EQUATIONS, INEQUATIONS AND INEQUALITIES CHARACTERIZING THE CONFIGURATIONS OF TWO REAL PROJECTIVE CONICS

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ABSTRACT. Couples of proper, non-empty real projective conics can be classified *modulo* rigid isotopy and ambient isotopy. We characterize the classes by equations, inequations and inequalities in the coefficients of the quadratic forms defining the conics. The results are well–adapted to the study of the relative position of two conics defined by equations depending on parameters.

1. INTRODUCTION

Couples of proper real projective conics, admitting real points, can be classified *modulo* ambient isotopy. The goal of this paper is to provide equations, inequations and inequalities characterizing each class.

This is particularly well-suited for the following problem: given two conics whose equations depend on parameters, for which values of the parameters are these conics in a given ambient isotopy class ?

Such problems are of interest in geometric modeling. They are considered for instance in the articles [9, 20] (and [21] for the similar problem for ellipsoids). We consider this paper as the systematization of their main ideas. Specially, in [9], an algorithm was proposed to determine the configuration of a pair of ellipses, by means of calculations of Sturm-Habicht sequences. Our approach is different: when there, computations were performed for each particular case, we perform the computations once for all in the most general case. The formulas obtained behave well under specialization.

Instead of working with ambient isotopy, we consider another equivalence relation, *rigid isotopy*[1], corresponding to real deformation of the equations of the conics that doesn't change the nature of the (complex) singularities (definition 1, following the ideas of [12]). Figures 1 and 2 provide a drawing for a representative of each class. Rigid isotopy happens to be an equivalence relation just slightly finer than ambient

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[1]We follow the terminology used in real algebraic geometry in similar situations.
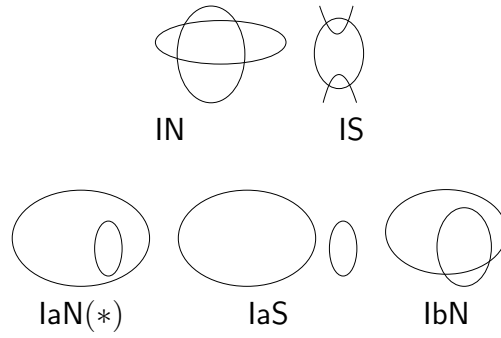
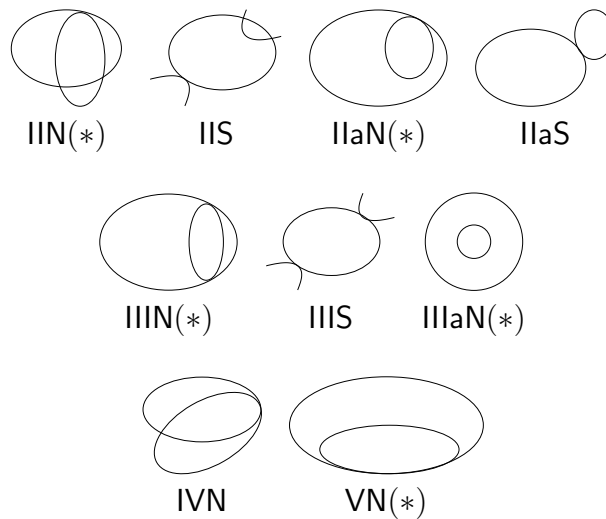FIGURE 1. The rigid isotopy classes for generic pairs of conics.

FIGURE 2. The rigid isotopy classes for non-generic pairs
of proper conics.

isotopy. So we get the classification under ambient isotopy directly from the one under rigid isotopy.

The classification of pairs of real projective conics under rigid isotopy was first obtained by Gudkov and Polotovskiy [13, 14, 15] in their work on quartic real projective curves. Nevertheless, we start (section 2) with re–establishing this classification. We emphasize the following key ingredient: that any rigid isotopy decomposes into a path in one orbit of the space of pencils of conics under projective transformations, and a rigid isotopy stabilizing some pencil of conic (Lemma 2). As a consequence, each rigid isotopy class is determined by an orbit of pencils of conics and the position of the two conics with respect to the degenerate conics in the pencil they generate. This has two direct applications.

First, because there are finitely many orbits of pencils of conics under projective transformations, we get easily a finite set of couples of conics meeting at least once each rigid isotopy class.

Second, it indicates clearly how to derive the equations, inequations, inequalities characterizing the classes, which is done in section 3. The determination of the position of the conics with respect to the degenerate conics in a pencil essentially reduces to problems of location of roots of univariate polynomials. They can be treated using standard tools from real algebra, namely Descartes' law of signs and subresultant sequences. This contributes both to the characterization of the orbits of pencils of conics, and the characterization of the rigid isotopy class associated to each orbit of pencils of conics. Classical invariant theory is also used for the first task.

Last, section 4 provides some examples of computations using the previous results.

Generalities and notations. The real projective space of dimension k will be denoted with $\mathbb{RP}^k$; in particular, $\mathbb{RP}^2$ denotes the projective plane. The space of real ternary quadratic forms will be denoted with $S^2\mathbb{R}^{3*}$. We will consider $\mathbb{P}(S^2\mathbb{R}^{3*})$, the associated projective space (see [6] for the definitions of the notions of projective geometry needed here). The term *conic* will be used with two meanings:

- an algebraic meaning: an element of $\mathbb{P}(S^2\mathbb{R}^{3*})$. The algebraic conic associated to the quadratic form f will be denoted with $[f]$.
- a geometric meaning: the zero locus, in $\mathbb{RP}^2$, of a non-zero quadratic form f. It will be denoted with $[f = 0]$.

A (geometric or algebraic) conic is said *proper* if it comes from a non-degenerate quadratic form; *degenerate* if it comes from a degenerate quadratic form. Note that, with this definition, the empty set is a proper (geometric) conic. Algebraic and geometric proper non-empty conics are in bijection, and can be identified.

We define the *discriminant of the quadratic form f* to be

$$\mathrm{Disc}(f) = \det(\mathrm{Matrix}(f)).$$

Any proper non-empty conic cuts out the real projective plane into two connected components. They are topologically non-equivalent: one is homeomorphic to a Möbius strip, the other to an open disk. The former is the *outside* of the conic, the latter is its *inside*. Let $f_0 = x^2 + y^2 - z^2$. The inside of $[f_0 = 0]$ is the solution set of the inequation $f_0 < 0$, or, equivalently, the set of points where f_0 has the sign of $\mathrm{Disc}(f_0)$. These signs change together under linear transformations. Now any proper, non-empty conic is obtained from $[f_0 = 0]$ by means of a transformation of $PGL(3, \mathbb{R})$. Thus *the inside of $[f = 0]$ is the set of points where f takes the sign of $\mathrm{Disc}(f)$.*

The *tangential quadratic form* associated to the quadratic form f on $\mathbb{R}^3$ is the quadratic form $\tilde{f}$ on $\mathbb{R}^{3*}$ whose matrix is the matrix of the cofactors of the matrix of f. The *tangential conic* associated to $[f]$ (resp. $[f = 0]$) is $[\tilde{f}]$ (resp. $[\tilde{f} = 0]$).

A *pencil of quadratic forms* is a plane (through the origin) in $S^2\mathbb{R}^{3*}$; the associated *(projective) pencil of conics* is the corresponding line in $\mathbb{P}(S^2\mathbb{R}^{3*})$. It is said to be *non-degenerate* if it contains proper conics[2]. The common points of all conics of a given pencil are called the *base points* of the pencil. They are also the common points of any two distinct conics of the pencil. A non-degenerate pencil of conics has always four common points in the complex projective space, when counted with multiplicities.

Note that we will distinguish between (ordered) *couples* of conics ((C_1, C_2) distinct from (C_2, C_1), except when $C_1 = C_2$) and (unordered) *pairs* of conics ($\{C_1, C_2\} = \{C_2, C_1\}$).

The *characteristic form* of the couple (f, g) of real ternary quadratic forms is the binary cubic in (t, u):

$$\Phi(f, g; t, u) := \mathrm{Disc}(tf + ug).$$

Its coefficients will be denoted as follows:

$$\Phi(f, g; t, u) = \Phi_{30}t^3 + \Phi_{21}t^2u + \Phi_{12}tu^2 + \Phi_{30}u^3.$$

We will also consider the de-homogenized polynomial obtained from Φ by setting $u = 1$. It will be denoted with $\phi(f, g; t)$, or $\phi(t)$ when there is no ambiguity about f, g. So:

$$\phi(t) := \mathrm{Disc}(tf + g)$$

Note that $\Phi_{30} = \mathrm{Disc}(f)$ and $\Phi_{03} = \mathrm{Disc}(g)$.

An *isotopy* of a manifold M is a continuous mapping $\theta : I \times M \to M$, where I is an interval containing 0, such that for each $t \in I$, the mapping $x \mapsto \theta(t, x)$ is an homeomorphism of M onto itself, and $x \mapsto \theta(0, x)$ is the identity of M.

Two subsets N_1, N_2 of M are *ambient isotopic* if there is an isotopy of M such that, at some instant $t \in I$, $\theta(t, N_1) = N_2$.

This definition is immediately generalized to couples of subsets: (N_1, N_1') and (N_2, N_2') are ambient isotopic if there is an isotopy of M such that, at some instant t, $\theta(t, N_1) = N_2$ and $\theta(t, N_1') = N_2'$.

2. Classification

2.1. **Rigid isotopy.** To classify the couples of conics up to ambient isotopy, we introduce a slightly finer equivalence relation, *rigid isotopy*, corresponding to a continuous path in the space of couples of distinct proper conics, that doesn't change the nature of the complex singularities of the union of the conics. Before stating formally the definition

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[2]Contrary, for instance, to the pencil of the zero loci of the $f(x, y, z) = \lambda xy + \mu xz$.

(definition 1 below), we clarify this point. The complex singularities of the union of the conics correspond to the (real and imaginary) intersections of the conics. For a given multiplicity, there is only one analytic type of intersection point of two conics. Thus the nature of the singularities for the union of two distinct proper conics is determined by the numbers of real and imaginary intersections of each multiplicity. This is narrowly connected to the projective classification of pencils of conics, that can be found in [7, 17]. The connection is the following theorem.

