Balanced Islands in Two Colored Point Sets in the Plane*

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Abstract

Let S be a set of n points in general position in the plane, r of which are red and b of which are blue. In this paper we prove that there exist: for every $\alpha \in \left[0, \frac{1}{2}\right]$, a convex set containing exactly $\lceil \alpha r \rceil$ red points and exactly $\lceil \alpha b \rceil$ blue points of S; a convex set containing exactly $\left\lceil \frac{r+1}{2} \right\rceil$ red points and exactly $\left\lceil \frac{b+1}{2} \right\rceil$ blue points of S. Furthermore, we present polynomial time algorithms to find these convex sets. In the first case we provide an $O(n^4)$ time algorithm and an $O(n^2 \log n)$ time algorithm in the second case. Finally, if $\lceil \alpha r \rceil + \lceil \alpha b \rceil$ is small, that is, not much larger than $\frac{1}{3}n$, we improve the running time to $O(n \log n)$.

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1 Introduction

Let S be a set of n points in the plane, r of which are red and b of which are blue. Without loss of generality, we assume that S (and any other finite point set in this paper) is in general position, that is, no three points lie on a common line. A large class of problems in Discrete and Computational Geometry involves partitioning such point sets. A typical question in this context is whether a given 2-colored point set may be partitioned into parts that satisfy certain predefined properties. In this paper, we present algorithms for computing convex sets that contain a balanced proportion of points of S of each color.

The Ham Sandwich theorem states that there exists a straight line that simultaneously partitions the red points and the blue points in half. As a consequence, there exists a convex set containing half of the red points and half of the blue points of S. This result can be generalized as follows.

Theorem 1 ***(The Balanced Island Theorem)*** *Let S be a set of r red points and b blue points in the plane. Then, for every $\alpha \in \left[0, \frac{1}{2}\right]$ there exists:*

1. *a convex set containing exactly $\lceil \alpha r \rceil$ red points and exactly $\lceil \alpha b \rceil$ blue points of S;*
2. *a convex set containing exactly $\left\lceil \frac{r+1}{2} \right\rceil$ red points and exactly $\left\lceil \frac{b+1}{2} \right\rceil$ blue points of S.*

An *island* of S is a subset I of S such that $\mathrm{Conv}(I) \cap S = I$. The first case of Theorem 1 implies in particular, that if $r = b$ then for every $k = 1, \ldots, \lceil \frac{r}{2} \rceil$ there exists an island containing k red points and k blue points of S; such an island contains a balanced number of red and blue points. See Figure 1 for an example.

On the other hand, consider the following construction. Place r red points at the vertices of a regular r-gon and place b blue points close to its center. Every convex set containing $\left\lceil \frac{r+1}{2} \right\rceil + 1$ red points contains all the blue points. The gap between $\left\lceil \frac{r}{2} \right\rceil$ and $\left\lceil \frac{r+1}{2} \right\rceil + 1$ is filled by the second case of Theorem 1.

In higher dimensions the Ham Sandwich theorem states that the color classes of a d-colored finite set of points in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane. Although the Ham Sandwich theorem can be easily proven in dimension two, already for dimension three, tools from Algebraic Topology are needed. This is also the case for Theorem 1. The first case of Theorem 1 can be derived from a theorem of Blagojević and Dimitrijević Blagojević (Theorem 3.2 in [4]), which is a prime example of the applications of Algebraic Topology in Combinatorial Geometry. This implication was also noted by Soberón as a remark in his PhD thesis [10].

The connection between Algebraic Topology and Combinatorial Geometry can sometimes be hard to understand without the proper background. For the sake of self-containment, we include an expository account of this connection in Section 2.

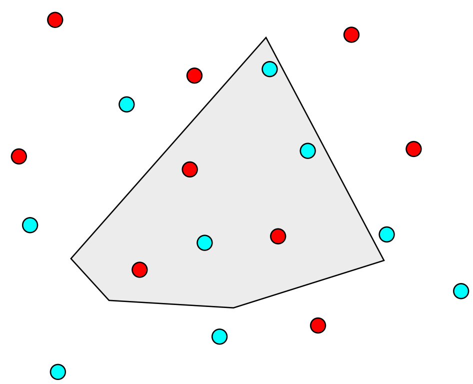

Figure 1: An example of a balanced island for $r = b = 9$ and $\alpha = 1/3$.

We prove Theorem 1 in Section 3. In Lemma 7 we show how to derive the first case of Theorem 1 from the result of [4]; the second case of Theorem 1 needs a separate proof, which we give in Lemma 8; the argument used is also topological in essence.

Finally, in Section 4, we consider the algorithmic facet of Theorem 1. In Theorem 11 we show that the convex set guaranteed by Theorem 1 can be found in $O(n^4)$ time in the first case, and in $O(n^2 \log n)$ time in the second case. We also show in Theorem 15 that if $\lceil \alpha r \rceil + \lceil \alpha b \rceil$ is small, that is, not much larger than $\frac{1}{3}n$, the running time can be improved to $O(n \log n)$.

2 Topological preliminaries

2.1 The Ham Sandwich and Borsuk-Ulam theorems

The statement that there exists a straight line simultaneously bisecting the color classes of a two-colored finite point set in the plane, is what many computational geometers would recognize as the Ham Sandwich theorem. It generalizes to higher dimensions as follows.

Theorem 2 ***(Discrete Ham Sandwich theorem).***
Let $S_1, \dots, S_d$ be finite point sets in $\mathbb{R}^d$. Then there exists a hyperplane that simultaneously bisects[1] each S_i.

At first sight it might be hard to see the connection of Theorem 2 with Topology. This connection perhaps is more apparent in the following continuous version of Theorem 2.

[1]Each of the two open half-spaces defined by the hyperplane contain at most $\lfloor |S_i|/2 \rfloor$ points of S_i.

Theorem 3 ***(Continuous Ham Sandwich theorem).***

Let $U_1, \ldots, U_d$ be bounded open sets in $\mathbb{R}^d$. Then there exists a hyperplane that simultaneously bisects[2] each U_i.

We point out that Theorem 3 is usually stated in the more general setting of finite Borel measures. To keep our exposition as self-contained as possible, we opted to use volumes of open sets instead.

The discrete version of the Ham Sandwich theorem can be proven using the continuous version. Given $S_1, \ldots, S_d$ finite point sets in $\mathbb{R}^d$, the first step is to replace each point with a ball of radius $\varepsilon > 0$ centered at the point. The continuous Ham Sandwich theorem ensures that there exists a hyperplane that simultaneously bisects these expanded S_i's. If we let ε tend to zero this hyperplane converges to a hyperplane that simultaneously bisects the original S_i's.

The continuous version of the Ham Sandwich theorem can be proven using the Borsuk-Ulam theorem. The Borsuk-Ulam theorem has many equivalent formulations. One of them states that for any map (continuous function) from the d-dimensional sphere S^d to $\mathbb{R}^d$, there exists a pair of antipodal points with the same image.

Theorem 4 ***(Borsuk-Ulam theorem A).***

For every map $f : S^d \to \mathbb{R}^d$ there exists a point $x \in S^d$ such that $f(x) = f(-x)$.

