Spec}(x(E)), \bar{x}(E)) \to \pi_1^{\text{ét}}(E, \bar{x}(E)) \overset{\pi_1^{\text{ét}}(\partial_1)}{\to} \pi_1^{\text{ét}}(X_v, \bar{x}_v) = G_v \rtimes \mathrm{Gal}_{k,v}$. Define also $\theta_{\sigma,E} \in G_w$ such that σ maps to $\theta_{\sigma,E}\sigma$ via $\mathrm{Gal}_k = \pi_1^{\text{ét}}(\mathrm{Spec}(x(E)), \bar{x}(E)) \to \pi_1^{\text{ét}}(E, \bar{x}(E)) \overset{\pi_1^{\text{ét}}(\partial_0)}{\to} \pi_1^{\text{ét}}(X_w, \bar{x}_w) = G_w \rtimes \mathrm{Gal}_{k,w}$. For $E \in T$, we will write $\delta_{\sigma,v\to w}$ and $\theta_{\sigma,v\to w}$ (as there is no confusion possible). Let $\eta_{\sigma,E} = \delta_{\sigma,E}^{-1}\theta_{\sigma,E} \in *_v G_v * D$. If $E \in T$ we will write $\eta_{\sigma,v\to w}$. We extend these notions in the following way: define $\delta_{\sigma,w\to v} = \theta_{\sigma,v\to w}$, $\delta_{\sigma,w\to v} = \theta_{\sigma,v\to w}$ and $\eta_{\sigma,w\to v} = \eta_{\sigma,v\to w}^{-1} = \delta_{\sigma,w\to v}^{-1}\theta_{\sigma,w\to v}$. Now, for v, w arbitrary (i.e. no longer being the initial and terminal vertices of some edge in T), let $v, v_1, \ldots, v_{n-1}, w$ be the unique (undirected) path in T that starts in v, ends in w and does not go two times through any vertex. Then $\eta_{\sigma,v\to w} = \eta_{\sigma,v\to v_1}\eta_{\sigma,v_1\to v_2}\ldots\eta_{\sigma,v_{n-1}\to w}$. The motivation for this definition is that $\sigma_v = \eta_{\sigma,v\to w}\sigma_w$ in $(*_v(G_v \rtimes \mathrm{Gal}_{k,v}) * D)/\langle R_1, R_2\rangle^{nc}$. Let l/k be a finite extension such that for each edge E starting in X_v and ending in X_w and for any $\sigma \in \mathrm{Gal}_l$, $\delta_{\sigma,E}$ acts trivially on c_v^1 and $\theta_{\sigma,E}$ acts trivially on c_w^1. This is possible, as there is only a finite number of edges E and it is easy to see that the maps $\sigma \mapsto \delta_{\sigma,E}$ and $\sigma \mapsto \theta_{\sigma,E}$ are continuous. We base-change to l and assume $l = k$ from now on. This does not affect what we have done so far, because $\pi_1^{\text{proét}}((X_v)_l) = \pi_1^{\text{ét}}((X_v)_l)$ and the sets c_v^N still satisfy the desired conditions when viewed as $G_v \rtimes \mathrm{Gal}_l$-sets. After this base-change, we have that all $\delta_{\sigma,E}$ and $\theta_{\sigma,E}$ act trivially on respective c_v^1's and c_w^1's, from which it follows that $\delta_{\sigma,E} \sim \Omega$ and $\theta_{\sigma,E} \sim \Omega$ and in turn $\eta_{\sigma,v\to w} \in V$ for any v, w and σ.

Let us define an action of $(*_v(G_v \rtimes \mathrm{Gal}_{k,v}) * D)/\langle R_1, R_2\rangle^{nc}$ on S'. It is equivalent to defining an action of $*_v(G_v \rtimes \mathrm{Gal}_{k,v}) * D$ such that the elements of R_1, R_2 act trivially. To give an action of $*_v(G_v \rtimes \mathrm{Gal}_{k,v}) * D$ is to give an action of $(G_v \rtimes \mathrm{Gal}_{k,v})$ and of D separately. The action of D will be simply by multiplication on the left (as $S' = ((*_v^{\text{top}} G_v *^{\text{top}} D)/\langle \bar{R}_1, \bar{R}_2\rangle^{nc})/U$). To give an action of $G_v \rtimes \mathrm{Gal}_{k,v}$, we declare G_v to act by the multiplication on the left and need to define an action of $\mathrm{Gal}_{k,v}$ compatibly with this of G_v.

We first define an action (as an abstract group) $\phi_{v_0} : \mathrm{Gal}_{v_0} \to \mathrm{Aut}_{\mathrm{Grp}}(*_v G_v * D)$ for any v_0. This action is intuitively supposed to be the conjugation in $\pi_1^{\text{proét}}(X)$. For $\sigma = \sigma_{v_0} \in \mathrm{Gal}_{v_0}$ we define

$$\phi_{v_0}(\sigma)_{|G_v} : G_v \ni g_v \mapsto \eta_{\sigma,v_0\to v}{}^{\sigma}g_v\eta_{\sigma,v\to v_0} \in *_v G_v * D$$

and

$$\phi_{v_0}(\sigma)_{|D} : D = \mathrm{Fr}(\vec{e}|e \in E_1\ T) \ni \vec{e}^k \mapsto (\eta_{\sigma,v_0\to\mathrm{vert}_-(\vec{e})}\delta_{\sigma,\vec{e}}^{-1}\vec{e}\theta_{\sigma,\vec{e}}\eta_{\sigma,\mathrm{vert}_+(\vec{e})\to v_0})^k \in *_v G_v * D.$$

Here ${}^{\sigma}g_v$ denotes the conjugation by $\sigma = \sigma_v \in \mathrm{Gal}_{k,v}$ in the group $G_v \rtimes \mathrm{Gal}_{k,v}$. This is immediate to check that, for each σ, $\phi_{v_0}(\sigma)_{|G_v}$ is a group homomorphism for any v and similarly $\phi_{v_0}(\sigma)_{|\langle\vec{e}\rangle}$ for any $\vec{e} \in E_1 \setminus T$, thus $\phi_{v_0}(\sigma)$ gives an endomorphism of $*_v G_v * D$. Let us check that $\phi_{v_0}(\tau) \circ \phi_{v_0}(\sigma) = \phi_{v_0}(\tau\sigma)$. It is enough to check it on each G_v (and on D) separately. Let $v \in \{1, \ldots, n\}$ and $g_v \in G_v$. Then

$$\begin{aligned}
&\phi_{v_0}(\tau)(\phi_{v_0}(\sigma)(g_v))\\
&\quad= \phi_{v_0}(\tau)(\eta_{\sigma,v_0\to v}{}^{\sigma}g_v\eta_{\sigma,v\to v_0})\\
&\quad= \phi_{v_0}(\tau)(\delta_{\sigma,v_0\to v_1}^{-1}\theta_{\sigma,v_0\to v_1}\delta_{\sigma,v_1\to v_2}^{-1}\theta_{\sigma,v_1\to v_2}\ldots\delta_{\sigma,v_m\to v}^{-1}\theta_{\sigma,v_m\to v}{}^{\sigma}g_v\delta_{\sigma,v\to v_m}^{-1}\theta_{\sigma,v\to v_m}\ldots\delta_{\sigma,v_2\to v_1}^{-1}\theta_{\sigma,v_2\to v_1}\delta_{\sigma,v_1\to v_0}^{-1}\theta_{\sigma,v_1\to v_0})\\
&\quad= (\eta_{\tau,v_0\to v_0}{}^{\tau}\delta_{\sigma,v_0\to v_1}^{-1}\eta_{\tau,v_0\to v_0})(\eta_{\tau,v_0\to v_1}{}^{\tau}\theta_{\sigma,v_0\to v_1}\eta_{\tau,v_1\to v_0})(\eta_{\tau,v_0\to v_1}{}^{\tau}\delta_{\sigma,v_1\to v_2}^{-1}\eta_{\tau,v_1\to v_0})(\eta_{\tau,v_0\to v_2}{}^{\tau}\theta_{\sigma,v_1\to v_2}\eta_{\tau,v_2\to v_0})\ldots\\
&\ldots(\eta_{\tau,v_0\to v_m}{}^{\tau}\delta_{\sigma,v_m\to v}^{-1}\eta_{\tau,v_m\to v_0})(\eta_{\tau,v_0\to v}{}^{\tau}\theta_{\sigma,v_m\to v}\eta_{\tau,v\to v_0})(\eta_{\tau,v_0\to v}{}^{\tau}({}^{\sigma}g_v)\eta_{\tau,v\to v_0})(\eta_{\tau,v_0\to v}{}^{\tau}\delta_{\sigma,v\to v_m}^{-1}\eta_{\tau,v\to v_0})(\eta_{\tau,v_0\to v_m}{}^{\tau}\theta_{\sigma,v\to v_m}\eta_{\tau,v_m\to v_0})\ldots\\
&\quad\ldots(\eta_{\tau,v_0\to v_2}{}^{\tau}\delta_{\sigma,v_2\to v_1}^{-1}\eta_{\tau,v_2\to v_0})(\eta_{\tau,v_0\to v_1}{}^{\tau}\theta_{\sigma,v_2\to v_1}\eta_{\tau,v_1\to v_0})(\eta_{\tau,v_0\to v_1}{}^{\tau}\delta_{\sigma,v_1\to v_0}^{-1}\eta_{\tau,v_1\to v_0})(\eta_{\tau,v_0\to v_0}{}^{\tau}\theta_{\sigma,v_1\to v_0}\eta_{\tau,v_0\to v_0})\\
&\quad= {}^{\tau}\delta_{\sigma,v_0\to v_1}^{-1}\eta_{\tau,v_0\to v_1}{}^{\tau}\theta_{\sigma,v_0\to v_1}{}^{\tau}\delta_{\sigma,v_1\to v_2}^{-1}\eta_{\tau,v_1\to v_2}{}^{\tau}\theta_{\sigma,v_1\to v_2}\ldots\\
&\quad\ldots{}^{\tau}\delta_{\sigma,v_m\to v}^{-1}\eta_{\tau,v_m\to v}{}^{\tau}\theta_{\sigma,v_m\to v}{}^{\tau\sigma}g_v{}^{\tau}\delta_{\sigma,v\to v_m}^{-1}\eta_{\tau,v\to v_m}{}^{\tau}\theta_{\sigma,v\to v_m}\ldots\\
&\quad\ldots{}^{\tau}\delta_{\sigma,v_2\to v_1}^{-1}\eta_{\tau,v_2\to v_1}{}^{\tau}\theta_{\sigma,v_2\to v_1}{}^{\tau}\delta_{\sigma,v_1\to v_0}^{-1}\eta_{\tau,v_1\to v_0}{}^{\tau}\theta_{\sigma,v_1\to v_0}\\
&\quad\overset{*}{=} \eta_{\tau\sigma,v_0\to v_1}\eta_{\tau\sigma,v_1\to v_2}\ldots\eta_{\tau\sigma,v_m\to v}{}^{\tau\sigma}g_v\eta_{\tau\sigma,v\to v_m}\ldots\eta_{\tau\sigma,v_2\to v_1}\eta_{\tau,v_1\to v_0}\\
&\quad= \phi_{v_0}(\tau\sigma)(g_v).
\end{aligned}$$

To see that the equality marked with $*$ above holds, it is enough to prove that for an edge $v \to w$, there is

$$ {}^{\tau}\delta^{-1}_{\sigma,v\to w}\eta_{\tau,v\to w}{}^{\tau}\theta_{\sigma,v\to w} = \eta_{\tau\sigma,v\to w}. $$

But this follows from the equality $\delta_{\tau\sigma,v\to w} = \delta_{\tau,v\to w}{}^{\tau}\delta_{\sigma,v\to w}$ (and an analogous equality for θ), which holds by the fact that $\tau\sigma \mapsto (\delta_{\tau,v\to w}\tau)(\delta_{\sigma,v\to w}\sigma) = \delta_{\tau,v\to w}{}^{\tau}\delta_{\sigma,v\to w}\tau\sigma$ via the homomorphism in the definition of δ. The computation for D is similar and we omit it. For $g \in *_v G_v * D$, we will denote $\phi_{v_0}(\sigma)(g)$ by ${}^{\sigma_{v_0}}g$. There will be no confusion, as for $g \in G_{v_0}$ the new notion agrees with the usual action, i.e. ${}^{\sigma_{v_0}}g = {}^{\sigma}g$. It is easy to see that for any v_0, v_1 and g, the following equality holds

$$ {}^{\sigma_{v_1}}g = \eta_{\sigma,v_1\to v_0}{}^{\sigma_{v_0}}g\eta^{-1}_{\sigma,v_1\to v_0} $$

We come back to defining an action of $G_v \rtimes \mathrm{Gal}_{k,v}$ on S'. We define

$$ g_v\sigma_v \cdot \bar{g}U = \overline{g_v{}^{\sigma_v}g}U, $$

where $\bar{g}$ denotes the image of $g \in *_v G_v * D$ in $(*_v G_v * D)/\langle \bar{R}_1, \bar{R}_2\rangle^{nc}$. We have to prove that this definition is independent of the choice of the representative of the coset and of its lift to $*_v G_v * D$. To see this first independence, let $u \in V < *_v G_v * D$ (recall that $U = \mathrm{im}(V)$) and write $u = u_1 u_2 \ldots u_t$ with $u_i \sim \Omega$. We have $\bar{g}U = \overline{gu}U$ and we see that it is enough to check that ${}^{\sigma_v}u \in V$. For this it is enough to check that ${}^{\sigma_v}u_i \in V$. Write $u_i = \mathrm{red}(\omega)$ with $\omega =$ "$g_1 \ldots g_m g_{m+1} g'_m \ldots g'_1$" $\sim \Omega$. From the definition of being looplike, we have that $g_j g'_j \in G_{v_j}$ acts trivially on $c^N_{v_j}$ for $j = 1, \ldots m$ and g_{m+1} acts trivially on $c^N_{v_{m+1}}$, where $N = N(u_i)$ was defined above. We have

$$ {}^{\sigma_v}u_i = \eta_{\sigma,v\to v_1}{}^{\sigma}g_1\eta_{\sigma,v_1\to v_2}{}^{\sigma}g_2 \ldots \eta_{\sigma,v_m\to v_{m+1}}{}^{\sigma}g_{m+1}\eta_{\sigma,v_{m+1}\to v_m} \ldots {}^{\sigma}g'_2\eta_{\sigma,v_2\to v_1}{}^{\sigma}g'_1\eta_{\sigma,v_1\to v}. $$

In this computation, we have assumed that all $v_j \neq e$. This is just to make the exposition simpler, but it is easy to check that the following argument still works if some $(\vec{e})^l$ appears in w. We arranged at the beginning so that each $\delta_{\sigma,\vec{e}}$ and $\theta_{\sigma,\vec{e}}$ act trivially on the suitable c^1_w's, which shows that $\eta_{\sigma,v\to v_1}$ (and $\eta_{\sigma,v_1\to v}$) are products of elements $\sim \Omega$ and so belong to V. Thus, it is enough to show that
$a := {}^{\sigma}g_1\eta_{\sigma,v_1\to v_2}{}^{\sigma}g_2 \ldots \eta_{\sigma,v_m\to v_{m+1}}{}^{\sigma}g_{m+1}\eta_{\sigma,v_{m+1}\to v_m} \ldots {}^{\sigma}g'_2\eta_{\sigma,v_2\to v_1}{}^{\sigma}g'_1 \in V$. Let ω' be a lift of the element a to an abstract word (i.e. $a = \mathrm{red}(\omega')$) defined in the following way:

$$ \omega' = \text{"}({}^{\sigma}g_1\delta^{-1}_{\sigma,v_1\to v_{i_1}})(\theta_{\sigma,v_1\to v_{i_1}}\delta^{-1}_{\sigma,v_{i_1}\to v_{i_2}})(\theta_{\sigma,v_{i_1}\to v_{i_2}}\delta^{-1}_{\sigma,v_{i_2}\to v_{i_3}}) \ldots (\theta_{\sigma,v_{i_r}\to v_2}{}^{\sigma}g_2\delta^{-1}_{\sigma,v_2\to v_{j_1}})(\theta_{\sigma,v_2\to v_{j_1}}\delta^{-1}_{\sigma,v_{j_1}\to v_{j_2}}) \ldots\text{"}, $$

where $v_1 \to v_{i_1} \to \ldots \to v_{i_r} \to v_2$ and $v_2 \to v_{j_1} \to \ldots \to v_3$ are the unique simple paths in T joining v_1 with v_2 and v_2 with v_3 respectively. In each of the parentheses, the elements belong to a single group, i.e. G_{v_1}, $G_{v_{i_1}}$ and so on and we view the product in each pair of parentheses (even if it is equal to the trivial element 1_{G_v}) as a single letter in the word ω'. Again, the computation was made in the case when the $\vec{e}$'s do not appear in ω, but it is easy to see how to extend it to a general case and also that ω' will remain plain.

