ON THE GROWTH OF THE KRONECKER COEFFICIENTS.

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ABSTRACT. We study the rate of growth experienced by the Kronecker coefficients as we add cells to the rows and columns indexing partitions. We do this by moving to the setting of the reduced Kronecker coefficients.

1. INTRODUCTION

The Kronecker coefficients $g_{\lambda,\mu,\nu}$ are fundamental constants in Representation Theory. They describe how irreducible representations of $GL(V \otimes W)$ split, when viewed as representations of $GL(V) \times GL(W)$. They are also the structural constants for the tensor products of irreducible representations of the symmetric groups.

In spite of their importance, very little is known about the Kronecker coefficients, and this leaves some fundamental questions unanswered. For example, are the Kronecker coefficients described by a positive combinatorial rule, akin to the Littlewood–Richardson rule [2, 3, 22]? How difficult is it, algorithmically, to compute Kronecker coefficients [39, 10, 7, 42], or to determine they are nonzero [44, 21]? Remarkably, this latter problem relates the Kronecker coefficients with the *quantum marginal problem* in Quantum Information Theory [30, 13, 12].

A feature of the Kronecker coefficients that has been studied recently is the *stability* phenomenon: the fact that some sequences of Kronecker coefficients are eventually constant. The first example of such a behavior was observed by Murnaghan in 1938 [40]. The Kronecker coefficients $g_{\lambda,\mu,\nu}$ are indexed by triples of partitions (λ, μ, ν), and Murnaghan's stable sequences are obtained by incrementing the first part of all three partitions at each step. Their limit values (the *reduced*, or *stable* Kronecker coefficients) are interesting objects in their own right. As seen in [8], they contain enough information to recover the value of the Kronecker coefficients and are believed to be simpler to understand. For example, it is conjectured that they satisfy the *saturation property* [28, 31], and they have been used to find efficient formulas for computing some Kronecker coefficients [7].

Date: October 14, 2018.
E. Briand and M.Rosas are partially supported by projects MTM2013–40455–P, FQM–333, P12–FQM–2696 and FEDER.

Many more sequences of Kronecker coefficients are stable: large families have been produced by means of methods from geometry [36, 37, 38], enumerative combinatorics [54, 55] or symmetric functions calculations [41]. These stable sequences of Kronecker coefficients have general term of the form $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$, where we add, and multiply a partition by a scalar, as it is usually done for vectors.

Murnaghan's case corresponds to $\alpha = \beta = \gamma = (1)$.

In [52], it was conjectured that $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ stabilizes (for any λ, μ, ν) if and only if $g_{\alpha,\beta,\gamma} = 1$. This was proved in [47]. These stability phenomena are, as an aside, the prototype for the very general *representation stability* phenomenon unveiled in algebraic topology; see [15, 14, 46].

In this paper we present two new results related to the stability of Kronecker coefficients. The first one is indeed a result of stability, but the sequence that we consider is not of the type $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$. At each step, we simultaneously increase the first row and the first column of the Young diagrams of all three indexing partitions. We call this phenomenon *hook stability*.

Note that this hook stability does not seem to fit straighforwardly in the representation theory of fixed general linear groups, since it involves sequences of Kronecker coefficients indexed by partitions with unbounded lengths.

The second result is about the *asymptotics* of some sequences of Kronecker coefficients of type $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$, that do not stabilize, but are shown to grow linearly.

We describe the relevant coefficients (the limits for hook stability, and the coefficients appearing in quasipolynomial formulas for the asymptotic estimates, for the result on linear growth) by means of generating series.

Our tools are the following:

(1) *Vertex operators on symmetric functions.* Vertex operators on symmetric functions provide generating functions for Schur functions. They have been used widely by Thibon and his collaborators to establish several properties of stability. See Section 2.4.1 for references and a basic treatment of vertex operators.
(2) *The λ–ring formalism for symmetric functions.* This formalism is in fact a calculus on morphisms from the algebra of symmetric functions. See Section 2.3 for basic definitions and references.
(3) *Schur generating series* will be used to encode families of constants indexed by several partitions by means of symmetric series in several sets of variables. Important structural constants for symmetric functions have very compact Schur generaing series when expressed within the Lambda–ring formalism: $\sigma[XY + XZ]$ for Littlewood–Richardson coefficients, and

$\sigma[XYZ]$ for Kronecker coefficients. The coefficients introduced in this paper also have have simple Schur generating series.

The two sets of results in this paper (hook stability and linear growth) are obtained by first considering stability properties and linear growth for families of reduced Kronecker coefficients, and then translating the results obtained to Kronecker coefficients.

The two sets of results for reduced Kronecker coefficients are obtained the same way: by simplifying Schur generating series for sequences of reduced Kronecker coefficients by means of vertex operators. Because we have at our disposal two conjugate vertex operators (one related to first row increasing, the other to first column increasing) we simultaneously get these two sets of results.

This article is structured as follows. In Section 2 we introduce the basic tools used in this article. In particular, we review the two vertex operators that allow us to increase the sizes of the first row and column of a partition.

Section 3 presents the reduced Kronecker coefficients: it includes an elementary proof of Brion's formula [9], which we have not seen in the literature, and an elementary derivation of the generating function for the reduced Kronecker coefficients indexed by one row (and one column) shapes.

Section 4 provides the main technical lemma that allows us to factor a symmetric function (polynomial) out of some symmetric series. This lemma is applied twice in Sections 5 and 6, once with each of the two conjugate vertex operators.

In Section 5, we prove stability for the sequences of reduced Kronecker coefficients whose indexing partitions have their first column growing (Section 5.1). We deduce in Section 5.2 the hook stability property for Kronecker coefficients. Another approach to proving this property is explored in Section 5.3. We are not able to get an alternative proof of the hook stability property through this approach, but are led to the conjecture that Kronecker coefficients weakly increase when incrementing at the same time the first row and the first column of the diagram of each of their three indexing partitions (Conjecture 5.12).

In Section 6, we study the effect of the growth of the first rows of the partitions indexing a reduced Kronecker coefficient and obtain linear quasipolynomial formulas when these first rows are big enough. This also provides asymptotic linear quasipolynomial formulas for some sequences of Kronecker coefficients $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ where, together with other conditions, the partitions α, β and γ have at most two parts.

Schur generating functions for the limit values $\overline{\overline{g}}_{\lambda,\mu,\nu}$ in the hook stability property, and the coefficients of the quasipolynomial formulas of Section 6, are derived in Section 7.

Section 8 reviews some examples appearing in the literature, that study the effect of increasing the sizes of the other rows and columns.

Two appendices round up the results. In Appendix A, we provide a table with some constants appearing in Theorem 7.3. Appendix B contains a proof of some stability bounds for the hook stability described by Theorem 5.6.

2. Preliminaries

2.1. **Partitions.** A *partition of* n is a weakly descending sequence of non-negative integers whose sum is n. Two partitions that differ by a string of zeros are considered to be the same. The positive terms in a partition are called its *parts*, and the *length* $\ell(\lambda)$ of the partition λ is defined as the number of parts. The weight $|\lambda|$ of the partition λ is the sum of its parts. The conjugate of a partition λ will be denoted λ', and with parts $\lambda'_1, \lambda'_2, \ldots$. The empty partition will be denoted by $\emptyset$.

Let $\cup$ and $+$ be the standard operations on partitions, as defined in [35, I.§1]. If n is a nonnegative integer and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, then $n\lambda$ is the dilation of λ by a factor n, that is the partition $(n\lambda_1, n\lambda_2, \ldots, n\lambda_k)$.

Let $\overline{\lambda}$ be the partition obtained after removing the first term of λ. This operation can be iterated: $\overline{\overline{\lambda}}$ is the partition obtained from λ by removing the first two terms.

Let $\widehat{\lambda}$ be the partition obtained after removing the first row and the first column in the diagram of λ.

The sequence defined by prepending a first term a to the partition λ will be denoted (a, λ). The resulting sequence $(a, \lambda_1, \lambda_2, \ldots)$ is not necessarily a partition since we may have that $a < \lambda_1$. Given an integer N, we denote by $\lambda[N]$ the sequence $(N - |\lambda|, \lambda)$, which is also not necessarily a partition.

Finally, for any non–empty partition λ, we will write $\lambda \oplus (a|b)$ for $\lambda + (a) \cup (1^b)$.

For example, if $\lambda = (8, 3, 3, 1)$, then we have that $\overline{\lambda} = (3, 3, 1)$, $\overline{\overline{\lambda}} = (3, 1)$, $\widehat{\lambda} = (2, 2)$, $\lambda[20] = (5, \lambda) = (5, 8, 3, 3, 1)$ (not a partition), $\lambda[25] = (10, \lambda) = (10, 8, 3, 3, 1)$ and $\lambda \oplus (7|4) = (15, 3, 3, 1, 1, 1, 1, 1)$.

2.2. **Symmetric functions, Schur functions and Jacobi–Trudi determinants.** For λ a finite sequence of integers of length n, we define

$$s_\lambda = \det\left(h_{\lambda_j+i-j}\right)_{i,j=1\ldots n},$$

where $h_0 = 1$ and $h_k = 0$ for $k < 0$.

The Jacobi-Trudi formula implies that when λ is a partition then s_λ is the Schur function indexed by λ. Since rearranging the columns of the above determinant suffices to order the parts of any sequence of integers, we always obtain for s_λ either a Schur function (up to sign), or zero. For example, $s_{\lambda_1,\lambda_2} = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_2-1} \\ h_{\lambda_1+1} & h_{\lambda_2} \end{vmatrix}$, $s_{1,2} = 0$, and $s_{1,3} = -s_{(2,2)}$.

Let $Sym_{\mathbb{Q}} = Sym_{\mathbb{Q}}(X)$ be the algebra of symmetric functions with rational coefficients, with underlying alphabets $X = \{x_1, x_2, \ldots\}$. We denote by $\langle\,|\,\rangle$ or $\langle\,|\,\rangle_X$ the scalar product on $Sym_{\mathbb{Q}}$ defined by saying that the Schur functions are an orthonormal basis. For any symmetric function f, $f^{\perp}$ will denote the adjoint of multiplication by f. The scalar product is conveniently extended whenever it makes sense. For instance

$$\left\langle \sum_{i=0}^{\infty} f_i t^i \,\middle|\, \sum_{j=0}^{\infty} g_j \right\rangle = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \langle f_i \,|\, g_j \rangle \right) t^i$$

if, for each i, f_i is a symmetric function and for each j, g_j is a homogeneous symmetric function of degree j.

We also consider symmetric functions in different alphabets (set of variables) X, Y, Z. The scalar product is canonically extended to the algebras $Sym_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} Sym_{\mathbb{Q}}(Y)$, $Sym_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} Sym_{\mathbb{Q}}(Y) \otimes_{\mathbb{Q}} Sym_{\mathbb{Q}}(Z)$ they generate, and denoted by $\langle\,|\,\rangle_{X,Y}$ and $\langle\,|\,\rangle_{X,Y,Z}$.

2.3. **The λ–ring formalism for symmetric functions, and specializations.** Let $\mathcal{A}$ be any commutative algebra over a field $\mathcal{K}$ of characteristic zero

Given a morphism of algebras A from $Sym_{\mathbb{Q}}$ to $\mathcal{A}$, the image of a symmetric function f under A will be denoted with $f[A]$ rather than $A(f)$ and called "specialization of f at A".

Since the power sum symmetric functions p_k $(k \geq 1)$ generate $Sym_{\mathbb{Q}}$ and are algebraically independent, the map

$$A \mapsto (p_1[A], p_2[A], \ldots) \tag{1}$$

is a bijection from the set of all morphisms of algebras from $Sym_{\mathbb{Q}}$ to $\mathcal{A}$ to the set of infinite sequences of elements from $\mathcal{A}$. This set of sequences is endowed with its operations of component-wise sum, product, and product by a scalar. The bijection (1) is used to lift these operations to the set of morphism from $Sym_{\mathbb{Q}}$ to $\mathcal{A}$. This defines expressions like $f[A+B]$, $f[-A]$, $f[AB]$, $f[A/B]$ ... where f is a symmetric function and A and B are two specializations, and more general expressions $f[P(A, B, \ldots)]$ where $P(A, B, \ldots)$ is a rational function in several specializations A, B ... with coefficients in $\mathcal{K}$. Note that, by definition, for any power sum p_k $(k \geq 1)$, specializations A and B and scalar z,

$$p_k[A + B] = p_k[A] + p_k[B], \quad p_k[AB] = p_k[A]p_k[B], \quad p_k[zA] = zp_k[A].$$

Here are some important specializations. The specialization at -1 is defined on power sums by $p_k[-1] = -1$ for all k. The specialization ε is defined by $p_k[\varepsilon] = (-1)^k$ for all k. The product of the two previous specializations is $-\varepsilon$ and fulfills $p_k[-\varepsilon X] = (-1)^{k+1} p_k[X]$ for all k. As a consequence, the transformation $f[X] \mapsto f[-\varepsilon X]$ coincides with the standard involution ω defined by $\omega s_{\lambda} = s_{\lambda'}$ for all λ. There is also the specialization $X^\perp$ such that for any symmetric function f, $f[X^\perp] = f^\perp$, the adjoint of the multiplication by f with respect to $\langle \,|\, \rangle_X$.

Lemma 2.1. *Let $\sigma[X] = \sum_{n\geq 0} h_n[X]$ be the generating function for the complete homogeneous symmetric functions in X. It has the following well-known properties:*

(1) *Given an alphabet X,*

$$\sigma[X] = \prod_{x\in X} \frac{1}{1-x} \text{ and } \sigma[-X] = \prod_{x\in X}(1-x).$$

In particular for a single variable t, $\sigma[t] = 1/(1-t)$ and $\sigma[-t] = 1-t$;

(2) *Cauchy's Identity : $\sigma[XY] = \sum_\lambda s_\lambda[X]s_\lambda[Y]$.*

(3) *Given any two alphabets A and B, $\sigma[A+B] = \sigma[A]\sigma[B]$.*

(4) *The adjoint of multiplication by $\sigma[AX]$ with respect to $\langle \,|\, \rangle_X$. It has the following effect: $\sigma[AX^\perp]f[X] = f[X+A]$.*

(5) *As a particular case, we have the* reproducing kernel property *of $\sigma[AX]$: for any symmetric function f, $\langle \sigma[AX] \,|\, f[X]\rangle = f[A]$.*

Standard references for these results are [35] *and* [33]. *See also* [5].

Finally, it is well-known that using operations on alphabet, we can recover the Littlewood–Richardson, $c_{\lambda,\mu,\nu}$ and the Kronecker coefficients, $g_{\lambda,\mu,\nu}$:

$$s_\lambda[X+Y] = \sum_{\mu,\nu} c_{\lambda,\mu,\nu} s_\mu[X]s_\nu[Y]$$

$$s_\lambda[XY] = \sum_{\mu,\nu} g_{\lambda,\mu,\nu} s_\mu[X]s_\nu[Y]. \tag{2}$$

While (2) can be used to define the Kronecker coefficients, they can also be defined as follows. Let λ and μ be partitions of some integers. Define a product $*$ on the ring of symmetric functions where the

$$p_\lambda * p_\mu = \delta_{\lambda,\mu} z_\lambda^{-1} p_\lambda, \tag{3}$$

where, as usual, $z_\lambda = 1^{m_1}m_1!\cdots n^{m_n}m_n!$ and m_i are the number of parts of λ equal to i. Then, if ν is also a partition, the Kronecker coefficients $g_{\lambda,\mu,\nu}$ are

$$s_\mu[X] * s_\nu[X] = \sum_\lambda g_{\lambda,\mu,\nu} s_\lambda[X].$$

It's clear from (3), that $g_{\lambda,\mu,\nu} = 0$ if λ, μ and ν are not partitions of the same integer.

2.4. **Vertex operators for symmetric functions.**

2.4.1. *Vertex operators.* The *vertex operator for symmetric functions* $\Gamma_{(t|X)}$ is defined on the basis of Schur functions in X by:

$$\Gamma_{(t|X)} : s_\alpha[X] \mapsto \sum_{n\in\mathbb{Z}} s_{(n,\alpha)}[X]t^n$$

where t is an additional variable. Recall that Schur functions are defined using the Jacobi–Trudi identity in terms of the complete homogeneous basis, and that h_n is equal to zero when $n < 0$.