To prove the continuous version of the Ham Sandwich theorem, one first chooses a set of a given family of bounded open sets $U_1, \ldots, U_d$ of $\mathbb{R}^d$. Say U_d is chosen. For every possible direction $\vec{v} \in S^{d-1}$, consider the set of oriented hyperplanes, orthogonal to $\vec{v}$, that bisect U_d. These hyperplanes form an interval along this direction. Let $\Pi_{\vec{v}}$ be the first such hyperplane.

It can be checked that the set $\{\Pi_{\vec{v}} : \vec{v} \in S^{d-1}\}$ is topologically equivalent (homeomorphic) to S^{d-1}. In this setting, pairs of antipodal points correspond to pairs of oriented planes with parallel supporting planes and with opposite orientation, such that the volume of U_d that is contained between them is equal to zero. A map f from this space of oriented planes to $\mathbb{R}^{d-1}$, is defined by mapping each such plane Π to the point $f(\Pi) \in \mathbb{R}^{d-1}$ whose i-th coordinate is the fraction of the volume of U_i that lies above Π. By the Borsuk-Ulam theorem, there exists a plane Π that has the same fraction of the volume of U_i above it as its antipodal plane $-\Pi$ has. Therefore, Π and $-\Pi$ simultaneously bisect every U_i.

For more applications of the Borsuk-Ulam theorem see Matoušek's book [8].

2.2 Equivariant maps

The Borsuk-Ulam theorem as stated in Theorem 4 is formulated in a positive way—it ensures the existence of a pair of antipodal points of S^d with a certain

[2] Each of the two open half-spaces defined by the hyperplane contain half of the volume of U_i.

property. It can also be formulated in a negative way—that no map from S^d to S^{d-1} with a certain property (antipodality) exists. A continuous function $f : S^d \to S^{d-1}$ is *antipodal* if $f(-x) = -f(x)$ for all $x \in S^d$. We explicitly give this negative formulation of the Borsuk-Ulam theorem.

Theorem 5 ***(Borsuk-Ulam theorem B).***
There is no antipodal map from S^d to S^{d-1}.

Antipodality and the negative formulation of the Borsuk-Ulam theorem are examples of a more general phenomena. In the case of antipodality, consider the map that sends every point $x \in S^d$ to its antipodal point. This map together with the identity on S^d form a group under function composition. This group is isomorphic to the group $\mathbb{Z}_2$ (the unique group with two elements). So $\mathbb{Z}_2$ is said to *act* on S^d. The formal and more general definition is the following.

Definition 1 *An* ***action*** *of a group G on a topological space X is an assignment of an homeomorphism ϕ_g of X to every element g of G, such that*

- *ϕ_e is the identity on X if and only if e is the identity element of G.*
- *$\phi_g \phi_h = \phi_{gh}$ for all $g, h \in G$.*

The *action* of an element $g \in G$ on a point $x \in X$ is defined as the point $\phi_g(x)$. If the action is already specified or implied, we simply write gx. We say that ϕ is *free* if $\phi_g(x) = x$ for some $x \in X$ implies that $g = e$. In other words, the only homeomorphism that maps some point to itself is the one assigned to the identity element.

For a given group G acting on two topological spaces X and Y, a *G-equivariant* map is a map $f : X \to Y$ such that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. An antipodal map from S^d to S^{d-1} is just a $\mathbb{Z}_2$-equivariant map. The negative formulation of the Borsuk-Ulam theorem can be reinterpreted as the statement that there is no $\mathbb{Z}_2$-equivariant map from S^d to S^{d-1}, where $\mathbb{Z}_2$ acts freely on both S^d and S^{d-1}.

The negative formulation of the Borsuk-Ulam theorem can be used to prove the Ham Sandwich theorem in the following way. Assume that there exists a family of bounded open sets $U_1, \ldots, U_d$ of $\mathbb{R}^d$ for which there is no hyperplane that simultaneously bisects all of them. To apply the negative version of the Borsuk-Ulam theorem, we use this assumption to define an antipodal ($\mathbb{Z}_2$-equivariant) map from S^{d-1} to S^{d-2}; thus arriving to a contradiction.

We proceed in a similar way as when using the positive formulation of the Borsuk-Ulam theorem. Again, we use the map f from the set of oriented planes that bisect U_d to $\mathbb{R}^{d-1}$. (Recall that $f(\Pi)$ is the point of $\mathbb{R}^{d-1}$ whose i-th coordinate is the fraction of the volume of U_i that lies above the hyperplane Π.) The image of f is actually the $(d-1)$-dimensional cube $[0,1]^{d-1}$ rather than all of $\mathbb{R}^d$; we will regard f as a map from S^{d-1} to $[0,1]^{d-1}$.

The topological space $[0,1]^{d-1}$ can be equipped with an "antipodal" function by mapping every point $(x_1, \ldots, x_{d-1}) \in [0,1]^{d-1}$ to the point whose i-th coordinate is $1-x_i$. The assumption that there is no hyperplane simultaneously bisecting all the U_i's, is equivalent to the assumption that no bisecting plane of U_d is mapped to the point $p := \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. Actually, f is a $\mathbb{Z}_2$-equivariant map from S^{d-1} to $[0,1]^{d-1} \setminus \{p\}$.

Let x be a point in $[0,1]^{d-1} \setminus \{p\}$ and let γ_x be the infinite ray with apex p that passes through x. Let g be the map that sends x to the intersection of γ_x and the boundary of $[0,1]^{d-1}$. The boundary of $[0,1]^{d-1}$ is homeomorphic to S^{d-2} and the antipodal function on $[0,1]^{d-1}$ defines a free $\mathbb{Z}_2$-action when restricted to it. The function $g \circ f$ is the desired antipodal map from S^{d-1} to S^{d-2}.

The method just described to prove the Ham Sandwich theorem is certainly more involved than the method described in Section 2.1. However, it is this approach that has been streamlined to prove many equipartition theorems in what has been called the "Configuration Space-Test Map" (CS-TM) scheme. For a nice survey of the CS-TM scheme see Živaljević's paper [12].

2.3 Configuration Space-Test Map Scheme

Suppose that we want to prove that an object with a certain property exists. (In the previous example we searched for a hyperplane that simultaneously bisects a given family $U_1, \ldots, U_d$ of bounded open sets of $\mathbb{R}^d$.) The approach of the CS-TM scheme is as follows.

A set of candidates for the solution is first defined. This set is then given a topology and a free action of a group G. This space X is called the *configuration space*. In our previous example, X was the space of oriented hyperplanes that bisect U_d.

Afterwards, a map f from X to a space Y is defined. The desired object is then shown to exist if and only if some point of X is mapped by f to a certain subspace Z of Y. The space Y and its subspace Z are called the *test space*; the map f is called the *test map*. In our previous examples the roles of Y, Z and f where played by the boundary of $[0,1]^{d-1}$, $\{p\}$, and f, respectively.