Main observation now, that follows directly from the definition of being looplike and of ω', is that

$$ N(\omega') = N(\omega). $$

It follows that $\omega' \sim \Omega$. Indeed, it boils down to checking that:

- $(\theta^{\sigma}g_j\delta^{-1})(\delta^{\sigma}g'_j\theta^{-1}) = \theta^{\sigma}g_j{}^{\sigma}g'_j\theta^{-1}$ acts trivially on $c^{N(\omega')}_{v_j}$. This is true, because $g_j g'_j$ acts trivially on $c^{N(\omega)}_{v_j}$ and $c^{N(\omega)}_{v_j}$ is in fact a $G_{v_j} \rtimes \mathrm{Gal}_{k,v_j}$-set, thus ${}^{\sigma}(g_j g'_j) = \sigma(g_j g'_j)\sigma^{-1}$ acts trivially on $c^{N(\omega)}_{v_j}$. Similarly for ${}^{\sigma}g_{m+1}$. Here, it is crucial that N did not increase, i.e. that $N(\omega') = N(\omega)$.

- various $(\theta_{\sigma,v_{i_j}\to v_{i_{j+1}}}\delta^{-1}_{\sigma,v_{i_{j+1}}\to v_{i_{j+2}}})(\theta_{\sigma,v_{i_{j+2}}\to v_{i_{j+1}}}\delta^{-1}_{\sigma,v_{i_{j+1}}\to v_{i_j}})$ act trivially on $c^{N'}_v$. But this is automatic, as $\delta_{\sigma,w\to v} = \theta^{-1}_{\sigma,v\to w}$ by definition.

We still have to show the independence of the action of σ_v from the choice of the lift of $\bar{g}$ to $*_v G_v * D$. If $g \in *_v G_v * D$ is one such lift, then any other lift is of the form $g h_1 r_1^{\pm 1} h_1^{-1} \dots h_t r_t^{\pm 1} h_t^{-1}$, where $h_i \in *_v G_v * D$ and $r_i \in \bar{R}_1 \cup \bar{R}_2$. To check the independence of the lift, we first show that if $r \in \bar{R}_1$, then (for any v) ${}^{\sigma_v} r_i$ is a conjugate of the element in $\bar{R}_1$. This is indeed the case, as shown by the following computation. Let $r = \pi_1^{\text{ét}}(\partial_1)(h)\vec{e}\pi_1^{\text{ét}}(\partial_0)(h)^{-1}\vec{e}^{-1}$. Then

$$\begin{aligned}
{}^{\sigma_v} r &= \eta_{\sigma, v \to \mathrm{vert}_-(\vec{e})} {}^{\sigma}\left(\pi_1^{\text{ét}}(\partial_1)(h)\right)\delta_{\sigma,\vec{e}}^{-1}\vec{e}\theta_{\sigma,\vec{e}} {}^{\sigma}\left(\pi_1^{\text{ét}}(\partial_0)(h)\right)^{-1}\theta_{\sigma,\vec{e}}^{-1}\vec{e}^{-1}\delta_{\sigma,\vec{e}}\eta_{\sigma,\mathrm{vert}_-(\vec{e})\to v} \\
&= \eta_{\sigma, v \to \mathrm{vert}_-(\vec{e})}\delta_{\sigma,\vec{e}}^{-1}\Big(\delta_{\sigma,\vec{e}} {}^{\sigma}\left(\pi_1^{\text{ét}}(\partial_1)(h)\right)\delta_{\sigma,\vec{e}}^{-1}\vec{e}\theta_{\sigma,\vec{e}} {}^{\sigma}\left(\pi_1^{\text{ét}}(\partial_0)(h)\right)^{-1}\theta_{\sigma,\vec{e}}^{-1}\vec{e}^{-1}\Big)\delta_{\sigma,\vec{e}}\eta_{\sigma, v \to \mathrm{vert}_-(\vec{e})}^{-1} \\
&= \eta_{\sigma, v \to \mathrm{vert}_-(\vec{e})}\delta_{\sigma,\vec{e}}^{-1}\Big(\pi_1^{\text{ét}}(\partial_1)({}^{\sigma}h)\vec{e}(\pi_1^{\text{ét}}(\partial_0)({}^{\sigma}h))^{-1}\vec{e}^{-1}\Big)\delta_{\sigma,\vec{e}}\eta_{\sigma, v \to \mathrm{vert}_-(\vec{e})}^{-1}.
\end{aligned}$$

In the above computation we have used that

$$\delta_{\sigma,\vec{e}} {}^{\sigma}\left(\pi_1^{\text{ét}}(\partial_1)(h)\right)\delta_{\sigma,\vec{e}}^{-1} = \pi_1^{\text{ét}}(\partial_1)({}^{\sigma}h) \text{ and } \theta_{\sigma,\vec{e}} {}^{\sigma}\left(\pi_1^{\text{ét}}(\partial_0)(h)\right)^{-1}\theta_{\sigma,\vec{e}}^{-1} = (\pi_1^{\text{ét}}(\partial_0)({}^{\sigma}h))^{-1}.$$

This is because

$$\pi_1^{\text{ét}}(\partial_0) : \pi_1^{\text{proét}}(e) \rtimes \mathrm{Gal}_k \ni {}^{\sigma}h = \sigma h \sigma^{-1} \mapsto \delta_{\sigma,e}\sigma\pi_1^{\text{ét}}(\partial_0)(h)\sigma^{-1}\delta_{\sigma,e}^{-1} = \delta_{\sigma,e} {}^{\sigma}\left(\pi_1^{\text{ét}}(\partial_0)(h)\right)\delta_{\sigma,e}^{-1}$$

and similarly for ∂_0 and $\theta_{\sigma,e}$. The proof of the independence of the lift will be finished, if we can show that, for $r \in \bar{R}_2$, ${}^{\sigma_v} r$ is a conjugate of an element of $\bar{R}_2$ modulo $\langle \bar{R}_1 \rangle^{nc}$. Let

$$r = \overrightarrow{(\partial_2 f)}\alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1}\overrightarrow{(\partial_0 f)}\alpha_{210}^{(f)}(\alpha_{201}^{(f)})^{-1}\left(\overrightarrow{(\partial_1 f)}\right)^{-1}\alpha_{021}^{(f)}(\alpha_{012}^{(f)})^{-1} \in R_2$$

for some face f. Denote $\eta = \eta_{\sigma, v \to \mathrm{vert}_-\left(\overrightarrow{\partial_2 f}\right)}$. Let us compute

$$\begin{aligned}
&{}^{\sigma_v}\left(\overrightarrow{(\partial_2 f)}\alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1}\overrightarrow{(\partial_0 f)}\alpha_{210}^{(f)}(\alpha_{201}^{(f)})^{-1}\left(\overrightarrow{(\partial_1 f)}\right)^{-1}\alpha_{021}^{(f)}(\alpha_{012}^{(f)})^{-1}\right) \\
&\quad = \eta\delta_{\sigma,\overrightarrow{\partial_2 f}}^{-1}\overrightarrow{(\partial_2 f)}\theta_{\sigma,\overrightarrow{\partial_2 f}} {}^{\sigma}\alpha_{102}^{(f)} {}^{\sigma}(\alpha_{120}^{(f)})^{-1}\delta_{\sigma,\overrightarrow{\partial_0 f}}^{-1}\overrightarrow{(\partial_0 f)}\theta_{\sigma,\overrightarrow{\partial_0 f}} {}^{\sigma}\alpha_{210}^{(f)} {}^{\sigma}(\alpha_{201}^{(f)})^{-1}\theta_{\sigma,\overrightarrow{\partial_1 f}}^{-1}\overrightarrow{(\partial_1 f)}^{-1}\delta_{\sigma,\overrightarrow{\partial_1 f}} {}^{\sigma}\alpha_{021}^{(f)} {}^{\sigma}(\alpha_{012}^{(f)})^{-1}\eta^{-1}
\end{aligned}$$

The proof would be over, if we could show that $\theta_{\sigma,\overrightarrow{\partial_2 f}} {}^{\sigma}\alpha_{102}^{(f)} {}^{\sigma}(\alpha_{120}^{(f)})^{-1}\delta_{\sigma,\overrightarrow{\partial_0 f}}^{-1}$ is equal to $\alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1}$ and similarly for other α's and δ's. It turns out that a slightly more complicated equality is true, but still enables us to conclude. Define $\gamma_i = \gamma_{\sigma,i}$ as the element of $\pi_1^{\text{ét}}((\partial_i f)_{\bar{k}})$ such that $\sigma \mapsto \gamma_{\sigma,i}\sigma$ via $\pi_1^{\text{ét}}(\partial_i) : \pi_1^{\text{ét}}(f_{\bar{k}}) \rtimes \mathrm{Gal}_{k,f} = \pi_1^{\text{ét}}(f) \to \pi_1^{\text{ét}}(\partial_i f) = \pi_1^{\text{ét}}((\partial_i f)_{\bar{k}}) \rtimes \mathrm{Gal}_{k,\partial_i f}$. Denote by v_1 the vertex that is terminal for $\partial_2 f$ and initial for $\partial_0 f$. Denote also by ψ the map $\pi_1^{\text{ét}}(f) \to \pi_1^{\text{ét}}(v_1)$. By the definition of α's, we have:

$$\alpha_{102}\psi(\sigma)\alpha_{102}^{-1} = \pi_1^{\text{ét}}(\partial_0)(\pi_1^{\text{ét}}(\partial_2)(\sigma)) \text{ and } \alpha_{120}\psi(\sigma)\alpha_{120}^{-1} = \pi_1^{\text{ét}}(\partial_1)(\pi_1^{\text{ét}}(\partial_0)(\sigma))$$

which gives

$$\alpha_{102}\psi(\sigma)\alpha_{102}^{-1} = \pi_1^{\text{ét}}(\partial_0)(\gamma_2)\theta_{\sigma,\partial_2 f}\sigma \text{ and } \alpha_{120}\psi(\sigma)\alpha_{120}^{-1} = \pi_1^{\text{ét}}(\partial_1)(\gamma_0)\delta_{\sigma,\partial_0 f}\sigma$$

or equivalently,

$$\alpha_{102}\psi(\sigma) = \pi_1^{\text{ét}}(\partial_0)(\gamma_2)\theta_{\sigma,\partial_2 f}\sigma\alpha_{102} \text{ and } \alpha_{120}\psi(\sigma) = \pi_1^{\text{ét}}(\partial_1)(\gamma_0)\delta_{\sigma,\partial_0 f}\sigma\alpha_{120}$$

from which we get (multiplying the left equation by the inverse of the right, side by side)

$$\alpha_{102}\alpha_{120}^{-1} = \pi_1^{\text{ét}}(\partial_0)(\gamma_2)\theta_{\sigma,\partial_2 f}\sigma\alpha_{102}\alpha_{120}^{-1}\sigma^{-1}\delta_{\sigma,\partial_0 f}^{-1}\pi_1^{\text{ét}}(\partial_1)(\gamma_0)^{-1}$$

i.e.

$$\alpha_{102}\alpha_{120}^{-1} = \pi_1^{\text{ét}}(\partial_0)(\gamma_2)\theta_{\sigma,\partial_2 f} {}^{\sigma}\alpha_{102} {}^{\sigma}\alpha_{120}^{-1}\delta_{\sigma,\partial_0 f}^{-1}\pi_1^{\text{ét}}(\partial_1)(\gamma_0)^{-1}$$

We have analogous equalities for the remaining α's and δ's:

$$\alpha_{210}\alpha_{201}^{-1} = \pi_1^{\text{ét}}(\partial_0)(\gamma_0)\theta_{\sigma,\partial_0 f}{}^{\sigma}\alpha_{210}{}^{\sigma}\alpha_{201}^{-1}\theta_{\sigma,\partial_1 f}^{-1}\pi_1^{\text{ét}}(\partial_0)(\gamma_1)^{-1}$$
$$\alpha_{021}\alpha_{012}^{-1} = \pi_1^{\text{ét}}(\partial_1)(\gamma_1)\delta_{\sigma,\partial_1 f}{}^{\sigma}\alpha_{021}{}^{\sigma}\alpha_{012}^{-1}\delta_{\sigma,\partial_2 f}^{-1}\pi_1^{\text{ét}}(\partial_1)(\gamma_2)^{-1}$$

This is basically the equality we wanted to get, but with the additional factors of $\pi_1^{\text{ét}}(\partial_0)(\gamma_2)$ and $\pi_1^{\text{ét}}(\partial_1)(\gamma_0)^{-1}$. But observe that

$$\overline{\overrightarrow{\partial_i f}} = \overline{\pi_1^{\text{ét}}(\partial_1)(\gamma_i)^{-1}\overrightarrow{(\partial_i f)}\pi_1^{\text{ét}}(\partial_0)(\gamma_i)}$$

in $(*_v G_v * D)/\langle \bar{R}_1\rangle^{nc}$. Putting this together, we see that

$$\overline{{}^{\sigma}r} = \overline{hrh^{-1}}$$

in $(*_v G_v * D)/\langle \bar{R}_1\rangle^{nc}$. More precisely, we can take $h = \pi_1^{\text{ét}}(\partial_1)(\gamma_2)\delta_{\sigma,\partial_2 f}\eta^{-1}$.