From [11, Lemma 3.1], this operator can be factorized as $\Gamma_{(t|X)} = \sigma[tX]\sigma\left[-\frac{1}{t}X^\perp\right]$ and therefore fulfills

$$\Gamma_{(t|X)}f[X] = \sigma[tX]f\left[X - \frac{1}{t}\right] \tag{4}$$

for any symmetric function f. In particular, given any partition α,

$$\sum_{n\in\mathbb{Z}} s_{(n,\alpha)}[X]t^n = \Gamma_{(t|X)}s_\alpha[X] = \sigma[tX]s_\alpha\left[X - \frac{1}{t}\right]. \tag{5}$$

Alternatively, using the index n for the weights of the partitions in the formal series instead of for the first parts, we have also

$$\sum_{n\in\mathbb{Z}} s_{(n-|\alpha|,\alpha)}[X]t^n = t^{|\alpha|}\Gamma_{(t|X)}s_\alpha[X] = \sigma[tX]s_\alpha\left[tX - 1\right]. \tag{6}$$

This follows from (4), and the fact that Schur functions are homogeneous.

This vertex operator is a classical tool in the theory of symmetric functions used in particular by Jing (see for instance [24]), and, for the study of various phenomena of stability, by Thibon and his collaborators [53, 49, 11, 48, 32]. It is the generating series for Bernstein's creation operators introduced in [56]. See also [35, I.§5 Ex. 29].

2.4.2. *Vertex operators for columns.* The vertex operator $\Gamma_{(t|X)}$ associates to any Schur function s_α a generating series for the Schur functions $s_{(n,\alpha)}$ obtained by prepending a first part n to α. Let us build another operator that associates to s_α a generating series for the Schur functions $s_{\alpha+(1^n)}$ obtained by adjoining to the Young diagram of α a first column (1^n). For this, apply the involution ω (that maps s_α to $s_{\alpha'}$), next the vertex operator $\Gamma_{(t|X)}$ (appends a first row to α') and then again ω (the new first row of α' becomes a new first column attached to α). That is, our new operator is $\omega\Gamma_{(t|X)}\omega$. It sends any Schur function s_α to $\omega\Gamma_{(t|X)}s_{\alpha'}$, which is equal to $\sum_n \omega s_{(n,\alpha')}t^n$.

Set $\lambda = (n, \alpha')$. Recall that $s_{(n,\alpha')}$ is the Jacobi–Trudi determinant $\det(h_{\lambda_j+i-j})_{i,j}$ of order $\ell(\alpha') + 1$. The involution ω exchanges h_k with e_k. Therefore $\omega s_{(n,\alpha')} = \det(e_{\lambda_j+i-j})_{i,j}$. We will denote with $\tilde{s}_{(1^n|\alpha)}$ the value of this determinant. When $n \geq \alpha'_1$, this determinant is equal to $s_{\alpha+(1^n)}$, the Schur function indexed by the partition obtained from α by adding a new column of size n to its diagram. For instance,

$\Gamma_{(-\varepsilon t|X)}(s_\emptyset) = \omega(\sigma[tX])$, the generating function for the elementary symmetric functions. We have thus

$$\begin{aligned}\omega\Gamma_{(t|X)}\omega s_\alpha[X] &= \sum_{n\in\mathbb{Z}} \tilde{s}_{(1^n|\alpha)}[X]t^n \\ &= \sum_{n\geq\alpha_1'} s_{\alpha+(1^n)}[X]t^n + \text{ terms of degree} < \alpha_1' \text{ in } t.\end{aligned}$$

Since ω coincides with $f[X] \mapsto f[-\varepsilon X]$,

$$\begin{aligned}(\omega\Gamma_{(t|X)}\omega)(f) = \omega\Gamma_{(t|X)} f[-\varepsilon X] &= \omega\sigma[tX] f\left[-\varepsilon\left(X - \frac{1}{t}\right)\right], \\ &= \sigma[-\varepsilon tX] f\left[-\varepsilon\left(-\varepsilon X - \frac{1}{t}\right)\right] = \sigma[-\varepsilon tX] f\left[X - \frac{1}{(-\varepsilon t)}\right],\end{aligned}$$

and thus

$$\omega\Gamma_{(t|X)}\omega = \sigma[-\varepsilon tX]\sigma\left[-\frac{1}{-\varepsilon t}X^\perp\right].$$

We will write $\Gamma_{(-\varepsilon t|X)}$ for $\omega\Gamma_{(t|X)}\omega$.

The operator $\Gamma_{(-\varepsilon t|X)}$ appears, for instance, in [24, 23, 25]. The operators $\Gamma_{(t|X)}$ and $\Gamma_{(-\varepsilon t|X)}$ are $V_\alpha(t)$ with $\alpha = 1$ and $\alpha = -1$ respectively in the notations of [23]. They are $S(t)$ and $S^*(t)$ in the notations of [25].

3. Reduced Kronecker coefficients

3.1. **Murnaghan Stability.** Murnaghan observed [40] that, for any triple of partitions λ, μ, ν of some positive integer n, the sequence of Kronecker coefficients $g_{\lambda+(m),\mu+(m),\nu+(m)}$ stabilizes (*i.e.* is eventually constant).

Several classical proofs exist of this fact. It has been shown by Littlewood using invariant theory [34], by Brion using geometric methods [9, §3.4, Corollary 1], and by Thibon by means of vertex operators [53, §3]. More recent proofs have been obtained by interpreting the Kronecker coefficients in the setting of representations of partition algebras [4], and by constructing appropriate frameworks for addressing stability in general [14, 46].

The stable value of the sequence $g_{\lambda+(m),\mu+(m),\nu+(m)}$ does not depend on the first part of λ, μ and ν. Accordingly, it will be denoted $\overline{g}_{\overline{\lambda},\overline{\mu},\overline{\nu}}$, and called here a *reduced Kronecker coefficient*. More precisely, the sequence with general term $g_{\overline{\lambda}[N],\overline{\mu}[N],\overline{\nu}[N]}$, beginning at some suitably large N, has a limit whose value we label with $\overline{g}_{\overline{\lambda},\overline{\mu},\overline{\nu}}$. Thus, while the reduced Kronecker coefficients are defined for any triple of partitions, the Kronecker coefficients are only defined for triples of partitions of the same integer.

M. Brion has shown that the sequence of Kronecker coefficients $g_{\lambda+(m),\mu+(m),\nu+(m)}$ is weakly increasing [9, §3.4, Corollary 1]. This implies in particular that $g_{\lambda,\mu,\nu} \leq \overline{g}_{\overline{\lambda},\overline{\mu},\overline{\nu}}$

It is shown in [8, Theorem 1.5] that $g_{\lambda+(m),\mu+(m),\nu+(m)} = \overline{g}_{\overline{\lambda},\overline{\mu},\overline{\nu}}$ holds for all m such that $|\lambda| + m \geq N_0(\alpha, \beta, \gamma)$, where

$$N_0(\alpha, \beta, \gamma) = \frac{|\alpha| + \alpha_1 + |\beta| + \beta_1 + |\gamma| + \gamma_1}{2}. \tag{7}$$

Moreover, Murnaghan showed that the reduced Kronecker coefficients are zero unless the following inequalities hold:

Lemma 3.1 (Murnaghan's inequalities)**.** *The reduced Kronecker coefficient $\overline{g}_{\lambda,\mu,\nu}$ are zero unless the following three conditions hold:*

$$\begin{cases} |\lambda| \leq |\mu| + |\nu| \\ |\mu| \leq |\lambda| + |\nu| \\ |\nu| \leq |\lambda| + |\mu| \end{cases} \tag{8}$$

3.2. **Brion's formula and the generating series for the Reduced Kronecker Coefficients.** In [9, §3.4, Corollary 1], M. Brion obtained the following formula for the reduced Kronecker coefficients.

Proposition 3.2. *For any three partitions α, β and γ,*

$$\overline{g}_{\alpha,\beta,\gamma} = \langle s_\alpha[X] s_\beta[Y] \,|\, \sigma[XY] s_\gamma[XY + X + Y] \rangle_{X,Y}\,. \tag{9}$$

We include an elementary proof of Brion's formula (and, at the same time, of Murnaghan's stability), based on the following elementary lemma.

Lemma 3.3. *A sequence with general term u_n stabilizes (is eventually constant) if and only if its generating series $g(t) = \sum_n u_n t^n$ takes the form $P(t)/(1-t)$ with $P(t)$ a polynomial. Moreover, the stable value of the sequence is $P(1)$.*

Proof of Proposition 3.2. All scalar products appearing in this proof will be taken with respect to X, Y, as in the statement of the proposition.

Let α, β and γ be three partitions. We have, for n big enough,

$$g_{\alpha[n],\beta[n],\gamma[n]} = \left\langle s_{\gamma[n]}[XY] \,\middle|\, s_{\alpha[n]}[X] s_{\beta[n]}[Y] \right\rangle.$$

We can write as well

$$g_{\alpha[n],\beta[n],\gamma[n]} = \left\langle s_{\gamma[n]}[XY] \,\middle|\, \sum_a s_{\alpha[a]}[X] \cdot \sum_b s_{\beta[b]}[Y] \right\rangle.$$

Indeed, the extra terms in the right–hand side of the scalar product do not contribute since they do not the same degree as the left–hand side. We simplify the series on the right–hand side using (5), to get

$$g_{\alpha[n],\beta[n],\gamma[n]} = \left\langle s_{\gamma[n]}[XY] \,\middle|\, \sigma[X] s_\alpha[X-1] \cdot \sigma[Y] s_\beta[Y-1] \right\rangle.$$

Let us introduce the generating series

$$g(t) = \sum_n \left\langle s_{\gamma[n]}[XY] \,\middle|\, \sigma[X]s_\alpha[X-1]\sigma[Y]s_\beta[Y-1] \right\rangle t^n$$
$$= \left\langle \sum_n s_{\gamma[n]}[XY]t^n \,\middle|\, \sigma[X]s_\alpha[X-1]\sigma[Y]s_\beta[Y-1] \right\rangle.$$

From (6), with XY instead of X, we have

$$g(t) = \langle \sigma[tXY]s_\gamma[tXY-1] \,|\, \sigma[X]s_\alpha[X-1]\cdot\sigma[Y]s_\beta[Y-1]\rangle$$

Using the adjoints of $\sigma[X]$ and $\sigma[Y]$ (see Lemma 2.1), we get

$$\begin{aligned} g(t) &= \langle \sigma[t(X+1)(Y+1)]s_\gamma[t(X+1)(Y+1)-1] \,|\, s_\alpha[X-1]s_\beta[Y-1]\rangle \\ &= \sigma[t]\,\langle \sigma[tXY]\sigma[tX]\sigma[tY]s_\gamma[t(X+1)(Y+1)-1] \,|\, s_\alpha[X-1]s_\beta[Y-1]\rangle \\ &= \sigma[t]\,\langle \sigma[tXY]s_\gamma[t(X+1)(Y+1)-1] \,|\, s_\alpha[X+t-1]s_\beta[Y+t-1]\rangle\,. \end{aligned}$$

(Note that we specialized $\sigma[AX^\perp]$ to $A=1$.) That is, $g(t) = \frac{1}{1-t}P(t)$, with $P(t)$ equal to

$$P(t) = \langle \sigma[tXY]s_\gamma[t(X+1)(Y+1)-1] \,|\, s_\alpha[X+t-1]s_\beta[Y+t-1]\rangle\,.$$

We expand $\sigma[tXY] = \sum_{k=0}^\infty h_k[XY]t^k$, and observe that the terms $h_k[XY]t^k$, for k big enough, do not contribute to the scalar product. Indeed, they are homogeneous of total degree $2k$ in X and Y, while the right–hand side has degree $|\alpha|+|\beta|$. The infinite series can thus be truncated, and, $P(t)$ is equal to

$$\left\langle \sum_{k=0}^{k_0} h_k[XY]t^k s_\gamma[t(X+1)(Y+1)-1] \,\middle|\, s_\alpha[X+t-1]\cdot s_\beta[Y+t-1] \right\rangle.$$

Under this form, it is manifest that $P(t)$ is a polynomial in t.

After Lemma 3.3, the sequence of coefficients of $g(t)$ is eventually constant. This sequence of coefficients coincides with the sequence of Kronecker coefficients $g_{\alpha[n],\beta[n],\gamma[n]}$ for $n \gg 0$. This proves that this sequence of Kronecker coefficients is eventually constant. Finally, substituting 1 for t in the expression for $P(t)$ we get Brion's Formula. □

Brion's formula is equivalent to the following identity:

$$\sigma[XY]s_\gamma[XY+X+Y] = \sum_{\alpha,\beta} \overline{g}_{\alpha,\beta,\gamma} s_\alpha[X]s_\beta[Y].$$

Introducing a third alphabet Z, multiplying by $s_\gamma[Z]$, and summing over all partitions γ, yields the following analogue of Cauchy's formula for reduced Kronecker coefficients, which makes manifest the symmetry in the three indexing partitions:

$$\sum_{\alpha,\beta,\gamma} \overline{g}_{\alpha,\beta,\gamma} s_\alpha[X]s_\beta[Y]s_\gamma[Z] = \sigma[XYZ+XY+XZ+YZ]. \tag{10}$$

Conversely, Brion's Formula is obtained from (10) by taking the scalar product with $s_\gamma[Z]$ and making use of the reproducing kernel property for $\sigma[AZ]$ with $A = XY + X + Y$.

3.3. **Reduced Kronecker coefficients indexed by three one–row shapes or three one–column shape.** Let x, y and z be three variables. By specializing, in the generating series for the reduced Kronecker coefficients (10), the alphabets X, Y and Z to x, y, and z, we get that

$$\sigma[xyz + xy + xz + yz] = \sum_{(a,b,c)\in\mathbb{N}^3} \overline{g}_{(a),(b),(c)} x^a y^b z^c,$$

which is the ordinary generating function for the reduced Kronecker coefficients indexed by three one–row shapes.

Similarly, we get the generating function for the reduced Kronecker coefficients indexed by three one–column shapes by specializing, in (10), the alphabets X, Y and Z to $-\varepsilon x$, $-\varepsilon y$, and $-\varepsilon z$:

$$\sigma[-\varepsilon xyz + xy + xz + yz] = \sum_{(a,b,c)\in\mathbb{N}^3} \overline{g}_{(1^a),(1^b),(1^c)} x^a y^b z^c.$$

From the properties of the series σ, one gets straightforwardly the following simple expressions for the generating series:

$$\sigma[-\varepsilon xyz + xy + xz + yz] = \frac{1 + xyz}{(1 - xy)(1 - xz)(1 - yz)}$$

and

$$\sigma[xyz + xy + xz + yz] = \frac{1}{(1 - xyz)(1 - xy)(1 - xz)(1 - yz)},$$

which is, as an aside,

$$\frac{1}{1 - (xyz)^2} \cdot \sigma[-\varepsilon xyz + xy + xz + yz].$$

Proposition 3.4. *The above generating series admit the following expansions:*

$$\sigma[-\varepsilon xyz + xy + xz + yz] = \sum_{(a,b,c)\in\mathcal{C}\cap\mathbb{N}^3} x^a y^b z^c \tag{11}$$

and

$$\sigma[xyz+xy+xz+yz] = \sum_{(a,b,c)\in\mathcal{C}\cap\mathbb{N}^3} \left(1 + \left\lfloor \frac{\min\{\ell_1, \ell_2, \ell_3\}}{2} \right\rfloor\right) x^a y^b z^c \tag{12}$$

where $\ell_i = \ell_i(a, b, c)$ *with*

$$\begin{aligned} \ell_1(a, b, c) &= b + c - a,\\ \ell_2(a, b, c) &= a + c - b,\\ \ell_3(a, b, c) &= a + b - c \end{aligned} \tag{13}$$

and $\mathcal{C}$ *is the cone in* $\mathbb{R}^3$ *with coordinates* a, b, c, *defined by*

$$\begin{aligned} \ell_1 &\geq 0, \\ \ell_2 &\geq 0, \\ \ell_3 &\geq 0. \end{aligned} \tag{14}$$

Remark 3.5. The inequalities (14), defining the cone $\mathcal{C}$, are precisely those described by Murnaghan's inequalities (8), as a non–vanishing condition for the reduced Kronecker coefficients.