An action of G on Y is defined so that f is a G-equivariant map; it is also required that G acts freely on $Y \setminus Z$. When restricted to $Y \setminus Z$, the map f becomes a G-equivariant map from X to $Y \setminus Z$. Finally, one shows that no such maps can exist—contradicting the assumption that our desired object does not exist. This last part of proving the non-existence of equivariant maps is where tools from Algebraic Topology typically come into play.

2.3.1 Blagojević's and Dimitrijević Blagojević's Theorem under the CS-TM Scheme

The theorem of Blagojević and Dimitrijević Blagojević is an equipartition result of 3-fans on the sphere; we now illustrate how their proof fits into the CS-TM scheme. For the following definitions regarding k-fans we follow the exposition in Bárány's and Matoušek's paper [2]. The study of equipartition results using k-fans was initiated by Akiyama Kaneko, Kano, Nakamura, Rivera-Campo,

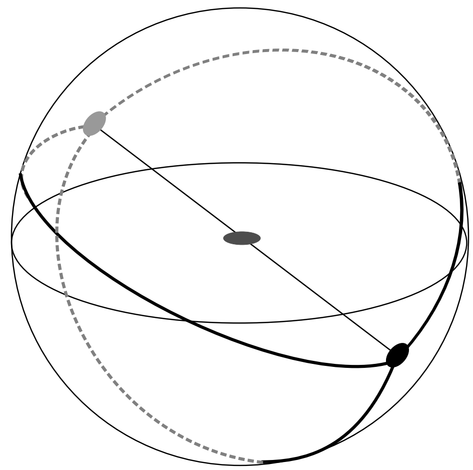

Figure 2: A 3-fan on S^2.

Tokunaga and Urrutia in [1].

A *k-fan* in the plane is a set of k infinite rays, that emanate from the same point. This point is called its *apex*. A k-fan in the plane can also be a set of k parallel lines. Given a k-fan γ in the plane, we call the connected open regions of $\mathbb{R}^2 \setminus \gamma$ the *wedges* of γ. In the case where γ consists of parallel lines, a wedge is also the union of the two open regions of $\mathbb{R}^d \setminus \{\gamma\}$ that are bounded by a single line.

The inclusion of k parallel lines in the definition of k-fans and the last exception in the definition of its wedges may seem awkward. However, k-fans consisting of rays and k-fans consisting of parallel lines are closely related. This connection will become clear once we consider k-fans in the sphere and their connection with k-fans in the plane.

A *k-fan* in the two dimensional *sphere* S^2 is a set of k great semicircles that emanate from the same two antipodal points. (Recall that S^2 is a *two*-dimensional surface, but it is normally regarded as embedded in $\mathbb{R}^3$.) These two points are called its apices. Given a k-fan γ in S^2, we call the connected open regions of $S^2 \setminus \gamma$ the *wedges* of γ. See Figure 2.

The connection between k-fans in the sphere and k-fans in the plane is given by the following map from the open southern hemisphere of S^2 to $\mathbb{R}^2$. Assume that S^2 lies on $\mathbb{R}^3$ and identify $\mathbb{R}^2$ with a horizontal plane lying below S^2. From the center of S^2 project every point on the southern hemisphere of S^2 to $\mathbb{R}^2$. Let π be this map. The image under π of a k-fan for which all its great semicircles intersect the open southern hemisphere of S^2 is a k-fan in $\mathbb{R}^2$. If the apices of this k-fan are on the equator of S^2 then the image of this k-fan corresponds to a set of k parallel lines in $\mathbb{R}^2$. Conversely the preimage under π of a k-fan in $\mathbb{R}^2$ corresponds to a k-fan in S^2 for which all its semicircles intersect the open southern hemisphere. This connection between k-fans in the sphere and in the plane allows us to translate partition results by k-fans in the sphere to partition results by k-fans in the plane.

Let R and B be two finite Borel measures on S^2 and $\alpha, \beta > 0$ be two real numbers such that $2\alpha + \beta = 1$. The result of [4] states that there exists a 3-fan such that two of its wedges have an α proportion of R and B, and the remaining wedge contains a β proportion of R and B. Keeping up with our convention we reformulate that statement in terms of areas of open sets.

Theorem 6 ***(Theorem 3.2 in [4])*** *Let R and B be two open sets on S^2, and let $\alpha, \beta > 0$ be two real numbers such that $2\alpha + \beta = 1$. Then there exists a 3-fan on S^2 such that its corresponding wedges W_1, W_2 and W_3 satisfy:*

- $\operatorname{Area}(W_1 \cap R)/\operatorname{Area}(R) = \operatorname{Area}(W_1 \cap B)/\operatorname{Area}(B) = \alpha$,
- $\operatorname{Area}(W_2 \cap R)/\operatorname{Area}(R) = \operatorname{Area}(W_2 \cap B)/\operatorname{Area}(B) = \beta$ *and*
- $\operatorname{Area}(W_3 \cap R)/\operatorname{Area}(R) = \operatorname{Area}(W_3 \cap B)/\operatorname{Area}(B) = \alpha$.

The proof of Theorem 6 in [4] is done when α and β are rational numbers. Afterwards, using standard arguments it is shown that Theorem 6 holds when α and β are real numbers. So assume that α and β are rational numbers. Let k, a_1, a_2 be natural numbers such that $\alpha = \frac{a_1}{k}$ and $\beta = \frac{a_2}{k}$. Note that $2a_1 + a_2 = k$.

Configuration Space

We consider k-fans rather than 3-fans. We need to "orient" the set of k-fans, similar to as we did with the halving planes in Section 2.1. Let γ be a k-fan in S^2. Recall that γ has two apices, and k great semicircles. The orientation is given by choosing a tuple (x, C) where x is one of its apices, and C is one of its great semicircles. Once an orientation (x, C) is chosen for γ, the k great semicircles $C_1, \dots, C_k$ of γ are assumed to be sorted counterclockwise around x, with $C_1 = C$ being the first one. For $1 \leq i \leq k-1$, let γ_i be the wedge of γ bounded by C_i and C_{i+1}, and let γ_k be the wedge bounded by C_k and C_1.

Let M be the set of all k-fans that equipartition R. That is, all k-fans γ such that $\operatorname{Area}(\gamma_i \cap R) = \operatorname{Area}(R)/k$ for $i = 1, \dots, k$. Let $\gamma \in M$ be such a k-fan and let ℓ be the straight line through its two apices. Note that each semicircle C_i of γ can be moved around a maximal interval I_i (of possibly one point) of great semicircles around ℓ, so that $\operatorname{Area}(\gamma_i \cap R)$ does not change. If I_i has more than one point, define φ_i as the wedge bounded by the great semicircles at the endpoints of I_i. In particular, if I_i has more than one point then $\operatorname{Area}(\varphi_i \cap R)$ is equal to zero. Let γ^* be the k-fan that is obtained from γ by placing each C_i at the midpoint of I_i. Note that $\gamma^* \in M$.