We have defined an action of $G_v \rtimes \mathrm{Gal}_{k,v}$ on S'. We want to show that it is continuous. It is enough to check it separately on G_v and $\mathrm{Gal}_{k,v}$ (this follows from the description of the topological semi-direct product as a quotient of the free topological product, see Lm. 3.25). It is clear in the case of G_v, as it is just (the composition of $G_v \to (*_v G_v * D)/\langle \bar{R}_1, \bar{R}_2\rangle^{nc}$ and) multiplication on the left (on $[U]$). Thus, we focus on $\mathrm{Gal}_{k,v}$. We see that that $\sigma \in \mathrm{Stab}_{\mathrm{Gal}_{k,v}}([\bar{g}U]) \Leftrightarrow \bar{g}^{-1}(\overline{{}^{\sigma}g}) \in U$. Thus, for a fixed g, we want to check that there is an open subgroup of σ's in $\mathrm{Gal}_{k,v}$ that satisfies $\bar{g}^{-1}(\overline{{}^{\sigma}g}) \in U$. Writing g in a plain form $g = \mathrm{red}(\text{"}g_1 \dots g_m\text{"})$ we have (in the computation below we assume $\forall_i g_i \notin D$ for clarity of exposition, but virtually the same proof works in full generality)

$$\begin{aligned} g^{-1}{}^{\sigma}g &= g_m^{-1}\dots g_1^{-1}\eta_{\sigma,v\to v_1}{}^{\sigma}g_1\eta_{\sigma,v_1\to v_2}\dots{}^{\sigma}g_m\eta_{\sigma,v_m\to v} \\ &= (g_m^{-1}\dots(g_1^{-1}({}^{\sigma}g_1))\dots{}^{\sigma}g_m)(h_{\sigma,1}\eta_{\sigma,v\to v_1}h_{\sigma,1}^{-1})\dots(h_{\sigma,m}\eta_{\sigma,v_{m-1}\to v_m}h_{\sigma,m}^{-1})\eta_{\sigma,v_m\to v}, \end{aligned}$$

where $h_{\sigma,i} = {}^{\sigma}g_m^{-1}{}^{\sigma}g_{m-1}^{-1}\dots{}^{\sigma}g_i^{-1}$. Observe that, for any σ, $N(\text{"}g_m^{-1}\dots g_1^{-1}{}^{\sigma}g_1\dots{}^{\sigma}g_m\text{"})$ (notation as in the definition of V) is bounded by $M = 2\sum_{1\le j\le m-1}\mathrm{dist}(g_j, g_{j+1}) + 1$ and thus independently of σ. So, if $\sigma \in \cap_{1\le j\le m}\ker(\mathrm{Gal}_k \to \mathrm{Aut}(c_{i_j}^M))$, then for any $1 \le j \le m$ and $t_0 \in c_{v_j}^M$, there is $g_j^{-1}({}^{\sigma}g_j)\cdot t_0 = g_j^{-1}\cdot(\sigma\cdot(g_j\cdot(\sigma^{-1}t_0))) = g_j^{-1}\cdot(\sigma\cdot(g_j\cdot t_0)) = g_j^{-1}\cdot(g_j\cdot t_0) = t_0$. From this we see that for such σ, $(g_1^{-1}\dots(g_k^{-1}({}^{\sigma}g_k))\dots{}^{\sigma}g_1) \sim \Omega$. We also know that $\eta_{\sigma,v_m\to v} \in V$. We have to take care of the elements $(h_{\sigma,i}\eta_{\sigma,v_{i-1}\to v_i}h_{\sigma,i}^{-1})$. We write $\eta_{\sigma,v_{i-1}\to v_i}$ as the product of suitable $\delta_{\sigma,w\to w'}^{-1}$'s and $\theta_{\sigma,w\to w'}$'s and see, that it suffices to show that (for small σ's) there is $(h_{\sigma,i}\delta_{\sigma,w\to w'}h_{\sigma,i}^{-1}) \sim \Omega$ for any i and edge $w \to w'$ (and similarly for θ's). But we can argue as before: we easily bound $N(\text{"}{}^{\sigma}g_m^{-1}{}^{\sigma}g_{m-1}^{-1}\dots{}^{\sigma}g_i^{-1}\delta_{\sigma,w\to w'}{}^{\sigma}g_i\dots{}^{\sigma}g_{m-1}{}^{\sigma}g_m\text{"})$ independently of σ by some M' and, as $\sigma \mapsto \delta_{\sigma,w\to w'}$ is continuous (as already explained), for small σ we have that $\delta_{\sigma,w\to w'}$ acts trivially on $c_w^{M'}$ which is the only non-trivial step in checking that $\text{"}{}^{\sigma}g_m^{-1}{}^{\sigma}g_{m-1}^{-1}\dots{}^{\sigma}g_i^{-1}\delta_{\sigma,w\to w'}{}^{\sigma}g_i\dots{}^{\sigma}g_{m-1}{}^{\sigma}g_m\text{"} \sim \Omega$. This finishes the proof of the continuity of $\mathrm{Gal}_{k,v}$ action.

So far, we have obtained a continuous action of $*_v^{\mathrm{top}}(G_v \rtimes \mathrm{Gal}_{k,v}) *^{\mathrm{top}} D$ on S'. If we prove that the elements of R_1 and R_2 act trivially on S', we will obtain that the group $(*_v^{\mathrm{top}}(G_v \rtimes \mathrm{Gal}_{k,v}) *^{\mathrm{top}} D)/\overline{\langle R_1, R_2\rangle^{nc}}$ acts continuously on S'. As $R_2 = \bar{R}_2$, it is clear that that elements of R_2 act trivially on S' by the very definition of

S'. Let now $\pi_1^{\text{ét}}(\partial_1)(g\sigma)\vec{E}\pi_1^{\text{ét}}(\partial_0)(g\sigma)^{-1}\vec{E}^{-1} \in R_1$ with $g \in \pi_1^{\text{ét}}(E_{\bar{k}})$. Then

$$\begin{aligned}
&\pi_1^{\text{ét}}(\partial_1)(g\sigma)\vec{E}\pi_1^{\text{ét}}(\partial_0)(g\sigma)^{-1}\vec{E}^{-1} \cdot \bar{h}U \\
&\qquad = \pi_1^{\text{ét}}(\partial_1)(g)\pi_1^{\text{ét}}(\partial_1)(\sigma)\vec{E}\pi_1^{\text{ét}}(\partial_0)(\sigma)^{-1}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1} \cdot \bar{h}U \\
&\qquad = \pi_1^{\text{ét}}(\partial_1)(g)\delta_{\sigma,\vec{E}}\sigma_{\mathrm{vert}_-(\vec{E})}\vec{E}\sigma^{-1}_{\mathrm{vert}_+(\vec{E})}\theta^{-1}_{\sigma,\vec{E}}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1} \cdot \bar{h}U \\
&\qquad = \pi_1^{\text{ét}}(\partial_1)(g)\delta_{\sigma,\vec{E}}\sigma_{\mathrm{vert}_-(\vec{E})}\vec{E} \cdot \overline{{}^{\sigma^{-1}_{\mathrm{vert}_+(\vec{E})}}(\theta^{-1}_{\sigma,\vec{E}}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1}h)U} \\
&\qquad = \pi_1^{\text{ét}}(\partial_1)(g)\delta_{\sigma,\vec{E}} \cdot \overline{{}^{\sigma_{\mathrm{vert}_-(\vec{E})}}(\vec{E}\,{}^{\sigma^{-1}_{\mathrm{vert}_+(\vec{E})}}(\theta^{-1}_{\sigma,\vec{E}}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1}h))U} \\
&\qquad = \pi_1^{\text{ét}}(\partial_1)(g)\delta_{\sigma,\vec{E}} \cdot \overline{\delta^{-1}_{\sigma,\vec{E}}\vec{E}\theta_{\sigma,\vec{E}}\eta_{\sigma,\mathrm{vert}_+(\vec{E})\to\mathrm{vert}_-(\vec{E})}{}^{\sigma_{\mathrm{vert}_-(\vec{E})}}({}^{\sigma^{-1}_{\mathrm{vert}_+(\vec{E})}}(\theta^{-1}_{\sigma,\vec{E}}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1}h))U} \\
&\qquad \overset{*}{=} \pi_1^{\text{ét}}(\partial_1)(g)\delta_{\sigma,\vec{E}} \cdot \overline{\delta^{-1}_{\sigma,\vec{E}}\vec{E}\theta_{\sigma,\vec{E}}{}^{\sigma_{\mathrm{vert}_+(\vec{E})}}({}^{\sigma^{-1}_{\mathrm{vert}_+(\vec{E})}}(\theta^{-1}_{\sigma,\vec{E}}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1}h))U} \\
&\qquad = \overline{\pi_1^{\text{ét}}(\partial_1)(g)\vec{E}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1}h}U \\
&\qquad = \bar{h}U.
\end{aligned}$$

The last equality is due to $\pi_1^{\text{ét}}(\partial_1)(g)\vec{E}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{E}^{-1} \in \bar{R}_1$. The equality "*" is due to

$$\eta_{\sigma,\mathrm{vert}_+(\vec{E})\to\mathrm{vert}_-(\vec{E})}{}^{\sigma_{\mathrm{vert}_-(\vec{E})}}q\,\eta^{-1}_{\sigma,\mathrm{vert}_+(\vec{E})\to\mathrm{vert}_-(\vec{E})} = {}^{\sigma_{\mathrm{vert}_+(\vec{E})}}q$$

for any q and the the fact that $\eta^{-1}_{\sigma,\mathrm{vert}_+(\vec{E})\to\mathrm{vert}_-(\vec{E})}V = V$.

Observe that the just defined action of $*_v(G_v \rtimes \mathrm{Gal}_{k,v}) * D$ on S' is compatible with the natural map $*_v G_v * D \to *_v(G_v \rtimes \mathrm{Gal}_{k,v}) * D$ and in turn the action of $\pi_1^{\text{proét}}(X_{\bar{k}})$ on S' comes via pullback of the action of $\pi_1^{\text{proét}}(X)$ on S' via $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$. This finishes the entire proof. □

As a corollary we obtain the following.

Theorem 4.16. *With the assumptions as in Thm. 4.2, the sequence of abstract groups*

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

is exact.

Moreover, the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding and the map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a quotient map of topological groups.

Proof. We already know the statements of the "moreover" part and the weak exactness in the middle of the sequence. All we have to prove is that $\pi_1^{\text{proét}}(X_{\bar{k}})$ is thickly closed in $\pi_1^{\text{proét}}(X)$. As $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding of Noohi groups and as Noohi groups are Raîkov complete, $\pi_1^{\text{proét}}(X_{\bar{k}})$ is a closed subgroup of $\pi_1^{\text{proét}}(X)$. By Lm. 2.67, the proof will be finished if we can show that $\pi_1^{\text{proét}}(X_{\bar{k}})$ is normal as a subgroup of $\pi_1^{\text{proét}}(X)$. Observe that checking whether $\overline{\pi_1^{\text{proét}}(X_{\bar{k}})} = \overline{\overline{\pi_1^{\text{proét}}(X_{\bar{k}})}}$ can be performed after replacing $\pi_1^{\text{proét}}(X)$ by any open subgroup U such that $\pi_1^{\text{proét}}(X_{\bar{k}}) < U <^\circ \pi_1^{\text{proét}}(X)$. Choosing a suitably large finite field extension l/k and looking at $U = \pi_1^{\text{proét}}(X_l)$, we can reduce to the situation as in the proof of Thm. 4.14, i.e. the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ can be seen via the identifications of the proof of Thm. 4.14 as the Noohi completion of the map

$$(*_v^{\mathrm{top}}\pi_1^{\text{ét}}(\overline{X}_v)) *^{\mathrm{top}} \pi_1(\Gamma, T))/\overline{\langle R_1, \overline{R}_2\rangle^{nc}} \xrightarrow{\psi} (*_v^{\mathrm{top}}(\pi_1^{\text{ét}}(\overline{X}_v) \rtimes \mathrm{Gal}_{k,v})) *^{\mathrm{top}} \pi_1(\Gamma, T))/\overline{\langle R_1, R_2\rangle^{nc}}$$

Using the fact that $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding and the fact that the image of the natural map from a topological group to its Noohi completion is dense (Lm. 2.49), we see that it is enough to check that the map ψ above has normal image. The only non-trivial thing to check is that conjugating by $\sigma_v \in \mathrm{Gal}_{k,v}$ preserves the image of ψ. But for $g \in (*_v^{\mathrm{top}} \pi_1^{\text{ét}}(\overline{X}_v)) *^{\mathrm{top}} \pi_1(\Gamma, T)$, we see that $\sigma_v \psi(\bar{g}) \sigma_v^{-1} = \psi(\overline{{}^{\sigma_v} g})$, where ${}^{\sigma_v} g = \phi_v(g)$ is the abstract action of $\mathrm{Gal}_{k,v}$ on $(*_v^{\mathrm{top}} \pi_1^{\text{ét}}(\overline{X}_v)) *^{\mathrm{top}} \pi_1(\Gamma, T)$ defined in the proof of Thm. 4.14 (and basically forced by the relations R_1, R_2). This finishes the proof. □

Chapter 5

Homotopy exact sequence over a general base

5.1 Statement of the main result, infinite Stein factorization and some examples

Let us state the aim of this chapter. As we will soon see, the main ingredient of the proof will be the construction of the "infinite Stein factorization" for geometric covers of Thm. 5.5 below.

Theorem 5.1. *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume that S is Nagata and connected. Let $\bar{s}$ be a geometric point of S. Let $\bar{x}$ be a geometric point on $X_{\bar{s}}$. Then the sequence induced on the pro-étale fundamental groups*

$$\pi_1^{\text{proét}}(X_{\bar{s}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(S, \bar{s}) \to 1$$

is weakly exact (see Defn. 2.65).

Moreover, the induced map

$$\left(\pi_1^{\text{proét}}(X, \bar{x}) / \overline{\overline{\overline{\mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{s}}, \bar{x}))}}}\right)^{\text{Noohi}} \to \pi_1^{\text{proét}}(S, \bar{s})$$

is a homeomorphism. Here, $\overline{\overline{\overline{H}}}$ denotes the "thick closure" introduced in Defn. 2.58.

We will usually omit the base points.

Remark 5.2. An example that "weakly exact" is needed in the statement (i.e. we have to look at thick closures): let R be a complete dvr with algebraically closed residue field k. Denote by K the field of fractions of R. Let X be a normal scheme proper over R such that X_K is an elliptic curve and X_k is a nodal curve. For example, one can take $R = \mathbb{C}[[t]]$ and $X = \{ZY^2 = W^3 + ZW^2 + Z^3 t\} \subset \mathbb{P}^2_R$. It is a normal scheme. To see this, one can use Serre's criterion: R_1 follows from smoothness of the generic fibre and at the generic point of the special fibre. As $\mathrm{Spec}(R)$ and the fibres of $X \to \mathrm{Spec}(R)$ satisfy property S_2, one can use the results of [Mat, §23] to check that X is S_2. We then have the following diagram

$$\begin{array}{ccccc}
\pi_1^{\text{proét}}(X_k) & \longrightarrow & \pi_1^{\text{proét}}(X) & \longrightarrow & \pi_1^{\text{proét}}(\mathrm{Spec}(R)) \\
\downarrow & & \| & & \| \\
\pi_1^{\text{ét}}(X_k) & \xrightarrow{\sim} & \pi_1^{\text{ét}}(X) & \longrightarrow & \pi_1^{\text{ét}}(\mathrm{Spec}(R)) = 0
\end{array}$$

where the equality $\pi_1^{\text{ét}}(\mathrm{Spec}(R)) = 0$ and the fact that $\pi_1^{\text{ét}}(X_k) \to \pi_1^{\text{ét}}(X)$ is an isomorphism follows from [SGA 1, Exp. X, Théorème 2.1] and the equality $\pi_1^{\text{proét}}(X) = \pi_1^{\text{ét}}(X)$ follows from the normality of the scheme X. On the other hand, we know that $\pi_1^{\text{proét}}(X_k) = \mathbb{Z}$ and so $\pi_1^{\text{ét}}(X) = \pi_1^{\text{ét}}(X_k) = \hat{\mathbb{Z}}$. So $\pi_1^{\text{proét}}(X_k) \to \pi_1^{\text{proét}}(X)$ is equal to the inclusion $\mathbb{Z} \to \hat{\mathbb{Z}}$ and so it is not surjective. This contrasts with the fact that $\pi_1^{\text{proét}}(\mathrm{Spec}(R)) = 0$.