Remark 3.6. The proposition amounts to explicit formulas for the reduced Kronecker coefficients indexed by three one–row shapes and three one–column shapes:

$$\overline{g}_{(1^a),(1^b),(1^c)} = \begin{cases} 1 & \text{if } (a,b,c) \in \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\overline{g}_{(a),(b),(c)} = \begin{cases} 1 + \left[\frac{\min\{\ell_1,\ell_2,\ell_3\}}{2}\right] & \text{if } (a,b,c) \in \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

They can be derived from [45, Corollary 5] and [45, Theorem 13] respectively. We give a different proof below.

Proof. We first prove (11). We begin with the expansion

$$\frac{1}{(1-xy)(1-xz)(1-yz)} = \sum_{(i,j,k)\in\mathbb{N}^3} (xy)^i (xz)^j (yz)^k = \sum_{(i,j,k)\in\mathbb{N}^3} x^{i+j} y^{i+k} z^{j+k}.$$

We solve the following system in i, j, k, over the rational numbers:

$$\begin{array}{ccccccc} a & = & i & + & j & & \\ b & = & i & & & + & k \\ c & = & & & j & + & k \end{array}$$

It has a unique solution, $i = (a+b-c)/2$, $j = (a+c-b)/2$, $k = (b+c-a)/2$. This solution is a triple of nonnegative integers if and only if $a+b+c \equiv 0 \mod 2$, and the inequalities (14) hold. Therefore,

$$\frac{1}{(1-xy)(1-xz)(1-yz)} = \sum x^a y^b z^c,$$

where the sum is over all $(a,b,c) \in \mathbb{N}^3 \cap \mathcal{C}$ that satisfy $a+b+c \equiv 0 \mod 2$. As a consequence, we have also

$$\frac{xyz}{(1-xy)(1-xz)(1-yz)} = \sum x^a y^b z^c$$

where the sum is over all $(a,b,c) \in \mathbb{N}^3 \cap \mathcal{C}$ that satisfy $a+b+c \equiv 1 \mod 2$. Formula (11) follows.

Let us prove now (12). We observe that

$$\sigma[xyz+xy+xz+yz] = \frac{1}{1-(xyz)^2}\sigma[-\varepsilon xyz+xy+xz+yz]$$
$$= \sum_{m\in\mathbb{N}} x^{2m}y^{2m}z^{2m} \cdot \sum_{(i,j,k)\in\mathbb{N}^3\cap\mathcal{C}} x^i y^j z^k.$$

Therefore, the coefficient $\overline{g}_{(a),(b),(c)}$ of $x^a y^b z^c$ in $\sigma[xyz+xy+xz+yz]$ is the number of integers $m \geq 0$ such that $(a,b,c)-2(m,m,m) \in \mathcal{C}$. It is obtained as the number of solutions $m \geq 0$ of

$$\forall i \in \{1,2,3\}, \quad \ell_i((a,b,c)-2(m,m,m)) \geq 0$$

But $\ell_i((a,b,c)-2(m,m,m)) = \ell_i(a,b,c) - 2\ell_i(m,m,m) = \ell_i(a,b,c) - 2m$. Therefore $\overline{g}_{(a),(b),(c)}$ is the number of integers $m \geq 0$ such that $m \leq \ell_i(a,b,c)/2$ for all $i=1,2,3$. This is $1+[\min_i \ell_i/2]$, as claimed. □

4. A factorization.

Given an alphabet X', define $\Gamma_{(X'|X)} = \sigma[X'X]\sigma\left[-\frac{1}{X'}X^{\perp}\right]$. This allows us to consider the two vertex operators we are working with in this paper simultaneously, as for $X'=t$ we recover the standard vertex operator, and for $X'=-\epsilon t$ the vertex operator for columns.

Let X, Y, Z, X', Y' and Z' be six independent alphabets. Let $F(X,Y,Z) = XYZ+XZ+YZ+XY$. For any triple of partitions α, β, γ, let $\Phi_{\alpha,\beta,\gamma}$ be the series

(15)

$$\Phi_{\alpha,\beta,\gamma} = \left\langle \sigma[F(X,Y,Z)] \,\middle|\, \Gamma_{(X'|X)}s_\alpha[X]\Gamma_{(Y'|Y)}s_\beta[Y]\Gamma_{(Z'|Z)}s_\gamma[Z] \right\rangle_{X,Y,Z}$$

Then $\Phi_{\alpha,\beta,\gamma}$ is a symmetric series in each of the sets of variables X', Y' and Z'. Lemma 4.1 below decribes this symmetric series in more detail.

Lemma 4.1. *For any partitions α, β and γ, there exists a symmetric function $Q_{\alpha,\beta,\gamma}$ (in the alphabets X', Y' and Z') such that*

$$\Phi_{\alpha,\beta,\gamma} = \sigma[X'Y'Z'+X'Y'+X'Z'+Y'Z'] \cdot Q_{\alpha,\beta,\gamma}.$$

The symmetric function $Q_{\alpha,\beta,\gamma}$ is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion in the Schur basis of $\sigma[H]$ (as a symmetric series in X, Y and Z), where

$$H = (X+X')(Y+Y')(Z+Z')+(X+X')(Y+Y')+(X+X')(Z+Z') + (Y+Y')(Z+Z') - (X'Y'Z'+X'Y'+X'Z'+Y'Z') - X/X' - Y/Y' - Z/Z'.$$

The main point of this lemma is that $Q_{\alpha,\beta,\gamma}$ is not just a symmetric series but a symmetric function; it has finitely many non–zero homogeneous components.

Proof. Fix three partitions α, β and γ.

In (15), we move the $\Gamma_{(|)}$ to the left–hand side of the scalar product by taking adjoints. The adjoint of $\Gamma_{(X'|X)}$ (with respect to the alphabet X) is the operator $\sigma[-\frac{1}{X'}X]\sigma[X'X^{\perp}]$ that sends $f[X]$ to $\sigma[-\frac{1}{X'}X]f[X+X']$ and, likewise, for $\Gamma_{(Y'|Y)}$ and $\Gamma_{(Z'|Z)}$.

As a result, $\Phi_{\alpha,\beta,\gamma} = \langle \sigma[G] \,|\, s_\alpha[X]s_\beta[Y]s_\gamma[Z]\rangle$ with

$$G = F(X + X', Y + Y', Z + Z') - X/X' - Y/Y' - Z/Z'.$$

Let us split G as $G = F(X', Y', Z') + H$. That is, H is obtained from G by deleting all monomials that do not involve X, Y nor Z. Then H is given by the formula in the lemma.

We have $\sigma[G] = \sigma[F(X', Y', Z')]\cdot\sigma[H]$ by Lemma 2.1. Since $\sigma[F(X', Y', Z')]$ does not depend on X, Y and Z, it can be factored out of the scalar product:

$$\Phi_{\alpha,\beta,\gamma} = \sigma[F(X', Y', Z')] \cdot \langle \sigma[H] \,|\, s_\alpha[X]s_\beta[Y]s_\gamma[Z]\rangle_{X,Y,Z}\,.$$

This gives the announced factorization, since the scalar product in the above formula is equal to $Q_{\alpha,\beta,\gamma}$.

We contend that the non–zero homogeneous components of $Q_{\alpha,\beta,\gamma}$ (in the variables in X', Y' and Z') have bounded degrees. Indeed, we have the expansion

$$\sigma[H] = \sum_{k=0}^{\infty} h_k[H].$$

But all terms in H have total degree at least 1, with respect to the variables X, Y and Z. Therefore for each k, $h_k[H]$ is a sum of homogeneous symmetric functions (in X, Y and Z) of total degrees $\geq k$. When $k > |\alpha| + |\beta| + |\gamma|$, the term $h_k[H]$ does not contribute to the scalar product with $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$. The sum can therefore be truncated, so that

$$Q_{\alpha,\beta,\gamma} = \left\langle \sum_{k=0}^{|\alpha|+|\beta|+|\gamma|} h_k[H] \,\middle|\, s_\alpha[X]s_\beta[Y]s_\gamma[Z] \right\rangle_{X,Y,Z}.$$

This makes clear that the homogeneous components of Q, as a symmetric series in X', Y' and Z', have bounded degree. □

5. Hook stability

In this section, we show the existence of a stability phenomenon reminiscent of the one described by Murnaghan, when we simultaneously increase the first row and first column of each of the three indexing partitions of a Kronecker coefficient. We call this "hook stability".

We start by establishing the corresponding property for reduced Kronecker coefficients (Section 5.1). The hook stability property for Kronecker coefficients is then deduced in Section 5.2. An alternative approach to proving this property, using only well–known properties of Kronecker coefficients, is explored in Section 5.3.

5.1. **Stability for reduced Kronecker coefficients under first column increasing.** In this section we show that reduced Kronecker coefficients *themselves* stabilize when we increase the first column of each of their three indexing partitions.

Example 5.1. The reduced Kronecker coefficients $\overline{g}_{(2,2)\cup 1^k,(3)\cup 1^k,(4)\cup 1^k}$ stabilize with stable value 204. Their first values for $k = 1,2,\dots$ are

$$1, 17, 66, 133, 180, 198, 203, 204, 204, 204\dots$$

and $\overline{g}_{(2,2)\cup 1^k,(3)\cup 1^k,(4)\cup 1^k} = 204$ for all $k \geq 8$.

More interesting examples are given in Remark 3.6 (where we combinatorially describe all possible situations that can obtained when we start with three empty shapes) and Example 5.4.

We now proceed to study the general situation. Let α, β and γ be three partitions.

Theorem 5.2. *For any triple of partitions α, β, γ, there exist integers k_1, k_2, k_3 and $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ such that whenever $a \geq \ell(\alpha)$, $b \geq \ell(\beta)$, $c \geq \ell(\gamma)$, and*

$$\begin{aligned} b+c-a &\geq k_1 \\ a+c-b &\geq k_2 \\ a+b-c &\geq k_3, \end{aligned} \tag{16}$$

we have $\overline{g}_{\alpha+(1^a),\beta+(1^b),\gamma+(1^c)} = \overline{\overline{g}}_{\alpha,\beta,\gamma}$.

In light of Theorem 5.2, we call the value $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ the *column stable value* of the reduced Kronecker coefficient.

The conditions $a \geq \ell(\alpha)$, $b \geq \ell(\beta)$, $c \geq \ell(\gamma)$ ensure us that, after adding cells to the new first columns of the three original partitions, we obtain proper partitions. Note that they define a translation of the cone described by Murnaghan's inequalities: see (8).

Proof. For any nonnegative integers a, b, c, set

$$\phi^-_{a,b,c} = \left\langle \sigma[F(X,Y,Z)] \,\middle|\, \tilde{s}_{(1^a|\alpha)}[X]\tilde{s}_{(1^b|\beta)}[Y]\tilde{s}_{(1^c|\gamma)}[Z] \right\rangle.$$

with $F(X,Y,Z) = XYZ + XY + XZ + YZ$. Comparing with (10) we obtain that, when $a \geq \ell(\alpha)$, $b \geq \ell(\beta)$ and $c \geq \ell(\gamma)$,

$$\phi^-_{a,b,c} = \overline{g}_{\alpha+(1^a),\beta+(1^b),\gamma+(1^c)}.$$

Let us consider the generating series $\Phi^-_{\alpha,\beta,\gamma} = \sum_{a,b,c} \phi^-_{a,b,c} x^a y^b z^c$. Then

$$\Phi^-_{\alpha,\beta,\gamma} = \langle \sigma[F(X,Y,Z)] | \Gamma_{(-\varepsilon x|X)} s_\alpha[X] \Gamma_{(-\varepsilon y|Y)} s_\beta[Y] \Gamma_{(-\varepsilon z|Z)} s_\gamma[Z] \rangle$$

That is, $\Phi^-_{\alpha,\beta,\gamma}$ is the specialization of the series $\Phi_{\alpha,\beta,\gamma}$ of Lemma 4.1 at $X' = -\varepsilon x$, $Y' = -\varepsilon y$ and $Z' = -\varepsilon z$. Let $Q^-_{\alpha,\beta,\gamma}(x,y,z)$ be the polynomial obtained from $Q_{\alpha,\beta,\gamma}$ by means of the same specialization. After Lemma 4.1, we have thus $\Phi^-_{\alpha,\beta,\gamma} = \sigma[-\varepsilon xyz+xy+xz+yz]\cdot Q^-_{\alpha,\beta,\gamma}$. Set $\mathbf{t}^{(a,b,c)} = x^a y^b z^c$. Let $\sum_{\omega\in\Omega} q_\omega \mathbf{t}^\omega = Q^-_{\alpha,\beta,\gamma}$ be the expansion of $Q^-_{\alpha,\beta,\gamma}$ in monomials, where Ω is the (finite) support of $Q^-_{\alpha,\beta,\gamma}$. It follows from (11) that

$$\sigma[-\varepsilon xyz + xy + xz + yz] = \sum_{\theta\in\mathcal{C}\cap\mathbb{N}^3} \mathbf{t}^\theta.$$

Therefore,

$$\Phi^-_{\alpha,\beta,\gamma}(x,y,z) = \sum_{\omega\in\Omega,\theta\in\mathcal{C}\cap\mathbb{N}^3} q_\omega \mathbf{t}^{\omega+\theta}.$$

It follows that, for any $\tau = (a,b,c) \in \mathbb{Z}^3$, $\phi^-_\tau = \sum_\omega q_\omega$, where the sum is over all ω such that $\tau - \omega \in \mathcal{C}$. Recall that the cone $\mathcal{C}$ is defined by the inequalities $\ell_i \geq 0$ (see Proposition 3.4). Therefore, the sum is over all ω such that $\ell_i(\tau-\omega) \geq 0$ for all i, or, equivalently, $\ell_i(\tau) \geq \ell_i(\omega)$ for all i.

Suppose now that $\ell_i(\tau) \geq \ell_i(\omega)$ for all i and all $\omega \in \Omega$, or, equivalently, that $\ell_i(\tau) \geq \max_{\omega\in\Omega} \ell_i(\omega)$ for all i. Then $\phi^-_\tau = \sum_{\omega\in\Omega} q_\omega$, a value that does not depend on τ.

This proves the theorem, with $k_i = \max_{\omega\in\Omega} \ell_i(\omega)$. □

Remark 5.3. One can show that in Theorem 5.2, one can take

$$\begin{aligned} k_1 &= |\alpha| + \alpha_1 + \beta'_1 + \gamma'_1, \\ k_2 &= |\beta| + \beta_1 + \alpha'_1 + \gamma'_1, \\ k_3 &= |\gamma| + \gamma_1 + \alpha'_1 + \beta'_1. \end{aligned}$$

Example 5.4. Let us compute some polynomials $Q^-_{\alpha,\beta,\gamma}$.