Theorem 1. ([7, 17]) *Two non-degenerate pencils of conics are equivalent* modulo $PGL(3, \mathbb{R})$ *if and only if they have the same numbers of real and imaginary base points of each multiplicity.*

The space of couples of distinct real conics is an algebraic fiber bundle over the variety of pencils, which is a grassmannian of the $\mathbb{RP}^1$'s in a $\mathbb{RP}^5$. The fibers are isomorphic to the space of couples of distinct points in $\mathbb{RP}^1$. The sets of couples of distinct conics with given numbers of real and imaginary intersections of each multiplicity are, after Theorem 1, exactly the inverse images of the orbits of the variety of pencils under $PGL(\mathbb{R}^3)$, and are thus also smooth real algebraic submanifolds.

We can now state the following definition.

Definition 1. *Two couples of distinct proper conics are* rigidly isotopic *if they are connected by a path in the space of couples of distinct proper conics, along which the numbers of real and imaginary intersections of each multiplicity don't change.*

We will now show that *rigidly isotopic* implies *ambient isotopic*. We first show it for some special rigid isotopies.

Definition 2. *Let f, g be two non-degenerate non-proportional quadratic forms. We define a* sliding *for $[f]$ and $[g]$ as a path of the form $t \mapsto ([f + tkg], [g])$ (or $t \mapsto ([f], [g + tkf])$) for t in a closed interval containing 0; no t with $f + tkg$ (resp. $g + tkf$) degenerate; and k some real number.*

Let α be an homeomorphism of $\mathbb{RP}^2$. For a couple $([f], [g])$ of non-empty proper conics, we write $\alpha([f], [g])$ for the couple of algebraic conics corresponding to $(\alpha([f = 0]), \alpha([g = 0]))$.

Lemma 1. *Any sliding, for a couple of non-empty conics, lifts to an ambient isotopy.*

For a sliding:
$$t \mapsto ([f + tkg], [g]), \quad t \in I$$
with $[f = 0]$ and $[g = 0]$ non-empty, this means that there exists a family of homeomorphisms β_t of $\mathbb{RP}^2$, with $\beta_0 = \text{id}$ and $\beta_t([f], [g]) = ([f + tkg], [g])$.

Proof. Let B be the set of the base points of the pencil of $[f]$ and $[g]$. A stratification of $\mathbb{RP}^2 \times I$ is given by:

$$
\begin{aligned}
S_1 &= B \times I \\
S_2 &= ([g = 0] \times I) \setminus S_1 \\
S_3 &= \{(\mathbf{p}; t) \mid (f + tkg)(\mathbf{p}) = 0\} \setminus S_1 \\
S_4 &= \left(\mathbb{RP}^2 \times I\right) \setminus (S_1 \cup S_2 \cup S_3)
\end{aligned}
$$

One checks this stratification is Whitney. The projection from $\mathbb{RP}^2 \times I$ to I is a proper stratified submersion. The lemma now follows, by direct application of Thom's isotopy lemma, as it is stated in [11].   □   □

Lemma 2. *Consider two couples of distinct proper non-empty conics. If they are rigidly isotopic, then they can also be connected by a rigid isotopy $\alpha_t(s_t)$ where*

- *s_t is a sequence of slidings along one given pencil.*
- *α_t is a path in $PGL(3, \mathbb{R})$ with $\alpha_0 = \mathrm{id}$*

Proof. Let (C_0, D_0) and (C_1, D_1) be the couples of conics, and $t \mapsto (C_t, D_t)$, $t \in [0; 1]$ be the rigid isotopy that connects them. It projects to a path in one $PGL(3, \mathbb{R})$-orbit of the variety of pencils. This path lifts to a path α_t of $PGL(3, \mathbb{R})$ with $\alpha_0 = \mathrm{id}$ (indeed, the group is a principal fiber bundle over each orbit; specially, it is a locally trivial fiber bundle: [3], ch. I, 4).

The mapping $t \mapsto \alpha_t^{-1}(C_t, D_t)$ is a rigid isotopy drawn inside one pencil of conics. Such an isotopy is easy to describe: the pencil is a space $\mathbb{RP}^1$ with a finite set Γ of degenerate conics. Let $E = \mathbb{RP}^1 \setminus \Gamma$. A rigid isotopy inside the pencil is exactly a path in $E \times E \setminus \mathrm{Diag}(E \times E)$. There exists a finite sequence s_t of horizontal and vertical paths, *i.e.* of slidings, having also origin (C_0, D_0) and extremity $\alpha_1^{-1}(C_1, D_1)$.

Consider now $\alpha_t(s_t)$. This is a rigid isotopy connecting (C_0, D_0) to (C_1, D_1).   □   □

Theorem 2. *Two couples of distinct proper non-empty conics that are rigidly isotopic are also ambient isotopic.*

Proof. Let (C_0, D_0) and (C_1, D_1) be rigidly isotopic. Consider a path $t \in [0, 1] \mapsto \alpha_t(s_t)$ connecting them, as in Lemma 2. After Lemma 1, s_t lifts to an ambient isotopy β_t with $\beta_0 = \mathrm{id}$. Then $\alpha_t \circ \beta_t$ is an ambient isotopy carrying (C_0, D_0) to (C_1, D_1).   □   □

2.2. **Orbits of pencils of conics.** After [7, 17], there are nine orbits of non-degenerate pencils of conics under the action of $PGL(3, \mathbb{R})$. We follow Levy's nomenclature [17] for them. It is presented in the following table, where the second and third lines display the multiplicities of the real and imaginary base points. For instance, 211 stands for one

Orbit	f_0	g_0
I	$x^2 - y^2$	$x^2 - z^2$
Ia	$x^2 + y^2 + z^2$	xz
Ib	$x^2 + y^2 - z^2$	xz
II	yz	$x(y - z)$
IIa	$y^2 + z^2$	xz
III	xz	y^2
IIIa	$x^2 + y^2$	z^2
IV	$xz - y^2$	xy
V	$xz - y^2$	x^2

TABLE 1. Levy's representatives for each orbit of pencils. Each representative is the pencil generated by $[f_0]$ and $[g_0]$.
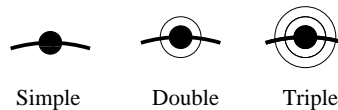

Simple   Double   Triple

FIGURE 3. Degenerate conics corresponding to multiple roots of the discriminant, in the representations of the pencils.

base point of multiplicity 2 and two base points of multiplicity 1.

Orbit	I	Ia	Ib	II	IIa	III	IIIa	IV	V
real points	1111	−	11	211	2	22	−	31	4
imaginary points	−	1111	11	−	11	−	22	−	−

We will also use the representatives of the orbits provided by Levy [17]. Each representative is given by a pair of generators of the corresponding pencil of quadratic forms. They are presented in Table 1. We provide, in figures 5 and 6, graphical representations of characteristic features of the pencils in each orbit. This has two goals: finding how to discriminate between the different orbits of pencils, and determining the possible rigid isotopy classes corresponding to each orbit of pencils. Each pencil is displayed as a circle, as it is topologically. In addition, the following information is represented:

- the degenerate conics of the pencil. They are given by roots of the characteristic form, so the multiplicity of this root is also indicated, following the encoding shown in Figure 3.
- the nature of the proper conics (empty or non-empty) and of the degenerate conics (pair of lines, line or isolated point). The nature of the proper conics is constant on each arc between two degenerate conics.
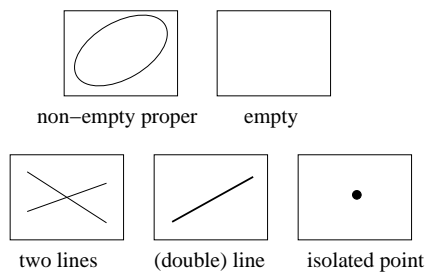
FIGURE 4. Nature of the conics, in the representations of the pencils.
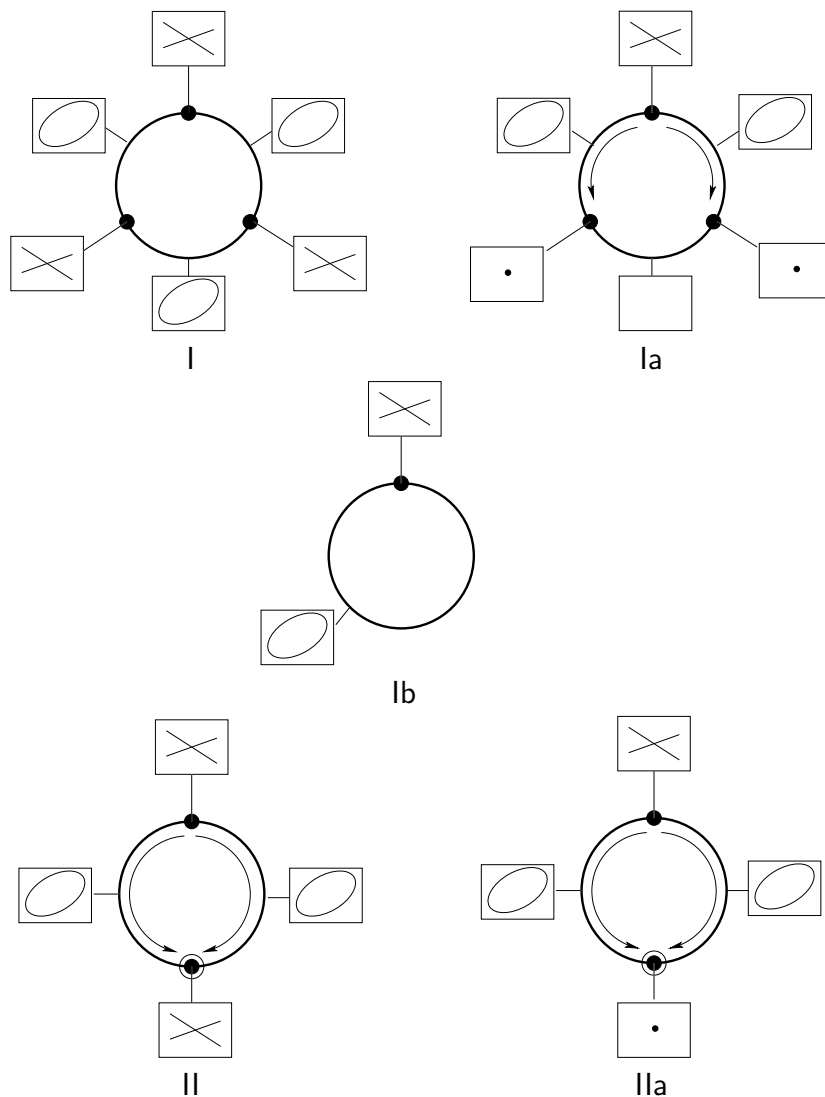

FIGURE 5. Pencils of conics up to projective equivalence (beginning).