Remark 5.3. If G is a Noohi group and $H = \overline{H} \lhd G$ is its closed normal subgroup, then the quotient group G/H still has the property that open subgroups form a basis of open neighbourhoods of the identity. This is because the image of some basis of neighbourhoods of the identity of a Hausdorff group via quotient map is a basis of neighbourhoods of 1 in the quotient group (see [Bou, §3.2, Prop. 17]). However, the quotient G/H might still fail to be complete and thus Noohi. The counterexample can be obtained by applying (the proof of) [RD, Prop. 11.1]) (which gives a way of producing many examples of complete groups with non-complete quotients) to a non-complete abelian group whose open subgroups form a basis of neigbourhoods of 1, e.g. $(\mathbb{Z}_{(p)}, +)$ with p-adic topology. More concretely, one can take $G = \oplus_{i\in\mathbb{N}}\mathbb{Z}_{(p)}$ with a topology induced from $(\prod_{i\in\mathbb{N}} \mathbb{Z}_{(p)}, \tau)$, where τ is generated by the sets of the form $\{x_1\} \times \ldots \times \{x_m\} \times U_{m+1} \times U_{m+2} \times \ldots$, where m varies and U_j are open subsets of $\mathbb{Z}_{(p)}$. We do not require almost all U_j to be full $\mathbb{Z}_{(p)}$ and thus the topology τ is much stronger than the usual product topology. One checks that in G open subgroups still form a basis of neighbourhoods of 1 and, moreover, G is complete. One then defines $\epsilon : G \to \mathbb{Z}_{(p)}$ to be $\epsilon((x_i)) = \sum_i x_i$. One checks that it is continuous, surjective and open and so a quotient map. Taking $H = \ker(\epsilon)$ finishes the example.

Observe that any basis of open neighbourhoods of the constructed group is uncountable and so the group is non-metrizable. This is necessary, as quotients of metrizable complete groups remain complete, see [Bou, §IX.3.1, Prop. 4.].

Proof. (of weak exactness on the right in Thm. 5.1, i.e. the image of $\pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S)$ is dense): by Prop. 2.64, it is enough to check that the pullback of a connected geometric cover of S remains connected. This follows directly from Lemma 5.32 below. Let us give a slightly different proof. Geometrically connected and reduced fibres imply that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ (see [Vak, Exc. 28.1.H]). Let $u : U \to S$ be a connected geometric cover. As $X \to S$ is in particular quasi-compact and separated and $U \to S$ is flat, [Vak, Thm. 24.2.8] applies (i.e. "cohomology commutes with flat base change"), and we get that in this situation $u^* f_* \mathscr{F} = f_{U*} u_X^* \mathscr{F}$ for a quasicoherent sheaf on X. Applying this to $\mathcal{O}_X$ and using our assumption, we get $f_{U*}\mathcal{O}_{X\times_S U} = \mathcal{O}_U$. Now, if $X \times_S U$ were disconnected, $\mathcal{O}_{X\times_S U}$ could be written as a product of two sheaves of algebras and the same would be true for $f_{U*}\mathcal{O}_{X\times_S U} = \mathcal{O}_U$, which would contradict connectedness of U. □

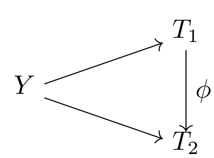
Let us remark that if S is normal, we get actual surjectivity on the right, see Lm. 5.13.

Definition 5.4. To make the statements shorter, we will call a morphism of schemes $f : X \to S$ a "morphism as in h.e.s." if S is connected and f is flat proper of finite presentation with geometrically connected and geometrically reduced fibres.

Let us state the main result of this chapter. It will imply the weak exactness in the middle in Thm. 5.1.

Theorem 5.5. *Let S be a Nagata scheme. Let $X \to S$ be as in h.e.s. and let $Y \in \mathrm{Cov}_X$ be connected. Then there exists a connected $T \in \mathrm{Cov}_S$ and a morphism $g : Y \to T$ over $X \to S$, such that g has geometrically connected fibres.*

Moreover, for any two T_1, T_2 and maps $g_i : Y \to T_i$, $i = 1, 2$, as in the statement, there exists a unique isomorphism $\phi : T_1 \simeq T_2$ in Cov_S making the diagram



commute.

Definition 5.6. In the situation of Thm. 5.5, we will refer to the scheme $T \in \mathrm{Cov}_S$ as the *infinite Stein factorization* of Y.

In the case when $Y \in \mathrm{Cov}_X$ is a finite étale cover, the "infinite Stein factorization" coincides with the usual Stein factorization of the map $Y \to S$. See [SP, Lemma 0BUN] or [SGA 1, Exp. X, Proposition 1.2].

Proposition 5.7. *Assume that Theorem 5.5 holds. Then Theorem 5.1 holds.*

Proof. We have already checked above that the image of the map $\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X)$ is dense. We have to prove the remaining statements.

Let $Y \in \mathrm{Cov}_X$ be connected and such that $Y_{\bar{s}}$ has a section $\sigma : X_{\bar{s}} \to Y_{\bar{s}}$. Let $T \in \mathrm{Cov}_S$ be the "infinite Stein factorization" of Y over S constructed in Thm. 5.5. The section σ gives that $Y_{\bar{s}}$ contains a copy $X'_{\bar{s}}$ of $X_{\bar{s}}$ as a connected clopen subset. Observe that $T_{\bar{s}} \simeq \sqcup_t \bar{s}$, and so $T_{X_{\bar{s}}} \simeq \sqcup_t X_{\bar{s}}$. As $Y_{\bar{s}} \to T_{\bar{s}}$ has connected fibres, one easily checks that that $X'_{\bar{s}}$ equals one of the fibres and the restriction of $Y_{\bar{s}} \to T_{X_{\bar{s}}}$ to $X'_{\bar{s}}$ is an isomorphism. Thus, one of the geometric fibres of $Y_{\bar{s}} \to T_{X_{\bar{s}}}$ is a singleton and so the same holds for $Y \to T_X$. As Y and T_X are connected geometric covers of X, we conclude that $Y \to T_X$ is an isomorphism. By Prop. 2.64, the proof is finished. More precisely, Prop. 2.64 (3) shows that $\overline{\mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X))}$ is a normal subgroup of $\pi_1^{\text{proét}}(X)$ and then Prop. 2.64 (5) implies that $\overline{\mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X))} = \ker(\pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S))$ and $\left(\pi_1^{\text{proét}}(X)/\ker(\pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S))\right)^{\text{Noohi}} \xrightarrow{\sim} \pi_1^{\text{proét}}(S)$. □

Let us also recall some facts about Nagata schemes.
Firstly, Nagata schemes are locally Noetherian (by definition).

Fact 5.8. ([SP, Lemma 035S]) Let X be a Nagata scheme. Then the normalization $\nu : X^\nu \to X$ is a finite morphism.

Fact 5.9. The spectra of the following rings are Nagata schemes: fields, Noetherian complete local rings, Dedekind rings of characteristic zero. Moreover, any scheme locally of finite type over a Nagata scheme is Nagata.

Proof. See [SP, Tag 035A] and [SP, Tag 035B]. □

Example in the case S is normal

As an example, let us apply the obtained homotopy exact sequence in the case S - normal. The direct analogue of the following result holds for the étale fundamental groups and can be checked using the usual homotopy exact sequence and diagram chasing. The point of the following proof is to show that we can redo this proof even if we have only weak exactness.

Corollary 5.10. *Let $f : X \to S$ be as in h.e.s. Assume S to be normal and locally noetherian. Let ξ be its generic point. Then the induced morphism*

$$\alpha : \pi_1^{\text{proét}}(X_\xi) \to \pi_1^{\text{proét}}(X)$$

has dense image.

Proof. Let $\bar{\xi}$ be a geometric point over ξ. Denote $K = \kappa(\xi)$. Applying Thm. 5.1 (see also Rmk. 5.12), we have the following diagram

$$\begin{array}{ccccccc}
\pi_1^{\text{proét}}(X_{\bar{\xi}}) & \longrightarrow & \pi_1^{\text{proét}}(X_\xi) & \longrightarrow & \mathrm{Gal}_K & \longrightarrow & 1 \\
\| & & \alpha\downarrow & & \downarrow & & \\
\pi_1^{\text{proét}}(X_{\bar{\xi}}) & \longrightarrow & \pi_1^{\text{proét}}(X) & \xrightarrow{\pi_1^{\text{proét}}(f)} & \pi_1^{\text{proét}}(S) & \longrightarrow & 1
\end{array}$$

with weakly exact rows. As S is normal, we have $\pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S)$ and the map $\mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$ is surjective by [SP, Prop. 0BQM].

Let $U \subset \pi_1^{\text{proét}}(X)$ be an open subgroup and let $g \in \pi_1^{\text{proét}}(X)$. We need to show, that $\mathrm{im}(\alpha) \cap gU \neq \emptyset$. By Thm. 5.1, $\left(\pi_1^{\text{proét}}(X)/\ker(\pi_1^{\text{proét}}(f))\right)^{\text{Noohi}} \simeq \pi_1^{\text{proét}}(S)$ and thus there is $h \in \mathrm{im}(\alpha) \cap (\ker(\pi_1^{\text{proét}}(f)) \cdot gU)$. It can be also seen more directly, using Lm. 5.13: it implies that the morphism $\pi_1^{\text{proét}}(f)$ is open. It is also surjective and thus it is a quotient map and $\pi_1^{\text{proét}}(X)/\ker(\pi_1^{\text{proét}}(f)) \simeq \pi_1^{\text{proét}}(S)$ as topological groups (thus in fact we do not need to pass to Noohi completions in this case). So, from the diagram, we know that α is surjective modulo $\ker(\pi_1^{\text{proét}}(f))$. But from the weak exactness of $\pi_1^{\text{proét}}(X_{\bar{\xi}}) \to \pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S)$, we have that $\ker(\pi_1^{\text{proét}}(f)) \cdot gU = \mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{\xi}})) \cdot gU$ and so $h \in \mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{\xi}})) \cdot gU$ implies that $h = xgu$ with $x \in \mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{\xi}}))$, $u \in U$ and so $x^{-1}h \in \mathrm{im}(\alpha) \cap gU$. □

Remark 5.11. Using Thm. 5.5 directly, one can give a short alternative proof of the above Corollary. Indeed, let $Y \in \mathrm{Cov}_X$ be connected. Let $T \in \mathrm{Cov}_S$ be the scheme obtained by applying Thm. 5.5 (see also Rmk. 5.12). It is connected and thus finite (as $\pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S)$). T_ξ is connected (by the surjectivity of $\mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$). The morphism $Y \to T$ (and so also $Y_\xi \to T_\xi$) has geometrically connected fibres. As $Y_\xi \to T_\xi$ is open and surjective, this implies that Y_ξ is connected, as desired.

Remark 5.12. Formally, in the proof of Cor. 5.10 we should assume S to be normal and Nagata to apply Thm. 5.1 or Thm. 5.5, but examining the proofs we see that we only use that a Nagata scheme is locally noetherian and its normalization $S^\nu \to S$ is finite.

The following lemma is independent, i.e. we do not assume Thm. 5.1 in the proof.

Lemma 5.13. *Let S be a locally noetherian normal domain and ξ its generic point. Let $f : X \to S$ be a quasi-separated morphism of finite type. Assume X is connected and the fibre X_ξ is geometrically connected. Then the induced morphism*

$$\pi_1^{\text{proét}}(f) : \pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S)$$

is open and surjective.

Proof. We have a following diagram

$$\begin{array}{ccc} \pi_1^{\text{proét}}(X_\xi) & \twoheadrightarrow & \mathrm{Gal}_K \\ \downarrow & & \downarrow\!\!\downarrow \\ \pi_1^{\text{proét}}(X) & \longrightarrow & \pi_1^{\text{ét}}(S) \end{array}$$

As $\pi_1^{\text{proét}}(X)$ is Noohi, it is enough to show that the image of any open subgroup $U \subset \pi_1^{\text{proét}}(X)$ is open. Fix such U. Let V be the preimage of U in $\pi_1^{\text{proét}}(X_\xi)$. Then the image $\pi_1^{\text{proét}}(f)(U)$ contains the image of V via $\pi_1^{\text{proét}}(X_\xi) \to \mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$. But both $\pi_1^{\text{proét}}(X_\xi) \to \mathrm{Gal}_K$ and $\mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$ are open (the first one by Thm. 4.2 and the second one follows from Fact 5.14) and thus $\pi_1^{\text{proét}}(f)(U)$ contains an open subset and so is open as desired. □

Fact 5.14. Let $f : G \to H$ be a surjective morphism of Hausdorff topological groups. Assume G compact. Then f is open.

Proof. As stated, this fact can be easily checked by hand. This is also a special case of a more general "Open mapping theorem", see [Dik, Thm. 7.2.8]. □

5.2 Preliminary results on connected components of schemes

π_0 of an qcqs scheme

We found Appendix A of [Sch17] to be a handy reference for dealing with connected components of fpqc schemes. We include two useful statements below.

Let X be a topological space and $a \in X$ a point. The *connected component* containing a, denoted C_a, is the union of all connected subsets containing a. This is the largest connected subset containing a, and it is closed. In contrast, the *quasicomponent* $Q = Q_a$ is defined as the intersection of all clopen neighborhoods of a, which is also closed. We list some handy facts below.

The following result is stated as [Sch17, Lemma A.1.]. As remarked there, it is due to Ferrand in the affine case, and Lazard in the general case (see [Laz, Prop. 6.1], [Laz, Cor. 8.5]).

Lemma 5.15. *Let X be a qcqs scheme and $a \in X$ be a point. Then we have an equality $C_a = Q_a$ between the connected component and the quasicomponent containing a.*

By [SP, Tag 0900], each quasi-compact space X satisfying the assertion of the lemma above has a profinite set of connected components $\pi_0(X)$.

Corollary 5.16. *Let X be a qcqs scheme. Then $\pi_0(X)$ is profinite.*

The proof of Lm. 5.15 relies on a useful fact on the behaviour of connected components under cofiltered limits. It is essentially [EGA IV 3, Proposition 8.4.1 (ii)], but as explained in [Sch17], in [EGA IV 3] the scheme is only assumed to be quasi-compact, while one needs to assume qcqs.

Fact 5.17. ([Sch17, Proposition A.2.]) Let X_0 be a quasi-compact and quasi-separated scheme, and X_λ a filtered inverse system of affine X_0-schemes, and $X = \varprojlim_{\lambda \in \Lambda} X_\lambda$. If $X = X' \sqcup X''$ is a decomposition into disjoint open subsets, then there is some $\lambda \in \Lambda$ and a decomposition $X_\lambda = X'_\lambda \sqcup X''_\lambda$ into disjoint open subsets so that $X', X'' \subset X$ are the respective preimages.

