GRAVITY INTERPRETATION AND INFORMATION THEORY II. SMOOTHING AND COMPUTATION OF REGIONALS

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SUMMARY

Some formulas proposed for smoothing and computation of the regional component of the original gravity field are discussed. It has been derived from the investigation of the transfer functions that the exponential smoothing formula possesses the best filtering properties. This formula has been generalized to form one- and two-dimensional sets of low-pass filters for performing the operations of smoothing and of the computation of regionals.

Introduction

In a previous paper, the author has begun a systematic investigation of linear transformations of gravity maps, using the concepts and relations of information theory (M e s k ó, 1966). He has given formulas for the computation of transfer functions, described some consequences of digital computation (aliasing etc.) and determined the frequency range to be investigated. The possibilities to represent the transfer functions have been treated, as well as examples concerning the details of averaging on circles and of grid methods.

The operation

$$f(x, y) = \sum_{k=1}^{n} c_k \overline{f(r_k)} \tag{1}$$

has the transfer function

$$S(\omega', \psi') \equiv S(\varrho') = \sum_{k=1}^{n} c_k J_0(\varrho' \mu_k); \tag{2}$$

where

$$\varrho'^2 = \omega'^2 + \psi'^2,$$

$$\omega' = \omega s,$$

$$\psi' = \psi s,$$

$$r_k = \mu_k s,$$

and J_0 is the zero-order Bessel function of the first kind.

The operation

$$f(x, y) = \sum_{k=1}^{n} c_k f(x + x_k, y + y_k) \tag{3}$$

has the transfer function

$$S(\omega', \psi') = \sum_{k=1}^{n} c_k e^{i(\omega' \xi_k + \psi' \eta_k)}; \tag{4}$$

where $\xi_k s = x_k,$

and $\eta_k s = y_k.$

The frequency range to be investigated is

$$0 \leqq |\omega'| \leqq 180°; \qquad 0 \leqq |\psi'| \leqq 180°. \tag{5}$$

Making use of the formulas listed above, we shall deal in the present paper with the smoothing and computation of regionals.

The most abrupt changes in the gravity field have often no connection at all with the geological structures to be investigated. They originate from small disturbing bodies which lie near the surface, or, on the other hand, from errors of measurement (or reduction). To remove or decrease these effects is obviously useful. The operation performed with this end in view is called smoothing.

The regional part of the field is attributed to effects whose sources are too deep or too large to be of interest. If we knew the shape, depth and density contrast of the disturbing bodies causing the regional part of the field, we could compute an exact expression for regionals. In general, however, we have only a rough estimate of these parameters; consequently, the regionals have to be derived from the data system itself.

Smoothing and computation of regionals both represent cases of low-pass filtering: only the removed frequency ranges differ. Smoothing has to remove the highest frequency components while it has to preserve the others in a form as free as possible of distortions. The computation of regionals, on the other hand, has to preserve the lowest frequency components only.

In the next two paragraphs, the filtering performed by some commonly used formulas will be investigated.

Smoothing

Smoothing of data sets derived from uniformly spaced grids of measurements will be treated. In the one-dimensional case the formula (4) may be simplified as follows:

$$S(\omega') = \sum_{k=1}^{n} c_k e^{i\omega' \xi_k} \tag{6}$$

Now let

$$n = 2m+1, \ \xi_k = k-m-1$$

and

$$c_k = c_{2m+2-k} = c_0 \cdot d_{m+1-k},$$

where c_0 is a constant factor.

On substituting these terms into (6) the transfer function becomes

$$S(\omega') = c_0 \left[d_0 + 2 \sum_{l=1}^{m} d_l \cos l\omega' \right]. \tag{7}$$

The formulas described by K. Jung (1961) have been investigated. Table I. contains the coefficient sets of the so called "simple formulas", as well as of binomial and exponential smoothing.