- In the case where the conics of one arc are nested, we indicate, by means of an arrow, which are the inner ones.
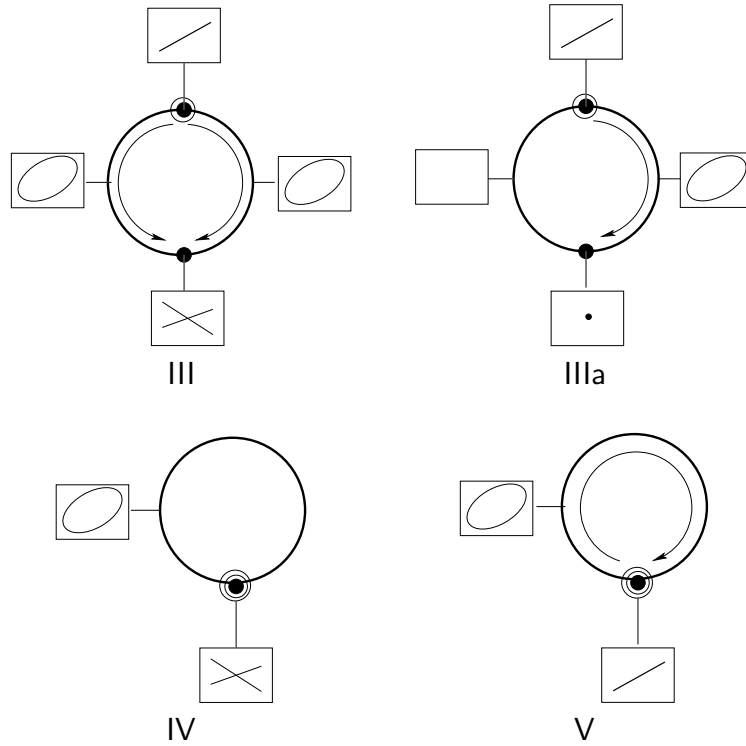
FIGURE 6. Pencils of conics up to projective equivalence (end).

These features are conserved under projective equivalence. Thus the representations are established by considering Levy's representative.

2.3. **Rigid isotopy classification for pairs.** We first classify *pairs* of proper conics, that is: couples of distinct proper conics, under rigid isotopy *and permutation of the two conics.* Later, for each pair class, we will check whether it is also a couple class (that is: the exchange of the two conics corresponds to a rigid isotopy) or it splits into two couple classes.

Any pencil of conics is cut into arcs by its degenerate conics. Two proper conics are either on a same arc, or on distinct arcs on the pencil they generate.

Lemma 3. *If a pencil of conics has (at least) two arcs of non-empty conics, then there are two equivalence classes for pairs of conics generating it. They correspond to the following situations:*

- *the conics are on a same arc.*
- *the conics are on distinct arcs.*

If the pencil has only one arc with proper non-empty conics, there is only one class.

Proof. The orbit of pencils is assumed to be fixed. Because of Lemma 1, to get (at least) one representative for each class, it is enough:

- to choose arbitrarily one conic on each arc and consider all the possible pairs of these conics.
- to choose arbitrarily two conics on each arc and consider these pairs for each arc.

But one observes that for a pencil in one of the orbits Ia, II, IIa, III, there is a projective automorphism that leaves it globally invariant and exchanges its two arcs (the two arcs bearing non-empty conics for orbit Ia). Again, this is proved by considering only Levy's representatives: the reflection $x \leftrightarrow -x$ is suitable. Similarly, a pencil in orbit I is left globally invariant by some projective automorphism that permutes cyclically the three arcs. For Levy's representative, one can take the cyclic permutation of coordinates: $x \mapsto y \mapsto z \mapsto x$.

Pencils in the four other orbits have only one arc with non-empty proper conics.

We have shown it is enough:

- to choose arbitrarily one arc with non-empty conics and two conics on this arc.
- to choose arbitrarily two arcs and one conic on each arc.

This gives nine representatives for the pairs of conics on a same arc, denoted with IN, ..., VN (N like *neighbors*) and five representatives for pairs of conics on distinct arcs, denoted with IS, IaS, IIS, IIaS, IIIS (S like *separated*).

Now it remains to check that for orbits I, Ia, II IIa and III, the S–representative and the N–representative are not equivalent. We use that a rigid isotopy conserves the topological type of $(\mathbb{RP}^2, [f = 0], [g = 0])$, after Theorem 2. To distinguish between IN and IS, one can count the number of connected components of the complement of $[f = 0] \cup [g = 0]$: there are 6 in the first case and 5 in the second, the topological types are different, so are the rigid isotopy classes. For the other four orbits of pencils, one conic lies in the inside of the other (at least at the neighborhood of the double point for II) for the N-representative, while there is no such inclusion for the S-representative.   □   □

Corollary 1. *There are* 14 *equivalence classes for pairs of proper non-empty conics under rigid isotopy and exchange. Representatives for them are given in Table 2. They correspond to the graphical representations displayed in Figures 1 and 2.*

2.4. **Rigid isotopy classification for couples.** We now derive from our classification for pairs of conics the classification for couples of conics.

Lemma 4. *For each of the following representatives: IN, IS, IaS, IbN, IIS, IIaS, IIIS, IVN, there is a rigid isotopy that swaps the two conics. As a consequence, each of these classes for pairs is also a class for couples.*

class	f	g
IN	$3x^2 - 2y^2 - z^2$	$3x^2 - y^2 - 2z^2$
IS	$3x^2 - 2y^2 - z^2$	$x^2 - 2y^2 + z^2$
IaN(∗)	$x^2 + y^2 + z^2 + 3xz$	$x^2 + y^2 + z^2 + 4xz$
IaS	$x^2 + y^2 + z^2 + 3xz$	$x^2 + y^2 + z^2 - 3xz$
IbN	$x^2 + y^2 - z^2 + xz$	$x^2 + y^2 - z^2 - xz$
IIN(∗)	$yz + xy - xz$	$yz + 2xy - 2xz$
IIS	$yz + xy - xz$	$yz - xy + xz$
IIaN(∗)	$y^2 + z^2 + xz$	$y^2 + z^2 + 2xz$
IIaS	$y^2 + z^2 + xz$	$y^2 + z^2 - xz$
IIIN(∗)	$xz+^2$	$xz + 2y^2$
IIIS	$xz + y^2$	$xz - y^2$
IIIaN(∗)	$x^2 + y^2 - z^2$	$x^2 + y^2 - 2z^2$
IVN	$xz - y^2 + xy$	$xz - y^2 - 2xy$
VN(∗)	$xz - y^2 - x^2$	$xz - y^2 + x^2$

TABLE 2. The rigid isotopy classes.

Proof. For IN, IS, IaS, IIS, IIaS, IIIS, it is enough to exhibit projective automorphisms that stabilize the corresponding Levy's representative and swap two arcs of non-empty proper conics. It was already done in the proof of Lemma 3, except for IN and IS. For them, the reflection $y \leftrightarrow z$ is convenient.

For IbN and IVN, it is enough to exhibit projective automorphisms that stabilize the corresponding Levy's representative and reverse the pencil's orientation. For IbN, the reflection $x \mapsto -x$ is convenient; for IVN, one may use the transformation $x \mapsto -x, z \mapsto -z$.   □   □

Lemma 5.

- *Each of the classes of pairs IaN, IIaN, IIIN, IIIaN, VN splits into two classes for couples, corresponding to one conic lying inside the other (except for the base points).*
- *The class of pairs IIN also splits into two classes for couples, corresponding to one conic lying inside the other in a neighborhood of the double point (except the double point itself).*

Proof. The property that one conic lies inside the second is conserved under ambient homeomorphism, and thus under rigid isotopy. The same holds for inclusion at the neighborhood of a double intersection point.

Thus it is enough to consider the representatives of the given pair classes and check the inclusion to show the theorem. The computations are trivial, hence we omit them.   □   □

Theorem 3. *There are* 20 *classes of couples under rigid isotopy. A set of representatives is given by Table 2, where the reader should add*

*the couple obtained by swapping f and g for each of the lines marked
with $(*)$.*

Corollary 2. *The ambient isotopy classes for couples of conics are the
following unions of rigid isotopy classes:*

- *classes where the two conics can be swapped:* **IN, IS, IaS, IbN** ∪
  IVN, IIS, IIaS *and* **IIIS.**
- *pair classes splitting into two classes for couples, one with* $[f =
  0]$ *inside* $[g = 0]$*, one with* $[g = 0]$ *inside* $[f = 0]$*:* **IaN** ∪ **IIIaN,
  IIN, IIaN** ∪ **VN** *and* **IIIN.**

Proof. (sketch) One shows that **IbN** and **IVN** are ambient isotopic by
building explicitly a homeomorphism[3] of $\mathbb{RP}^2$ sending a representative
of the first to a representative of the second. Details on how to do it
are tedious, we skip them. *Idem* (with $[f = 0]$ inside $[g = 0]$) for **IaN**
and **IIIaN**, and for **IIaN** and **VN**.

Next, one shows the displayed rigid isotopy classes or unions of rigid
isotopy classes are not equivalent *modulo* ambient isotopy. This is done
by considering topological invariants of the triples $(\mathbb{RP}^2, [f = 0], [g =
0])$ which take different values on the 15 representatives.

Let C be the conic $[f = 0]$ (resp. D the conic $[g = 0]$), I (resp. J)
its inside and $\bar{I}$ (resp. $\bar{J}$) the topological closure of this inside.

Then one checks that the numbers of connected components of the
four following sets are suitable for separating the ambient isotopy classes:

$$C \cap D, \quad \mathbb{RP}^2 \setminus (C \cup D), \quad I \setminus \bar{J}, \quad J \setminus \bar{I}$$

□   □

Remark : It is legitimate curiosity to compare these isotopy classes of
couples of conics with the isotopy classes of projective quartic curves
presented in [16], the union of two conics being a quartic. One then
observes that **IN** corresponds to $17p$, **IS** to $16p$, **IaS** to $22p$, **IIS** to $34p$,
IIaS to $44p$, **IIIS** to $38p$, **IaN** ∪ **IIIaN** to $21p$, **IIN** to $36p$ and **IIaN** ∪ **VN** to
$43p$. Finally both **IbN** ∪ **IVN** and **IIIN** correspond to $18p$.   ⊡

3. Characterizing the isotopy classes by equations, inequations and inequalities

3.1. **Preliminaries.**

3.1.1. *The invariants and covariants of two ternary quadratic forms.*
Invariants and covariants (see [18] for a modern reference about clas-
sical invariant theory) are the convenient objects to discriminate, by
means of equations and inequalities, between the different orbits for
couples of complex conics under the group $PGL(3, \mathbb{C})$.