However, as explained below the proof of [Sch17, Proposition A.2.], both assumptions, quasi-compact and quasi-separated, are needed in general. All the spaces we are going to deal with will be quasi-separated. But non-finite geometric covers are not quasi-compact and thus some extra care is needed when dealing with them. Thus, we devote some time to study connected components of (often) non-quasi-compact schemes.

Some aspects of Galois action on π_0

We proceed to discuss some results on Galois action on connected components of schemes. The results in [SP, Tag 0361], especially [SP, Tag 038D], are very handy for this discussion.

Lemma 5.18. *Let k be a field. Let $k \subset k'$ be a (possibly infinite) Galois extension. Let X be a connected scheme over k. Let $\overline{T}_0 \subset \pi_0(X_{k'})$ be a closed subset preserved by the $\mathrm{Gal}(k'/k)$-action. Then $\overline{T}_0 = \pi_0(X_{k'})$.*

Proof. Let $\overline{T}$ be the preimage of $\overline{T}_0$ in $X_{k'}$ (with the reduced induced structure). By [SP, Lemma 038B], $\overline{T}$ is the preimage of a closed subset $T \subset X$ via the projection morphism $p : X_{k'} \to X$. On the other hand, by [SP, Lemma 04PZ], the image $p(\overline{T})$ equals the entire X. Thus, $T = X$ and $\overline{T} = X_{k'}$, and so $\overline{T}_0 = \pi_0(X_{k'})$. □

Lemma 5.19. *Let X be a connected scheme over a field k with an l'-rational point with l'/k a finite field extension. Then $\pi_0(X_{k^{\mathrm{sep}}})$ is finite, the Gal_k action on $\pi_0(X_{k^{\mathrm{sep}}})$ is continuous and there exists a finite separable extension l/k such that the induced map $\pi_0(X_{k^{\mathrm{sep}}}) \to \pi_0(X_l)$ is a bijection. Moreover, there exists the smallest field (contained in k^{sep}) with this property and it is Galois over k.*

Proof. Let us first show the continuity of the Gal_k-action. The morphism $\mathrm{Spec}(l') \to X$ gives a Gal_k-equivariant morphism $\mathrm{Spec}(l' \otimes_k k^{\mathrm{sep}}) \to X_{k^{\mathrm{sep}}}$ and a Gal_k-equivariant map $\pi_0(\mathrm{Spec}(l' \otimes_k k^{\mathrm{sep}})) \to \pi_0(X_{k^{\mathrm{sep}}})$. Denote by $M \subset \pi_0(X_{k^{\mathrm{sep}}})$ the image of the last map. It is finite and Gal_k-invariant, and by Lm. 5.18, $M = \pi_0(X_{k'})$. We have tacitly used that M is closed, as $\pi_0(X_{k'})$ is Hausdorff (as the connected components are closed). As Gal_k acts continuously on $\pi_0(\mathrm{Spec}(l' \otimes_k k^{\mathrm{sep}}))$ (for example by [SP, Lemma 038E]), we conclude that it acts continuously on $\pi_0(X_{k^{\mathrm{sep}}})$ as well. From Lm. 5.18 again and from [SP, Tag 038D], we easily see that the fields $l \subset k^{\mathrm{sep}}$ such that $\pi_0(X_{k^{\mathrm{sep}}}) \to \pi_0(X_l)$ is a bijection are precisely those that Gal_l acts trivially on $\pi_0(X_{k^{\mathrm{sep}}})$. To get the minimal field with this property we choose l such that $\mathrm{Gal}_l = \ker(\mathrm{Gal}_k \to \mathrm{Aut}(\pi_0(X_{k^{\mathrm{sep}}})))$. □

We continue the discussion by noting the following lemma, which is not used in the rest of the article.

Lemma 5.20. *Let X be a connected scheme over a field k such that $\pi_0(X_{k^{\mathrm{sep}}})$ is finite and the action of Gal_k on it is continuous. Then there is a finite separable extension l/k and a morphism $X \to \mathrm{Spec}(l)$ such that X is geometrically connected over l. Moreover, this association is functorial.*

Proof. $\pi_0(X_{k^{\mathrm{sep}}})$ is a finite and connected Gal_k-set (connectedness by Lm. 5.18), thus corresponds to a finite connected cover of $\mathrm{Spec}(k)$. This must be of a form $\mathrm{Spec}(l)$ for a finite separable extension l/k. Moreover, $\mathrm{Spec}(l)$ can be seen as obtained via Galois (or fpqc) descent from $\sqcup_{\pi_0(X_{k^{\mathrm{sep}}})}\mathrm{Spec}(k^{\mathrm{sep}})$ with descent datum given by the Galois action on $\pi_0(X_{k^{\mathrm{sep}}})$ and we have an obvious morphism $X_{k^{\mathrm{sep}}} \to \sqcup_{\pi_0(X_{k^{\mathrm{sep}}})}\mathrm{Spec}(k^{\mathrm{sep}})$ respecting the descent datum and thus a morphism $X \to \mathrm{Spec}(l)$. This association is functorial by construction. X is geometrically connected over l because any embedding (over k) $l \subset k^{\mathrm{sep}}$ factorizes through $l \otimes_k k^{\mathrm{sep}}$ and thus $X \times_{\mathrm{Spec}(l)} \mathrm{Spec}(k^{\mathrm{sep}}) = (X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{\mathrm{sep}})) \times_{\mathrm{Spec}(l\otimes_k k^{\mathrm{sep}})} \mathrm{Spec}(k^{\mathrm{sep}})$ and the last scheme is equal to one of the connected components of $X_{k^{\mathrm{sep}}}$. ☐

Connected components, fibres and geometric covers

Lemma 5.21. *Let X be a topologically noetherian scheme and $Y \to X$ be in Cov_X. Let Z be an irreducible component of Y. Then Z is quasi-compact.*

Proof. The image of Z in X sits in an irreducible component of X. We can base-change the situation to that component and assume that X is irreducible. Let $\eta \in Z \subset Y$ be the generic point of Z. Let $\widetilde{X} \to X$ be a cover in $X_{\text{proét}}$ by a qcqs scheme such that $\widetilde{Y} = Y \times_X \widetilde{X}$ represents a constant sheaf, i.e. $\widetilde{Y} \simeq \sqcup_{i\in I}\widetilde{X}$, where the indexing set I is possibly infinite. The morphism $\widetilde{X} \to X$ is qcqs. Thus, the same is true for $\widetilde{Y} \to Y$ and in turn the preimage $\tilde{E}$ of η in $\widetilde{Y}$ is quasi-compact. So there is a finite subset $I' \subset I$ such that $\tilde{E} \subset \sqcup_{i\in I'}\widetilde{X} \subset \widetilde{Y}$. Let $\tilde{Z}$ be the preimage of Z in $\widetilde{Y}$. Any point of Z generalizes to η and so, by flatness of $\widetilde{Y} \to Y$, the going-down property implies that any point of $\tilde{Z}$ generalizes to a point in $\tilde{E}$. It follows that the closure of $\tilde{E}$ in $\widetilde{Y}$ contains $\tilde{Z}$. But this closure is contained in $\sqcup_{i\in I'}\widetilde{X} \subset \widetilde{Y}$. This last set is quasi-compact. As $\widetilde{Z}$ is closed in $\widetilde{Y}$, it is quasi-compact as well. As $\widetilde{Z} \to Z$ is surjective, Z is quasi-compact, as desired. ☐

Remark 5.22. An alternative proof of the last lemma can be given if the normalization X^ν of X is topologically noetherian. Let $X^\nu \to X$ be the normalization map. Then the base-change $Y \times_X X^\nu$ is the normalization Y^ν of Y (see Cor. 2.74 or [SP, Lm. 03GV]) and we have a diagram

$$\begin{array}{ccc} Y^\nu & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X^\nu & \longrightarrow & X. \end{array}$$

Each irreducible component of Y is the image of a connected component of Y^ν. Thus, it is enough to show that the connected components of Y^ν are quasi-compact. As X^ν is topologically noetherian, we can apply Lm. 2.18 to get that each connected component of Y^ν is finite (étale) over X^ν and thus quasi-compact.

Lemma 5.23. *Let X be a connected reduced topologically noetherian scheme and let $Y \in \mathrm{Cov}_X$ be connected. Then there exist open immersions $U_n \xrightarrow{i_n} Y$ and closed immersions $Z_n \xrightarrow{j_n} Y$, $n \in \mathbb{Z}_{\geq 0}$ such that:*

1. *U_n, Z_n are of finite type over X, j_n factorizes through i_n and i_n factorizes through j_{n+1}, i.e. we have $Z_n \to U_n \to Z_{n+1} \to U_{n+1} \to \dots \to Y$,*
2. *$\bigcup_n U_n = Y$,*
3. *Each Z_n is a finite union of irreducible components of Y.*

Proof. Observe, that as Y is locally topologically noetherian and locally of finite type over X, it is enough to ensure that Z_n and U_n are quasi-compact, to obtain that they are of finite type over X. By Lm. 5.21, every irreducible component of Y is quasi-compact. Let us define Z_1 to be any irreducible component of Y and U_1 to be a connected quasi-compact open neighbourhood of Z_1 (it exists as Z_1 is quasi-compact and connected and Y is locally topologically noetherian and so locally connected). Now, let

$$Z_2 = \bigcup_{\text{irr. comp. } Z \text{ of } Y \text{ s.t. } Z \cap U_1 \neq \emptyset} Z$$

As U_1 is quasi-compact subset of a locally noetherian space it is noetherian and we see that the indexing set in the above sum is finite. By Lm. 5.21, we see that Z_2 is quasi-compact. Z_2 is a closed subset of Y and we put the reduced induced structure on it. Moreover, $U_1 \subset Z_2$ and Z_2 is connected. We can now take U_2 to be a connected quasi-compact open containing Z_2. Repeating this procedure we produce connected schemes Z_n, U_n satisfying (1) and (3). To check (2) we need to show that $U_\infty = \bigcup_n U_n$ is equal to Y. From connectedness of Y, it is enough to show that U_∞ is clopen. It is obviously open. In particular constructible. Thus, it is enough to show that it is closed under specialization (see [SP, Tag 0542]. It is stated for noetherian topological spaces but clearly locally noetherian is enough, as for any $y \in \overline{U_\infty}$ we can check whether $y \in U_\infty$ by restricting to a topologically noetherian neighbourhood V of y and working with the intersection $U_\infty \cap V$). Let $\xi \in U_\infty$ and assume ξ specializes to a point $y \in Y$. Let m be such that $\xi \in U_m \subset U_\infty$. There exist an irreducible component Z of Y containing ξ. It is closed and so $y \in Z$. But $Z \subset Z_{m+1}$ by construction. Thus, $y \in Z_{m+1} \subset U_{m+1} \subset U_\infty$ as desired.

□

Remark 5.24. One can use the above result to check that the closure of a quasi-compact subset of Y remains quasi-compact.

Observation 5.25. Let Y be a scheme and let $U_1 \subset U_2 \subset U_3 \subset \ldots \subset Y$ be an increasing sequence of open subschemes such that $\bigcup_n U_n = Y$. Then, directly from the sheaf property, it follows that

$$\Gamma(Y, \mathcal{O}_Y) = \varprojlim \Gamma(U_n, \mathcal{O}_{U_n}).$$

Lemma 5.26. *Let Y be a reduced scheme over an algebraically closed field k having a filtration $Z_0 \subset U_0 \subset Z_1 \subset U_1 \subset \ldots \subset Y$ with U_i open and Z_i connected and proper over k. Then $\Gamma(U, \mathcal{O}_U) = k$*

Proof. By Obs. 5.25, $\Gamma(U, \mathcal{O}_U) = \varprojlim \Gamma(U_n, \mathcal{O}_{U_n})$. But every map $\Gamma(U_{n+1}, \mathcal{O}_{U_{n+1}}) \to \Gamma(U_n, \mathcal{O}_{U_n})$ factorizes through $\Gamma(Z_{n+1}, \mathcal{O}_{Z_{n+1}}) = k$ (from the properness of Z_n and $k = \bar{k}$). Thus, $\varprojlim \Gamma(U_n, \mathcal{O}_{U_n}) = \varprojlim \Gamma(Z_n, \mathcal{O}_{Z_n}) = k$. □

The following lemma makes precise the statement that "a flat degeneration of a disconnected scheme is either disconnected or nonreduced".

Lemma 5.27. *Let R be a dvr and let X be a connected scheme flat over R. If the special fibre X_s is reduced, then the generic fibre X_ξ is connected.*

Proof. This is [SP, Tag 055J]. □

The following two facts are not used later, but they gives us a bit more intuition.

Proposition 5.28. *Let X be a connected noetherian scheme such that the normalization $X^\nu \to X$ is finite (e.g. X Nagata). Let $Y \in \mathrm{Cov}_X$ be connected and $\bar{x}$ be a geometric point on X. Then $Y_{\bar{x}}$ is countable.*

Proof. Thanks to the above assumptions, we can write $\pi_1^{\text{proét}}(X, \bar{x})$ as a Noohi completion of the quotient of $*_v^{\text{top}} G_v * D$, v running over a finite set, G_v - profinite and $D \simeq \mathbb{Z}^{*r}$ a discrete and countable group. Fix some v_0. Then, with the notation as in Thm. 4.14, the sets $O_{v_0}^N$ are finite and $Y_{\bar{x}} = \bigcup_{N>0} \bigcup_{o \in O_{v_0}^N} o$, which finishes the proof, as the G_{v_0}-orbits o are finite. □

Lemma 5.29. *Let R be a Dedekind domain and let Y be a reduced scheme flat and locally of finite type over R. Let $Z_1, \ldots, Z_m$ be a finite subset of irreducible components of Y and let $Z = \cup_i Z_i \subset Y$. Then Z (with the reduced induced structure) is flat over R.*

Proof. Observe that Z is closed, so the statement makes sense. Let $\mathcal{I}$ be the sheaf of ideals cutting Z. Cover Y with affine schemes $V_j = \mathrm{Spec}(A_j)$, $j = 1, \ldots, n$. $V_j \cap Z$ are affine and cover Z, thus it is enough to show they are flat over R. For any i, j, $Z_i \cap V_j$ is either empty or is an irreducible component of $Z_i \cap V_j$. Thus, we can reduce to the affine case $Y = \mathrm{Spec}(A)$ and $Z_i = \mathrm{Spec}(A/I_i)$. The ideals I_i are among minimal primes of A and, as A is noetherian, are associated primes. Thus, for every i, $I_i = \mathrm{Ann}_A(f_i)$ for some $f_i \in A$. As $Z = \mathrm{Spec}(A/I)$ with $I = \cap_i I_i = \cap_i \mathrm{Ann}_A(f_i)$, we want show that A/I is flat over R. R being Dedekind, it is enough to show that A/I is torsion-free over R. Assume that $r \in R$, $a \in A \setminus I$ are such that $ra \in I$. Thus, $\forall_i raf_i = 0$. But $a \notin I$ implies $\exists_{i_0} af_{i_0} \neq 0$. As $r \cdot (af_{i_0}) = 0$ and A is torsion-free over R, we get that $r = 0$. Thus, A/I is torsion-free over R. ☐