We will consider the case when α, β and γ are one–row shapes, (p), (q) and (r) respectively. The coefficient of $s_{(p)}[X]s_{(q)}[Y]s_{(r)}[Z]$ in $\sigma[H(-\varepsilon x, -\varepsilon y, -\varepsilon z)]$ can be obtained by specializing the alphabets to only one letter: $X = \{x_1\}$, $Y = \{y_1\}$, $Z = \{z_1\}$, and taking the coefficient of $x_1^p y_1^p z_1^r$. That is, the generating function $\sigma[H(-\varepsilon x, -\varepsilon y, -\varepsilon z)]$ becomes an ordinary generating function:

$$\sum Q^-_{(p),(q),(r)} x_1^p y_1^q z_1^r = \frac{(1+xyz_1)(1+xzy_1)(1+yzx_1)}{(1-x_1y_1z_1)} \times \frac{(1+yx_1)(1+xy_1)(1+zx_1)(1+xz_1)(1+zy_1)(1+yz_1)}{(1-x_1y_1)(1-x_1z_1)(1-y_1z_1)(1+\frac{x_1}{x})(1+\frac{y_1}{y})(1+\frac{z_1}{z})}$$

From this, it follows, for instance,

$$
\begin{aligned}
&Q^-_{\emptyset,\emptyset,\emptyset} = 1,\\
&Q^-_{\emptyset,\emptyset,(1)} = x + y + xy - 1/z,\\
&Q^-_{\emptyset,\emptyset,(2)} = x^2y^2 + x^2y + xy^2 + xy - xy/z - x/z - y/z + 1/z^2,\\
&Q^-_{\emptyset,(1),(1)} = x^2yz + x^2y + x^2z + 2\,xyz + x^2,\\
&\qquad\qquad + xy + xz + yz - x - x/y - x/z + 1/(yz) - 1,
\end{aligned}
$$

Let us consider more closely the case $\emptyset$, $\emptyset$, (1). This case corresponds to the reduced Kronecker coefficients $\overline{g}_{(1^a),(1^b),(2,1^{c-1})}$. From the description $\overline{g}_{(1^a),(1^b),(2,1^{c-1})} = \sum q_\omega$, with the sum over the ω in the support of Q^- such that $(a, b, c) \in \omega + \mathcal{C}$, we obtain the following explicit description (it is assumed that $c \geq 1$):

$$
\overline{g}_{(1^a),(1^b),(2,1^{c-1})} = \begin{cases} 1 & \text{for } c = |a-b| \text{ with } a+b > c+1 \\ & \text{and for } c > |a-b| \text{ with } a+b = c+1, \\ 2 & \text{for } c > |a-b| \text{ with } a+b > c+1, \\ 0 & \text{otherwise.} \end{cases}
$$

This is the Kronecker coefficient $g_{(n-a,1^a),(n-b,1^b),(n-c-1,2,1^{c-1})}$ for $n \geq (a+b+c+5)/2$.

This result also follows from the computations in [45] and [53].

5.2. **Towards hook stability for the Kronecker coefficients.** We discuss how, combining our results, with the classical stability phenomena of Murnaghan, we obtain that the Kronecker coefficients are stable when we increase the first row and first column of the three indexing partitions *simultaneously*. We will be using the notations for $\overline{\lambda}$, $\widehat{\lambda}$, $\lambda \oplus (a|b)$ as defined in Section 2.1.

Example 5.5. Table 1 presents the Kronecker coefficients $g_{\lambda\oplus(i|j),\lambda\oplus(i|j),\lambda\oplus(i|j)}$ for $\lambda = (3,3)$ and i and j between 0 and 9. We know that each column of the table is stable because of Murnaghan's result, and that each row is eventually zero because these sequences will eventually fail a condition for positivity described by Dvir, Klemm, and Clausen–Meier in [20, 29, 16]. But we observe a more general stability phenomenon. There is a grey region where the coefficients are 145.

Let λ, μ and ν be three non–empty partitions of the same weight. Let a, b, c and m be nonnegative integers, such that a, b and c do not exceed m. Under certain conditions, made precise in Theorem 5.6 below, we will have:

$$
g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)} = \overline{g}_{\overline{\lambda}\cup(1^a),\overline{\mu}\cup(1^b),\overline{\nu}\cup(1^c)} = \overline{\overline{g}}_{\widehat{\lambda},\widehat{\mu},\widehat{\nu}}.
$$

This is made precise in the following theorem.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9
0	0	1	5	5	1	0	0	0	0	0
1	1	8	27	40	30	11	1	0	0	0
2	1	15	53	89	91	64	33	11	1	0
3	2	19	62	108	129	122	97	64	33	11
4	2	19	63	112	138	141	135	122	97	64
5	2	19	63	112	139	145	144	141	135	122
6	2	19	63	112	139	145	145	145	144	141
7	2	19	63	112	139	145	145	145	145	145
8	2	19	63	112	139	145	145	145	145	145
9	2	19	63	112	139	145	145	145	145	145

TABLE 1. The Kronecker coefficients $g_{(3,3)\oplus(i|j),(3,3)\oplus(i|j),(3,3)\oplus(i|j)}$.

Theorem 5.6. *For any triple of non–empty partitions λ, μ, ν of the same weight, there exists integers d_1, d_2, d_3 and d such that for all $(a,b,c,m) \in \mathbb{N}^4$ with*

$$\begin{aligned} \ell_i(a,b,c) &\geq d_i \quad \text{for all } i \in \{1,2,3\}, \\ m-(a+b+c)/2 &\geq d, \\ m \geq a,b,c. & \end{aligned} \tag{17}$$

we have,

$$g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)} = \overline{g}_{\overline{\lambda}\cup(1^a),\overline{\mu}\cup(1^b),\overline{\nu}\cup(1^c)} = \overline{\overline{g}}_{\widehat{\lambda},\widehat{\mu},\widehat{\nu}}. \tag{18}$$

The linear forms $\ell_i(a,b,c)$ in the theorem are those defined in (13).

Proof. Let N be the weight of λ, μ and ν. The second equality in (18) holds when $\ell_i(a+\lambda'_1-1, b+\mu'_1-1, c+\nu'_1-1) \geq k_i(\widehat{\lambda},\widehat{\mu},\widehat{\nu})$ for all i, where k_i are defined in Theorem 5.2. We have $\ell_i(a+\lambda'_1-1, b+\mu'_1-1, c+\nu'_1-1) = \ell_i(a,b,c)+\ell_i(\lambda'_1,\mu'_1,\nu'_1)-1$. Therefore the second equality holds when, for all i, we have $\ell_i(a,b,c) \geq d_i$, with $d_i = k_i(\widehat{\lambda},\widehat{\mu},\widehat{\nu}) - \ell_i(\lambda'_1,\mu'_1,\nu'_1)+1$.

On the other hand, the first equality in (18) holds when $m+N \geq N_0(\overline{\lambda}\cup(1^a),\overline{\mu}\cup(1^b),\overline{\nu}\cup(1^c))$ (the number N_0 as defined in (7)). Lemma 5.8, that comes just below, shows that

$$N_0(\overline{\lambda}\cup(1^a),\overline{\mu}\cup(1^b),\overline{\nu}\cup(1^c)) \leq N_0(\widehat{\lambda},\widehat{\mu},\widehat{\nu}) + \frac{\lambda'_1+\mu'_1+\nu'_1}{2} + \frac{a+b+c}{2}.$$

From this we conclude that the first equality holds when $m-(a+b+c)/2 \geq d$ with $d = N_0(\widehat{\lambda},\widehat{\mu},\widehat{\nu}) + \frac{\lambda'_1+\mu'_1+\nu'_1}{2} - N$. □

Example 5.7. Let us go back to Table 1. The reduced Kronecker coefficients $\overline{g}_{(3)\cup(1^j),(3)\cup(1^j),(3)\cup(1^j)}$ are $2, 19, 63, 112, 139$ and then, for $j \geq 5$, to $\overline{\overline{g}}_{(2),(2),(2)} = 145$. Moreover, the sequences are stable when $j \geq 5$ and $i-5 \geq (j-5)/2$.

The Kronecker coefficients of the main diagonal are 0, 8, 53, 108, 138, and finally 145 for all $m \geq 5$. The values of the bounds d and d_i for the stability degrees given in the proof of Theorem 5.6 are $d = 3$ and $d_i = 5$. This corresponds to stability for $j \geq 5$ and $i - j/2 \geq 3$, which is not far from being sharp.

Lemma 5.8. *Let λ, μ and ν be three non–empty partitions with the same weight. We have* $N_0(\overline{\lambda}, \overline{\mu}, \overline{\nu}) \leq N_0(\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}) + \frac{\lambda'_1+\mu'_1+\nu'_1}{2}$.

Proof. Recall from (7) that $N_0(\overline{\lambda}, \overline{\mu}, \overline{\nu}) = \frac{|\overline{\lambda}|+\overline{\lambda}_1+|\overline{\mu}|+\overline{\mu}_1+|\overline{\nu}|+\overline{\nu}_1}{2}$. Observe that $|\overline{\lambda}| = |\widehat{\lambda}| + (\lambda'_1 - 1)$ and

$$\overline{\lambda}_1 = \begin{cases} \widehat{\lambda}_1 \quad +1 & \text{if } \ell(\lambda) \geq 2, \\ \widehat{\lambda}_1 & \text{if } \ell(\lambda) = 1. \end{cases}$$

Likewise for μ and ν instead of λ. The lemma follows. Additionally we see that the inequality is actually an equality, except when at least one of the partitions has only one row. □

Corollary 5.9. *Let λ, μ and ν be non–empty partitions of the same weight. The sequence of Kronecker coefficients* $g_{\lambda\oplus(n|n),\mu\oplus(n|n),\nu\oplus(n|n)}$ *stabilizes to* $\overline{\overline{g}}_{\widehat{\lambda},\widehat{\mu},\widehat{\nu}}$.

Proof. This corresponds to $(a, b, c, m) = (n, n, n, 2n)$ and fulfills all inequalities in (17) for $n \gg 0$. □

Remark 5.10. For $(a, b, c, m) = n \cdot (1, 1, 1, 2)$ we have $\ell_i = n$ and $m - (a + b + c)/2 = n/2$. Therefore the stable behavior in Corollary 5.9 takes place already for $n \geq \max(2\,d, d_1, d_2, d_3)$.

5.3. **Another approach to the hook stability property, derived from Murnaghan's stability and conjugation.** In this section we show that using only the well–known invariance of the Kronecker coefficients under conjugating two of their three indexing partitions (see for instance [35, 50]),

$$g_{\lambda,\mu,\nu} = g_{\lambda',\mu',\nu} = g_{\lambda',\mu,\nu'} = g_{\lambda,\mu',\nu'}, \tag{19}$$

it is not difficult to prove Theorem 5.6 in a special case.

Namely, one derives from the symmetry property in (19), in an elementary way, that for any three partitions λ, μ, ν of the same weight, there exists integers d, d_1, d_2, d_3 such that (18) holds when (17) holds with *additional condition* that $a + b + c \equiv 0 \mod 2$. To recover the full theorem, it would be enough to establish that there exists m big enough such that

$$g_{\lambda\oplus(2m|2m),\mu\oplus(2m|2m),\nu\oplus(2m|2m)} = g_{\lambda\oplus(2m+1|2m+1),\mu\oplus(2m+1|2m+1),\nu\oplus(2m+1|2m+1)}. \tag{20}$$

Conjecture 5.11. *For any three partitions λ, μ and ν of the same weight, and any (a,b,c,m) fulfilling the inequalities*

$$\begin{aligned} \ell_i &\geq 0 \qquad \text{for all } i \in \{1,2,3\}, \\ m &\geq a+b+c. \end{aligned} \tag{21}$$

there is

$$g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)}. \tag{22}$$

Again, using the symmetries of the Kronecker coefficients, it is not difficult to prove this conjecture in a restricted case, namely that (22) holds for all partitions λ, μ, ν of the same weight and all (a,b,c,m) fulfilling (21) and, additionally, that $a+b+c \equiv 0 \mod 2$.

Therefore Conjecture 5.11 is equivalent to the following seemingly much weaker statement.

Conjecture 5.12 (Equivalent form of Conjecture 5.11)**.** *For any three partitions λ, μ and ν of the same weight,*

$$g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(1|1),\mu\oplus(1|1),\nu\oplus(1|1)}.$$

Remark 5.13. Conjecture 5.12 was checked by computer, with SAGE [51], for all triples of partitions of weight at most 16.

Remark 5.14. A proof of Conjecture 5.12 would provide an alternative proof of Theorem 5.6.

Indeed, assuming Conjecture 5.12, we have the inequalities

$$\begin{aligned} &g_{\lambda\oplus(2m|2m),\mu\oplus(2m|2m),\nu\oplus(2m|2m)} \\ &\qquad \leq g_{\lambda\oplus(2m+1|2m+1),\mu\oplus(2m+1|2m+1),\nu\oplus(2m+1|2m+1)} \\ &\qquad\qquad \leq g_{\lambda\oplus(2m+2|2m+2),\mu\oplus(2m+2|2m+2),\nu\oplus(2m+2|2m+2)}. \end{aligned} \tag{23}$$

The two bounds in this inequality are equal for m big enough by the hook stability property proved using only the invariance of the Kronecker coefficients under conjugation in (19). Then (20) would follow.

6. The second row

In this section, we describe the asymptotic behavior of some sequences of Kronecker coefficients $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ where the integer n varies, and the partitions α, β and γ have at most two parts. To this end, we move to the setting of the reduced Kronecker coefficients.

We first consider in Section 6.1 the family of reduced Kronecker coefficients $\overline{g}_{(a,\alpha),(b,\beta),(c,\gamma)}$ where the first parts a, b and c vary arbitrarily, while the remaining parts α, β, γ are fixed. We obtain for these coefficients, when a, b and c are big enough, quasipolynomial formulas in a, b, c, of degree at most 1 and period at most 2. This generalizes Proposition 3.4 corresponding to α, β, γ equal to the empty partition.

We determine in Section 6.2 the vanishing of the generic leading coefficient $A_{\alpha,\beta,\gamma}$ in these formulas. In Section 6.3, we describe the asymptotic behavior of sequences of reduced Kronecker coefficients $\overline{g}_{\lambda+n\cdot(a),\mu+n\cdot(b),\nu+n\cdot(c)}$ with the partitions λ, μ, ν and the integers a, b, c fixed, while n varies.

The asymptotic behaviors of the corresponding sequences of Kronecker coefficients is then derived in Section 6.4.

6.1. **For reduced Kronecker coefficients.** In this section, we obtain quasipolynomial formulas in a, b, c for some reduced Kronecker coefficients $\overline{g}_{(a,\alpha),(b,\beta),(c,\gamma)}$, with α, β, γ fixed.

Theorem 6.1. *Let α, β and γ be three partitions. There exists integers k'_1, k'_2, k'_3 and $A_{\alpha,\beta,\gamma}$, $B_{\alpha,\beta,\gamma}$ and $C_{\alpha,\beta,\gamma}$, such that whenever $a \geq \alpha_1$, $b \geq \beta_1$, $c \geq \gamma_1$ and*

$$\begin{array}{rcl} a-b & \geq & k'_1, \\ a-c & \geq & k'_2, \\ b+c-a & \geq & k'_3 \end{array} \tag{24}$$

we have

$$\overline{g}_{(a,\alpha),(b,\beta),(c,\gamma)} = \frac{1}{2}A_{\alpha,\beta,\gamma}\cdot(b+c-a)+B_{\alpha,\beta,\gamma}+\begin{cases} 0 & \text{for } b+c-a \text{ even,} \\ C_{\alpha,\beta,\gamma}/2 & \text{for } b+c-a \text{ odd.} \end{cases}$$

Proof. For any a, b, c, we set from (10)

$$\phi^+_{a,b,c} = \left\langle \sigma[F(X,Y,Z)] \,\middle|\, s_{(a,\alpha)}[X]s_{(b,\beta)}[Y]s_{(c,\gamma)}[Z] \right\rangle$$

where $F(X,Y,Z) = XYZ + XY + XZ + YZ$. When a, b and c are at least α_1, β_1 and γ_1 respectively, we have that $\phi^+_{a,b,c} = \overline{g}_{(a,\alpha),(b,\beta),(c,\gamma)}$.

Consider $\Phi^+_{\alpha,\beta,\gamma}(x,y,z) = \sum_{a,b,c} \phi^+_{a,b,c} x^a y^b z^c$. Then, $\Phi^+_{\alpha,\beta,\gamma}(x,y,z)$ is equal to

$$\left\langle \sigma[F(X,Y,Z)] \,\middle|\, \Gamma_{(x|X)} s_\alpha[X] \Gamma_{(y|Y)} s_\beta[Y] \Gamma_{(z|Z)} s_\gamma[Z] \right\rangle.$$

This is the specialization of $\Phi_{\alpha,\beta,\gamma}$ (see Lemma 4.1) at $X' = x$, $Y' = y$ and $Z' = z$.