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[3]This is enough. Indeed, the set of the homeomorphisms of $\mathbb{RP}^2$ is connected, so
any homeomorphism is the extremity of some ambient isotopy.

Invariants and covariants of a couple of quadratic ternary forms have been calculated by the classics, and can be found in Glenn's book [10] or Casey's treatise [5].

Proposition 1. *The algebra of invariants of a couple of ternary quadratic forms is freely generated by the coefficients of the characteristic form*[4].

The invariants alone are not sufficient to discriminate between the complex orbits. One has to consider the covariants. Some remarkable covariants of a couple of ternary quadratic forms are:

- The *apolar covariant of the tangential quadratic forms* $\tilde{f}$ and $\tilde{g}$. We will denote it with F. This is a quadratic form that depends quadratically on f, as on g.
- The *autopolar triangle covariant* G, a cubic form that is also cubic in f, as in g, and that always factorizes as a product of three linear forms. When $[f = 0]$ and $[g = 0]$ have four distinct intersections, they are equations of the sides of the unique autopolar triangle associated to them (see [2], 14.5.4 and 16.4.10).

Proposition 2. *The algebra of covariants of a couple of ternary quadratic forms (f, g) is generated by the invariants, the ground forms f and g, the apolar covariant F and the autopolar triangle covariant G.*

The covariant F will not be needed in this paper, but G will be used. We now explain how to derive a formula for it. Consider a generic couple of forms f, g. Let t_1, t_2, t_3 be the three roots of $\mathrm{Disc}(tf + g)$. Each of the $t_i f + g$ has rank two. Their respective associated tangential quadratic forms have all rank one: they are the squares of three linear forms p_1, p_2, p_3 of the dual space, and the associated points $[p_1], [p_2], [p_3]$ of $\mathbb{P}(\mathbb{R}^3)$ are exactly the vertices of the autopolar triangle. The sides of the triangle are obtained as the zero loci of the product of determinants:

$$\det(p_1, p_2, p) \det(p_1, p_3, p) \det(p_2, p_3, p).$$

Working in coordinates, with $p_i = p_{i1}X + p_{i2}Y + p_{i3}Z$, where X, Y, Z are coordinates on $\mathbb{R}^{3*}$ dual to x, y, z, one expands this and replace the products $p_{ij}p_{ik}$ by the corresponding term given by the equality

$$p_i^2 = \widetilde{ft_i + g}.$$

This product is antisymmetric in t_1, t_2, t_3, and thus can be divided by the Vandermonde determinant $(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$. The quotient happens to be free of t_i's: it is, up to a rational number in factor, the covariant G.

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[4]The analogue assertion is still true for a couple of quadratic forms in n variables, for any n.

One finds that the formula for this covariant can be displayed shortly. Denote:

$$\widetilde{tf + g} = \tilde{f}t^2 + \Omega(f,g)t + \tilde{g}$$

and

$$\tilde{f} = \sum \tilde{a}_{ijk}X^iY^jZ^k, \quad \Omega(f,g) = \sum \omega_{ijk}X^iY^jZ^k, \quad \tilde{g} = \sum \tilde{b}_{ijk}X^iY^jZ^k.$$

Consider the matrix of their coefficients:

$$M = \begin{bmatrix} \tilde{a}_{200} & \tilde{a}_{020} & \tilde{a}_{002} & \tilde{a}_{011} & \tilde{a}_{101} & \tilde{a}_{110} \\ \omega_{200} & \omega_{020} & \omega_{002} & \omega_{011} & \omega_{101} & \omega_{110} \\ \tilde{b}_{200} & \tilde{b}_{020} & \tilde{b}_{002} & \tilde{b}_{011} & \tilde{b}_{101} & \tilde{b}_{110} \end{bmatrix}$$

and label its columns with $1, 2, 3, \bar{1}, \bar{2}, \bar{3}$. Label the maximal minor corresponding to columns i, j, k with $[ijk]$. Then the autopolar triangle covariant is, up to a rational number in factor,

$$- [\bar{1}\,2\,3]x^3 - [1\,\bar{2}\,3]y^3 - [1\,2\,\bar{3}]z^3 + ([1\,\bar{1}\,3] + 2[\bar{3}\,\bar{2}\,3])xy^2$$
$$+ ([1\,2\,\bar{1}] + 2[\bar{2}\,2\,\bar{3}])xz^2 + ([\bar{2}\,2\,3] + 2[\bar{1}\,\bar{3}\,3])yx^2 + ([1\,2\,\bar{2}] + 2[1\,\bar{1}\,\bar{3}])yz^2$$
$$+ ([\bar{3}\,2\,3] + 2[\bar{1}\,2\,\bar{2}])zx^2 + ([1\,\bar{3}\,3] + 2[1\,\bar{2}\,\bar{1}])zy^2 + ([1\,2\,3] + 4[\bar{1}\,\bar{2}\,\bar{3}])xyz.$$

3.1.2. *Resultants.* Let $U(t)$ and $V(t)$ be two univariate polynomials. Remember that their resultant $\mathrm{Res}(U,V)$ is the determinant of their Sylvester matrix: the matrix of the coefficients of degree $\deg(U) + \deg(V) - 1$ down to 0 of

$$t^{\deg(V)-1}U, t^{\deg(V)-2}U, \ldots, U, t^{\deg(U)-1}V, t^{\deg(U)-2}V, \ldots, V.$$

A few classical formulas about resultants will be needed.

Lemma 6. *One has*

$$\mathrm{Res}(U,V) = (-1)^{\deg(U)\deg(V)}\mathrm{Res}(V,U).$$

Lemma 7. *Let c be the leading coefficient of U. Then*

$$\mathrm{Res}(U,V) = c^{\deg(V)}\prod V(\rho)$$

where the product is carried over the complex roots ρ of U, counted with multiplicities[5].

Lemma 8. *Let $U(t), V(t), W(t)$ be three univariate polynomials. Then*

$$\mathrm{Res}(U,VW) = \mathrm{Res}(U,V)\mathrm{Res}(U,W).$$

And last:

Lemma 9. *Let $U(t), V(t)$ be two univariate polynomials, and W the remainder in the euclidean division of U by V. Let c be the leading coefficient of V. Then*

$$\mathrm{Res}(U,V) = (-1)^{\deg(U)\deg(V)}c^{\deg(U)-\deg(W)}\mathrm{Res}(V,W).$$

This is Lemma 4.27 in [1], where a proof is provided.

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[5]*e.-g.* a double real root should be here counted as two roots.

3.1.3. *Descartes' law of signs.* Let $U(t)$ be an univariate polynomial. Then Descartes' law of signs give some insight about its number $\mathcal{N}(U)$ of positive real roots, counted with multiplicities.

Consider the sequence of the signs ($+$'s and $-$'s) of the (non-zero) coefficients of U and denote with $\mathcal{V}(U)$ the number of changes in consecutive terms. The following lemma is Descartes' Law of signs. It can be found as Theorem 2.34 in [1].

Lemma 10. *One has $\mathcal{V}(U) \geq \mathcal{N}(U)$, and $\mathcal{V}(U) - \mathcal{N}(U)$ is even.*

Only the following particular consequence will be needed in the sequel:

Lemma 11. *Let $U(t) = u_3 t^3 + u_2 t^2 + u_1 t + u_0$ of degree 3. Suppose U has all its roots real and non-zero. Then they have all the same sign if and only if: $u_3 u_1 > 0$ and $u_2 u_0 > 0$.*

It is obtained by applying Descartes' law of signs to $U(t)$ and $U(-t)$.

3.1.4. *Subresultant sequences.* Here we briefly introduce another tool: subresultant sequences. More details about them can be found in the book [1].

Let $U(t), V(t)$ be two univariate polynomials. One wants to know on how many[6] of the (real) roots of V the polynomial U is positive, negative, and zero. The *Sturm query of U for V* is defined as the number of roots of V making $U > 0$, *minus* the number of roots of V making $U < 0$. This information is easily accessible once one knows the signs of the $\deg(V) + 1$ *signed subresultants principal coefficients* of V and W, where W is the remainder in the euclidean division of $U \cdot V'$ by V.

We give the formulas for these signed subresultant principal coefficients, and the procedure for getting the Sturm query from their signs, only for the particular case needed: when V has degree 3. Write

$$\begin{aligned} V &= v_3 t^3 + v_2 t^2 + v_1 t + v_0 \\ W &= w_2 t^2 + w_1 t + w_0. \end{aligned}$$

Then

$$\mathrm{sr}_3(V, W) = v_3, \qquad\qquad\qquad \mathrm{sr}_2(V, W) = w_2,$$

$$\mathrm{sr}_1(V, W) = \begin{vmatrix} v_3 & v_2 & v_1 \\ 0 & w_2 & w_1 \\ w_2 & w_1 & w_0 \end{vmatrix}, \qquad \mathrm{sr}_0(V, W) = \begin{vmatrix} v_3 & v_2 & v_1 & v_0 & 0 \\ 0 & v_3 & v_2 & v_1 & v_0 \\ 0 & 0 & w_2 & w_1 & w_0 \\ 0 & w_2 & w_1 & w_0 & 0 \\ w_2 & w_1 & w_0 & 0 & 0 \end{vmatrix}.$$

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[6]When dealing with subresultant sequences, the multiplicities of the roots are not taken into account, *e.-g.* a double root will be counted as one root.

Note that $\mathrm{sr}_0(V, W) = -\mathrm{Res}(V, W)$, the opposite of the resultant of V and W. The Sturm query is obtained from the sequence of the signs of $\mathrm{sr}_3, \mathrm{sr}_2, \mathrm{sr}_1, \mathrm{sr}_0$ the following way[7]:

(1) If there is a pair of consecutive zeros, remove it and change the signs that were following to their opposites.