Some topology involving π_0's of non-noetherian schemes

Lemma 5.30. *Let $f : W \to T$ be a qcqs morphism of schemes. Assume that each connected component of T is locally connected (e.g. each connected component is topologically noetherian). Assume that the image of W is dense in every connected component of T. Then the induced map $\pi_0(f) : \pi_0(W) \to \pi_0(T)$ is a topological quotient map.*

Proof. The map $\pi_0(f)$ is surjective by the assumption of dense images. Let $U_0 \subset \pi_0(T)$. Assume that $\pi_0(f)^{-1}(U_0)$ is open. We want to show that U_0 is open. As the topology on $\pi_0(T)$ is the quotient topology from T, it is enough to show that $U = \pi^{-1}(U_0) \subset T$ is open. We have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & T \\ \pi\downarrow & & \pi\downarrow \\ \pi_0(W) & \xrightarrow{\pi_0(f)} & \pi_0(T). \end{array}$$

Thus, $f^{-1}(U) = \pi^{-1}(\pi_0(f)^{-1}(U_0))$ is open. To prove that U is open, it is enough to show that, for each affine open V of T, the intersection $U \cap V$ is open in V. Fix such V and denote $W_V = f^{-1}(V)$. Observe that $f|_{W_V}^{-1}(U \cap V) = f^{-1}(V) \cap W_V$ is open in W_V. Consider the commutative diagram of topological spaces

$$\begin{array}{ccccc} W_V & \xrightarrow{f|_{W_V}} & V & \xrightarrow{\subset} & T \\ \pi\downarrow & & \pi\downarrow & & \pi\downarrow \\ \pi_0(W_V) & \xrightarrow{\pi_0(f|_{W_V})} & \pi_0(V) & \longrightarrow & \pi_0(T). \end{array}$$

It follows from the diagram that there exists a subset $U_0' \subset \pi_0(V)$ such that $V \cap U = \pi^{-1}(U_0')$. Moreover, as V is affine and f is qcqs, $\pi_0(f|_{W_V})$ is a (continuous) surjective map of compact spaces and so a quotient map. Surjectivity of $\pi_0(f|_{W_V})$ follows from the assumptions: local connectedness of connected components of T implies that each connected component of V is an open subset of a connected component of T and by the assumption that the image of f is dense in every connected component of T, we get the desired surjectivity. As $\pi^{-1}(\pi_0(f|_{W_V})^{-1}(U_0')) = f^{-1}(V \cap U)$ is open and both π and $\pi_0(f|_{W_V})$ are quotient maps, we conclude that U_0' is open and thus $V \cap U$ is open as desired. ☐

Lemma 5.31. *Let $f : W \to T$ be a continuous map of topological spaces. Assume that f is a topological quotient map (e.g. surjective and open or surjective and closed). Then $\pi_0(f)$ is a topological quotient map.*

Proof. Let $U_0 \subset \pi_0(T)$ be such that $\pi_0(f)^{-1}(U_0)$ is open. We want to show that U_0 is open. It is equivalent to checking that $U = \pi^{-1}(U_0)$ is open. But $f^{-1}(U) = \pi^{-1}(\pi_0(f)^{-1}(U_0))$ and so is open. As f is a quotient map, U is open as well, which finishes the proof. ☐

Lemma 5.32. *Let $f : X \to S$ be a universally open and surjective morphism of schemes (e.g. f faithfully flat locally of finite presentation) with geometrically connected fibres. Then for any morphism of schemes $\widetilde{S} \to S$, the map induced on π_0's by the base-change of f to $\widetilde{S}$*

$$\pi_0(f) : \pi_0(\widetilde{X}) \to \pi_0(\widetilde{S})$$

is a homeomorphism.

Proof. We can obviously assume $\widetilde{S} = S$. Let $t \in \pi_0(S)$. We can see it as a closed subscheme $t \hookrightarrow S$ and obtain $f_t : X_t \to t$ via base-change. Let us first show that $\pi_0(f)$ is bijective. As f is surjective, $\pi_0(f)$ is surjective as well and thus we only need to show that X_t is connected. But this follows easily from the fact that t is connected, f_t is open, surjective and has connected fibres. Thus, $\pi_0(f)$ is a continuous bijection and by Lm. 5.31, it is a homeomorphism. □

π_0, normalization and w-contractible covers

The following result is not used in the main proof and can be skipped. However, it provides some insight into behaviour of π_0's of the schemes involved. It is essentially [BS, Lm 2.4.10] made explicit. We followed [SP, Tag 0986].

Proposition 5.33. *Let S be a quasi-compact Nagata scheme. Let $S^\nu \to S$ be the normalization of S and let $\widetilde{S}$ be a pro-étale cover of S by an affine w-contractible scheme. Then*

1. *The base-change $\widetilde{S^\nu} = S^\nu \times_S \widetilde{S}$ is w-contractible.*
2. *There exists a finite partition into clopen subspaces $\pi_0(\widetilde{S}) = \sqcup_{i=1}^k Z_i$ such that, for each i, there is a homeomorphism $\pi_0(\widetilde{S^\nu}) \times_{\pi_0(\widetilde{S})} Z_i \simeq \sqcup_{j=1}^m Z_i$ for some integer $m = m_i$ (depending on i).*

In particular, there exists a continuous section $\tau : \pi_0(\widetilde{S}) \to \pi_0(\widetilde{S^\nu})$ of $\pi_0(\widetilde{S^\nu}) \to \pi_0(\widetilde{S})$ that maps $\pi_0(\widetilde{S})$ homeomorphically onto a clopen subset of $\pi_0(\widetilde{S^\nu})$.

Proof. As S is Nagata, S is noetherian and $S^\nu \to S$ is finite. Thus, the first statement follows directly from [SP, Tag 0986]. The second statement follows from the proof of [SP, Tag 0986] as we are going to sketch. By w-locality, the subset $\widetilde{S}^c \subset \widetilde{S}$ of closed points is closed and the composition $\widetilde{S}^c \to \widetilde{S} \to \pi_0(\widetilde{S})$ is a homeomorphism. Moreover, as $\widetilde{S}$ is strictly local, all the residue fields of points of $\widetilde{S}^c$ are separably closed (and $\widetilde{S}^c$ is Hausdorff as a topological space!). From the finiteness assumption, it follows that the preimage of the closed points of $\widetilde{S}$ via $\widetilde{S^\nu} \to \widetilde{S}$ is the set of closed points of $\widetilde{S^\nu}$, i.e. $|(\widetilde{S^\nu})^c| = |\widetilde{S^\nu} \times_{\widetilde{S}} \widetilde{S}^c| = |\widetilde{S^\nu}| \times_{|\widetilde{S}|} |\widetilde{S}^c|$, where $|\dots|$ denotes the underlying topological space. By [SP, Tag 095K], there exists a finite partition $\widetilde{S}^c = \sqcup_{i=1}^k (\widetilde{S}^c)_i$ by locally closed, constructible, affine strata, and surjective finite locally free morphisms $(\widetilde{S}^c)'_i \to (\widetilde{S}^c)_i$ such that the reduction of $((\widetilde{S^\nu})^c)'_i = (\widetilde{S^\nu})^c \times_{\widetilde{S}^c} (\widetilde{S}^c)'_i \to (\widetilde{S}^c)'_i$ is isomorphic to $\sqcup_{j=1}^{m_i} ((\widetilde{S}^c)'_i)_{\mathrm{red}} \to ((\widetilde{S}^c)'_i)_{\mathrm{red}}$ for some m_i. Note that, in the terminology of the Stacks Project, a subset E of a topological space W is constructible if it is a finite union of sets of the form $U \cap (W \setminus V)$ with U, V open and retrocompact. Thus, if W is (Hausdorff) compact, the constructible subsets of W are clopen. It follows that to show (2) we can and do assume $k = 1$ and remove i from the notation. Using finiteness assumptions and the fact that all the residue fields of $\widetilde{S}^c$ are separably closed, we have that

$$\begin{array}{ccc} \sqcup_{j=1}^m |(\widetilde{S}^c)'| & \longrightarrow & |(\widetilde{S}^c)'| \\ \downarrow & & \downarrow \\ |(\widetilde{S^\nu})^c| & \longrightarrow & |\widetilde{S}^c| \end{array}$$

is a diagram of topological spaces that is fibred as a diagram of underlying sets. As $\widetilde{S}$ is w-contractible, $|\widetilde{S}^c|$ is extremally disconnected, i.e. the closure of every open is open. By theorem of Gleason (see [SP, Thm 2.4.5]), extremally disconnected compact spaces are exactly the projective objects in the category of all compact Hausdorff spaces. By [SP, Tag 0978], $(\widetilde{S}^c)'$ is a compact Hausdorff space. Thus, there exists a continuous section $\sigma : |\widetilde{S}^c| \to |(\widetilde{S}^c)'|$ of the map $|(\widetilde{S}^c)'| \to |\widetilde{S}^c|$. The above diagram implies, that $\sqcup_{j=1}^m \sigma(|\widetilde{S}^c|) \to |(\widetilde{S^\nu})^c|$ is a continuous bijection and thus a homeomorphism (as the involved spaces are compact). This finishes the proof, as from w-locality of $\widetilde{S}$ and $\widetilde{S^\nu}$ we have that $|\widetilde{S}^c|$ and $|(\widetilde{S^\nu})^c|$ are homeomorphic to $\pi_0(\widetilde{S})$ and $\pi_0((\widetilde{S^\nu})^c)$ respectively. The existence of the section τ follows immediately. □

5.3 Proof of Theorem 5.5

Let us start with two results that essentially give the homotopy exact sequence in the case S equal to a spectrum of a strictly henselian ring. Recall that in this case $\pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S) = 1$ by [BS, Lemma 7.3.8.].

Proposition 5.34. *Let S be a spectrum of a strictly henselian noetherian ring, let $X \to S$ be proper with X connected and let $Y \in \mathrm{Cov}_X$ be connected. Let $\bar{s}$ be a geometric point over the closed point s of S. Then the geometric fibre $Y_{\bar{s}}$ is connected. In other words, the morphism*

$$\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X)$$

has dense image.

As a result, for any $Y \in \mathrm{Cov}_X$, the natural map $\pi_0(Y_{\bar{s}}) \to \pi_0(Y)$ is a bijection (of discrete sets).

Proof. As the residue field $\kappa(s)$ is separably closed, it is enough to show that Y_s is connected ([SP, Tag 0387]). Assume the contrary and write $Y_s = W_1 \sqcup W_2$, where W_1, W_2 are clopen subsets of Y_s. Apply Lm. 5.23 to Y and produce a sequence of connected quasi-compact closed $Z_n \subset Y$ such that $\bigcup_n Z_n = Y$. There exist $N \geq 0$ such that $Z_N \cap W_1 \neq \emptyset$ and $Z_N \cap W_2 \neq \emptyset$. Thus, the fibre $Z_{N,s}$ is not connected. But Z_N is of finite type over X and satisfies the valuative criterion of properness (as $Y \in \mathrm{Cov}_X$). Thus, Z_N is proper over X and so also over S. But for such scheme it is known (by the special case of the proper base-change theorem: see e.g. [SP, Lm. 0A3S] or [SP, Lm. 0A0B]) that the induced map $\pi_0(Z_{N,s}) \to \pi_0(Z_N)$ is bijective. This is a contradiction. ☐

Lemma 5.35. *Let S be the spectrum of a strictly henselian dvr R and let $X \to S$ be a morphism as in h.e.s. Let $\bar{\xi}$ be a geometric point over the generic point ξ of S. Then the morphism*

$$\pi_1^{\text{proét}}(X_{\bar{\xi}}) \to \pi_1^{\text{proét}}(X)$$

has dense image. In other words, for a connected $Y \in \mathrm{Cov}_X$, $Y_{\bar{\xi}}$ remains connected. As a result, for any $Y \in \mathrm{Cov}_X$, the natural map $\pi_0(Y_{\bar{\xi}}) \to \pi_0(Y)$ is a bijection (of discrete sets).

Proof. Denote by s the closed point of S. Let $Y \in \mathrm{Cov}_X$ be connected. Using Prop. 5.34, we know that Y_s is connected. The scheme Y_ξ is locally of finite type over $K = \kappa(\xi)$. Thus, it has an L'-point with L'/K a finite field extension. Applying Lm. 5.19, we get that there exists a finite separable extension L/K such that $Y_{\xi,L}$ has a finite number of connected components and each of them is geometrically connected. Let R' be an integral closure of R in L. It is a local (as R is henselian) algebra finite over R (as the extension L/K was separable). R' is thus a dvr. Let π' be its uniformizer. The field $k' = R'/\pi'$ is a finite (purely inseparable) extension of $R/\pi = k$. Thus, we can assume $k' \subset \kappa(\bar{s})$. Thus, we can see $\bar{s}$ as a geometric point of $\mathrm{Spec}(R')$ lying over the special point s'. Denoting $Y' = Y \times_S \mathrm{Spec}(R')$, we have $Y'_{\bar{s}} = Y_{\bar{s}}$ and so it is reduced and connected. We conclude that the fibre $Y'_{s'}$ is connected and reduced as well (because $Y'_{\bar{s}} \to Y'_{s'}$ is faithfully flat). Thus, also Y' is connected (this is clear when thinking of $Y'_{s'}$ and Y' as elements of $\pi_1^{\text{proét}}(X'_{s'}) - \mathrm{Sets}$ and $\pi_1^{\text{proét}}(X') - \mathrm{Sets}$). By Lm. 5.27 we conclude that the generic fibre $Y_{\mathrm{Frac}(R')}$ is connected. But $Y'_{\mathrm{Frac}(R')} = Y' \times_{\mathrm{Spec}(R')} \mathrm{Spec}(L) = Y \times_{\mathrm{Spec}(R)} \mathrm{Spec}(L) = Y_{\xi,L}$. Combining it with what we observed at the beginning of the proof, we conclude that $Y_{\xi,L}$ is geometrically connected. This finishes the proof. ☐

Theorem 5.36. *Let S be a connected noetherian scheme. Let $X \to S$ be as in h.e.s. Let $\bar{\xi}$ and $\bar{s}$ be two geometric points on S with images $\xi, s \in S$. Then, for any $Y \in \mathrm{Cov}_X$, there is a bijection*

$$\pi_0(Y_{\bar{\xi}}) \simeq \pi_0(Y_{\bar{s}}).$$

It depends on the choice of a "path" between $\bar{s}$ and $\bar{\xi}$, i.e. chain of maps from strictly henselian dvrs (see the proof). When S is the spectrum of a strictly henselian dvr and $\bar{\xi}$ and $\bar{s}$ lie over the special and the generic point, the bijection is the obvious one given by combining Lm. 5.34 with Lm. 5.35.