Let $Q^+_{\alpha,\beta,\gamma}(x,y,z)$ be the image of $Q_{\alpha,\beta,\gamma}$ from Lemma 4.1 under the same specialization. From Lemma 4.1, $\Phi^+_{\alpha,\beta,\gamma} = \sigma[xyz + xy + xz + yz] \cdot Q^+_{\alpha,\beta,\gamma}$. Set $\mathbf{t}^{(a,b,c)} = x^a y^b z^c$. Write $Q^+_{\alpha,\beta,\gamma}$ as a sum of monomials, $Q^+_{\alpha,\beta,\gamma} = \sum_{\omega \in \Omega} q_\omega \mathbf{t}^\omega$, with Ω the support of $Q^+_{\alpha,\beta,\gamma}$. From (12),

$$\sigma[xyz + xy + xz + yz] = \sum_{\theta \in \mathcal{C} \cap \mathbb{N}^3} r_\theta \mathbf{t}^\theta$$

with $r_\theta = 1 + [\min\{\ell_1(\theta), \ell_2(\theta), \ell_3(\theta)\}/2]$, and it follows that

$$\Phi^+_{\alpha,\beta,\gamma}(x,y,z) = \sum_{\omega \in \Omega, \theta \in \mathcal{C} \cap \mathbb{N}^3} q_\omega r_\theta \mathbf{t}^{\omega+\theta}.$$

For any $\tau = (a,b,c) \in \mathbb{Z}^3$, we therefore have $\phi^+_\tau = \sum q_\omega r_{\tau-\omega}$ where the sum is over all $\omega \in \Omega$ such that $\tau - \omega \in \mathcal{C}$. Let $\mathcal{C}_1$ be the cone defined by the inequalities

$$(25)\qquad \begin{array}{c} \ell_1 \geq 0, \\ \ell_2 \geq \ell_1, \\ \ell_3 \geq \ell_1, \end{array} \qquad \text{or, equivalently,} \qquad \begin{array}{c} b+c \geq a, \\ a \geq b, \\ a \geq c. \end{array}$$

where, as usual, the parameters ℓ_i are defined as in (13). If ω is such that $\tau - \omega \in \mathcal{C}_1$, then we have

$$\begin{aligned} r_{\tau-\omega} &= 1 + \left[\frac{\ell_1(\tau-\omega)}{2}\right] \\ &= 1 + \ell_1(\tau)/2 - \ell_1(\omega)/2 - \begin{cases} 0 & \text{if } \ell_1(\omega) \equiv \ell_1(\tau) \pmod 2 \\ 1/2 & \text{if } \ell_1(\omega) \not\equiv \ell_1(\tau) \pmod 2 \end{cases} \end{aligned}$$

Therefore, if τ fulfills $\tau - \omega \in \mathcal{C}_1$ for all ω in Ω, then we have

$$\begin{aligned} \phi_\tau^+ &= \sum_{\omega\in\Omega} q_\omega r_{\tau-\omega} = \sum_{\omega\in\Omega} q_\omega \left(1 + \left[\frac{\ell_1(\tau-\omega)}{2}\right]\right) \\ &= \sum_{\omega\in\Omega} q_\omega + \frac{1}{2}\sum_{\omega\in\Omega} q_\omega \ell_1(\tau) - \frac{1}{2}\sum_{\omega\in\Omega} q_\omega \ell_1(\omega) - \frac{1}{2} \sum_{\substack{\omega\,:\,\ell_1(\omega)\not\equiv \\ \ell_1(\tau) \mod 2}} q_\omega. \end{aligned}$$

Note that $|\omega| \equiv \ell_1(\omega) \pmod 2$ for all $\omega \in \mathbb{Z}^3$. The condition $\ell_1(\omega) \not\equiv \ell_1(\tau) \pmod 2$ in the last sum can therefore be replaced with $|\omega| \not\equiv |\tau| \pmod 2$.

Set

$$A = \sum_{\omega\in\Omega} q_\omega = Q^+(1,1,1), \quad K = \sum_{\omega\in\Omega} q_\omega \ell_1(\omega)$$

and

$$A^+ = \sum_{\omega:|\omega| \text{ even}} q_\omega, \quad A^- = \sum_{\omega:|\omega| \text{ odd}} q_\omega.$$

We have obtained

$$\begin{aligned} \phi_\tau^+ &= A + \frac{A}{2}\ell_1(\tau) - \frac{K}{2} - \begin{cases} A^-/2 & \text{if } \ell_1(\tau) \text{ is even,} \\ A^+/2 & \text{if } \ell_1(\tau) \text{ is odd.} \end{cases} \\ &= A + \frac{A}{2}\ell_1(\tau) - \frac{K}{2} - \frac{A^-}{2} - \begin{cases} 0 & \text{if } \ell_1(\tau) \text{ is even,} \\ (A^+ - A^-)/2 & \text{if } \ell_1(\tau) \text{ is odd.} \end{cases} \end{aligned}$$

Set $A_{\alpha,\beta,\gamma} = A$, $B_{\alpha,\beta,\gamma} = A - K/2 - A^-/2$ and $C_{\alpha,\beta,\gamma} = A^+ - A^-$. The formula in the theorem is obtained. Note that $B_{\alpha,\beta,\gamma}$ is an integer since $K \equiv A^- \pmod 2$. Indeed,

$$K = \sum_\omega q_\omega \ell_1(\omega) \equiv \sum_\omega q_\omega |\omega| \equiv \sum_{\omega:|\omega| \text{ odd}} q_\omega \pmod 2$$

To conclude, observe that the condition $\tau - \omega \in \mathcal{C}_1$ for all ω in Ω, can be rewritten as

$$\begin{aligned} \ell_1(\tau) &\geq \max\nolimits_{\omega\in\Omega} \ell_1(\omega), \\ \ell_1(\tau) - \ell_2(\tau) &\geq \max\nolimits_{\omega\in\Omega} \left(\ell_1(\omega) - \ell_2(\omega)\right), \\ \ell_1(\tau) - \ell_3(\tau) &\geq \max\nolimits_{\omega\in\Omega} \left(\ell_1(\omega) - \ell_3(\omega)\right), \end{aligned}$$

which is equivalent to (24). This proves the theorem. ∎

Remark 6.2. Further computations show that one can take for the k'_i in Theorem 6.1

$$\begin{aligned} k'_1 &= |\alpha|+|\beta|+|\gamma|+\beta_1, \\ k'_2 &= |\alpha|+|\beta|+|\gamma|+\gamma_1, \\ k'_3 &= |\alpha|+|\beta|+|\gamma|+\alpha_1+\beta_1+\gamma_1. \end{aligned}$$

Theorem 6.6 has the following immediate corollary for Kronecker coefficients.

Corollary 6.3. *Let k'_1, k'_2 and k'_3 be as in Theorem 6.1.*

For all partitions λ, μ, ν of the same weight N, fulfilling the conditions

$$\begin{aligned} \lambda_2-\mu_2 &\geq k'_1, \\ \lambda_2-\nu_2 &\geq k'_2, \\ \mu_2+\nu_2-\lambda_2 &\geq k'_3, \\ N-\lambda_2-\mu_2-\nu_2 &\geq (|\overline{\overline{\lambda}}|+|\overline{\overline{\mu}}|+|\overline{\overline{\nu}}|)/2, \end{aligned}$$

we have that

$$g_{\lambda,\mu,\nu} = \frac{1}{2}A_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}}\cdot(\mu_2+\nu_2-\lambda_2)+B_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}}+\begin{cases} 0 & \text{for } \lambda_2+\mu_2+\nu_2 \text{ even,} \\ C_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}}/2 & \text{for } \lambda_2+\mu_2+\nu_2 \text{ odd.} \end{cases}$$

Proof. The condition

$$N-\lambda_2-\mu_2-\nu_2 \geq (|\overline{\overline{\lambda}}|+|\overline{\overline{\mu}}|+|\overline{\overline{\nu}}|)/2$$

ensures that $N \geq N_0(\overline{\lambda},\overline{\mu},\overline{\nu})$, so that $g_{\lambda,\mu,\nu} = \overline{g}_{\overline{\lambda},\overline{\mu},\overline{\nu}}$, as in Section 3.1. Applying Theorem 6.6 gives the result. ∎

6.2. **When is $A_{\alpha,\beta,\gamma}$ equal to zero?** From Theorem 6.1, the coefficient $A_{\alpha,\beta,\gamma}$ is the generic leading term of the expression of $\overline{g}_{(a,\alpha),(b,\beta),(c,\gamma)}$ that is quasipolynomial of degree 1 in a, b and c. It is relevant to ask when it vanishes.

We will need the following lemma.

Lemma 6.4. *Let λ, μ, ν and α, β, γ be partitions.*

1. *If $g_{\alpha,\beta,\gamma} \neq 0$ then $g_{\lambda+\alpha,\mu+\beta,\nu+\gamma} \geq g_{\lambda,\mu,\nu}$.*
2. *If $\overline{g}_{\alpha,\beta,\gamma} \neq 0$ then $\overline{g}_{\lambda+\alpha,\mu+\beta,\nu+\gamma} \geq \overline{g}_{\lambda,\mu,\nu}$.*

Proof. For the first assertion see [37].

The second assertion follows from the first one as follows. Suppose that $\overline{g}_{\alpha,\beta,\gamma} \neq 0$. There exist integers a, b and c such that $\overline{g}_{\alpha,\beta,\gamma} = g_{(a,\alpha),(b,\beta),(c,\gamma)}$. In particular this Kronecker coefficient is non–zero. Let p, q and r be integers such that (p,λ), (q,μ), (r,ν) are partitions of the same weight. Then we have, for all $n \geq 0$,

$$g_{(a+p+n,\lambda+\alpha),(b+q+n,\mu+\beta),(c+r+n,\nu+\gamma)} \geq g_{(p+n,\lambda),(q+n,\mu),(r+n,\nu)}.$$

Taking n big enough, so that both Kronecker coefficients coincide with the corresponding reduced Kronecker coefficient, we get $\overline{g}_{\lambda+\alpha,\mu+\beta,\nu+\gamma} \geq \overline{g}_{\lambda,\mu,\nu}$. ∎

Proposition 6.5. *Let α, β, γ be partitions. The coefficient $A_{\alpha,\beta,\gamma}$ is zero if and only if all Kronecker coefficients $g_{(a_1,a_2,\alpha),(b_1,b_2,\beta),(c_1,c_2,\gamma)}$ are zero for all $a_1 \geq a_2 \geq \alpha_1$, $b_1 \geq b_2 \geq \beta_1$, and $c_1 \geq c_2 \geq \gamma_1$.*

Proof. It is enough to show that $A_{\alpha,\beta,\gamma}$ is zero if and only if all reduced Kronecker coefficients $\overline{g}_{(a_2,\alpha),(b_2,\beta),(c_2,\gamma)}$ are zero since, on the first hand, we have always $g_{(a_1,a_2,\alpha),(b_1,b_2,\beta),(c_1,c_2,\gamma)} \leq \overline{g}_{(a_2,\alpha),(b_2,\beta),(c_2,\gamma)}$, as in see Section 3.1; and, on the other hand, any reduced Kronecker coefficient $\overline{g}_{(a_2,\alpha),(b_2,\beta),(c_2,\gamma)}$ is equal to some Kronecker coefficient $g_{(a_1,a_2,\alpha),(b_1,b_2,\beta),(c_1,c_2,\gamma)}$.

Assume that all reduced Kronecker coefficients $\overline{g}_{(a_2,\alpha),(b_2,\beta),(c_2,\gamma)}$ are zero. This is the case, in particular, for the coefficients $\overline{g}_{(n,\alpha),(n,\beta),(n,\gamma)}$. But, from Theorem 6.6, for n big enough.

$$\overline{g}_{(n,\alpha),(n,\beta),(n,\gamma)} = \frac{A_{\alpha,\beta,\gamma}}{2} n + \text{ a bounded term.}$$

Then $A_{\alpha,\beta,\gamma}$ must be zero.

Assume now that there exists some reduced Kronecker coefficient $\overline{g}_{(a,\alpha),(b,\beta),(c,\gamma)}$ that is non–zero. After Lemma 6.4, we have, for all $n \geq 0$, $\overline{g}_{(a+n,\alpha),(b+n,\beta),(c+n,\gamma)} \geq \overline{g}_{(n),(n),(n)}$ On the other hand, for n big enough, $\overline{g}_{(a+n,\alpha),(b+n,\beta),(c+n,\gamma)} = \frac{A_{\alpha,\beta,\gamma}}{2} n +$ a bounded term. But $\overline{g}_{(n),(n),(n)} \sim \frac{n}{2}$ from Proposition 3.4. Whence, necessarily $A_{\alpha,\beta,\gamma} \neq 0$. □

6.3. **Asymptotics of some sequences of reduced Kronecker coefficients $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)}$.** We consider the asymptotic behavior of the sequence with general term $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)}$, where (λ,μ,ν) is a fixed triple of partitions, and $((a),(b),(c))$ a fixed triple of partitions with at most one part, and $n(a)$ is n times the one–part partition (a).

We will set $\theta_0 = (\lambda_1,\mu_1,\nu_1)$ and $\theta = (a,b,c)$, so that $\theta_0 + n\theta = (\lambda_1 + na, \mu_1 + nb, \nu_1 + nc)$.

Three cases will be examined, corresponding to the position of $\theta = (a,b,c)$ with respect to the cone $\mathcal{C}$: outside, on the border or in the interior. We will be able to say even more when θ is in the interior of the smaller cone $\mathcal{C}_1$ defined in (25).

We will use the following decomposition from the proof of Theorem 6.1:

$$\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)} = \phi_\theta^+ = \sum_{\omega\in\Omega} q_\omega r_{\theta_0+n\theta-\omega}$$

where $Q^+_{\overline{\lambda},\overline{\mu},\overline{\nu}} = \sum_{\omega\in\Omega} q_\omega \mathbf{t}^\omega$ and $r_\tau = 1 + [\min_i \ell_i(\tau)/2)]$ if $\tau \in \mathcal{C}$, and 0 else. The formula writes more explicitly:

$$\overline{g}_{\theta_0+n\theta} = \sum q_\omega \left(1 + \left[\min_i \frac{\ell_i(\theta_0-\omega) + n\ell_i(\theta)}{2}\right]\right), \tag{26}$$

where now the sum is over all ω such that: $\ell_i(\theta_0 - \omega) + n\ell_i(\theta) \geq 0$ for all i.

When θ is outside $\mathcal{C}$. This means that $\ell_i(\theta) < 0$ for some i. Then $\ell_i(\theta_0 - \omega) + n\ell_i(\theta)$ is < 0 for $n \gg 0$. The sum (26) becomes empty. As a consequence, in this case, $\overline{g}_{\theta_0+n\theta} = 0$ for $n \gg 0$.

When θ is on the border of $\mathcal{C}$. This means that all inequalities $\ell_i(\theta) \geq 0$ are fulfilled, but at least one of them is an equality. For i and j such that $\ell_i(\theta) > 0$ and $\ell_j(\theta) = 0$, we have $\ell_i(\theta_0 - \omega) + n\ell_i(\theta) > \ell_j(\theta_0 - \omega) + n\ell_j(\theta) = \ell_j(\theta_0 - \omega)$ for $n \gg 0$. Therefore all terms $r_{\theta_0+n\theta-\omega}$ are independent on n. Also the sum is restricted to all ω such that $\ell_i(\theta_0 - \omega) + n\ell_i(\theta) \geq 0$ for all i. For i such that $\ell_i(\theta) > 0$, this is automatically fulfilled for $n \gg 0$; there only remains the condition $\ell_j(\theta_0 - \omega) + n\ell_j(\theta) \geq 0$ for all j such that $\ell_j(\theta) = 0$. This condition is actually independent on n. This shows that $\overline{g}_{\theta_0+n\theta}$ is eventually constant in this case.

When θ is in the interior of $\mathcal{C}$. This means that $\ell_i(\theta) > 0$ for all i. For $n \gg 0$, the inequalities $\ell_i(\theta_0 - \omega) + n\ell_i(\theta) \geq 0$ are fulfilled for all $\omega \in \Omega$.