The general form of binomial smoothing reads

$$f_{sm}(x_0) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{n-k} [f(x_0+ks)+f(x_0-ks)]\varepsilon_k; \tag{8}$$

where

$$\varepsilon_0 = \frac{1}{2}, \quad \varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_h =$$ [illegible]

The exponential formula has the general form

$$f_{sm}(x_0) = \frac{1}{\sqrt{n\pi}} \sum_{k=1}^{n} e^{-\frac{1}{n} \cdot k^2} [f(x_0+ks)+f(x_0-ks)]. \tag{9}$$

A further procedure sometimes employed is smoothing with the fourth differences, as defined by

$$f_{sm}(x_0) = f(x_0) - c \cdot \Delta^{(4)}(x_0), \tag{10}$$

where $\Delta^{(4)}(x_0)$, the fourth difference is evaluated from

$$\Delta^{(4)}(x_0) = 6f(x_0) - 4\,[f(x_0+s)+f(x_0-s)] + f(x_0+2s) + f(x_0-2s),$$

and $c_1 = 3/35$ or $c_2 = 1/12$.

The transfer functions of the "simple formulas" and of smoothing with the fourth differences are plotted in Figs. 1 and 2, respectively. The transfer functions of exponential and binomial formulas are shown in Figs. 3–6. Increasing n means an increasing degree of smoothing, i.e. more and more high-frequency components will be eliminated. At the same time, the differences between both the coefficients and the transfer functions of the exponential and binomial formulas decrease.

Table I.

Formulas	Coefficients (according to K. Jung)							
	c_0	d_0	d_1	d_2	d_3	d_4	d_5	d_6
"simple formula" 1	1/2	2	1					
"simple formula" 2	1/25	5	4	3	2	1		
"simple formula" 3	1/125	25	24	21	7	3	−2	−3
binomial (n = 1)	1	0,5	0,250					
binomial (n = 2)	1	0,375	0,250	0,062				
binomial (n = 3)	1	0,312	0,234	0,094	0,016			
binomial (n = 4)	1	0,273	0,219	0,109	0,031	0,004		
exponential (n = 1)	1	0,564	0,208	0,010				
exponential (n = 2)	1	0,399	0,242	0,054	0,004			
exponential (n = 3)	1	0,326	0,233	0,086	0,016	0,002		
exponential (n = 4)	1	0,282	0,220	0,104	0,030	0,005		

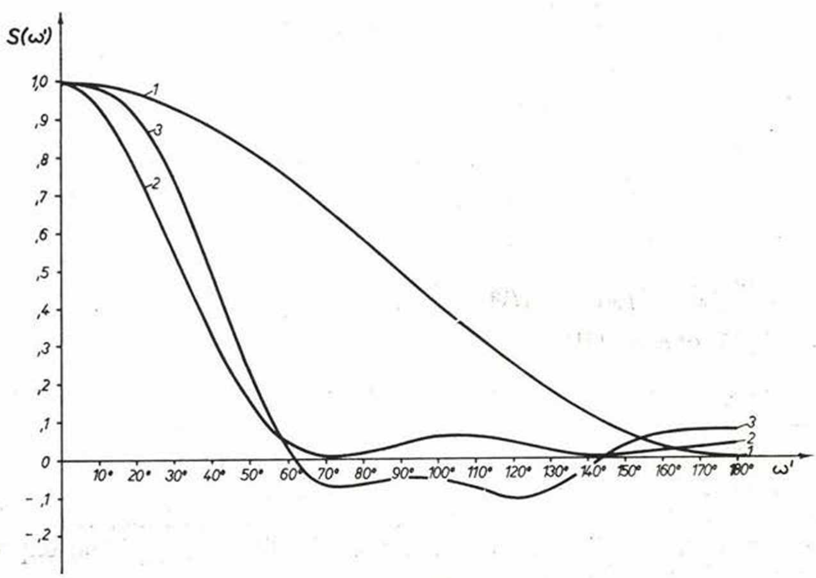

Fig. 1. Transfer functions of the "simple formulas".