Proof. As S is connected and noetherian, we can join s with ξ by a finite sequence of specializations and generizations of points on S (every point lies on one of the finitely many irreducible components $Z_1, \ldots, Z_m$ of S and within a fixed irreducible component every point is a specialization of the generic point. It follows that the set of points reachable via sequence of specializations and generizations from a given point is a union of some irreducible components and thus closed. There is only finitely many of such "path components", and thus it is also open). Thus, we can and do reduce to the case where s is a specialization of ξ. By [SP, Tag 054F] we can find a dvr R and a morphism $\mathrm{Spec}(R) \to S$ such that the generic point of $\mathrm{Spec}(R)$ maps to ξ and the special point maps to s. The strict henselization R^{sh} of R is a strictly henselian dvr by [SP, Tag 0AP3]. Let ξ', s' be the generic and the special point of $\mathrm{Spec}(R^{sh})$ respectively and let $\bar{\xi}'$ and $\bar{s}'$ be some geometric points over them. By Lm. 5.35 and Prop. 5.34, we conclude that $\pi_0(Y_{\bar{s}'}) \simeq \pi_0(Y_{\mathrm{Spec}(R^{sh})}) \simeq \pi_0(Y_{\bar{\xi}'})$. Choosing geometric points $\bar{s}''$ and $\bar{\xi}''$ on S such that $\bar{s}''$ factors both through $\bar{s}$ and $\bar{s}'$ and $\bar{\xi}''$ factors both through $\bar{\xi}$ and $\bar{\xi}'$ and using the fact that the base-change of a connected scheme over an algebraically closed field to another algebraically closed field remains connected, we finish the proof. □

Corollary 5.37. *Let S be the spectrum a strictly henselian noetherian ring and let $X \to S$ be as in h.e.s. Let $\bar{s}$ be any geometric point on S. Then the morphism*

$$\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X)$$

has dense image. In other words, for a connected $Y \in \mathrm{Cov}_X$, the base-change $Y_{\bar{s}} \in \mathrm{Cov}_{X_{\bar{s}}}$ remains connected. As a result, for any $Y \in \mathrm{Cov}_X$, the natural map $\pi_0(Y_{\bar{s}}) \to \pi_0(Y)$ is a bijection (of discrete sets).

Proof. If $\bar{s}$ lies over the closed point of S, then the statement was proven in Prop. 5.34. But now Thm. 5.36 tells us that the statement holds for any $\bar{s}$. □

Let us also mention a technical lemma used later in the proof.

Lemma 5.38. *Let X be a compact topological space. Let $W = \sqcup_i X_i$ be a disjoint union, indexed by i, of copies of X. Let $g : W \to X$ be the obvious structural map. Let $g_i : X_i \subset W \to X$ be the structural (iso-)morphism. Let $\phi \in \mathrm{Aut}_X(W)$ be an automorphism of W **over** X. Then*

1. *there exists a decomposition of W into clopen subsets W_{ij}, such that:*
   - *$W_{ij} \subset X_i$,*
   - *X_i is a sum of finitely many W_{ij},*
   - *ϕ preserves this decomposition, i.e. for each i, j, $\phi(W_{ij}) = W_{i'j'}$ for some i', j'.*
2. *Assume that $X = \pi_0(X')$ for some topological space X'. Let $W' = \sqcup_i X'_i$ be the disjoint onion of copies of X'. Then ϕ lifts uniquely to an automorphism of W' over X', i.e. there exist an automorphism $\phi' : W' \to W'$ such that $\phi = \pi_0(\phi')$.*
3. *Assume that $X = \pi_0(X')$ for some scheme X'. Let $W' = \sqcup_i X'_i$ be the disjoint onion of copies of X'. Then ϕ lifts uniquely to an automorphism of W' over X', i.e. there exist an automorphism $\phi' : W' \to W'$ of X'-schemes such that $\phi = \pi_0(\phi')$.*

Proof. We omit the proof of the first part and only comment on the other two. If X' is a topological space, the first part of the Lemma produces the clopen subsets W_{ij} of W. Observe that if $\phi(W_{ij}) = W_{i'j'}$, then $\phi|_{W_{ij}} : W_{ij} \to W_{i'j'}$ is equal to $g_i'^{-1} \circ g_i|_{W_{ij}}$. This means that ϕ can be described purely combinatorically (in terms of the indexing set $\{i\}$ and clopen decompositions of the topological space X). The crucial thing here is that we assume ϕ to respect the structural morphism $W \to X$. Taking the preimages W'_{ij} of W_{ij} via the projection $W' \to \pi_0(W') = W$, we get a clopen decomposition of W'. The combinatorial description of ϕ can be lifted in an obvious (and unique) way to an automorphism of W'. Moreover, for X' a scheme, ϕ' will be an automorphism of a scheme (i.e. we get a map of the structure sheaves). This is because if ϕ' maps a clopen subset $W'_1 \subset X'_1$ homeomorphically onto a clopen subset $W'_2 \subset X'_2$, then denoting $g'_1 : X'_1 \simeq X'$, $g'_2 : X'_2 \simeq X'$ (the structural isomorphisms), $\phi'|_{W'_1}$ is given by $g_2'^{-1} \circ g'_1|_{W'_1}$. □

Proof of Theorem 5.5:

Proof. Let us start with the proof of uniqueness. Let $\bar{x}$ be a geometric point on X and $\bar{s}$ its image on S. Let $Y \in \mathrm{Cov}_X$ be connected. Given two connected $T_i \in \mathrm{Cov}_S$ and maps $g_i : Y \to T_i$ over $X \to S$ that have geometrically connected fibres, we easily see that the maps g_i induce bijections $b_i : \pi_0(Y_{\bar{s}}) \to (T_i)_{\bar{s}}$. Let "$\phi_{\bar{s}}$" : $(T_1)_{\bar{s}} \xrightarrow{\simeq} (T_2)_{\bar{s}}$ be the bijection given by $b_2 \circ b_1^{-1}$. We have to check, that the bijection "$\phi_{\bar{s}}$" is a map of $\pi_1^{\text{proét}}(S, \bar{s}) - \mathrm{Sets}$, i.e. that it is $\pi_1^{\text{proét}}(S, \bar{s})$-equivariant. It is easy to check that the following diagram of $\pi_1^{\text{proét}}(X, \bar{x}) - \mathrm{Sets}$ is commutative

$$\begin{array}{ccc} & & (T_1)_{\bar{s}} \\ & \nearrow\!\!\!\nearrow & \\ Y_{\bar{x}} & & \Big\downarrow \text{"}\phi_{\bar{s}}\text{"} \\ & \searrow\!\!\!\searrow & \\ & & (T_2)_{\bar{s}} \end{array}$$

Thus, "$\phi_{\bar{s}}$" is $\pi_1^{\text{proét}}(X, \bar{x})$-equivariant and so also $\pi_1^{\text{proét}}(S, \bar{s})$-equivariant, as $\pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(S, \bar{s})$ has dense image (by the part of Thm. 5.1 that was already proven). The equivalence of categories $\mathrm{Cov}_S \simeq \pi_1^{\text{proét}}(S, \bar{s}) - \mathrm{Sets}$ gives us ϕ. This finishes the proof of uniqueness. Let us proceed with the proof of existence.

Reduction of the proof of existence to the case S - normal:
Let $S^\nu \to S$ be the normalization morphism. It is finite by the fact that S is Nagata. Thus, it is a morphism of effective descent for Cov_S by Prop. 3.12. Assume that the statement of the theorem holds for normal schemes. Let $Y \in \mathrm{Cov}_X$ be connected. Denote by Y' and X' the base-changes of Y and X to S^ν (we do not denote it Y^ν and X^ν to avoid the confusion with the normalizations). Applying the theorem to (each one of the discrete set of connected components of) S^ν we obtain $T' \in \mathrm{Cov}_S$ and a surjective morphism $Y' \to T'$ over $X' \to S$. The proof will be finished if we equip $Y' \to T'$ with a descent datum with respect to $S^\nu \to S$. Let $p, q : S'_2 = S^\nu \times_S S^\nu \to S^\nu$ be the two projections. We get two morphisms $Y'_2 = Y \times_S S'_2 \to p^*T'$ and $Y'_2 = Y \times_S S'_2 \to q^*T'$ (over $X'_2 \to S'_2$) that are surjective with geometrically connected fibres. Thus, by the uniqueness part of the theorem (that we have proven over any scheme) applied to (each connected component of) S'_2, we get an isomorphism $\phi : p^*T' \to q^*T'$. Then the cocycle condition holds by applying the uniqueness statement (over $S'_3 = S^\nu \times_S S^\nu \times_S S^\nu$) again. Observe that the morphism $Y' \to T'$ descends as well by the following reasoning: the schemes Y', T' descent to Y and T and then we look at Y and $T_X = T \times_S X \in \mathrm{Cov}_X$, and see that $Y' \to T' \times_{S^\nu} X'$ descends to a morphism $Y \to T_X$. This finishes the reduction. Similarly, we see that the problem is Zariski local on S and we can assume S to be affine.

Thus, we can and do assume that S is normal and affine in the rest of the proof.

Let $\widetilde{S} \to S$ be a pro-étale cover, such that $\widetilde{S}$ is affine w-strictly local (as defined in [BS, Def. 2.2.1]). This is possible by [BS, Cor. 2.2.14.] or [SP, Tag 097R]). All local rings at closed points of $\widetilde{S}$ are strictly henselian by [BS, Lm. 2.2.9]. They are equal to the strict henselizations of the local rings at corresponding (geometric) points of S. By [BS, Lm. 2.1.4], if $\widetilde{S}^c$ denotes the set of closed points of $\widetilde{S}$, the composition map $\widetilde{S}^c \to \widetilde{S} \to \pi_0(\widetilde{S})$ is a homeomorphism (as $\widetilde{S}$ is w-local). From this we see that each localization at a closed point of $\widetilde{S}$ is equal to a connected component of $\widetilde{S}$ containing this point. Let us denote by $\widetilde{X}$ and $\widetilde{Y}$ the base-changes of X and Y to $\widetilde{S}$. By Cor. 5.37, if $\tilde{s} \in \widetilde{S}$ is a closed point and if $c_{\tilde{s}}$ denotes the connected component of $\tilde{s}$, then $Y_{c_{\tilde{s}}}$ is a disjoint union of connected schemes and $\pi_0(Y_{c_{\tilde{s}}})$ (it is a discrete set, as $Y_{c_{\tilde{s}}}$ is locally noetherian and so the connected components are open) can be canonically identified with $\pi_0(Y_{\tilde{s}})$. The point $\tilde{s}$ can be seen as a geometric point over its image $s \in S$. By Thm. 5.36, we have a bijection $\pi_0(Y_{\bar{s}}) \simeq \pi_0(Y_{\bar{\xi}})$ for any two geometric points $\bar{s}$ and $\bar{\xi}$ on S. Thus, we see that π_0 of restrictions of $\widetilde{Y}$ to two connected components of $\widetilde{S}$ can be identified, i.e. if $c_1, c_2 \subset \widetilde{S}$ are two connected components, there is a bijection of discrete sets $\pi_0(\widetilde{Y}_{c_1}) \simeq \pi_0(\widetilde{Y}_{c_2})$. Thus, as sets, we can write $\pi_0(\widetilde{Y}) \simeq \sqcup_{t \in \pi_0(Y_{\bar{s}})} \pi_0(\widetilde{S})_t$ and this identification is compatible with the natural maps $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{S})$ and $\pi_0(\widetilde{S})_t \xrightarrow{\mathrm{id}} \pi_0(\widetilde{S})$. Here $\bar{s}$ is some fixed (arbitrarily chosen) geometric point of S and the subscript notation in $\pi_0(\widetilde{S})_t$ denotes different copies of the set $\pi_0(\widetilde{S})$ ($\pi_0(\widetilde{S})_t$ does not denote a fibre over some $t \in S$!). As we explain below, it can be upgraded to a homeomorphism. Observe that $\pi_0(\widetilde{S})$ is a profinite set (as $\widetilde{S}$ is qcqs), but usually it will not be finite.

Claim 1: There is a homeomorphism $\alpha : \sqcup_{t\in\pi_0(Y_{\bar{s}})}\pi_0(\widetilde{S})_t \to \pi_0(\widetilde{Y})$ over $\pi_0(\widetilde{S})$.
Proof of the claim: Let η be the generic point of S (recall that S is now assumed to be normal). By Lm. 2.34, we can identify $\pi_0(\widetilde{S}_\eta) = \pi_0(\widetilde{S})$. Let us show that we can identify $\pi_0(\widetilde{Y}_\eta) = \pi_0(\widetilde{Y})$ as well. Here $\widetilde{Y}_\eta = \widetilde{Y} \times_S \eta = \widetilde{Y} \times_{\widetilde{S}} \widetilde{S}_\eta = Y_\eta \times_S \widetilde{S}$.

Lemma. The map $\pi_0(\widetilde{Y}_\eta) \to \pi_0(\widetilde{Y})$ is a homeomorphism.

To show that $\pi_0(\widetilde{Y}_\eta) \to \pi_0(\widetilde{Y})$ is bijective it is enough to look fibre by fibre over $\pi_0(\widetilde{S})$. Thus, we can fix a connected component $c \in \pi_0(\widetilde{S})$ and base-change to c (keep in mind that, as a morphism of schemes, $c \to S$ is (among other properties) a closed immersion. In particular, $\pi_0(\widetilde{Y} \times_{\widetilde{S}} c)$ is equal to the preimage of c under $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{S})$. Similarly for $\widetilde{Y}_\eta$.). The component c is a strict henselization at a (geometric) point on S (Lm. 2.34) and so is noetherian. The fibre c_η consists of a single point: the generic point of c (c is a normal connected scheme), let us call it ξ. Then the map $(\widetilde{Y}_\eta)_c \to \widetilde{Y}_c$ (where the subscript c denotes the base-change from $\widetilde{S}$ to c) is equal to the embedding of the fibre over ξ to $\widetilde{Y}_c$ (i.e the base-change of $\xi \to c$ to $\widetilde{Y}_c$). By Cor. 5.37, $\pi_0((\widetilde{Y}_c)_{\bar{\xi}}) \to \pi_0(\widetilde{Y}_c)$ is a bijection. But we have a factorization $\pi_0((\widetilde{Y}_c)_{\bar{\xi}}) \twoheadrightarrow \pi_0((\widetilde{Y}_c)_\xi) \to \pi_0(\widetilde{Y}_c)$, where the first map is surjective. It follows that $\pi_0((\widetilde{Y}_c)_\xi) \to \pi_0(\widetilde{Y}_c)$ is a bijection. But $(\widetilde{Y}_c)_\xi \simeq (\widetilde{Y}_c)_\eta$ and the proof of bijectivity is finished. By Lm. 5.30, $\pi_0(\widetilde{Y}_\eta) \to \pi_0(\widetilde{Y})$ is a homeomorphism and the proof of the lemma is finished.

Thus, we can focus on understanding $\pi_0(\widetilde{Y}_\eta)$. The scheme Y_η is a (possibly infinite in this proof, but see Rmk. 5.39) disjoint union of connected components belonging to Cov_{X_η}. These components are clopen and so $\pi_0(Y_\eta)$ and $\pi_0(\widetilde{Y}_\eta)$ split accordingly. We can thus restrict attention to one connected component and assume Y_η connected in the proof of the claim. Let $c \in \pi_0(\widetilde{S}_\eta)$. The component c is the spectrum of a separable algebraic field extension L_c of $K = \kappa(\eta)$. By Cor. 5.37 and Lm. 5.19, the base-change $\widetilde{Y}_c$ is a disjoint union of finitely many components and each of them is geometrically connected. Thus, Y_{L_c} has geometrically connected components. Moreover, the number of these connected components is constant when c varies (by Thm. 5.36), let us say equal M. The scheme $\widetilde{S}_\eta$ is pro-étale over η and so a cofiltered limit $\lim S_\lambda$ of finite unions of spectra of finite separable extensions of K.