We can assume, without loss of generality, that $\ell_1(\theta) \leq \ell_2(\theta)$ and $\ell_1(\theta) \leq \ell_3(\theta)$. Then, for all ω, we have for $n \gg 0$, that

$$\begin{aligned} &q_\omega \cdot \left(1 + \left[\min_i(\ell_i(\theta_0 - \omega) + n\ell_i(\theta))/2\right]\right) \\ &\qquad = q_\omega \ell_1(\theta)/2 \cdot n + \text{ a periodic term in } n \text{ with period at most } 2. \end{aligned}$$

Summing over all $\omega \in \Omega$ we get

$$\overline{g}_{\theta_0+n\theta} = \frac{A_{\overline{\lambda},\overline{\mu},\overline{\nu}}}{2}\ell_1(\theta) \cdot n + \text{ a periodic term in } n \text{ with period at most } 2.$$

When θ is in the interior of $\mathcal{C}_1$. Then we can apply Theorem 6.1 with $(\lambda_1 + na, \mu_1 + nb, \nu_1 + nc)$ instead of (a, b, c).

Let us state the results obtained in a theorem.

Theorem 6.6. *Let λ, μ and ν be three partitions and $(a, b, c) \in \mathbb{N}^3$. Without loss of generality, we can assume that $\max(a, b, c) = a$. Suppose that there exists n such that $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)}$ is non–zero. Then $A_{\overline{\lambda},\overline{\mu},\overline{\nu}}$ is nonzero, and*

(1) *if (a, b, c) is in the interior of $\mathcal{C}$, then*

$$\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)} \sim_{n\to\infty} \frac{A_{\overline{\lambda},\overline{\mu},\overline{\nu}} \cdot (b + c - a)}{2} \cdot n$$

and the difference is a periodic term in n with period at most 2.

(2) *if, besides, (a, b, c) is in the interior of $\mathcal{C}_1$, then the periodic term is*

$$\begin{cases} B_{\overline{\lambda},\overline{\mu},\overline{\nu}} \quad -C_{\overline{\lambda},\overline{\mu},\overline{\nu}}/2 & \text{if } n \text{ and } a+b+c \text{ are both odd,} \\ B_{\overline{\lambda},\overline{\mu},\overline{\nu}} & \text{otherwise.} \end{cases}$$

(3) *if (a, b, c) is on the border of $\mathcal{C}$ then $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)}$ is eventually constant.*

(4) *if $(a, b, c) \notin \mathcal{C}$ then $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)} = 0$ for $n \gg 0$.*

6.4. **Asymptotics of some sequences of Kronecker coefficients.** Set $\mathcal{L}$ for the set of triples of partitions (λ, μ, ν) such that

$$|\lambda| = |\mu| = |\nu| \geq N_0(\overline{\lambda}, \overline{\mu}, \overline{\nu}).$$

Note that the set $\mathcal{L}$ is stable under sum, and that $(\lambda, \mu, \nu) \in \mathcal{L}$ implies that $g_{\lambda,\mu,\nu} = \overline{g}_{\overline{\lambda},\overline{\mu},\overline{\nu}}$. Theorem 6.6 has the following immediate consequence for Kronecker coefficients.

Corollary 6.7. *Let (λ, μ, ν) and (α, β, γ) be two triples of partitions in $\mathcal{L}$, with α, β and γ with at most two parts.*

Without loss of generality, we assume that $\max(\alpha_2, \beta_2, \gamma_2) = \alpha_2$.

Assume that there exists n such that $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ is non–zero.

Then, $A_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}}$ is nonzero, and

(1) *if $(\alpha_2, \beta_2, \gamma_2)$ is in the interior of $\mathcal{C}$ then*

$$g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma} \sim_{n\to\infty} \frac{A_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}} \cdot (\beta_2 + \gamma_2 - \alpha_2)}{2} \cdot n$$

and the difference is periodic in n with period at most 2.

(2) *if besides, $(\alpha_2, \beta_2, \gamma_2)$ is in the interior of $\mathcal{C}_1$, then the periodic term is*

$$\begin{cases} B_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}} - C_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}}/2 & \text{if } n \text{ and } a+b+c \text{ are both odd,} \\ B_{\overline{\overline{\lambda}},\overline{\overline{\mu}},\overline{\overline{\nu}}} & \text{otherwise.} \end{cases}$$

(3) *if $(\alpha_2, \beta_2, \gamma_2)$ is on the border of $\mathcal{C}$ then $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ is eventually constant.*

(4) *if $(\alpha_2, \beta_2, \gamma_2) \notin \mathcal{C}$ then $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma} = 0$ for $n \gg 0$.*

Proof. Since (λ, μ, ν) and (α, β, γ) are in $\mathcal{L}$, so are all triples of partitions $(\lambda+n\alpha, \mu+n\beta, \nu+n\gamma)$ for all $n \geq 0$. Therefore, $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma} = \overline{g}_{\overline{\lambda}+n(\alpha_2),\overline{\mu}+n(\beta_2),\overline{\nu}+n(\gamma_2)}$ for all $n \geq 0$. Then we can apply Theorem 6.6. □

Remark 6.8. We make a few remarks about Corollary 6.7.

(1) Statement (2) is a particular case of a much more general statement: given any three partitions α, β and γ, such that $g(\alpha, \beta, \gamma) > 0$, the sequence with general term $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ is eventually constant, for all triples of partitions (α, β, γ) of the same weight, if and only if $g_{n\alpha,n\beta,n\gamma} = 1$ for all $n \geq 0$. See [52, 46].
(2) For partitions with length at most 2 of the same weight, $\alpha = (m - \alpha_2, \alpha_2)$, $\beta = (m - \beta_2, \beta_2)$ and $\gamma = (m - \gamma_2, \gamma_2)$, we have that $(\alpha, \beta, \gamma) \in \mathcal{L} \Leftrightarrow m \geq \alpha_2 + \beta_2 + \gamma_2$.

7. Generating series

Four families of constants were defined in the previous sections: the limits $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ under "hook stability" (Section 5) and the coefficients

$A_{\alpha,\beta,\gamma}$, $B_{\alpha,\beta,\gamma}$ and $C_{\alpha,\beta,\gamma}$ appearing in the quasipolynomial formulas of Section 6.

In this section, we provide, for these families of constants, generating series akin to the generating series for the Littlewood–Richardson coefficients

$$\sigma[XY + XZ] = \sum_{\lambda,\mu,\nu} c_{\lambda,\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z],$$

for the Kronecker coefficients

$$\sigma[XYZ] = \sum_{\lambda,\mu,\nu} g_{\lambda,\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z],$$

and for the reduced Kronecker coefficients in (10).

7.1. **Generating series for the coefficients $\overline{\overline{g}}$.** We give now a generating series for the limit coefficients $\overline{\overline{g}}_{\alpha,\beta,\gamma}$.

Theorem 7.1. *The limit $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ in Theorem 5.2 is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion, in the Schur basis, of*

$$\sigma\left[XYZ + (1-\varepsilon)(XY + XZ + YZ + X + Y + Z)\right].$$

Proof. It follows from the proof of Theorem 5.2 that $\overline{\overline{g}}_{\alpha,\beta,\gamma} = \sum_{\omega\in\Omega} q_\omega = Q^-_{\alpha,\beta,\gamma}(1,1,1)$. After the proof of Theorem 5.2, $Q^-_{\alpha,\beta,\gamma}(x,y,z)$ is the specialization of the symmetric function $Q_{\alpha,\beta,\gamma}$ at $X' = -\varepsilon x$, $Y' = -\varepsilon y$, $Z' = -\varepsilon z$. Therefore, $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ is the specialization of $Q_{\alpha,\beta,\gamma}$ at $X' = -\varepsilon$, $Y' = -\varepsilon$, $Z' = -\varepsilon$. By definition of $Q_{\alpha,\beta,\gamma}$ (see Lemma 4.1), this is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion of $\sigma[H(-\varepsilon,-\varepsilon,-\varepsilon)]$. Finally, it is straightforward to compute that

$$H(-\varepsilon,-\varepsilon,-\varepsilon) = XYZ + (1-\varepsilon)(XY + XZ + YZ + X + Y + Z).$$

□

Remark 7.2. We have used that $Q^-_{\alpha,\beta,\gamma}(1,1,1) = \overline{\overline{g}}_{\alpha,\beta,\gamma}$. Interestingly, with the specialization $x=-1$, $y=-1$, $z=-1$ we get $Q^-_{\alpha,\beta,\gamma}(-1,-1,-1) = g_{\alpha,\beta,\gamma}$.

7.2. **Generating series for the coefficients A, B and C.** Here we will prove the following.

Theorem 7.3. *Let α, β, γ be three partitions.*

Let $\chi = \sum_{n=1}^{\infty} p_n$, the formal sum of all power sum symmetric functions.

Let $W = XY + XZ + YZ + X + Y + Z$.

The coefficients $A_{\alpha,\beta,\gamma}$, $C_{\alpha,\beta,\gamma}$, and $B_{\alpha,\beta,\gamma}$ in Theorem 6.1 are the coefficients of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansions in the Schur basis of, *respectively,*

$$\sigma[XYZ+2W],\ \sigma[XYZ+(1+\varepsilon)W],\ \textit{and}$$
$$\sigma[XYZ+2W]\cdot\left(\frac{3}{4}+\frac{1}{4}\sigma[(\varepsilon-1)W]-\frac{1}{2}\chi[W]+\chi[YZ-X]\right)$$

Proof. Let us drop in this proof the indices (α,β,γ) of the coefficients involved: A, B, C, Q^+ stand for $A_{\alpha,\beta,\gamma}$, $B_{\alpha,\beta,\gamma}$, $C_{\alpha,\beta,\gamma}$ and $Q^+_{\alpha,\beta,\gamma}$.

With the notations of the proof of Theorem 6.1, we have $A = Q^+(1,1,1)$, $B = A - K/2 - A^-/2$ and $C = A^+ - A^-$.

Remember that Q^+ is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion in the Schur basis of $\sigma[H(x,y,z)]$. We have

$$\begin{aligned}H(x,y,z) &= XYZ+(1+z)XY+(1+y)XZ+(1+x)YZ\\ &+(yz+y+z-1/x)X+(xz+x+z-1/y)Y+(xy+x+y-1/z)Z.\end{aligned}$$

Specializing x,y,z at 1 we get that A is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion in the Schur basis of $\sigma[XYZ+2W]$.

Let us get now a generating series for the coefficients C. We have $C = A^+ - A^-$, where A^+ (resp. A^-) is the sum of all coefficients q_ω of Q^+ such that $\ell_1(\omega)$ is even (resp. odd). Note that for any $\omega\in\mathbb{Z}^3$, we have $\ell_1(\omega)\equiv|\omega| \mod 2$. Therefore, A^+ (resp. A^-) is also the sum of all coefficients q_ω of Q^+ such that $|\omega|$ is even (resp. odd). Thus

$$Q^+(-1,-1,-1)=\sum_\omega q_\omega\,(-1)^{|\omega|}=A^+-A^-=C.$$

Specializing the variables x, y and z at -1 in $\sigma[H(x,y,z)]$ (this corresponds to specializations at ε as alphabets), we get that C is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion in the Schur basis of $\sigma[XYZ+(1+\varepsilon)W]$.

Now $B = A - K/2 - A^-/2$. Since $C = A^+ - A^-$ and $A = A^+ + A^-$, we have also $B = 3A/4 + C/4 - K/2$. The generating series for the coefficients A and C have just been obtained. Let us focus on the generating series for the coefficients K.

Note that

$$K=\sum_{(a,b,c)\in\Omega} q_{a,b,c}(b+c-a)$$

This can be obtained as $\frac{\partial Q^+(1/t,t,t)}{\partial t}_{|t=1}$. Therefore K is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion in the Schur basis of

$$\frac{\partial\sigma\left[H(1/t,t,t)\right]}{\partial t}_{|t=1}$$

We compute that

$$H(1/t,t,t)=t^2X+t(W-YZ)+(XYZ+W-X)+1/tYZ.$$

We now use that $\sigma = \exp\left(\sum_{n=1}^{\infty} p_n/n\right)$, where the p_n are the power sum symmetric functions (see [35,]). We get that $\sigma[H(1/t,t,t)]$ is equal to

$$\exp\left(\sum_{n=1}^{\infty} \frac{p_n[X]t^{2n} + p_n[W-YZ]t^n + p_n[XYZ+W-X] + p_n[YZ]t^{-n}}{n}\right).$$

Derivating with respect to t and specializing t at 1, we obtain

$$\frac{\partial\sigma[H(1/t,t,t)]}{\partial t}_{|t=1} = \sigma[H(1,1,1)]\cdot\sum_{n=1}^{\infty}\left(2p_n[X] + p_n[W-YZ] - p_n[YZ]\right)$$

Let $\chi = \sum_{n=1}^{\infty} p_n$. Since $H(1,1,1) = XYZ + 2W$, we get

$$\frac{\partial\sigma[H(1/t,t,t)]}{\partial t}_{|t=1} = \sigma[XYZ+2W]\cdot\left(\chi[W] + 2\chi[X-YZ]\right).$$

The generating series for the coefficients $B_{\alpha,\beta,\gamma}$ is thus

$$\frac{3}{4}\sigma[XYZ+2W]+\frac{1}{4}\sigma[XYZ+(1+\varepsilon)W]-\frac{1}{2}\sigma[XYZ+2W]\cdot\left(\chi[W] + 2\chi[X-YZ]\right)$$

which is equal to

$$\sigma[XYZ+2W]\left(\frac{3}{4} + \frac{1}{4}\sigma[(\varepsilon-1)W] - \frac{1}{2}\chi[W] + \chi[YZ-X]\right).$$

□

Remark 7.4. It is possible to rewrite the formula for the generating function of the coefficients $B_{\alpha,\beta,\gamma}$ in such a way that it is clearly a combination of Schur functions with integer coefficients (as we know it is, after Theorem 6.1). There are many ways of doing this. One of them is:

$$\sigma[XYZ+2W]\cdot\left(1 - \sum_{a \text{ even},b}(-1)^b s_{(a|b)}[W] + \sum_{a,b}(-1)^b s_{(a|b)}[YZ-X]\right)$$

where $(a|b)$ is the partition $(1+a, 1^b)$ ("Frobenius notation" for partitions, see [35, I. §1])

Example 7.5. One can derive from Theorem 7.3 the following formulas for the coefficients in the paper, when two of the three indices are the empty partition.

$$A_{(\alpha_1,\alpha_2),\emptyset,\emptyset} = \alpha_1 - \alpha_2 + 1,$$

$$C_{(\alpha_1,\alpha_2),\emptyset,\emptyset} = \begin{cases} (-1)^{\alpha_2} & \text{if } \alpha_1 \equiv \alpha_2 \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$B_{(\alpha_1,\alpha_2),\emptyset,\emptyset} \text{ is the nearest integer from } -3\cdot\frac{(\alpha_1)^2 - (\alpha_2-1)^2}{4}$$

and, when (α_1,α_2) is not the empty partition,

$$B_{\emptyset,(\alpha_1,\alpha_2),\emptyset} \text{ is the nearest integer from } -3\cdot\frac{(\alpha_1-1)^2 - (\alpha_2-2)^2}{4}.$$

Similarly, one derives from Theorem 7.1 that

$$\overline{\overline{g}}_{\alpha,\emptyset,\emptyset} = \begin{cases} 2 & \text{if } \alpha \text{ is a hook,} \\ 1 & \text{if } \alpha = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

8. Final remarks

The rate of growth experienced by the reduced Kronecker coefficients as we add cells to remaining row is harder to understand. The stretched Kronecker coefficients are known to be described by a quasi-polynomial; see [37, 39, 1]. In particular, the following corollary holds.

Corollary 8.1 (Manivel, [37])**.** *For any triple λ, μ, and ν, the stretched Kronecker coefficient $g(k\lambda, k\mu, k\nu)$ is a quasi-polynomial function of $k \geq 0$.*

Examples of these quasi-polynomial functions have been computed in [6, 1]. All specializations of the form $\lambda := \lambda + k\mu$ will then be described by quasi-polynomials in k.