(2) From the resulting sequences of consecutive non-zero terms, compute the difference: number of sign permanences (identical consecutive terms, $++$ or $--$) *minus* number of sign exchanges (opposite consecutive terms, $+-$ or $-+$). This gives the Sturm query[8].

3.2. Discriminating between the orbits of pencils.

In this section, we give the equations and inequations characterizing the couples (f, g) of non-degenerate quadratic forms generating a pencil of each of the orbits.

We first use invariants and covariants whose vanishing depends only of the generated pencil, that is those C that, besides the good behavior with respect to the action of $SL(3, \mathbb{C})$:

$$C(f \circ \theta, g \circ \theta; x, y, z; t, u) = C(f, g; \theta(x, y, z); t, u) \qquad \forall \theta \in SL(3, \mathbb{C})$$

are covariant with respect to combinations of f and g:

$$C(\theta(f, g); x, y, z; \theta(t, u)) = C(f, g; x, y, z; t, u) \qquad \forall \theta \in SL(2, \mathbb{C}).$$

Such objects are called *combinants*.

Obviously, the characteristic form Φ and its covariants are combinants. Remember that the algebra of the covariants of a binary cubic form $\Phi(t, u)$ is generated by the ground form Φ, its discriminant[9], and its Hessian determinant, which are

$$\mathrm{Disc}(\Phi) = \frac{\mathrm{Res}(\phi, \phi')}{27\Phi_{30}}, \quad H(t, u) = \begin{vmatrix} \frac{\partial^2 \Phi}{dt^2} & \frac{\partial^2 \Phi}{dt\,du} \\ \frac{\partial^2 \Phi}{dt\,du} & \frac{\partial^2 \Phi}{du^2} \end{vmatrix}$$

(the division is a simplification in the definition of the discriminant, that is: there remains no Φ_{30} at the denominator). The covariant G is also a combinant.

The vanishing or non-vanishing of each of the combinants are properties of the orbits of pencils of conics. The sign of $\mathrm{Disc}(\Phi)$ is also invariant on each orbit of pencils of conics (because $\mathrm{Disc}(\Phi)$ has even degree in f as well as in g). Thus we just evaluate the combinants on Levy's representatives, and we get the following result:

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[7]for this specific case with 4 terms in the sign sequence.

[8]So for instance, the sign sequence $+0-0$ has no sign permanence, nor sign change (because there are no consecutive non-zero terms). For the sign sequence $+00-$, the Sturm query is computed as for $++$: one permanence, no change, this gives 1.

[9]The discriminant of the characteristic form, $\mathrm{Disc}(\Phi)$, is called the *Tact invariant* by the classics, because it vanishes exactly when the two conics are tangent [5].

Proposition 3. *Let f, g be two non-proportional non-degenerate ternary quadratic forms.*

- *If $\mathrm{Disc}(\Phi) < 0$ then f, g generate a pencil in orbit I or Ia.*
- *If $\mathrm{Disc}(\Phi) > 0$ then f, g generate a pencil in orbit Ib.*
- *If $\mathrm{Disc}(\Phi) = 0$ then f, g generate a pencil in one of the six other orbits. The following table indicates how the vanishings of H and G discriminate further between the orbits of pencils (under the hypothesis that the discriminant vanishes):*

	$H \neq 0$	$H = 0$
$G \neq 0$	II, IIa	IV
$G = 0$	III, IIIa	V

Remark : The fact that the coefficients of G are linear combinations of maximal minors of the matrix M defined in 3.1.1 suggests that the vanishing of G is equivalent to: *M takes rank two.* This is true. To see this, consider the image of the (complex) pencil generated by $[f]$ and $[g]$ by the quadratic mapping "tangential quadratic form" from $\mathbb{P}(S^2\mathbb{C}^{3*})$ to $\mathbb{P}(S^2\mathbb{C}^3)$. It is an irreducible conic, thus either a proper conic or a line. One checks on Levy's representative that it is a line exactly when $G = 0$ (see also [2], 16.5.6.2). Finally, remark that the rows of M are the coordinates of generators of the linear span of this conic.   ⊡

It remains now to discriminate between I and Ia, between II and IIa and between III and IIIa. For this we use that the numbers of degenerate conics of each type (pair of lines, isolated point or double line) in a pencil characterize its orbit, as shown in the table below, established by considering figures 5 and 6.

orbit of pencils	I	Ia	II	IIa	III	IIIa
num. pairs of line	3	1	2	1	1	0
num. isolated points	0	2	0	1	0	1
num. (double) lines	0	0	0	0	1	1

This has an algebraic translation. Consider

$$\det(v \cdot I - \mathrm{Matrix}(tf + g))$$

that expands into

$$v^3 - \mu(t)v^2 + \psi(t)v - \phi(t).$$

A degenerate conic of the pencil corresponds to a parameter t that annihilates ϕ, and is

- a pair of lines when $\mathrm{Matrix}(tf + g)$ has one eigenvalue positive, one negative, and one zero. Then $\psi(t) < 0$.
- an isolated point when the matrix has an eigenvalue zero and the two other both positive or both negative. Then $\psi(t) > 0$.

- a single line when the matrix has two eigenvalues zero, and one non-zero. Then $\psi(t) = 0$.

Thus the discriminations can be performed by a Sturm query of ψ for ϕ. In order not to introduce denominators, we consider the euclidean division of $\Phi_{30}\psi\phi'$ by ϕ, instead of the division of $\psi\phi'$ by ϕ suggested by 3.1.4. Set

$$P = \text{Remainder}(\Phi_{30} \cdot \Psi \cdot \phi', \phi)$$
$$= p_2 t^2 + p_1 t + p_0$$

and

$$A_i = \text{sr}_i(\phi, P)$$

for i between 0 and 3. The consideration of the sign permanences and sign exchanges in $\Phi_{30} = A_3, A_2, A_1, A_0$ gives the Sturm query of $\Phi_{30}\psi$ for ϕ. The Sturm query of ψ for ϕ is the same as the Sturm query of $\Phi_{30}^2\psi$ for ϕ. Using that $\text{sr}_i(\phi, \Phi_{30}P) = \Phi_{30}^{4-i}\text{sr}_i(\phi, P)$, we get that this Sturm query is obtained by considering the sign permanences and sign exchanges in $\Phi_{30}, \Phi_{30}A_2, A_1, \Phi_{30}A_0$; or, simpler, those in $1, A_2, \Phi_{30}A_1, A_0$.

The polynomial A_1 is

$$\begin{vmatrix} \Phi_{30} & \Phi_{21} & \Phi_{12} \\ 0 & p_2 & p_1 \\ p_2 & p_1 & p_0 \end{vmatrix}.$$

And

$$A_0 = -\text{Res}(\phi, P).$$

This simplifies. Applying Lemma 9, one gets

$$\text{Res}(\Phi_{30}\psi\phi', \phi) = \Phi_{30}^2 \text{Res}(\phi, P).$$

And, on the other hand, from Lemma 8 and the definition of $\text{Disc}(\Phi)$:

$$\text{Res}(\Phi_{30}\psi\phi', \phi) = \Phi_{30}^4 \cdot \text{Res}(\psi, \phi) \cdot \text{Disc}(\Phi).$$

Thus

$$A_0 = -\Phi_{30}^2 \text{Res}(\psi, \phi) \text{Disc}(\Phi).$$

3.2.1. *Discriminating between I and Ia.* Suppose f and g generate a pencil in orbit I or Ia. The Sturm query of ψ for ϕ is -3 for orbit I and 1 for orbit Ia.

The assumption that f, g generate a pencil in orbit I or Ia gives more information:

Lemma 12. *If f, g generate a pencil in orbit I or Ia, then $A_0 < 0$.*

Proof. We had established that $A_0 = -\Phi_{30}^2 \text{Res}(\psi, \phi) \text{Disc}(\Phi)$. For orbit I or Ia, one has $\text{Disc}(\Phi) < 0$. Moreover, $\text{Res}(\psi, \phi) < 0$ because, from lemmas 6 and 7,

$$\text{Res}(\psi, \phi) = \text{Res}(\phi, \psi) = \Phi_{30}^2 \prod \psi(\rho),$$

where the product is carried over the three roots ρ of ϕ. They make either ψ three times negative (orbit I), either one time negative and two times positive (orbit Ia). In both cases, the product is negative. $\square$ $\square$

There is only one sign sequence giving Sturm query -3 and beginning with $+$ and finishing with $-$, that is $+ - +-$. There are several sign sequences giving Sturm query 1, beginning with $+$, finishing with $-$:

$$+ + + - \qquad + + - - \qquad + + 0 - \qquad + - - - \qquad + 0 - - \qquad + 00 - \,.$$

We deduce from this the criterion stated in the following proposition.

Proposition 4. *Let f, g be non-degenerate quadratic forms generating a pencil in orbit I or Ia.*

- *if it is orbit I then $p_2 < 0$ and $\Phi_{30} A_1 > 0$.*
- *if it is orbit Ia, then $p_2 > 0$, or $\Phi_{30} A_1 < 0$, or $p_2 = A_1 = 0$.*

3.2.2. *Discriminating between II and IIa.* Suppose f and g generate a pencil in orbit II or IIa. Note first that $A_0 = 0$. The Sturm query of ψ for ϕ is -2 for orbit II and 0 for orbit IIa.

There is only one sign sequence with beginning with $+$, finishing with 0 that gives Sturm query -2, this is $+ - +0$. Those giving Sturm query 0 are

$$+ + -0 \quad + - - 0 \quad + 0 + 0 \quad + 0 - 0 \quad + 000.$$

Proposition 5. *Let f, g be non-degenerate quadratic forms generating a pencil in orbit II or IIa.*

- *if it is in orbit II then $p_2 < 0$ and $\Phi_{30} A_1 > 0$.*
- *if it is in orbit IIa, then $p_2 = 0$ or $\Phi_{30} A_1 < 0$.*

3.2.3. *Discriminating between III and IIIa.* Suppose f and g generate a pencil in orbit III or IIIa. Once again, $A_0 = 0$. The Sturm query of ψ for ϕ is -1 for orbit III, 1 for orbit IIIa.

The only sign sequence (beginning with $+$, terminating with 0) giving Sturm query -1 is $+ - 00$. There is also only one giving Sturm query 1, that is $+ + 00$.