The configuration space is the subset of M, given by $M^* := \{\gamma^* : \gamma \in M\}$. We parametrize this space by assigning a pair of orthogonal unit vectors to each $\gamma \in M^*$ as follows. Assume that S^2 is of radius one. Assign to γ the tuple $(\vec{x}, \vec{y})$ where $\vec{x}$ is the unit vector with endpoint at x, and $\vec{y}$ is the unit vector orthogonal to $\vec{x}$ whose endpoint lies in C_1. Note that since γ equipartitions R, and γ is in M^*, x and C_1 determine all of γ. Thus, a tuple $(\vec{x}, \vec{y})$ of orthogonal vectors corresponds to exactly one of these k-fans. With this correspondence in mind, the configuration space is the space, $V_2(\mathbb{R}^3)$, of all tuples of unit orthogonal vectors in $\mathbb{R}^d$. (In the literature $V_2(\mathbb{R}^3)$ is known as the Stiefel manifold [11].)

We specify a group action on $V_2(\mathbb{R}^3)$. Let ϕ be the homeomorphism of $V_2(\mathbb{R}^3)$ that sends γ to $(-x, C)$, and let ψ be the homeomorphism that sends γ to (x, C_2). The group generated by ϕ and ψ is isomorphic to the dihedral group $\mathbb{D}_{2k}$—the group of symmetries of the regular k-gon. This is clear once one realizes that ϕ corresponds to a reflection of the regular k-gon, and that ψ corresponds to a clockwise rotation of the k-gon by one. We assume that $\mathbb{D}_{2k}$ acts on $V_2(\mathbb{R}^3)$ via ψ and ϕ; note that this action is free.

Test Space and Test Map

Let Y be the linear subspace of $\mathbb{R}^k$ defined by

$$Y := \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 + \cdots + x_k = 0\}.$$

Let Z be the subspace of Y defined by the equations

$$\begin{aligned} x_1 &+ \cdots + x_{a_1} &&= 0, \\ x_{a_1+1} &+ \cdots + x_{a_1+a_2} &&= 0, \\ x_{a_1+a_2+1} &+ \cdots + x_k &&= 0. \end{aligned}$$

Let $f : V_2(\mathbb{R}^3) \to Y$ be the map defined by

$$f(\gamma) = (\operatorname{Area}(\gamma_1 \cap B) - \operatorname{Area}(B)/k, \ldots, \operatorname{Area}(\gamma_k \cap B) - \operatorname{Area}(B)/k).$$

Let γ' be the 3-fan with apex x and with great semicircles C_1, C_{a_1} and $C_{a_1+a_2}$. Note that if $f(\gamma) \in Z$ then:

$$\begin{aligned} \operatorname{Area}(\gamma'_1 \cap B)/\operatorname{Area}(B) &= \sum_{i=1}^{a_1} \operatorname{Area}(\gamma_i \cap B)/\operatorname{Area}(B) = \alpha, \\ \operatorname{Area}(\gamma'_2 \cap B)/\operatorname{Area}(B) &= \sum_{i=a_1+1}^{a_1+a_2} \operatorname{Area}(\gamma_i \cap B)/\operatorname{Area}(B) = \beta, \text{ and} \\ \operatorname{Area}(\gamma'_3 \cap B)/\operatorname{Area}(B) &= \sum_{i=a_1+a_2}^{k} \operatorname{Area}(\gamma_i \cap B)/\operatorname{Area}(B) = \alpha. \end{aligned}$$

In this case, γ' is the desired 3-fan of Theorem 6. We now equip Y with a free $\mathbb{D}_{2k}$-action. Let ϕ' be the homeomorphism of Y that sends $(x_1, x_2, \ldots, x_{k-1}, x_k)$ to $(x_k, x_{k-1}, \ldots, x_2, x_1)$, and let ψ' be the homeomorphism of Y that sends $(x_1, x_2, \ldots, x_k)$ to $(x_2, \ldots, x_k, x_1)$. The group generated by ϕ' and ψ' is isomorphic to $\mathbb{D}_{2k}$. We assume that $\mathbb{D}_{2k}$ acts on Y via ϕ' and ψ'. Note that this action is free and f is a $\mathbb{D}_{2k}$-equivariant map from $V_2(\mathbb{R}^3)$ to Y. If no k-fan is mapped by f to Z then f is an equivariant map from $V_2(\mathbb{R}^3)$ to $Y \setminus Z$. In [4], Blagojević and Dimitrijević Blagojević prove that no such map exists. The proof of this last part is far from trivial. Indeed, it is the gist of the proof of Theorem 6–we have merely presented the prelude. Unfortunately, a detailed account of this is beyond an expository account.

3 Proof of the Balanced Island Theorem

We are now ready to prove Theorem 1. We show the first case in Lemma 7 and the second case in Lemma 8.

A way to prove partition theorems on points sets from similar partition theorems on finite Borel measures (or areas of open sets in our case) is the following. First enlarge each point to a disk of radius $\varepsilon > 0$, and use the measure theorem to find a solution. Then let ε tend to zero and show that the limit of these solutions exists and that it is the desired solution for point sets. We applied this approach in Section 2.1, when we sketched how to obtain the discrete version (Theorem 2) of the Ham Sandwich theorem from its continuous version (Theorem 3). We follow this approach again in the proof of Lemma 7.

Lemma 7 *Let S be a set of r red points and b blue points in the plane. Then for every $\alpha \in \left[0, \frac{1}{2}\right]$ there exists a convex set containing exactly $\lceil \alpha r \rceil$ red points and exactly $\lceil \alpha b \rceil$ blue points of S. Moreover, this convex set is either a convex wedge or a strip.*

Proof. Assume without loss of generality that α and β are rationals. We project the open southern hemisphere of S^2 to $\mathbb{R}^2$ as in Section 2.3.1. Assume that S^2 lies on $\mathbb{R}^3$ and identify $\mathbb{R}^2$ with an horizontal plane lying below S^2. From the center of S^2 project every point on the southern hemisphere of S^2 to $\mathbb{R}^2$. Let π be this map.

Let $\varepsilon > 0$. On every red point place an open disk of radius ε centered at this point; let R_ε be the union of all these disks. Likewise, on every blue point place an open disk of radius ε centered at this point; let B_ε be the union of all these disks. Let γ_ε be the 3-fan of S^2 given by Theorem 6 for $R := \pi^{-1}(R_\varepsilon)$, $B := \pi^{-1}(B_\varepsilon)$, α and $\beta := 1 - 2\alpha$. Choose ε small enough so that R and B lie on the southern hemisphere of S^2; if necessary, perturb γ_ε so that the three semicircles of γ_ε intersect the southern hemisphere of S^2. Note that $\pi(\gamma_\varepsilon)$ is a 3-fan in $\mathbb{R}^2$.

The 3-fan γ_ε defines three wedges: two of which contain an α proportion of the area of R and B; the remaining wedge, W_β, contains a β proportion of the area R and B. Assume that γ_ε is oriented so that W_β is bounded by the first and the second semicircle. At least one of the two wedges that contain an α proportion of the area of R and B is convex when projected under π. Of these two wedges, let W_ε be the first wedge clockwise from W_β that is convex when projected under π.