Some particular instances of this problem have been studied in the literature. Recall that in [40] Murnaghan observed that the reduced Kronecker coefficients $\overline{g}^{\lambda}_{\mu,\nu}$ such that $|\lambda| = |\mu| + |\nu|$ coincide with the Littlewood-Richardson coefficients $c^{\lambda}_{\mu,\nu}$. For the stretched Littlewood-Richardson coefficients, it has been shown in [43, 19] that $\overline{g}(k\lambda, k\mu, k\nu)$ is described by a polynomial (and not just a quasi-polynomial). Moreover, the degree of the stretched Littlewood-Richardson polynomials have been studied in [26, 27].

Other families (that are not Littlewood-Richardson coefficients) have appeared in the literature. For example, from the calculations appearing in [18, 17] we know that the sequence $\bar{g}^{(k)}_{(k^a),(k^a)}$ is described by a quasipolynomial of degree $2a-1$. However, the sequence $\bar{g}^{(k)}_{(2k-j,k^{a-1}),(k^a)}$, with $k \geq 2j$ is described by a quasipolynomial of degree $3a-2$. Note that the period of both quasipolynomials divides ℓ, the least common multiple of $1, 2, \dots, a, a+1$. In fact, it has been checked that the period is exactly ℓ for $a \leq 10$ for the first family, and for $a \leq 7$ for the second one.

Note that for $a = 1$, both sequences are described by a linear quasipolynomial of period 2, as predicted by our work. For $a = 2$ the first sequence is described by a quasipolynomial of degree 3, whereas the second sequence is described by a quasipolynomial of degree 4. The two resulting quasi polynomials are copied here.

Example 8.2. The coefficients $\overline{g}^{(k)}_{(k^2),(k^2)}$ are given by the following quasipolynomial of degree 3 and period 6:

$$\overline{g}^{(k)}_{(k^2),(k^2)} = \begin{cases} 1/72\,(k+6)\,(k^2+6k+12) & \text{if } k \equiv 0 \mod 6 \\ 1/72\,(k+5)\,(k^2+7k+4) & \text{if } k \equiv 1 \mod 6 \\ 1/72\,(k+4)^3 & \text{if } k \equiv 2 \mod 6 \\ 1/72\,(k+3)\,(k^2+9k+12) & \text{if } k \equiv 3 \mod 6 \\ 1/72\,(k+2)\,(k^2+10k+28) & \text{if } k \equiv 4 \mod 6 \\ 1/72\,(k+1)\,(k+4)\,(k+7) & \text{if } k \equiv 5 \mod 6 \end{cases}$$

The factorizations obtained for this families resemble those observed and studied for the stretched Littlewood-Richardson coefficients in [26, 27].

Example 8.3. The coefficients $\bar{g}^{(k)}_{(2k-j,k),(k^2)}$, with $k \geq 2j$, are given by the following quasipolynomial of degree 4 and period 6:

$$\bar{g}^{(k)}_{(2k-j,k),(k^2)} = \begin{cases} 1/288\,(j+6)\,(j^3+12j^2+40j+48) & \text{if } j \equiv 0 \mod 6 \\ 1/288\,(j+5)^2\,(j+1)\,(j+7) & \text{if } j \equiv 1 \mod 6 \\ 1/288\,(j+4)^2\,(j+2)\,(j+8) & \text{if } j \equiv 2 \mod 6 \\ 1/288\,(j+3)\,(j^3+15j^2+67j+69) & \text{if } j \equiv 3 \mod 6 \\ 1/288\,(j+2)\,(j+4)^2\,(j+8) & \text{if } j \equiv 4 \mod 6 \\ 1/288\,(j+1)\,(j+5)^2\,(j+7) & \text{if } j \equiv 5 \mod 6 \end{cases}$$

Interestingly, for both families, the sequences obtained as the result of incrementing the parameter a (the sizes of the columns) is weakly increasing, and bounded. This follows easily from the combinatorial interpretations in terms of plane partitions provided in [18, 17].

Acknowledgments

We thank Laura Colmenarejo and Rosa Orellana for some interesting comments. The second and third authors want to thank Birkbeck College and the Imperial College Library for hosting us during our visit to London where most of this research was conducted.

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Appendix A. Table of coefficients

Tables 2 and 3 display the coefficients $\overline{\overline{g}}_{\alpha,\beta,\gamma}$, $A_{\alpha,\beta,\gamma}$, $B_{\alpha,\beta,\gamma}$ and $C_{\alpha,\beta,\gamma}$ for all partitions α, β and γ with weight at most 3. Note that $\overline{\overline{g}}_{\alpha,\beta,\gamma}$, $A_{\alpha,\beta,\gamma}$ and $C_{\alpha,\beta,\gamma}$ are invariant under permutation of their three indices. This is why the table gives their values only for $\alpha \geq \beta \geq \gamma$, where the order $\geq$ is the degree lexicographic ordering. The coefficients $B_{\alpha,\beta,\gamma}$ is only invariant under permuting its last two indices.

These coefficients where calculated by series expansion of the generating series and using SAGE [51].

Appendix B. Bounds

We prove here the assertions made in Remarks 5.3 and 6.2 about the values of the constants k_i (in Theorem 5.2) and k'_i (in Theorem 6.1). These technical and less central results do not appear in the printed version of this work.

B.1. **Hook stability, reduced Kronecker coefficients.** In this section, we find explicitly bounds for the quantities k_1, k_2, k_3 appearing in Theorem 5.2.

Theorem B.1. *In Theorem 5.2, one can take*

$$\begin{cases} k_1 &=& |\alpha| + \alpha_1 + \beta'_1 + \gamma'_1, \\ k_2 &=& |\beta| + \beta_1 + \alpha'_1 + \gamma'_1, \\ k_3 &=& |\gamma| + \gamma_1 + \alpha'_1 + \beta'_1 \end{cases}$$

Proof. After the proof of Theorem 5.2, one can take $k_i = \max_{\omega\in\Omega} \ell_i(\omega)$, where Ω is the support of $Q^- = Q^-_{\alpha,\beta,\gamma}(x, y, z)$.

Let us perform the change of variables $x = \frac{vw}{u}$, $y = \frac{uw}{v}$, $z = \frac{uv}{w}$, so that the identity $x^a y^b z^c = u^{\ell_1} v^{\ell_2} w^{\ell_3}$ holds. Then k_1 (resp. k_2, k_3), as defined above, is the degree of P with respect to the variable u (resp. v, w).

After this change of variables, $H(-\varepsilon x, -\varepsilon y, -\varepsilon z)$ equals

$$XYZ + XY + XZ + YZ - \varepsilon \cdot \left(\frac{uv}{w}XY + \frac{uw}{v}XZ + \frac{vw}{u}YZ\right) + u^2X + v^2Y + w^2Z + \varepsilon\frac{1}{vw}\left(u^2 - uv^2 - uw^2\right)X + \varepsilon\frac{1}{uw}\left(v^2 - vu^2 - vw^2\right)Y + \varepsilon\frac{1}{uv}\left(w^2 - wu^2 - wv^2\right)Z.$$

We reorder the terms as follows:

$$H(-\varepsilon x, -\varepsilon y, -\varepsilon z) = u^2X + \varepsilon uXH_1 - \varepsilon u\left(\frac{v}{w}Y + \frac{w}{v}Z\right) + H_0.$$

where H_0 is a sum of monomials with non-positive degree in u, and H_1 is free of u and X. We now factorize $\sigma[H(-\varepsilon x, -\varepsilon y, -\varepsilon z)]$ as

$$\sigma[u^2X] \cdot \sigma[\varepsilon uXH_1] \cdot \sigma\left[-\varepsilon u\frac{v}{w}Y\right] \cdot \sigma\left[-\varepsilon u\frac{w}{v}Z\right] \cdot \sigma[H_0]$$

α	β	γ	$\overline{\overline{g}}_{\alpha,\beta,\gamma}$	$A_{\alpha,\beta,\gamma}$	$B_{\alpha,\beta,\gamma}$	$B_{\beta,\alpha,\gamma}$	$B_{\gamma,\alpha,\beta}$	$C_{\alpha,\beta,\gamma}$
$\emptyset$	$\emptyset$	$\emptyset$	1	1	1	1	1	1
(1)	$\emptyset$	$\emptyset$	2	2	0	1	1	0
(1)	(1)	$\emptyset$	6	6	0	0	3	0
(1)	(1)	(1)	21	21	0	0	0	1
(2)	$\emptyset$	$\emptyset$	2	3	−2	1	1	1
(2)	(1)	$\emptyset$	8	10	−7	−2	3	0
(2)	(1)	(1)	34	40	−25	−5	−5	0
(2)	(2)	$\emptyset$	14	20	−14	−14	6	2
(2)	(2)	(1)	66	86	−57	−57	−14	0
(2)	(2)	(2)	145	203	−133	−133	−133	5
(2)	(2)	(1, 1)	144	150	−84	−84	−84	−4
(2)	(2)	(1, 1, 1)	204	134	−54	−54	−121	0
(2)	(1, 1)	$\emptyset$	14	12	−8	−8	4	−2
(2)	(1, 1)	(1)	66	62	−33	−33	−2	0
(2)	(1, 1)	(1, 1)	145	131	−55	−55	−55	5
(2)	(1, 1, 1)	$\emptyset$	16	6	−3	−6	3	0
(2)	(1, 1, 1)	(1)	84	46	−19	−42	4	0
(2)	(1, 1, 1)	(1, 1)	206	144	−45	−117	−45	0
(2)	(1, 1, 1)	(1, 1, 1)	326	240	−48	−168	−168	0
(1, 1)	$\emptyset$	$\emptyset$	2	1	−1	0	0	−1
(1, 1)	(1)	$\emptyset$	8	6	−3	0	3	0
(1, 1)	(1)	(1)	34	28	−13	1	1	0
(1, 1)	(1, 1)	$\emptyset$	14	12	−4	−4	8	2
(1, 1)	(1, 1)	(1)	66	54	−21	−21	6	0
(1, 1)	(1, 1)	(1, 1)	144	110	−38	−38	−38	−4
(3)	$\emptyset$	$\emptyset$	2	4	−6	0	0	0
(3)	(1)	$\emptyset$	8	14	−20	−6	1	0
(3)	(1)	(1)	38	59	−78	−19	−19	1
(3)	(2)	$\emptyset$	16	30	−42	−27	3	0
(3)	(2)	(1)	84	138	−178	−109	−40	0
(3)	(2)	(2)	206	348	−435	−261	−261	0
(3)	(2)	(1, 1)	204	258	−299	−170	−170	0
(3)	(2)	(1, 1, 1)	320	250	−250	−125	−250	0
(3)	(1, 1)	$\emptyset$	16	18	−24	−15	3	0
(3)	(1, 1)	(1)	84	98	−118	−69	−20	0
(3)	(1, 1)	(1, 1)	206	220	−235	−125	−125	0
(3)	(3)	$\emptyset$	22	50	−72	−72	3	0
(3)	(3)	(1)	122	240	−321	−321	−81	2
(3)	(3)	(2)	326	640	−820	−820	−500	0
(3)	(3)	(1, 1)	320	478	−574	−574	−335	0
(3)	(3)	(3)	565	1243	−1597	−1597	−1597	5
(3)	(3)	(2, 1)	1056	1632	−1888	−1888	−1888	0
(3)	(3)	(1, 1, 1)	544	506	−521	−521	−521	−4

TABLE 2. Table of the coefficients of the paper, for three indexing partitions with weight at most 3 (part 1 of 2).

α	β	γ	$\overline{\overline{g}}_{\alpha,\beta,\gamma}$	$A_{\alpha,\beta,\gamma}$	$B_{\alpha,\beta,\gamma}$	$B_{\beta,\alpha,\gamma}$	$B_{\gamma,\alpha,\beta}$	$C_{\alpha,\beta,\gamma}$
(3)	(2,1)	∅	38	50	−66	−66	9	0
(3)	(2,1)	(1)	224	288	−344	−344	−56	0
(3)	(2,1)	(2)	610	824	−938	−938	−526	0
(3)	(2,1)	(1,1)	610	700	−738	−738	−388	0
(3)	(2,1)	(2,1)	2037	2465	−2515	−2515	−2515	1
(3)	(2,1)	(1,1,1)	1056	928	−832	−832	−832	0
(3)	(1,1,1)	∅	22	10	−12	−12	3	0
(3)	(1,1,1)	(1)	122	80	−85	−85	−5	−2
(3)	(1,1,1)	(1,1)	326	260	−240	−240	−110	0
(3)	(1,1,1)	(1,1,1)	565	451	−355	−355	−355	5
(2,1)	∅	∅	2	2	−3	0	0	0
(2,1)	(1)	∅	12	12	−15	−3	3	0
(2,1)	(1)	(1)	64	64	−72	−8	−8	0
(2,1)	(2)	∅	28	30	−36	−21	9	0
(2,1)	(2)	(1)	152	164	−181	−99	−17	0
(2,1)	(2)	(2)	382	442	−477	−256	−256	0
(2,1)	(2)	(1,1)	382	378	−371	−182	−182	0
(2,1)	(2)	(1,1,1)	610	472	−394	−158	−394	0
(2,1)	(1,1)	∅	28	26	−28	−15	11	0
(2,1)	(1,1)	(1)	152	140	−139	−69	1	0
(2,1)	(1,1)	(1,1)	382	330	−293	−128	−128	0
(2,1)	(2,1)	∅	74	74	−81	−81	30	0
(2,1)	(2,1)	(1)	428	428	−433	−433	−5	0
(2,1)	(2,1)	(2)	1168	1242	−1218	−1218	−597	0
(2,1)	(2,1)	(1,1)	1168	1094	−982	−982	−435	0
(2,1)	(2,1)	(2,1)	3933	3933	−3470	−3470	−3470	1
(2,1)	(2,1)	(1,1,1)	2037	1609	−1221	−1221	−1221	1
(2,1)	(1,1,1)	∅	38	26	−24	−24	15	0
(2,1)	(1,1,1)	(1)	224	160	−136	−136	24	0
(2,1)	(1,1,1)	(1,1)	610	444	−338	−338	−116	0
(2,1)	(1,1,1)	(1,1,1)	1056	736	−480	−480	−480	0
(1,1,1)	∅	∅	2	0	0	0	0	0
(1,1,1)	(1)	∅	8	2	−2	0	1	0
(1,1,1)	(1)	(1)	38	17	−15	2	2	−1
(1,1,1)	(1,1)	∅	16	10	−8	−3	7	0
(1,1,1)	(1,1)	(1)	84	54	−42	−15	12	0
(1,1,1)	(1,1)	(1,1)	204	134	−97	−30	−30	0
(1,1,1)	(1,1,1)	∅	22	18	−12	−12	15	0
(1,1,1)	(1,1,1)	(1)	122	88	−59	−59	29	2
(1,1,1)	(1,1,1)	(1,1)	320	206	−130	−130	−27	0
(1,1,1)	(1,1,1)	(1,1,1)	544	322	−175	−175	−175	−4

TABLE 3. Table of the coefficients of the paper, for three indexing partitions with weight at most 3 (part 2 of 2).

and expand each series, except $\sigma[H_0]$. We get that $\sigma[H(-\varepsilon x,-\varepsilon y,-\varepsilon z)]$ is equal to

$$\sum u^{2i} h_i[X]\, u^j e_j[XH_1] \left(\frac{v}{w}u\right)^k e_k[Y] \left(\frac{w}{v}u\right)^\ell e_\ell[Z]\sigma[H_0],$$

where the sum ranges over all nonnegative integers i, j, k, ℓ. Therefore,

$$\begin{aligned} Q^- &= \langle \sigma[H(-\varepsilon x,-\varepsilon y,-\varepsilon z)] \mid s_\alpha[X]s_\beta[Y]s_\gamma[Z]\rangle \\ &= \sum u^{2i+j+k+\ell} v^{k-\ell} w^{\ell-k} \left\langle e_j[XH_1]\sigma[H_0] \,\middle|\, (h_i^\perp s_\alpha)[X](e_k^\perp s_\beta)[Y](e_\ell^\perp s_\gamma[Z])\right\rangle. \end{aligned}$$

We have $h_i^\perp s_\alpha = 0$ unless $i \leq \alpha_1$, and that $e_k^\perp s_\beta = 0$ (resp. $e_\ell^\perp s_\gamma = 0$) unless $j \leq \beta'_1$ (resp. $k \leq \gamma'_1$). Finally, since $e_j[XH_1]$ is homogeneous of degree j in X and $h_i^\perp s_\alpha$ has degree $|\alpha|-i$, the summand corresponding to i, j, k, ℓ can be non zero only if $i \leq \alpha_1$, $j \leq |\alpha| - i$, $k \leq \beta'_1$ and $\ell \leq \gamma'_1$. Therefore $2i + j + k + \ell \leq |\alpha| + \alpha_1 + \beta'_1 + \gamma'_1$.