Proposition 6. *Let f, g be non-degenerate quadratic forms generating a pencil in orbit III or IIIa.*

- *if it is in orbit III, then $p_2 < 0$.*
- *if it is in orbit IIIa, then $p_2 > 0$.*

3.3. **Characterizing the rigid isotopy classes for pairs inside each pencil.** Given f, g two non-proportional non-degenerate quadratic forms, we suppose we know the orbit of the pencil they generate. After Lemma 3, one decides to which class belongs $\{[f], [g]\}$ by looking whether or not $[f]$ and $[g]$ are on a same arc of their pencil. This corresponds to $\phi(t)$ having, or not, all its real roots of the same sign. The simplest way to translate it into algebraic identities is by using

Descartes' law of signs (precisely Lemma 11, because ϕ has all its roots real and non-zero in the considered cases).

Proposition 7. *Let* $[f], [g]$ *be two distinct proper non-empty conics, generating a pencil in one of the orbits:* **I, Ia, II IIa, III.** *Then* $\{[f], [g]\}$ *is in the class* **N** *if and only if*

$$\Phi_{30}\Phi_{12} > 0 \wedge \Phi_{03}\Phi_{21} > 0$$

3.4. **Which is inside ?** Suppose the pair of conics is in one of the classes: **IaN, IIN, IIaN, IIIN, IIIaN, VN.** Which conic lies inside the other ? Otherwise stated, for any given class of pairs inside, we want to characterize the corresponding classes of couples.

3.4.1. *The antisymmetric invariant solves the problem for pair classes* **IIN, IIaN, IIIN, IIIaN.** *The* antisymmetric invariant *is*

$$\mathcal{A} = \Phi_{30}\Phi_{12}^3 - \Phi_{03}\Phi_{21}^3.$$

First it is homogeneous of even degree, 6, in f, as well as in g. So its sign depends only on the algebraic conics, not on the quadratic forms defining them.

Consider again Table 1. Set

(1) $$f = f_0 + t_1 g_0, \qquad g = f_0 + t_2 g_0.$$

From figures 5 and 6, for the cases **Ia,II IIa, III, IIIa**, the inner conic is the one nearer from f_0, that is the one whose parameter (t_1 or t_2) has smaller absolute value. For case **V**, it is the one with whose parameter is smaller. Evaluate the antisymmetric invariant on (f, g). For **II, IIa, III, IIIa**, we get each time a positive rational number times

$$(t_1 t_2 (t_1 - t_2))^2 (t_1^2 - t_2^2).$$

This proves the following proposition.

Proposition 8. *Suppose* $([f], [g])$ *is a couple of distinct proper non-empty conics, such that* $\{[f], [g]\}$ *is in class* **IIN, IIaN, IIIN** *or* **IIIaN.** *Then* $[f = 0]$ *lies inside*[10] *of* $[g = 0]$ *if and only if* $\mathcal{A}(f, g) < 0$.

For **VN**, the evaluation of the antisymmetric invariant gives zero, and for **IaN** it gives the expression

$$(t_1^2 - t_2^2)(t_1 - t_2)^2 \left((t_1 + t_2)^2 - (t_1 t_2 - 3)^2\right)$$

whose sign is not clear. We need other methods to solve the question in these two cases.

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[10] only at the neighborhood of the double intersection point for class **IIN**.

3.4.2. *The antisymmetric covariant solves the problem for class* VN. Instead of considering the antisymmetric invariant, we can consider the following *antisymmetric covariant*[11]:

$$\mathcal{B}(f,g) = \Phi_{12}f - \Phi_{21}g.$$

We consider its value on f, g generating a pencil in orbit V. It is enough to look at Levy's representative. Consider f, g as in (1) for Levy's representative of orbit V. Then

$$\mathcal{B}(xz - y^2 + t_1x^2, xz - y^2 + t_2x^2) = \frac{t_1 - t_2}{4}x^2.$$

Thus $\mathcal{B}(f,g)$ is a semi-definite quadratic form, negative when $t_1 < t_2$ (that is $[f = 0]$ lies inside $[g = 0]$) and positive in the opposite case.

For the purpose of calculation, we use that one decides if a semi-definite quadratic form is negative or positive merely by considering the sign of the trace of its matrix. Define $T = \mathrm{tr}(\mathcal{B}(f,g))$.

Proposition 9. *If f, g generate a pencil in orbit* V, *then the conic* $[f = 0]$ *lies in inside the conic* $[g = 0]$ *if and only if $T < 0$.*

3.4.3. *Case* IaN. This case is more difficult than the previous ones.

Suppose (f, g) is in class IaN. After Figure 5, $\phi(t)$ has three roots of the same sign, two making $\psi > 0$ (conics of the pencil degenerating into isolated points) and one making $\psi < 0$ (conic degenerating into a double line). Denote them with t_1, t_2, t_3, such that $|t_1| < |t_2| < |t_3|$. Denote also with ν their common sign (note it is obtained as the sign of $-\Phi_{30}\Phi_{03}$). The sign of $\Phi_{30}\phi''(t_1)$ is $-\nu$ and the sign of $\Phi_{30}\phi''(t_3)$ is ν (because $\Phi_{30}\phi''$ is linear, with leading coefficient 6 Φ_{30}^2, positive, so it is increasing; its root lies between t_1 and t_3).

The sign of $\Phi_{30}\phi''(t_2)$ is unknown, denote it with ε.

After Figure 5, $[f = 0]$ (resp. $[g = 0]$) is inside the other *iff* $\psi(t_1) < 0$ (resp. $\psi(t_3) < 0$). Thus we have the following table of signs:

		t_1	t_2	t_3
$\Phi_{30}\phi''$		$-\nu$	ε	ν
ψ	$[f = 0]$ inside	$-$	$+$	$+$
	$[g = 0]$ inside	$+$	$+$	$-$
$\Phi_{30}\phi''\psi$	$[f = 0]$ inside	ν	ε	ν
	$[g = 0]$ inside	$-\nu$	ε	$-\nu$

One sees that a Sturm query of $\Phi_{30}\phi''\psi$ for ϕ will give 3 or 1 in one case, -3 or -1 in the other, allowing to obtain the relative position of the conics. Precisely, the Sturm queries corresponding to the situations

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[11]This is a quadratic form, and actually the antisymmetric invariant of the previous paragraph is its discriminant.

$[f = 0]$ *inside* vs. $[g = 0]$ *inside* are given by the following table:

	$\nu = +$	$\nu = -$
$\varepsilon = +$	3 vs. -1	-1 vs. 3
$\varepsilon = -$	1 vs. -3	-3 vs. 1
$\varepsilon = 0$	2 vs. -2	-2 vs. 2

Let

$$Q = \frac{1}{2}\,\text{Remainder}(\Phi_{30}\phi''\phi'\psi, \phi)$$
$$= q_2 t^2 + q_1 t + q_0.$$

Note that Q can be defined in a simpler way from the already introduced polynomial $P = \text{Remainder}(\Phi_{30}\phi'\psi, \phi)$, that is: $Q = \frac{1}{2}\,\text{Remainder}(\phi''P, \phi)$.

Define $B_i = \text{sr}_i(\phi, Q)$ for i between 0 and 3. Then

$$B_3 = \Phi_{30}$$
$$B_2 = q_2$$
$$B_1 = \begin{vmatrix} \Phi_{30} & \Phi_{21} & \Phi_{12} \\ 0 & q_2 & q_1 \\ q_2 & q_1 & q_0 \end{vmatrix}.$$

Finally

$$B_0 = -\,\text{Res}(\phi, Q).$$

This last polynomial simplifies. Using Lemma 9, one gets:

$$\text{Res}(P\phi'', \phi) = -8\Phi_{30}\,\text{Res}(\phi, Q) = 8\Phi_{30}B_0.$$

On the other hand, from Lemma 8,

$$\text{Res}(P\phi'', \phi) = \text{Res}(P, \phi)\,\text{Res}(\phi'', \phi).$$

From Lemma 6, $\text{Res}(P, \phi) = \text{Res}(\phi, P)$, and this is $-A_0$, which was proved to be equal to:

$$\Phi_{30}^2\,\text{Res}(\psi, \phi)\,\text{Disc}(\Phi).$$

Gathering this information, we get that:

$$B_0 = \frac{1}{8}\Phi_{30}\,\text{Res}(\psi, \phi)\,\text{Res}(\phi'', \phi)\,\text{Disc}(\Phi).$$

It is convenient to remark here that Φ_{30} divides $\text{Res}(\phi'', \phi)$. We will define

$$R := \frac{\text{Res}(\phi'', \phi)}{8\Phi_{30}}$$

Thus

$$B_0 = \Phi_{30}^2\,\text{Res}(\psi, \phi)R\,\text{Disc}(\Phi).$$

From lemmas 6 and 7, it comes that $\text{Res}(\psi, \phi) = \text{Res}(\phi, \psi) < 0$ and $\text{Res}(\phi'', \phi) = -\text{Res}(\phi, \phi'')$ has the sign ε. Last, $\text{Disc}(\Phi) < 0$. Thus B_0 has the sign of $\varepsilon\Phi_{30}$.

The sign sequences s_1, s_2, s_3, s_4 giving 3 or -3 are characterized by $s_1 s_3 > 0$ with $s_2 s_4 > 0$.

The sign sequences giving 2 are $+++0$ and $---0$, those giving -2 are $+-+0$ and $-+-0$. The first are characterized with respect to the second by $s_1 s_2 > 0$.

If $\varepsilon\nu > 0$, then $[f = 0]$ is inside iff $\Phi_{30}B_1 > 0$ with $q_2\varepsilon\Phi_{30} > 0$. If $\varepsilon\nu < 0$, then this characterizes $[g = 0]$ *inside*. If $\varepsilon = 0$, $[f = 0]$ is inside iff $\nu\Phi_{30}q_2 > 0$.

Using that ε is obtained as the sign of $\mathrm{Res}(\phi'', \phi)$:

Proposition 10. *Suppose f, g are two non-degenerate quadratic forms generating a pencil in orbit* **Ia**. *Suppose that their zero locus are nested, that is (f, g) is in class $N1$. The following are the necessary and sufficient conditions for $[f = 0]$ lies inside $[g = 0]$:*

- *when $\Phi_{03}R < 0$, it is*

$$\Phi_{30}B_1 > 0 \ and \ \Phi_{03}q_2 < 0;$$

- *when $\Phi_{03}R > 0$, it is*

$$\Phi_{30}B_1 \leq 0 \ or \ \Phi_{03}q_2 \leq 0;$$

- *when $R = 0$, it is*

$$\Phi_{03}q_2 < 0.$$

3.5. **Recapitulation.** Here we display the explicit definitions of the polynomials appearing in the description of the rigid isotopy classes. Note that all these formulas are short: the complicated polynomials express simply in terms of the less complicated ones. We also display the explicit description of the rigid isotopy classes.