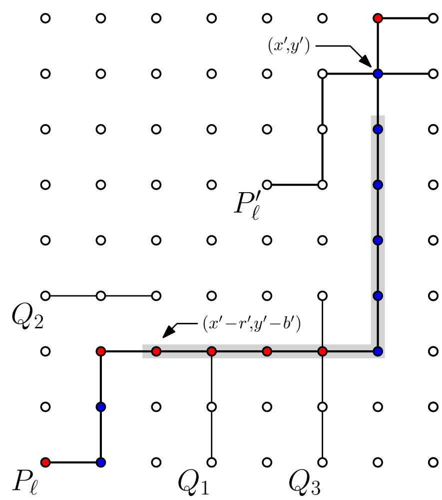

Figure 3: Illustration of the proof of Lemma 8 for $r = 7$ and $b = 8$. The points of P_ℓ are colored like the according points of S. The part of P_ℓ that corresponds to the desired subset of S is marked with a gray background.

Note that once the apex, p, and the first semicircle, C_1, of γ_ε are fixed, all of γ_ε is determined. This implies that once γ_ε is oriented we can identify it with a tuple of unit orthogonal vectors. Assume that S^2 is the unit sphere centered at the origin. Set the first vector of the tuple to be p and the second to be the unit vector orthogonal to p that lies on C_1. It can be verified that this space, $V_2(\mathbb{R}^3)$, of tuples of orthogonal unit vectors in $\mathbb{R}^3$ is compact.

Consider any sequence of ε's that converges to zero; the corresponding sequence of γ_ε's has a limit point γ. Let $\{\gamma_{\varepsilon_i}\}_{i=1}^\infty$ be a subsequence of this sequence that converges to γ and let W be the wedge of γ that $\{W_{\varepsilon_i}\}_{i=1}^\infty$ converges to. W is convex since each of the terms in $\{\gamma_{\varepsilon_i}\}_{i=1}^\infty$ is convex when projected under π.

Note that since S is in general position no three points $\pi^{-1}(S)$ are in a common great semicircle of S^2. Furthermore, each of the terms in $\{W_{\varepsilon_i}\}_{i=1}^\infty$ contain an α proportion of the area of R and B. This implies that the closure of W (in S^2) contains at least $\lceil \alpha r \rceil$ and at most $\lceil \alpha r \rceil + 1$ red points of $\pi^{-1}(S)$. By the same token, it contains at least $\lceil \alpha b \rceil$ and at most $\lceil \alpha b \rceil + 1$ blue points of $\pi^{-1}(S)$. If W has one more red point than $\lceil \alpha r \rceil$, then a red point lies in one of the bounding semicircles of W. Similarly, if W has one more blue point than $\lceil \alpha b \rceil$, then a blue point lies in one of the bounding semicircles of W. In both cases a small perturbation of W ensures that it contains exactly $\lceil \alpha r \rceil$ red points and exactly $\lceil \alpha b \rceil$ blue points. The convex wedge $\pi(W)$ is the desired convex set and the result follows. $\square$

Lemma 8 *Let S be a set of r red points and b blue points in the plane. Then there exists a strip containing exactly $\lceil \frac{r+1}{2} \rceil$ red points and exactly $\lceil \frac{b+1}{2} \rceil$ blue points of S.*

Proof. Let $r' := \lceil \frac{r+1}{2} \rceil$ and $b' := \lceil \frac{b+1}{2} \rceil$. For a given oriented line ℓ, let S_ℓ be the orthogonal projection of S to ℓ. Assume that ℓ is such that no two points of S are projected to the same point in S_ℓ. Further assume that the points of S_ℓ are sorted by the order in which they appear on ℓ. Note that if P_ℓ contains an interval (i.e., a contiguous subsequence of points) of exactly r' red points and exactly b' blue points, then there exists a strip bounded by two lines orthogonal to ℓ and containing exactly r' red points and b' blue points of S.

Let G be the $(r+1) \times (b+1)$ integer grid graph with vertex set $\{(i,j) : 0 \leq i \leq r \text{ and } 0 \leq j \leq b\}$, in which two vertices are adjacent if in one of their coordinates they are equal and in the other they differ by one. We assign a path $P_\ell := ((i_1, j_1), \ldots, (i_{r+b}, j_{r+b}))$ in G to ℓ. We define it as follows. The first vertex (i_1, j_1) is equal to $(0,0)$. For $k \geq 2$, the k-th vertex (i_k, j_k) is equal to $(i_{k-1}+1, j_{k-1})$ if the $(k-1)$-st element of S_ℓ is red, and equal to $(i_{k-1}, j_{k-1}+1)$ if it is blue. Note that P_ℓ always ends at (r,b).

Let P'_ℓ be the translation of P_ℓ by the vector (r', b'). That is, P'_ℓ has length $r+b-r'-b'$, and the k-th vertex of P'_ℓ has coordinates $(i_k + r', j_k + b')$. If P_ℓ and P'_ℓ have a common vertex (x', y') then P_ℓ contains the vertex $(x'-r', y'-b')$. This implies that in the contiguous sequence from the $(x'-r'+y'-b')$-th to the $(x'+y'-1)$-st point of S_ℓ there are exactly r' red and b' blue points. Hence it suffices to show that P_ℓ and P'_ℓ intersect.

We may assume that P_ℓ does not contain the vertex $(r-r', b-b')$, since it would imply that P'_ℓ also ends at (r,b). Therefore, P_ℓ must intersect the path $Q_1 := (r-r', 0), (r-r', 1), \ldots, (r-r', b-b'-1)$ or the path $Q_2 := (0, b-b')$, $(1, b-b'), \ldots, (r-r'-1, b-b')$. Without loss of generality, assume that P_ℓ intersects Q_1 (if not we interchange the colors), and let (x,y) be an intersection point of P_ℓ and Q_1.

Suppose first that P_ℓ intersects the path $Q_3 := (r-r'+2, 0), (r-r'+2, 1), \ldots, (r-r'+2, b-b')$. Then, since P_ℓ ends at (r,b) it must intersect the subpath of P'_ℓ that starts at (r', b') and ends at $(r'+x, b'+y)$, see Figure 3.

Assume now that P_ℓ does not intersect Q_3. Therefore, P_ℓ contains the vertices $(r-r'+1, b-b'-1)$, $(r-r'+1, b-b')$ and $(r-r'+1, b-b'+1)$. In particular, the $(r+b-r'-b')$-th and the $(r+b-r'-b'+1)$-th point of S_ℓ are blue. Move ℓ continuously until it reaches a line parallel to it but with opposite orientation. Throughout the motion, contiguous elements in S_ℓ exchange their positions until S_ℓ is inverted. During these exchanges, P_ℓ also changes somewhat continuously: at each step, one rightwards-upwards corner may become upwards-rightwards, or vice versa. This implies that at some line ℓ^* the $(r+b-r'-b')$-th or the $(r+b-r'-b'+1)$-th point of P_{ℓ^*} is red and P_{ℓ^*} and P'_{ℓ^*} intersect. $\square$

4 Algorithms

A drawback of using topological methods is that they tend to provide existential rather than constructive proofs. This is the case for the proof of Theorem 1. In this section we give polynomial time algorithms to find balanced islands.