This proves that in Theorem 5.2, one can take $k_1 = |\alpha| + \alpha_1 + \beta'_1 + \gamma'_1$. By symmetry, it follows that one can also take $k_2 = |\beta| + \beta_1 + \alpha'_1 + \gamma'_1$ and $k_3 = |\gamma| + \gamma_1 + \alpha'_1 + \beta'_1$. □

Remark B.2. More detailed computations show that the coefficient of $u^{|\alpha|+\alpha_1+\beta'_1+\gamma'_1}$ in $Q^-_{\alpha,\beta,\gamma}$ is

$$s_{\overline{\alpha}'}\left[\frac{1+v^2+w^2}{vw}\right] \cdot s_{\beta'}[1]s_{\gamma'}[1]$$

where $\overline{\alpha}$ is the partition obtained from α by removing its first row (and $\overline{\alpha}'$ is the conjugate of $\overline{\alpha}$). This is non–zero if and only if β and γ have at most one column, and $\overline{\alpha}$ has at most three columns. This is, the only case when the bound is reached.

B.2. **First row for reduced Kronecker coefficients.** We give bounds for the constants k'_1, k'_2 and k'_3 appearing In Theorem 6.1.

Theorem B.3. *In Theorem 6.1, one can take*

$$\begin{cases} k'_1 &= |\alpha| + |\beta| + |\gamma| + \beta_1, \\ k'_2 &= |\alpha| + |\beta| + |\gamma| + \gamma_1, \\ k'_3 &= |\alpha| + |\beta| + |\gamma| + \alpha_1 + \beta_1 + \gamma_1. \end{cases}$$

Proof. After the proof of Theorem 6.1, one can take

$$\begin{cases} k'_1 &= \max_{\omega\in\Omega} \ell_1(\omega), \\ k'_2 &= \max_{\omega\in\Omega} (\ell_1(\omega) - \ell_2(\omega)), \\ k'_3 &= \max_{\omega\in\Omega} (\ell_1(\omega) - \ell_3(\omega)). \end{cases}$$

Let us perform the change of variables $x = uvw$, $y = vw$, $z = uw$, so that $x^a y^b z^c = u^{\ell_1-\ell_2} v^{\ell_1-\ell_3} w^{\ell_1}$. Then the constants k'_1, k'_2, k'_3 are the degrees of $Q^+_{\alpha,\beta,\gamma}$ in the variables, respectively, u, v and w.

Let us bound the degree in u of Q^+. After the change of variables, we obtain that $H(x,y,z) = u^2vw^2Y + uH_1 + H_0$ where H_1 is free of u

and has all its terms of degree at least 1 in X, Y and Z, and H_0 has all its term of degree ≤ 0 in u. Thus,

$$\sigma[H] = \sigma[u^2vw^2Y]\sigma[uH_1]\sigma[H_0] = \sum_{i,j} u^{2i+j}(vw^2)^i h_i[Y]h_j[H_1]\sigma[H_0]$$

and, therefore,

$$\begin{aligned} Q^+ &= \sum_{i,j} u^{2i+j}(vw^2)^j \langle h_i[Y]h_j[H_1]\sigma[H_0] \mid s_\alpha[X]s_\beta[Y]s_\gamma[Z]\rangle \\ &= \sum_{i,j} u^{2i+j}(vw^2)^j \left\langle h_j[H_1]\sigma[H_0] \,\middle|\, s_\alpha[X](h_i^\perp s_\beta)[Y]s_\gamma[Z]\right\rangle \end{aligned}$$

Note that $h_i^\perp s_\beta = 0$ unless $i \leq \beta_1$. Moreover the left–hand side of each scalar product in the sum is now a sum of homogeneous symmetric functions all of total degree at least j, while the right–hand side has degree $|\alpha| + |\beta| + |\gamma| - i$. Thus, the non-zero summands fulfill $j \leq |\alpha|+|\beta+|\gamma|-i$. We conclude that for all non–zero summands, $2i+j \leq |\alpha| + |\beta| + |\gamma| + \beta_1$. □

Appendix C. Another approach to the hook stability property, derived from Murnaghan's stability and conjugation

C.1. **Half of Theorem 5.6.** It is well–known that the Kronecker coefficients are invariant under conjugation of any two of their arguments:

$$g_{\lambda,\mu,\nu} = g_{\lambda',\mu',\nu} = g_{\lambda',\mu,\nu'} = g_{\lambda,\mu',\nu'}.$$

We have (conjugating the arguments in position 1 and 2), $g_{\lambda,\mu,\nu} = g_{\lambda',\mu',\nu}$. Assuming that (λ', μ', ν) is stable, that is, that the value of the Kronecker coefficient does not change by adding one to the first row of each, we have

$$g_{\lambda',\mu',\nu} = g_{\lambda'\oplus(1|0),\mu'\oplus(1|0),\nu\oplus(1|0)}. \tag{27}$$

Conjugating again the arguments in position 1 and 2, we have that $g_{\lambda'\oplus(1|0),\mu'\oplus(1|0),\nu\oplus(1|0)} = g_{\lambda\oplus(0|1),\mu\oplus(0|1),\nu\oplus(1|0)}$. We conclude that $g_{\lambda,\mu,\nu} = g_{\lambda\oplus(0|1),\mu\oplus(0|1),\nu\oplus(1|0)}$. From Lemma 5.8 we get the following sufficient condition for (27) to hold:

$$N \geq N_0(\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}) + (\lambda_1' + \mu_1 + \nu_1)/2, \tag{28}$$

where, again, N is the weight of the partitions λ, μ and ν. Likewise, by conjugating the partitions at positions 1 and 3, or 2 and 3, or the Kronecker coefficients, we would get that $g_{\lambda,\mu,\nu} = g_{\lambda\oplus(0|1),\mu\oplus(1|0),\nu\oplus(0|1)}$ and $g_{\lambda,\mu,\nu} = g_{\lambda\oplus(1|0),\mu\oplus(0|1),\nu\oplus(0|1)}$, under the assumptions that, respectively

$$N \geq N_0(\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}) + (\lambda_1 + \mu_1' + \nu_1)/2, \tag{29}$$

$$N \geq N_0(\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}) + (\lambda_1 + \mu_1 + \nu_1')/2. \tag{30}$$

Of course we have also $g_{\lambda,\mu,\nu} = g_{\lambda\oplus(1|0),\mu\oplus(1|0),\nu\oplus(1|0)}$ when

$$N \geq N_0(\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}) + (\lambda'_1 + \mu'_1 + \nu'_1)/2. \tag{31}$$

Assume that all four hypotheses (28),(29), (30) and (31) hold for (λ, μ, ν). One can check that they hold as well for all triples of partitions $(\lambda \oplus (m-a|a), \mu \oplus (m-b|b), \nu \oplus (m-c|c))$ such that (a, b, c, m) is in the semigroup $\mathcal{S}$ generated by the four vectors $(1, 1, 0, 1)$, $(1, 0, 1, 1)$, $(1, 1, 0, 1)$ and $(0, 0, 0, 1)$. It follows, by induction, that

$$g_{\lambda,\mu,\nu} = g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)}$$

for all $(a, b, c, m) \in \mathcal{S}$. It is not difficult to establish (using again the change of variables ℓ_1, ℓ_2, ℓ_3) that the semigroup $\mathcal{S}$ is the set of points $(a, b, c, m) \in \mathbb{N}^4$ that fulfill the condition (21) *and, additionally, the congruence* $a + b + c \equiv 0 \mod 2$.

Let us consider also the case when the triple (λ, μ, ν) is not assumed to fulfill (28),(29), (30) and (31). Then one can check that $(\lambda \oplus (m - a|a), \mu \oplus (m-b|b), \nu \oplus (m-c|c))$ fulfill these conditions when a system of type (17) holds, with

$$\begin{cases} d_1 &= 2\,(N_0 - N) + \lambda'_1 + \mu_1 + \nu_1, \\ d_2 &= 2\,(N_0 - N) + \lambda_1 + \mu'_1 + \nu_1, \\ d_3 &= 2\,(N_0 - N) + \lambda_1 + \mu_1 + \nu'_1, \\ d &= N_0 - N + (\lambda'_1 + \mu'_1 + \nu'_1)/2. \end{cases}$$

Therefore, we *nearly* recover the stability property of Theorem 5.6. To recover the full theorem, it would be enough to establish that there exists m big enough, such that

$$g_{\lambda\oplus(2m|2m),\mu\oplus(2m|2m),\nu\oplus(2m|2m)} = g_{\lambda\oplus(2m+1|2m+1),\mu\oplus(2m+1|2m+1),\nu\oplus(2m+1|2m+1)}.$$

See next subsection for a possible approach to this question.

C.2. **A monotonicity conjecture.** Remember that Murnaghan's sequences of Kronecker coefficients are weakly increasing (see Section 3.1): for any three partitions λ, μ, ν of the same weight,

$$g_{\lambda,\mu,\nu} \leq g_{\lambda+(1),\mu+(1),\nu+(1)}. \tag{32}$$

Here is, conjecturally, a more general monotonicity property.

Conjecture C.1 (Conjecture 5.11 restated)**.** *For any three partitions λ, μ and ν of the same weight, and any (a, b, c, m) fulfilling* (21)*,*

$$g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)}.$$

Again, using the symmetries of the Kronecker coefficients, it is not difficult to prove part of this conjecture.

Proposition C.2. *Let λ, μ and ν be three non–empty partitions of the same weight. If $g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(1|1),\mu\oplus(1|1),\nu\oplus(1|1)}$, then*

$$g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)}$$

for all (a, b, c, m) fulfilling (21).

Proof. We use again the symmetry of the Kronecker coefficients under conjugating two arguments. We have the identity $g_{\lambda,\mu,\nu} = g_{\lambda',\mu',\nu}$. Using (32) we get $g_{\lambda',\mu',\nu} \leq g_{\lambda'\oplus(1|0),\mu'\oplus(1|0),\nu\oplus(1|0)}$. Conjugating again the arguments in position 1 and 2, we have that $g_{\lambda'\oplus(1|0),\mu'\oplus(1|0),\nu\oplus(1|0)}$ is equal to $g_{\lambda\oplus(0|1),\mu\oplus(0|1),\nu\oplus(1|0)}$. Therefore, $g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(0|1),\mu\oplus(0|1),\nu\oplus(1|0)}$. Likewise $g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(0|1),\mu\oplus(1|0),\nu\oplus(0|1)}$ and $g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(1|0),\mu\oplus(0|1),\nu\oplus(0|1)}$.

Using these 3 identities, together with (32), we see that $g_{\lambda,\mu,\nu} \leq g_{\lambda,\mu,\nu}(\tau)$ for all τ in the semigroup $\mathcal{S}$ generated by $(1,1,0,1)$, $(1,0,1,1)$, $(0,1,1,1)$ and $(0,0,0,1)$ (here we use again the notation introduced after (31)). This semigroup was determined earlier: the points (a,b,c,N) that fulfill (21) are exactly the elements of $\mathcal{S}$ and the elements $(1,1,1,1)+\tau$ for $\tau \in \mathcal{S}$. This proves the proposition. □

Proposition C.2 shows that Conjecture 5.11 is equivalent to the following seemingly much weaker statement.

Conjecture C.3 (Equivalent form of Conjecture 5.11; this is Conjecture 5.12 restated)**.** *For any three partitions λ, μ and ν of the same weight,*

$$g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(1|1),\mu\oplus(1|1),\nu\oplus(1|1)}.$$

Remark C.4. Conjecture 5.12 was checked by computer, with SAGE [51], for all triples of partitions of weight at most 16.

Remark C.5. A proof of Conjecture 5.12 would provide an alternative proof of Theorem 5.6.

Appendix D. One the generating function for the coefficients $B_{\alpha,\beta,\gamma}$

D.1. **Expression involving Schur functions indexed by hooks.** It is proved in Theorem 7.3 that the Schur generating function for the coefficients $B_{\alpha,\beta,\gamma}$ is

$$\sigma[XYZ + 2W] \cdot \left(\frac{3}{4} + \frac{1}{4}\sigma[(\varepsilon - 1)W] - \frac{1}{2}\chi[W] + \chi[YZ - X] \right)$$

The following result is stated in a remark, with no proof,

Proposition D.1. *Fix partitions α, β, γ. The coefficient $B_{\alpha,\beta,\gamma}$ in Theorem 6.1 is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion in the Schur basis of*

$$\sigma[XYZ+2W] \cdot \left(1 - \sum_{a\,even\ ,b} (-1)^b s_{(a|b)}[W] + \sum_{a,b} (-1)^b s_{(a|b)}[YZ - X] \right).$$

Proof. From Cauchy Formula,

$$\sigma[(\varepsilon - 1)W] = \sigma[(1-\varepsilon)(-W)] = \sum_\lambda s_\lambda[1-\varepsilon] s_\lambda[-W] = \sum_\lambda s_\lambda[1-\varepsilon](-1)^{|\lambda|} s_{\lambda'}[W]. \tag{33}$$

After [50, Ex. 7.43 with $t=1$], $s_\lambda[1-\varepsilon]$ is 1 if λ is the empty partition, 2 if λ is a hook and 0 otherwise. Therefore,

$$\sigma[(\varepsilon-1)W] = 1 + 2 \sum_{a,b\geq 0} (-1)^{1+a+b} s_{(a|b)}[W].$$

Thus,

$$\frac{3}{4} + \frac{1}{4}\sigma[(\varepsilon-1)W] = 1 + \frac{1}{2}\sum_{a,b}(-1)^{1+a+b} s_{(a|b)}[W].$$

After [35, I.§3. Ex. 11 (2) with $\mu=\emptyset$], we have

$$\chi = \sum_{a,b} (-1)^b s_{(a|b)}. \tag{34}$$

Therefore,

$$\begin{aligned}\frac{3}{4} + \frac{1}{4}\sigma[(\varepsilon-1)W] - \frac{1}{2}\chi[W] =& \\ 1 + \frac{1}{2}\sum_{a,b}(-1)^{1+a+b} s_{(a|b)}[W] - \frac{1}{2}\sum_{a,b}(-1)^b s_{(a|b)}& \\ = 1 - \sum_{a \text{ even}, b} (-1)^b s_{(a|b)}[W]&\end{aligned}$$

Using again (34) to rewrite $\chi[YZ-X]$, we get the following Formula for the generating function of the coefficients of B:

$$\sigma[XYZ+2W]\cdot\left(1 - \sum_{a \text{even }, b} (-1)^b s_{(a|b)}[W] + \sum_{a,b}(-1)^b s_{(a|b)}[YZ-X]\right)$$

□

D.2. **Toolbox for other expressions.** In order to write in other ways the generating function for the coefficients $B_{\alpha,\beta,\gamma}$, the following formulas may be useful:

$$\sigma[X]\cdot\chi[X] = \sum_k k\, h_k[X],$$

$$\sigma[2X]\cdot\chi[X] = \sum_{\lambda:\ell(\lambda)\leq 2} \frac{(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2)}{2} s_\lambda[X].$$

They follow from the fact that $\sigma[tX]\chi[tX]$ is the derivative of $\sigma[tX]$ (for the first one), and that $\sigma[2tX]\chi[2tX]$ is the derivative of $\sigma[2tX]$. Last, by Cauchy formula,

$$\sigma[2tX] = \sum_\lambda s_\lambda[2] s_\lambda[X] t^{|\lambda|},$$

and $s_\lambda[2] = (\lambda_1 - \lambda_2 + 1)$ is λ has at most two parts, and is equal to 0 otherwise.