3.5.1. *Formulas.* We will denote the two forms as follows:

$$f(x, y, z) = a_{200}x^2 + a_{020}y^2 + a_{002}z^2 + a_{110}xy + a_{101}xz + a_{011}yz$$
$$g(x, y, z) = b_{200}x^2 + b_{020}y^2 + b_{002}z^2 + b_{110}xy + b_{101}xz + b_{011}yz.$$

We will denote similarly the coefficients of $\tilde{f}, \tilde{g}, \Omega$ with $\tilde{a}_{ijk}, \tilde{b}_{ijk}, \omega_{ijk}$ respectively. One has:

$$\tilde{a}_{200} = \begin{vmatrix} a_{020} & a_{011}/2 \\ a_{011}/2 & a_{002} \end{vmatrix}, \qquad \tilde{a}_{011} = -2\begin{vmatrix} a_{200} & a_{110}/2 \\ a_{101}/2 & a_{011}/2 \end{vmatrix},$$

$$\tilde{a}_{020} = \begin{vmatrix} a_{200} & a_{101}/2 \\ a_{101}/2 & a_{002} \end{vmatrix}, \qquad \tilde{a}_{101} = -2\begin{vmatrix} a_{020} & a_{110}/2 \\ a_{011}/2 & a_{101}/2 \end{vmatrix},$$

$$\tilde{a}_{002} = \begin{vmatrix} a_{200} & a_{110}/2 \\ a_{110}/2 & a_{020} \end{vmatrix}, \qquad \tilde{a}_{110} = -2\begin{vmatrix} a_{002} & a_{011}/2 \\ a_{101}/2 & a_{110}/2 \end{vmatrix}.$$

Similarly the $\tilde{b}_{ijk}$'s are defined from the b_{ijk}'s, and

$$\omega_{200} = a_{020}b_{002} + a_{002}b_{020} - a_{011}b_{011}/2,$$
$$\omega_{020} = a_{002}b_{200} + a_{200}b_{002} - a_{101}b_{101}/2,$$
$$\omega_{002} = a_{020}b_{200} + a_{200}b_{020} - a_{110}b_{110}/2,$$
$$\omega_{011} = a_{200}b_{011} + a_{011}b_{200} - a_{110}b_{101}/2 - a_{101}b_{110}/2,$$
$$\omega_{101} = a_{020}b_{101} + a_{101}b_{020} - a_{011}b_{110}/2 - a_{110}b_{011}/2,$$
$$\omega_{110} = a_{002}b_{110} + a_{110}b_{002} - a_{011}b_{110}/2 - a_{110}b_{011}/2.$$

The (de-homogenized) characteristic form is

$$\phi(t) = \Phi_{30}t^3 + \Phi_{21}t^2 + \Phi_{12}t + \Phi_{03} = \mathrm{Disc}(tf+g).$$

Note that:

$$\Phi_{30} = a_{200}\tilde{a}_{200} + a_{110}\tilde{a}_{110} + a_{101}\tilde{a}_{101},$$

and

$$\Phi_{21} = b_{200}\tilde{a}_{200} + b_{002}\tilde{a}_{002} + b_{020}\tilde{a}_{020} + b_{110}\tilde{a}_{110} + b_{101}\tilde{a}_{101} + b_{011}\tilde{a}_{011}.$$

There are similar formulas for Φ_{03} and Φ_{12}, by exchanging a and b.

The discriminant of the characteristic form can be obtained as

$$\mathrm{Disc}(\Phi) = \frac{1}{81} \begin{vmatrix} 3\Phi_{30} & 2\Phi_{21} & \Phi_{12} & 0 \\ 0 & 3\Phi_{30} & 2\Phi_{21} & \Phi_{12} \\ \Phi_{21} & 2\Phi_{12} & 3\Phi_{03} & 0 \\ 0 & \Phi_{21} & 2\Phi_{12} & 3\Phi_{03} \end{vmatrix},$$

and its Hessian determinant as

$$(2) \quad H = H_{20}t^2 + H_{11}tu + H_{02}u^2$$

$$(3) \quad = 4\begin{vmatrix} 3\Phi_{30} & \Phi_{21} \\ \Phi_{21} & \Phi_{12} \end{vmatrix} t^2 + 4\begin{vmatrix} 3\Phi_{30} & \Phi_{12} \\ \Phi_{21} & 3\Phi_{03} \end{vmatrix} t\,u + 4\begin{vmatrix} \Phi_{21} & \Phi_{12} \\ \Phi_{12} & 3\Phi_{03} \end{vmatrix} u^2.$$

The autopolar triangle covariant is:

$$G = -[\bar{1}\,2\,3]x^3 - [1\,\bar{2}\,3]y^3 - [1\,2\,\bar{3}]z^3 + ([1\,\bar{1}\,3] + 2[\bar{3}\,\bar{2}\,3])xy^2$$
$$+ ([1\,2\,\bar{1}] + 2[\bar{2}\,2\,\bar{3}])xz^2 + ([\bar{2}\,2\,3] + 2[\bar{1}\,\bar{3}\,3])yx^2 + ([1\,2\,\bar{2}] + 2[1\,\bar{1}\,\bar{3}])yz^2$$
$$+ ([\bar{3}\,2\,3] + 2[\bar{1}\,2\,\bar{2}])zx^2 + ([1\,\bar{3}\,3] + 2[1\,\bar{2}\,\bar{1}])zy^2 + ([1\,2\,3] + 4[\bar{1}\,\bar{2}\,\bar{3}])xyz.$$

where $[i\,j\,k]$ denote the maximal minors of the matrix

$$M = \begin{bmatrix} \tilde{a}_{200} & \tilde{a}_{020} & \tilde{a}_{002} & \tilde{a}_{011} & \tilde{a}_{101} & \tilde{a}_{110} \\ \omega_{200} & \omega_{020} & \omega_{002} & \omega_{011} & \omega_{101} & \omega_{110} \\ \tilde{b}_{200} & \tilde{b}_{020} & \tilde{b}_{002} & \tilde{b}_{011} & \tilde{b}_{101} & \tilde{b}_{110} \end{bmatrix}$$

whose columns have been labeled $1, 2, 3, \bar{1}, \bar{2}, \bar{3}$.

Denote the coefficients of ψ as follows:

$$\psi(t) = \Psi_{20}\,t^2 + 2\,\Psi_{11}\,t + \Psi_{02},$$

(beware the coefficient of t is $2\Psi_{11}$) then

$$\Psi_{20} = \tilde{a}_{200} + \tilde{a}_{020} + \tilde{a}_{002},$$

Ψ_{02} is the corresponding expression with b instead of a, and

$$\Psi_{11} = \frac{1}{2}\left(\omega_{200} + \omega_{020} + \omega_{002}\right).$$

There is also $\mu = \mu_{10}t + \mu_{01}$. Then

$$\mu_{10} = a_{200} + a_{020} + a_{002}$$

and μ_{01} is defined by the corresponding formula with b instead of a.
The polynomial P for the Sturm query of ψ for ϕ is

$$P = \mathrm{Remainder}(\Phi_{30}\phi'\psi, \phi) = p_2\, t^2 + p_1\, t + p_0$$

with

$$p_2 = 3\Phi_{30}^2\Psi_{02} - 2\Phi_{21}\Phi_{30}\Psi_{11} - 2\Phi_{12}\Phi_{30}\Psi_{20} + \Phi_{21}^2\Psi_{20},$$
$$p_1 = 2\Phi_{21}\Phi_{30}\Psi_{02} - 4\Phi_{12}\Phi_{30}\Psi_{11} + \Phi_{12}\Phi_{21}\Psi_{20} - 3\Phi_{03}\Phi_{30}\Psi_{20},$$
$$p_0 = \Phi_{12}\Phi_{30}\Psi_{02} - 6\Phi_{03}\Phi_{30}\Psi_{11} + \Phi_{03}\Phi_{21}\Psi_{20}.$$

The subresultant A_1 is

$$A_1 = \begin{vmatrix} \Phi_{30} & \Phi_{21} & \Phi_{12} \\ 0 & p_2 & p_1 \\ p_2 & p_1 & p_0 \end{vmatrix}.$$

The antisymmetric invariant is

$$\mathcal{A} = \Phi_{30}\Phi_{12}^3 - \Phi_{03}\Phi_{21}^3.$$

and the trace of the antisymmetric covariant is

$$T = \Phi_{12}\mu_{10} - \Phi_{21}\mu_{01}.$$

The polynomial Q for the Sturm query of $\Phi_{30}\phi''\psi$ for ϕ is:

$$Q = \mathrm{Remainder}(P\,\phi'', \phi) = P\,\phi'' - 6p_2\phi = q_2\, t^2 + q_1\, t + q_0.$$

Its coefficients are

$$q_2 = 3p_1\Phi_{30} - 2p_2\Phi_{21},$$
$$q_1 = 3p_0\Phi_{30} + p_1\Phi_{21} - 3p_2\Phi_{12},$$
$$q_0 = p_0\Phi_{21} - 3p_2\Phi_{03}.$$

The subresultant B_1 is

$$B_1 = \begin{vmatrix} \Phi_{30} & \Phi_{21} & \Phi_{12} \\ 0 & q_2 & q_1 \\ q_2 & q_1 & q_0 \end{vmatrix}.$$

The last quantity to consider is

$$R = 27\Phi_{30}^2\Phi_{03} + 2\Phi_{21}^3 - 6\Phi_{30}\Phi_{21}\Phi_{12}.$$

Each of these expressions is homogeneous in the coefficients of f and as well in the coefficients of g. The following table gives their bi-degree.

$$
\begin{array}{llllll}
\tilde{a}_\alpha & : (2,0) & H_{i,j} & : (2+i, 2+j) & \mathcal{A} & : (6,6) \\
\omega_\alpha & : (1,1) & G & : (3,3) & T & : (2,2) \\
\tilde{b}_\alpha & : (0,2) & \Psi_{i,j} & : (i,j) & q_2 & : (8,3) \\
\Phi_{ij} & : (i,j) & p_2 & : (6,2) & B_1 & : (17,8) \\
\mathrm{Disc}(\Phi) & : (6,6) & A_1 & : (13,6) & R & : (6,3)
\end{array}
$$