Our strategy is to use the fact that the convex sets in Theorem 1 are either strips or wedges (Lemmas 7 and 8). We discretize the space of candidates (wedges or strips) and efficiently visit them. If we are looking for a wedge we first discretize the set of possible apices; if we are looking for a strip we first discretize the set of possible directions for the boundaries of the strip. These two steps are very similar. Indeed, strips and wedges are the same object on the sphere; fixing a direction of a strip is equivalent to fixing an apex for the corresponding wedge on the sphere (see Section 2.3.1).

Suppose that we are looking for a wedge. We know that the solution must have exactly $k := \lceil \alpha r \rceil + \lceil \beta b \rceil$ points of S (regardless of color). Suppose that a finite set of candidates apices has been chosen, and that p is the first candidate apex visited. The wedges with apex p containing exactly k points of S can be discretized as follows. Sort S clockwise by angle around p in $O(n \log n)$ time. Consider the set of all intervals of k contiguous points of S in this ordering; the set of candidate wedges with apex p are those wedges whose bounding rays pass through the endpoints of these intervals. Note that for every wedge with apex p and containing k points of S, there exists a candidate wedge containing exactly the same points of S. This set of wedges around p can be constructed in $O(n \log n)$ time, and as we construct them we can check whether any of them is balanced and convex. The candidate apices will then be visited in such a way so as to minimize the changes in this set of candidate wedges; the only possible changes are that two points transpose in the order around the apex, or that a wedge ceases to be or becomes convex. In particular, at most two wedges can change: the two wedges whose associated intervals have endpoints at the points that transpose or the wedge that ceases to be or becomes convex. As a result, we can update our set of candidates in constant time when we go from one candidate apex to the next.

When we are looking for strips we proceed in a similar way. First we discretize the the set of possible directions for the strips. We visit the first direction ℓ and sort the points in the orthogonal direction to ℓ in $O(n \log n)$ time. Like before, the candidate strips are the set of intervals containing k points in this ordering. When we visit the next direction the only change that occurs is that two points transpose in this ordering. Again, we can update our set of candidate strips in constant time. In the proof of Lemmas 9 and 10, we detail how to compute and visit these sets of candidates.

Lemma 9 *Let S be a set of n points in general position in the plane, r of which are red and b of which are blue. Then for any $r' \leq r$ and $b' \leq b$, finding a convex wedge containing exactly r' red points and exactly b' points of S, or determining that no such wedge exists, can be done in $O(n^4)$ time.*

Proof. Let $\mathcal{A}$ be the line arrangement generated by the set of lines that pass through every pair of points in S. As there are $O(n^2)$ such lines, $\mathcal{A}$ can be constructed in $O(n^4)$ time using standard algorithms for constructing line arrangements. Also note that $\mathcal{A}$ has $O(n^4)$ cells.

Let C be a cell of $\mathcal{A}$ and let p be a point in its interior. Starting at an arbitrary point of C, sort the points of S clockwise by angle around p. This is done in $O(n \log n)$ time; let S_p be this sorted set.

Let $k := r' + b'$. For $1 \leq i \leq n$, let I_i be the interval of S_p that starts at the i-th point and ends at the $(i + k - 1)$-th point of S_p (modulo n). We compute the number of red and blue points in each I_i in $O(n)$ time, by computing this parameter for I_1 and updating it when we move from I_i to I_{i+1}. We keep pointers from the first and last vertices of I_i to itself. Let W_i be the wedge with apex p, that contains I_i, and whose bounding rays pass through the first and last vertex of I_i. We also keep a record of whether W_i is convex. Note that neither the number of red and blue points of S inside W_i, nor whether it is convex, depend on the choice of p within C.

We visit all the cells in $\mathcal{A}$ by doing a DFS search in $O(n^4)$ time on its dual graph, so that we move between adjacent cells. We choose a point p in the interior of each of them. The only changes than can occur is that two consecutive points of S_p transpose or that a wedge ceases to be or becomes convex. This is determined by which line of $\mathcal{A}$ was crossed when visiting the next cell. By knowing which pair of points define this line, we can update the corresponding W_i's in constant time.

Consider any convex wedge γ with apex q that contains k points of S. Some cell of $\mathcal{A}$ contains q and at some point in our algorithm we chose a point p in this cell. One of the candidate wedges with apex p contains exactly the same points as γ and the result follows. $\square$

Lemma 10 *Let S be a set of n points in general position in the plane, r of which are red and b of which are blue. Then for any $r' \leq r$ and $b' \leq b$, finding a strip containing containing exactly r' red points and exactly b' blue points of S, or determining that no such strip exists, can be done in $O(n^2 \log n)$ time.*

Proof. Let $\mathcal{L}$ be the set of lines generated by every pair of points in S. Sort the lines in $\mathcal{L}$ by slope in $O(n^2 \log n)$ time. Let ℓ_1 be the first line of $\mathcal{L}$. Project S orthogonally to ℓ_1 and let S_{ℓ_1} be this set. Note that strictly speaking, the pair of points that define ℓ_1 are mapped to the same point. We wish to avoid this, so we actually choose a line with a slightly larger slope than ℓ_1 (but smaller than the next line in $\mathcal{L}$). We will make this choice each time we visit a line in $\mathcal{L}$.

Sort in $O(n \log n)$ time the points in S_{ℓ_1} by the order in which they appear on ℓ_1. Note that if S_{ℓ_1} contains an interval of exactly r' red points and exactly b' blue points then there exists a strip bounded by two lines, orthogonal to ℓ_1, containing exactly r' red points and exactly b' blue points of S.

Let $k := r' + b'$. For $1 \leq i \leq n - k + 1$, let I_i be the interval of S_{ℓ_1} that starts at the i-th point and ends at the $(i + k - 1)$-th point of S_{ℓ_1}. We compute the number of red and blue points in each I_i in $O(n)$ time, by computing this parameter for I_1 in $O(n)$ time and updating it when we move from I_i to I_{i+1}. We keep pointers from the first and last vertices of I_i to itself.

We visit the ℓ_j's in order, while maintaining the I_i's and their respective number of red and blue points. This can be done in constant time per line. At each step only two consecutive points of S_{ℓ_j} interchange their positions. The only intervals that change their endpoints—and thus their number of red and blue points—are precisely the two intervals that start and end at these points.

Note that every strip can be rotated without changing the points of S it contains until its bounding lines are orthogonal to a line in $\mathcal{L}$. Therefore, finding a strip containing exactly r' red points and exactly b' blue points of S, or determining that no such a strip exists, can be done in $O(n^2 \log n)$ time. $\square$

We now state the algorithmic version of Theorem 1.

Theorem 11 *Let S be a set of n points in general position in the plane, r of which are red and b of which are blue; let $\alpha \in \left[0, \frac{1}{2}\right]$, then:*

1. *a convex set containing exactly $\lceil \alpha r \rceil$ red points and exactly $\lceil \alpha b \rceil$ blue points of S can be found in $O(n^4)$ time;*
2. *a convex set containing exactly $\left\lceil \frac{r+1}{2} \right\rceil$ red points and exactly $\left\lceil \frac{b+1}{2} \right\rceil$ blue points of S can be found in $O(n^2 \log n)$ time.*

Proof. The existence of these convex sets follows from Lemmas 7 and 8. The running times of the algorithms to find them follow from Lemmas 9 and 10. $\square$

4.1 Balanced Islands in $O(n \log n)$ Time

The running times of the previous algorithms can be improved significantly for many values of α. For example, Lo and Steiger [7] gave an optimal $O(n)$ time algorithm for finding a Ham Sandwich cut for S, by this giving an optimal algorithm for $\alpha = \frac{1}{2}$. Using the results from the following lemmas, we will present a significantly improved algorithm for a large range of values α in Theorem 15.