Lemma. For some λ_0, there is $S_{\lambda_0} = \sqcup_i \mathrm{Spec}(L_i)$ with L_i/K finite separable and such that the connected components of Y_{L_i} are geometrically connected (or, in other words, Y_{L_i} has precisely M connected components).

Indeed, for each S_λ, let $W_\lambda \subset S_\lambda$ be the union of those $\mathrm{Spec}(L_i) \subset S_\lambda$ that do not have this property. This forms a sub-inverse system of S_λ. We want to show that, for some λ_0, W_{λ_0} is empty. Assume the contrary. The maps between $W_\lambda' s$ are affine and thus $\widetilde{W} = \lim W_\lambda$ exists in the category of schemes and moreover $\widetilde{W}_{\mathrm{top}} = \lim W_{\lambda,\mathrm{top}}$ ([SP, Tag 0CUF]), where W_{top} denotes the underlying topological space of a scheme W. As $W_{\lambda,\mathrm{top}}$ are finite and non-empty, the inverse limit is non-empty as well (see [SP, Lm. 086J]). The image of any point $w \in \widetilde{W}$ in $\widetilde{S}$ gives a point of $\widetilde{S}$ which has as the residue field a separable extension L/K such that Y_L is a disjoint union of geometrically connected components, but L can be written as a filtered colimit of fields L_α with L_α/K finite separable and such that the connected components of Y_{L_α} are not all geometrically connected. But L must contain the smallest field L_{smallest} of Lm. 5.19 and consequently (using that L_{smallest}/K is finite) one of L_α must contain L_{smallest} as well, which (by Lm. 5.19 again) contradicts the fact that Y_{L_α} has a component that is not geometrically connected. Thus, we proved that there exists λ_0 such that $S_{\lambda_0} = \sqcup_{i=1}^{m_0} \mathrm{Spec}(L_i)$ with L_i/K finite separable and such that the connected components of Y_{L_i} are geometrically connected. This finishes the proof of the lemma.

Now, we have an equality $\pi_0(Y_{L_i}) = \pi_0(Y_{\bar{\eta}})$. Taking the preimages of the connected components of each Y_{L_i} in $\widetilde{Y}_\eta$ we see that $\widetilde{Y}_\eta$ decomposes as a disjoint union of clopen subsets $\widetilde{Z}_t$, parametrized by $t \in \pi_0(Y_{\bar{\eta}})$, such that each $\widetilde{Z}_t$ maps surjectively onto $\widetilde{S}_\eta$ and induces a continuous bijection $\pi_0(\widetilde{Z}_t) \to \pi_0(\widetilde{S}_\eta)$. More precisely, we have a diagram

$$
\begin{array}{ccc}
\widetilde{Y}_\eta & \longrightarrow & \widetilde{S}_\eta \\
\downarrow & & \downarrow \\
\sqcup_i Y_{L_i} & \longrightarrow & S_{\lambda_0} = \sqcup_i \mathrm{Spec}(L_i) \\
\downarrow & & \downarrow \\
Y_\eta & \longrightarrow & \eta
\end{array}
$$

and we know that for each i, $Y_{L_i} = \sqcup_{t\in\pi_0(Y_{\bar{\eta}})} Z_{i,t}$, with $Z_{i,t} \to \mathrm{Spec}(L_i)$ geometrically connected. We define $\widetilde{Z}_{i,t}$ to be the preimage of $Z_{i,t}$ in $\widetilde{Y}_\eta$ and put $\widetilde{Z}_i = \sqcup_{t\in\pi_0(Y_{\bar{\eta}})} \widetilde{Z}_{i,t}$. As the map $\widetilde{Z}_t \subset \widetilde{Y}_\eta \to \widetilde{S}_\eta$ is open, we get by Lm. 5.32 that $\pi_0(\widetilde{Z}_t) \to \pi_0(\widetilde{S}_\eta)$ is actually a homeomorphism. Thus, we get a homeomorphism $\pi_0(\widetilde{Y}_\eta) \simeq \bigsqcup_{t\in\pi_0(Y_{\bar{\eta}})} \pi_0(\widetilde{S}_\eta)$ as desired.

By the last claim, there is an isomorphism $\widetilde{Y} \simeq \bigsqcup_{t\in\pi_0(Y_{\bar{s}})} \widetilde{Y}_t$, i.e. $\widetilde{Y}$ splits as a union of clopen subsets parametrized by $t \in \pi_0(Y_{\bar{s}})$. Define $\widetilde{T} = \bigsqcup_{t\in\pi_0(Y_{\bar{s}})} \widetilde{S}_t$, where $\widetilde{S}_t$ is a copy of $\widetilde{S}$. There is an obvious morphism $\widetilde{Y} \to \widetilde{T}$, which restricted to a fixed $\widetilde{Y}_t$ factorizes through $\widetilde{S}_t$. The scheme $\widetilde{T}$ is in $\mathrm{Cov}_{\widetilde{S}}$ and we want to show that it descends to a cover of T. The morphism $\widetilde{Y} \to \widetilde{T}$ is surjective and has geometrically connected fibres. Indeed, to see the surjectivity, observe that by Lm. 5.32 and by the construction of $\widetilde{T}$, $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{T}_{\widetilde{X}})$ is a homeomorphism and the connected components of $\widetilde{T}_{\widetilde{X}}$ are isomorphic to connected components of $\widetilde{X}$ and thus noetherian. Restricting to such a component, $\widetilde{Y} \to \widetilde{T}_{\widetilde{X}}$ becomes a geometric cover with dense image and thus surjective, e.g. by [BS, Lm. 7.3.9.]. To see the connectedness of geometric fibres, observe that, by the construction, $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{T})$ is a homeomorphism and $\widetilde{T} = \pi_0(Y_{\bar{\eta}}) \times \widetilde{S}$. We can restrict the situation to a fixed connected component c of $\widetilde{S}$. Identifying $Y_{\bar{\eta}} \simeq \widetilde{Y}_c \times_c \bar{\eta}$ (using some lift of $\bar{\eta}$ from S to c), we see that the $\bar{\eta}$-fibre of $\widetilde{Y} \to \widetilde{T}$ is connected. But now it follows quite easily from Cor. 5.37 that in fact every geometric fibre is connected. The proof of existence of the descent datum and checking the cocycle condition follows essentially from the uniqueness of the infinite Stein factorization. We cannot, however, simply apply the uniqueness statement proven above, as $\pi_0(\widetilde{S})$ is not discrete (and we cannot simply argue by restricting to the connected components). Thus, we have to be slightly more careful. We need to equip $\widetilde{T}$ with a descent datum. Denote for brevity $\widetilde{S}_2 = \widetilde{S} \times_S \widetilde{S}$, $\widetilde{Y}_2 = \widetilde{Y} \times_Y \widetilde{Y}$, $\widetilde{X}_2 = \widetilde{X} \times_X \widetilde{X}$ and let $p, q : \widetilde{S}_2 \to \widetilde{S}$ be the canonical projections. We need to define an isomorphism over $\widetilde{S}_2$ between the two base-changes $p^*\widetilde{T}$ and $q^*\widetilde{T}$. We have diagrams

$$
\begin{array}{ccccc}
\widetilde{Y}_2 & \xrightarrow{\alpha} & p^*\widetilde{T} & \longrightarrow & \widetilde{S}_2 \\
{\scriptstyle p}\downarrow & & \downarrow & & {\scriptstyle p}\downarrow \\
\widetilde{Y} & \longrightarrow & \widetilde{T} & \longrightarrow & \widetilde{S}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\widetilde{Y}_2 & \xrightarrow{\beta} & q^*\widetilde{T} & \longrightarrow & \widetilde{S}_2 \\
{\scriptstyle q}\downarrow & & \downarrow & & {\scriptstyle q}\downarrow \\
\widetilde{Y} & \longrightarrow & \widetilde{T} & \longrightarrow & \widetilde{S}
\end{array}
$$

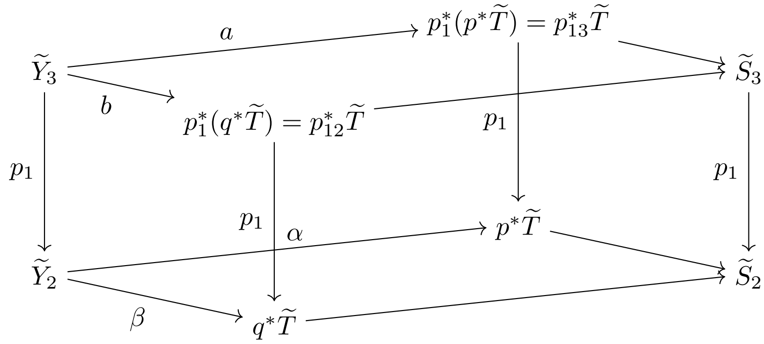
with all squares Cartesian. We claim that: α and β induce homeomorphisms on π_0's. Indeed, $\widetilde{Y} \to \widetilde{T}$ is universally open, surjective with geometrically connected fibres (we are using [EGA IV 2, Thm. 2.4.6] here to check openness. To check that $\widetilde{Y} \to \widetilde{T}$ is flat and locally of finite presentation, it is enough to these properties for $\widetilde{Y} \to \widetilde{T}_{\widetilde{X}} = \widetilde{T} \times_{\widetilde{S}} \widetilde{X}$, as $\widetilde{T}_{\widetilde{X}} \to \widetilde{T}$ has the desired properties. But $\widetilde{Y} \to \widetilde{T}_{\widetilde{X}}$ is a morphism of étale $\widetilde{X}$-schemes, so the properties follows. The surjectivity was proven above). Thus, the same is true for α and β. These assumptions imply that $\pi_0(\alpha)$ and $\pi_0(\beta)$ are continuous bijections and in fact homeomorphisms by Lm. 5.32.

From this claim we obtain a homeomorphism $\phi_0 = \pi_0(\beta) \circ \pi_0(\alpha)^{-1} : \pi_0(p^*\widetilde{T}) \to \pi_0(q^*\widetilde{T})$ over $\pi_0(\widetilde{S}_2)$.

Claim 2: ϕ_0 lifts uniquely to an isomorphism $\phi : p^*\widetilde{T} \to q^*\widetilde{T}$ over $\widetilde{S}_2$.

Proof of the claim: $p^*\widetilde{T}$ and $q^*\widetilde{T}$ are both isomorphic over $\widetilde{S}_2$ to a disjoin union $\sqcup_{t\in\pi_0(Y_{\bar{s}})} \widetilde{S}_2$. Fixing these isomorphisms, we can view ϕ_0 as a homeomorphism of $\sqcup_{t\in\pi_0(Y_{\bar{s}})} \pi_0(\widetilde{S}_2)$ with itself and we want to show, that it lifts to an isomorphism of $\sqcup_{t\in\pi_0(Y_{\bar{s}})}\widetilde{S}_2$. This follows by Lm. 5.38, as $\pi_0(\widetilde{S}_2)$ is compact and ϕ_0 is over the base $\pi_0(\widetilde{S}_2)$.

We need to show that ϕ satisfies the cocycle condition. Let $\widetilde{S}_3 = \widetilde{S} \times_S \widetilde{S} \times_S \widetilde{S}$ and analogously for $\widetilde{Y}$. For $i \in \{1, 2, 3\}$ let $p_i : \widetilde{S}_3 \to \widetilde{S}_2$ be the projection forgetting the i-th factor and for $i \neq j$ in $\{1, 2, 3\}$ denote $p_{ij} : \widetilde{S}_3 \to \widetilde{S}$ the projection forgetting the i-th and j-th factors. As in the case of double products, there are morphisms $\widetilde{Y}_3 \to p_{ij}^*\widetilde{T}$ fitting into suitable Cartesian diagrams. Denoting by a and b the morphisms $\widetilde{Y}_3 \to p_{13}^*\widetilde{T}$ and $\widetilde{Y}_3 \to p_{12}^*\widetilde{T}$ respectively, we have a commutative diagram



and analogous diagrams for the projections p_2 and p_3.
Claim 3: the induced maps $\pi_0(a) : \pi_0(\widetilde{Y}_3) \to \pi_0(p_{13}^*\widetilde{T})$ and $\pi_0(b) : \pi_0(\widetilde{Y}_3) \to \pi_0(p_{12}^*\widetilde{T})$ are homeomorphisms. The homeomorphism $\psi_0 = \pi_0(b) \circ \pi_0(a)^{-1}$ lifts uniquely to an isomorphism $\psi : p_{13}^*\widetilde{T} \to p_{12}^*\widetilde{T}$ over $\widetilde{S}_3$ and is equal to $p_1^*(\phi)$. Analogous statements hold respectively for the diagrams involving projections p_2 and p_3.
Proof of the claim: The proofs that $\pi_0(a), \pi_0(b)$ are homeomorphisms and that ψ_0 lifts canonically is virtually the same as the proof of the Claim 2. above. To see the last part of the claim we use the commutativity of the last diagram and the fact that $\widetilde{T} = \sqcup_t\widetilde{S}$, from which we easily conclude that $\psi_0 = \pi_0(p_1)^*(\phi_0)$ and from the definitions of ϕ and ψ we see that also $p_1^*(\phi) = \psi$.

Having the claim, the cocycle condition for ϕ follows and thus we have constructed a descent datum on $\widetilde{T}$. Moreover, by construction it is compatible with $\widetilde{Y} \to \widetilde{T}$. Thus, by fpqc descent, we obtain an sheaf T on S_{fpqc} that becomes constant on $\widetilde{S}$. Thus, we can view T as an element of Loc_S ([BS, Def. 7.3.1.]) and by the equivalence $\mathrm{Loc}_S = \mathrm{Cov}_S$ of [BS, Lm. 7.3.9.], T is representable by a geometric cover of S. The descent datum on $\widetilde{Y} \to \widetilde{T}$ gives a morphism $Y \to T$ over S by fpqc descent for morphisms of schemes (by [SP, Rmk. 040L] and [SP, Lm. 02W0]). See also Lm. 2.76 and the preceeding discussion. Let $\bar{t} \in T$ be a geometric point. It lifts to a geometric point on $\widetilde{T}$ (as $\widetilde{T} \to T$ is a base-change of $\widetilde{S} \to S$, and so weakly étale) and $Y_{\bar{t}} = \widetilde{Y} \times_{\widetilde{T}} \bar{t}$. Thus, $Y_{\bar{t}}$ is connected, as $\widetilde{Y} \to \widetilde{T}$ had geometrically connected fibres. Similarly, $Y \to T$ is surjective because $\widetilde{Y} \to \widetilde{T}$ was. Thus, T is is connected as an image of Y.

□

Remark 5.39. Having finished the above proof, one can use Cor. 5.10 together with Lm. 5.19 to conclude that, when S is normal, the indexing set that appeared many times in the proof of Thm. 5.5, namely $\pi_0(Y_{\bar{\eta}})$, is finite.

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Selbstständigkeitserklärung

Hiermit versichere ich, Marcin Lara,

- dass ich alle Hilfsmittel und Hilfen angegeben habe,
- dass ich auf dieser Grundlage die Arbeit selbständig verfasst habe,
- dass diese Arbeit nicht in einem früheren Promotionsverfahren eingereicht worden ist.

Berlin, den 03.12.2018
Marcin Lara

July 2019: a revised version with minor changes, some typos corrected and improved exposition.