3.5.2. *The decision procedure.*

First step: decide the orbit of pencils. Here are the descriptions of the sets of couples of distinct proper conics generating a pencil in a given orbit.

$$
\begin{array}{rl}
\mathsf{I}: & \mathrm{Disc}(\Phi) < 0 \wedge p_2 < 0 \wedge \Phi_{30}A_1 > 0 \\
\mathsf{Ia}: & \mathrm{Disc}(\Phi) < 0 \wedge [p_2 > 0 \vee \Phi_{30}A_1 < 0 \vee [A_1 = 0 \wedge p_2 = 0]] \\
\mathsf{Ib}: & \mathrm{Disc}(\Phi) > 0 \\
\mathsf{II}: & \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G \neq 0 \wedge p_2 < 0 \wedge \Phi_{30}A_1 > 0 \\
\mathsf{IIa}: & \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G \neq 0 \wedge [p_2 = 0 \vee \Phi_{30}A_1 < 0] \\
\mathsf{III}: & \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G = 0 \wedge p_2 < 0 \\
\mathsf{IIIa}: & \mathrm{Disc}(\Phi) = 0 \wedge H \neq 0 \wedge G = 0 \wedge p_2 > 0 \\
\mathsf{IV}: & H = 0 \wedge G \neq 0 \\
\mathsf{V}: & H = 0 \wedge G = 0.
\end{array}
$$

Second step: decide the class of pairs. There is only one rigid isotopy class for pairs (class N) corresponding to each of the orbits of pencils Ib, IIIa, IV, V.

There are two classes (N or S) corresponding to I Ia, II, IIa, III. The criterion for being in the class N is:

$$
\Phi_{30}\Phi_{12} > 0 \wedge \Phi_{03}\Phi_{21} > 0.
$$

Third step (nested cases): decide which of the conics is inside the other. The classes of pairs splitting into two classes of couples are: IaN, IIaN, IIIN, IIIaN, VN.

The criteria for $[f = 0]$ lies inside $[g = 0]$ are the following:

- IIN, IIaN, IIIN, IIIaN: $\mathcal{A} < 0$.
- VN: $T < 0$.
- IaN:

	sign of $\Phi_{03}R$		
	$-$	$+$	0
criterion for $[f=0]$ inside	$\Phi_{30}B_1 > 0$ $\wedge$ $\Phi_{03}q_2 < 0$	$\Phi_{03}B_1 \leq 0$ $\vee$ $\Phi_{03}q_2 \leq 0$	$\Phi_{03}q_2 < 0$

4. Examples and applications

We consider examples and applications for our work. In all of them, we specialize the above general formulas to pairs of quadratic forms depending on parameters. We obtain a complicated description of the partition of the parameters space into the subsets corresponding to the isotopy classes. We then use Christopher Brown's program *SLFQ* of simplification of large quantifier-free formulas [4] to get simpler descriptions.

4.1. **Two ellipsoids.**
We consider two ellipsoids given by the equations (example 2 in [21]):

$$x^2 + y^2 + z^2 - 25 = 0,$$
$$\frac{(x-6)^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} - 1 = 0.$$

We consider then as equations in x, y of two affine conics depending on a parameter z. This corresponds to using a sweeping plane to explore the two ellipsoids. We homogenize the equations in x, y with t, thus considering:

$$f = \quad x^2 + y^2 + t^2(z^2 - 25),$$
$$g = \quad \frac{(x-6t)^2}{9} + \frac{y^2}{4} + t^2\left(\frac{z^2}{16} - 1\right).$$

The quantity $\mathrm{Disc}(\Phi)$ is here $h = 49z^4 + 2516z^2 - 229376$. One checks easily that h has two single real roots $z_0, -z_0$ with $0 < z_0 < 4$. When the two conics are proper and non-empty, that is when $-4 < z < 4$, one finds, using our formulas, that the following classes can occur:

- IaS when $h > 0$, that is $-4 < z < -z_0$ or $z_0 < z < 4$.
- IIaS when $h = 0$, that is $z = \pm z_0$.
- IbN when $h < 0$, that is $-z_0 < z < z_0$.

The ellipsoids go each through the other.

4.2. **A paraboloid and an ellipsoid.**
Our equations, inequations, inequalities can tell the relative position of any two conics, not only ellipses, because of the choice of working in the projective plane.

Thus we can apply also the method of the previous example to any kind of quadric. In the following example, one considers a paraboloid and an ellipsoid:

$$4x^2 - 4xy + 2y^2 - 4xz + 14x - 6y + 2z^2 - 10z + 12 = 0,$$
$$3x^2 - 4xy + 2y^2 - 4xz + 16x + 2yz - 12y + 2z^2 - 16z + 39 = 0$$

As before, we consider z as a parameter and homogenize the equations in x, y with t, thus considering:

$$f = \quad 4x^2 - 4xy + 2y^2 + t(-4xz + 14x - 6y) + t^2(2z^2 - 10z + 12),$$
$$g = \quad 3x^2 - 4xy + 2y^2 + t(-4xz + 16x + 2yz - 12y) + t^2(2z^2 - 16z + 39)$$

We specialize our equations, inequations and inequalities and run $SLFQ$. Let $z_0 = -1/4$ and $z_1 < z_2$ be the two roots of $z^2 - 12z + 34$. One finds that $z_0 < z_1$, and $[f = 0]$ is proper non-empty when $z > z_0$, $[g = 0]$ is proper non-empty when $z_1 < z < z_2$. When both are proper and non-empty, one finds that the isotopy class is always IaN. Thus the ellipsoid is inside the paraboloid.

4.3. **Uhlig's canonical forms.** In [19], Uhlig presented representatives for the orbits under $GL(n, \mathbb{R})$ of couples of quadratic forms generating a non-degenerate pencil.

For conics $(n = 3)$, it follows from Uhlig's presentation that any couple of conics can be transformed, by means of $PSL(3, \mathbb{R})$, into one with associated couple of matrices among:

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \quad (U_{11}); \qquad \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & -\lambda_3 \end{bmatrix} \quad (U_{12});$$

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \begin{bmatrix} & \lambda_1 & \\ \lambda_1 & 1 & \\ & & \lambda_2 \end{bmatrix} \quad (U_{21}); \qquad \begin{bmatrix} & 1 & \\ 1 & & \\ & & -1 \end{bmatrix}, \begin{bmatrix} & \lambda_1 & \\ \lambda_1 & 1 & \\ & & -\lambda_2 \end{bmatrix} \quad (U_{22});$$

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \begin{bmatrix} b & a & \\ a & -b & \\ & & \lambda \end{bmatrix} \quad (U_{31}); \qquad \begin{bmatrix} & 1 & \\ 1 & & \\ & & -1 \end{bmatrix}, \begin{bmatrix} b & a & \\ a & -b & \\ & & -\lambda \end{bmatrix} \quad (U_{32});$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \begin{bmatrix} & & \lambda \\ & \lambda & 1 \\ \lambda & 1 & \end{bmatrix} \quad (U_4).$$

To which configuration corresponds each of these normal forms ?

We find simple description for the subsets of each parameters space corresponding to the isotopy classes.

As an illustration, we show the result for U_{21}.

- g is degenerate when $\lambda_1 = 0$ or $\lambda_2 = 0$.
- the conics are in class VN when $\lambda_1 = \lambda_2 \neq 0$. In this case, $[g = 0]$ lies inside $[f = 0]$.
- in the other cases, the conics are in class IIN, IIS, IIaN or IIaS as shown in Figure 7.

5. FINAL REMARKS

For clarity of the exposition, we have not considered the case when one conic, or both conics, are degenerated; but it is easy to list the corresponding isotopy classes and describe them with equations, inequations and inequalities.
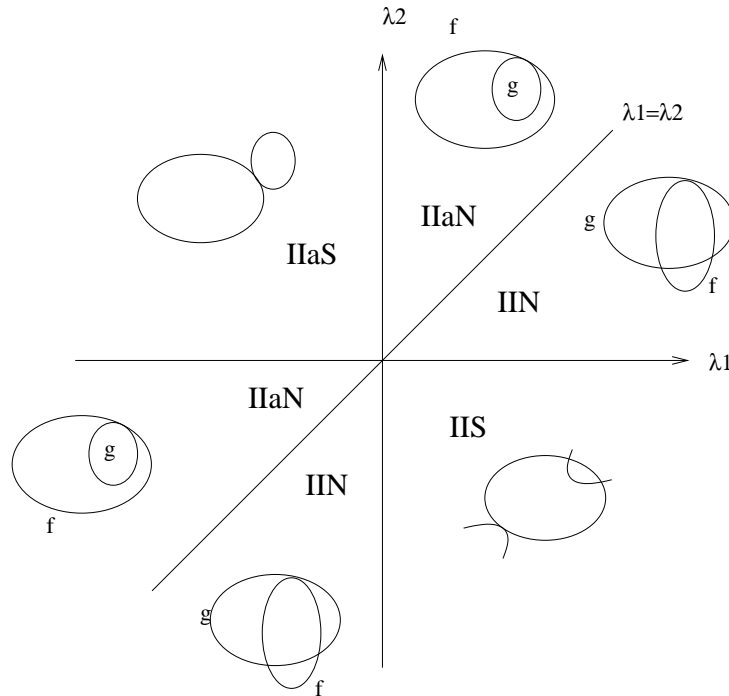
FIGURE 7. isotopy classes for representatives U_{21}.

Remark that the polynomials involved in the description of the classes, specially invariants and covariants, have often very compact expressions in function of the smaller ones. Thus they can be evaluated with substantial saving of arithmetic operations, as was pointed out in [8].

The following, more ambitious, step is the classification of couples of quadrics drawn in $\mathbb{RP}^3$. Hopefully some of the methods developed in the present paper will be useful in this task, on which we wish to return in another paper.

It follows from our study that the rigid isotopy classes for couples of conics are characterized nearly totally by the behavior of the signature function on the pencil generated by the quadratic forms. For the non-generic classes, this provides a precise answer to a question formulated in [20]. We plan also to develop this point in a forthcoming paper with B. Mourrain.

Finally, the reader will find some implementations and complements on the subject on the author's web page devoted to the paper:
http://emmanuel.jean.briand.free.fr/publications/twoconics

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