To this end, we first introduce the concept of weighted islands. Assume that every red point in S is given a positive weight of $1/r$ and every blue point in S is given a negative weight of $-1/b$. For a given island of S let its weight be the sum of the weight of its points. To obtain a balanced island for a given $\alpha \in [0, \frac{1}{2}]$, we want an island of weight $\lceil \alpha r \rceil / r - \lceil \alpha b \rceil / b$ and containing $\lceil \alpha r \rceil + \lceil \alpha b \rceil$ points. If αr and αb are both integers then the weight of this island is equal to zero. Otherwise, it is as close to zero as possible, among the islands of S with $\lceil \alpha r \rceil + \lceil \alpha b \rceil$ points. If an island has weight larger than $\lceil \alpha r \rceil / r - \lceil \alpha b \rceil / b$, we call it *positive*; if it has weight smaller than this value we call it *negative*.

Suppose that we have found a positive island I and a negative island J, both with $\lceil \alpha r \rceil + \lceil \alpha b \rceil$ points of S. A promising approach would be to move "continuously" from I to J, so that somewhere in the middle we find a balanced island of $\lceil \alpha r \rceil + \lceil \alpha b \rceil$ points. This indeed can be done. In [3] Bautista-Santiago et al. defined a graph whose vertices are all the islands of S of a given size k, where two of them are adjacent if their symmetric difference has a fixed cardinality ℓ. They showed that under mild assumptions on k and ℓ that this graph is connected. In our case we have $k = \lceil \alpha r \rceil + \lceil \alpha b \rceil$ and $\ell = 2$. A path from I to J in the resulting graph is our desired sequence; somewhere in the middle of such a sequence there is a balanced island. We show how to compute this sequence.

Lemma 12 *Let S be a set of n points in the plane and let I and J be islands of S of k points each. Then there exists a sequence of $O(n)$ islands, starting at I and ending at J, such that the symmetric difference between two consecutive islands is a pair of points. This sequence can be computed in $O(n \log n)$ time.*

Proof. Let G be the graph whose vertices are all the islands of S with k points, two of which are adjacent in G if their symmetric difference is a pair of points. We look for a path of linear length from I to J in G. Without loss of generality, assume that no two points of S have the same x-coordinate. We sort the points in S by their x-coordinate. The subsets of S consisting of k consecutive points in this ordering are islands of S (thus vertices of G). Let G' be the subgraph of G induced by these islands. Note that G' is a path of length at most n, and can be computed in $O(n \log n)$ time. To complete the proof, we show how to compute a path of linear length from I to a vertex of G' in $O(n \log n)$ time. A path from J to a vertex of G' can be computed in a similar way.

Compute the convex hull, C, of I in $O(n \log n)$ time. Let S' be the points of $S \setminus I$ that lie in the vertical strip between the leftmost and rightmost point of I. Initialize a priority min-queue Q with the vertices of S'. Store the points in Q according to their shortest distance to C. This distance can be computed in $O(\log n)$ time per point, by doing a binary search on C. Set $I_1 := I$.

Assume that I_i has been computed and that p_i is its rightmost point. We extract points from Q until we find a point q that is to the left of p_i. Note that q is the point to the left of p_i closest to C. Set $I_{i+1} := I_i \cup \{q\} \setminus p_i$. The rightmost point p_{i+1} of I_{i+1} can be computed in constant time, since it is either q or the first point of I to the left of p_i. If Q is empty or no such point is found then I_i is an island of G' and we are done. Note that by construction the symmetric difference between I_i and I_{i+1} is a pair of points. It remains to show that the I_i's are islands of S.

Suppose that some I_i is not an island of S. Then there exists a point $p \in S \setminus I_i$ contained in the convex hull of I_i. By Caratheodory's theorem there exist three points q_1, q_2 and q_3 of I_i that contain p in their convex hull. One of these points, say q_1, is farther away from C than p. In particular q_1 is not in I. Therefore, q_1 was added to create some I_j with $j < i$. When I_j was created, q_1 was chosen because it was the point of $S' \setminus I$ closest to C and to the left of p_{j-1}. This is a contradiction since p is also a point of $S' \setminus I$ to the left of p_{j-1}, and it is closer to C than q_1. □

The algorithm to find a balanced wedge in Theorem 11 computes a set of candidate apices, such that one of them is guaranteed to be the apex of a balanced wedge. The set of candidates however is quite large; it has size $\Theta(n^4)$.

If $\lceil \alpha r \rceil + \lceil \alpha b \rceil$ is not too large, we can somehow reduce this set of candidates to a single point p. This point has the property that either there is a balanced convex wedge with apex p, or there exists both a negative convex wedge with apex p and a positive convex wedge with apex p. In the latter case we can apply Lemma 12. This point p is given by the following lemma.

Lemma 13 *(**[6, 9]**) There exist three lines, concurrent at a point p, that divide the plane into six open regions, with the property that every region contains at least $\frac{1}{6}n - 1$ points of S. Moreover, p can be found in $O(n \log n)$ time.*

The existence of the point p given in Lemma 13 was shown by Ceder [6] using a theorem of Buck and Buck [5] on equipartitions of convex sets in the plane. The algorithm for finding p is due to Sambuddha and Steiger [9]. The point p has the property that many of the wedges with apex p and whose bounding rays pass through points of S are convex. This is quantified in Lemma 14.

Lemma 14 *Let S be a set of n points in the plane, and p a point given by Lemma 13. Let $k < \frac{5}{12}n$ be a positive integer. If $k < \frac{1}{3}n$ then all the wedges with apex p that contain k points of S and whose bounding rays pass through points of S are convex; if $k \geq \frac{1}{3}n$ then at least $2n - 3k - 3$ of them are convex.*

Proof. The six regions in Lemma 13 are all wedges with apex p. Let $W_0, \ldots, W_5$ be these wedges sorted clockwise around p. Note that since each W_i contains at least $\frac{1}{6}n - 1$ points of S, any wedge with apex p that contains less than $\frac{1}{3}n$ points is contained in the union of at most three consecutive W_i's. Thus any such wedge is convex.

Assume that $\frac{1}{3}n \leq k < \frac{5}{12}n$ and set $t := k - \frac{1}{3}n + 1$. Sort the points of S clockwise by angle around p. Let P_i and R_i be the first and last t points of W_i, respectively, in this order. Let W be a wedge with apex p, containing k points of S and whose bounding rays pass through points of S. Note that if the first vertex of W lies in $W_i \setminus R_i$ then its last vertex lies in $W_{i+1} \cup W_{i+2}$ (modulo 6). In this case W is convex. Therefore, for W to be non-convex, its first vertex must be in some R_i. Now, if the first vertex of W is in R_i then its last vertex is in P_{i+3} (modulo 6). Let V_i be the set of wedges with apex p that contain k points of S, whose bounding rays pass through points of S, and whose first vertex is in R_i. Since $k \leq \frac{5}{12}n$, P_i and R_i are disjoint. Therefore, all the wedges in V_i are convex or all the wedges in V_{i+3} (modulo 6) are convex. This gives $3t$ extra convex wedges. Summarizing, we have $n - 6t$ convex wedges not starting at some R_i plus at least $3t$ extra convex wedges starting at some R_i. So in total we have at least $2n - 3k - 3$ convex wedges with apex p that contain k points of S. $\square$

Theorem 15 *Let S be a set of r red and b blue points in the plane and let $\alpha \in \left[0, \frac{1}{2}\right]$ be such that*

$$\lceil \alpha r \rceil + \lceil \alpha b \rceil < \frac{1}{3}n + \frac{2}{3}\sqrt{n + r\lceil \alpha b \rceil - b\lceil \alpha r \rceil} - \frac{4}{3}.$$

Then an island containing exactly $\lceil \alpha r \rceil$ red and exactly $\lceil \alpha b \rceil$ blue points of S can be found in $O(n \log n)$ time.

Proof. Let $k := \lceil \alpha r \rceil + \lceil \alpha b \rceil$. Find p as in Lemma 13 in $O(n \log n)$ time. We compute the set of wedges, $\mathcal{W}$ that have apex p, whose bounding rays pass through points of S, and that contain k points of S in $O(n \log n)$ time. While computing the wedges $\mathcal{W}$, we also compute their weights. We are done if there is a convex wedge of weight $z := \lceil \alpha r \rceil / r - \lceil \alpha b \rceil / b$ in $\mathcal{W}$. So assume that every convex wedge in $\mathcal{W}$ has weight greater or less than z (that is, has positive or negative weight).

We show that $\mathcal{W}$ contains a convex negative wedge and a convex positive wedge. Afterwards, we apply Lemma 12 to find a balanced island of exactly $\lceil \alpha r \rceil$ red and exactly $\lceil \alpha b \rceil$ blue points of S in $O(n \log n)$ time. We only give the proof that $\mathcal{W}$ contains a positive convex wedge. The proof that it contains a negative convex wedge is similar.

Let P be the number of non-negative wedges in $\mathcal{W}$ and let M be the sum of the weights of the non-negative wedges. Note that the sum over all wedges in $\mathcal{W}$ is zero as every point appears in the same number of wedges. Therefore, the sum of the weights of the negative wedges is equal to $-M$. In particular, this implies that there exist both negative and positive wedges. If $k < \frac{1}{3}n$ then by Lemma 14 all the wedges in $\mathcal{W}$ are convex and we are done. Assume that $k \geq \frac{1}{3}n$.

Note that the weight of two consecutive wedges in $\mathcal{W}$ differs by at most $\delta := \frac{1}{r} + \frac{1}{b}$. For M to achieve its largest possible value, the non-negative wedges must lie consecutively as follows. The first wedge has weight z. Subsequent wedges increase in weight by δ until they reach a maximum. Afterwards, subsequent wedges decrease in weight by δ until they reach z again. Depending on whether P is even or odd the sequence will stay at this maximum value for one or two wedges of the sequence. In both cases, simple arithmetic shows that $M \leq \frac{1}{4}\delta(P^2 - 2P + 1) + Pz$.

The largest possible negative weight is $z - \delta$. Therefore the sum of the negative weights is at most $(z - \delta)(n - P)$. Thus, $M \geq (\delta - z)(n - P)$, and

$$\frac{1}{4}\delta(P^2 - 2P + 1) + Pz - (\delta - z)(n - P) \geq 0.$$

Solving for P, we have that

$$P \geq 2\sqrt{n - \frac{nz}{\delta}} - 1.$$

Given that $n = r + b$, $\delta = \frac{1}{r} + \frac{1}{b}$ and $z = \lceil \alpha r \rceil / r - \lceil \alpha b \rceil / b$, this implies that

$$P \geq 2\sqrt{n + r\lceil \alpha b \rceil - b\lceil \alpha r \rceil} - 1.$$

By Lemma 14, at least $2n - 3k - 3$ of the wedges of $\mathcal{W}$ are convex. Thus, since $k < \frac{1}{3}n + \frac{2}{3}\sqrt{n + r\lceil \alpha b \rceil - b\lceil \alpha r \rceil} - \frac{4}{3}$, the number of non-convex wedges of $\mathcal{W}$ is at most $3k + 3 - n < 2\sqrt{n - b\lceil \alpha r \rceil - r\lceil \alpha b \rceil} - 1 \leq P$. Therefore, at least one of the convex wedges of $\mathcal{W}$ is non-negative. Since all balanced wedges are non-convex by assumption, this wedge must be positive and the result follows. $\square$

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References

[1] J. Akiyama, A. Kaneko, M. Kano, G. Nakamura, E. Rivera-Campo, S. Tokunaga, and J. Urrutia. Radial perfect partitions of convex sets in the plane. In *Discrete and computational geometry (Tokyo, 1998)*, volume 1763 of *Lecture Notes in Comput. Sci.*, pages 1–13. Springer, Berlin, 2000.

[2] I. Bárány and J. Matoušek. Simultaneous partitions of measures by k-fans. *Discrete Comput. Geom.*, 25(3):317–334, 2001.

[3] C. Bautista-Santiago, J. Cano, R. Fabila-Monroy, D. Flores-Peñaloza, H. González-Aguilar, D. Lara, E. Sarmiento, and J. Urrutia. On the connectedness and diameter of a geometric Johnson graph. *Discrete Math. Theor. Comput. Sci.*, 15(3):21–30, 2013.

[4] P. V. M. Blagojević and A. S. Dimitrijević Blagojević. Using equivariant obstruction theory in combinatorial geometry. *Topology Appl.*, 154(14):2635–2655, 2007.

[5] R. C. Buck and E. F. Buck. Equipartition of convex sets. *Math. Mag.*, 22:195–198, 1949.

[6] J. G. Ceder. Generalized sixpartite problems. *Bol. Soc. Mat. Mexicana (2)*, 9:28–32, 1964.

[7] C.-Y. Lo and W. Steiger. An optimal time algorithm for ham-sandwich cuts in the plane. In *Proc. Canad. Conf. Comput. Geom.(CCCG 90)*, pages 5–9, 1990.

[8] J. Matoušek. *Using the Borsuk-Ulam theorem.* Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.

[9] R. Sambuddha and W. Steiger. Some combinatorial and algorithmic applications of the Borsuk-Ulam theorem. *Graphs Combin.*, 23(suppl. 1):331–341, 2007.

[10] P. Soberón. *Partition problems in discrete geometry.* PhD thesis, University College London, 2013.

[11] E. Stiefel. Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten. *Comment. Math. Helv.*, 8(1):305–353, 1935.

[12] R. T. Živaljević. User's guide to equivariant methods in combinatorics. *Publ. Inst. Math. (Beograd) (N.S.)*, 59(73):114–130, 1996.