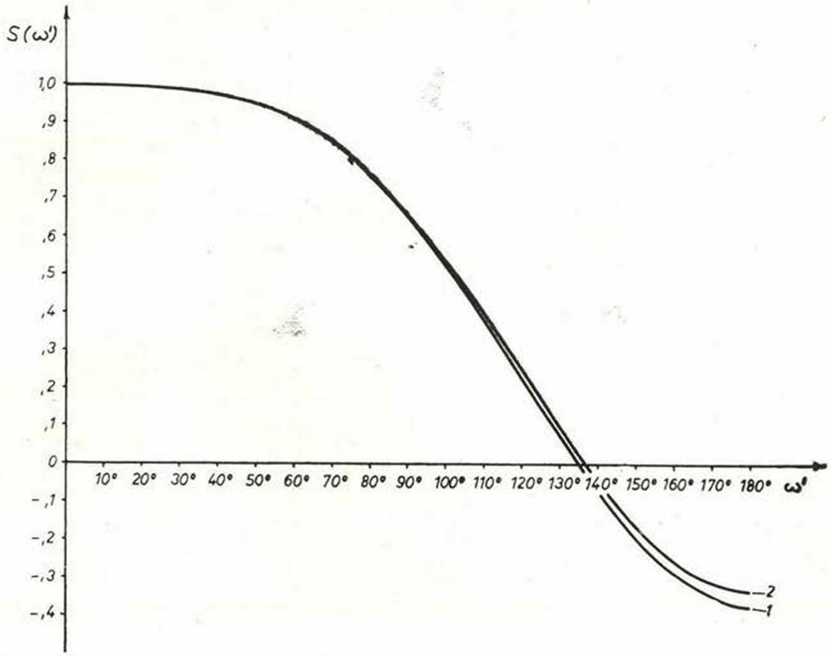

Fig. 2. Smoothing with the fourth differences. Transfer functions plotted for parameters $c_1 = 3/35$ (graph 1), and $c_2 = 1/12$ (graph 2).

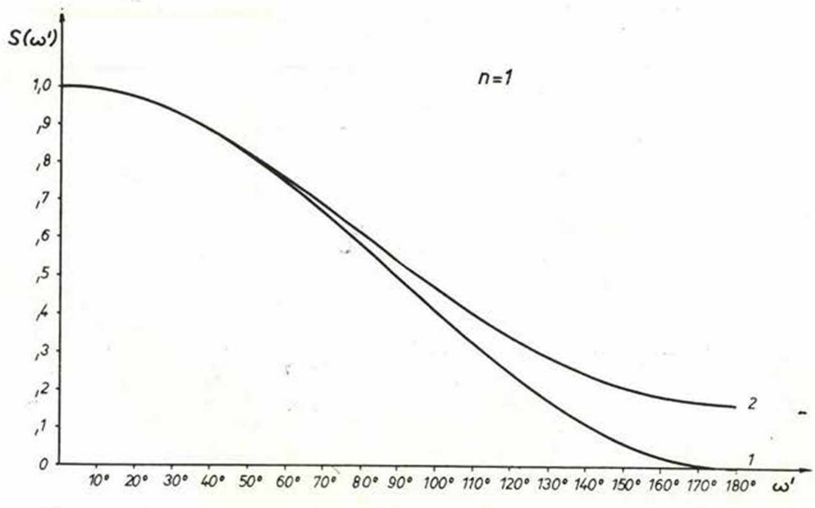

Fig. 3. Transfer functions of the binomial (1) and exponential (2) smoothing formula ($n = 1$).

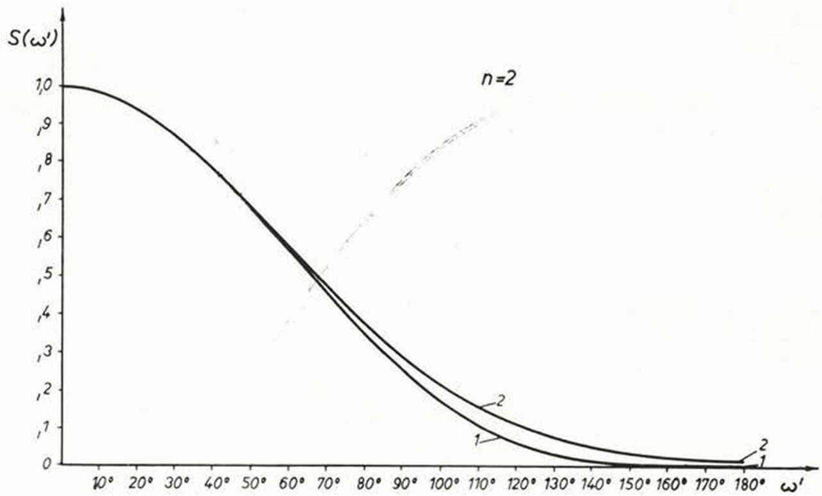

Fig. 4. Transfer functions of the binomial (1) and exponential (2) smoothing formula ($n = 2$)

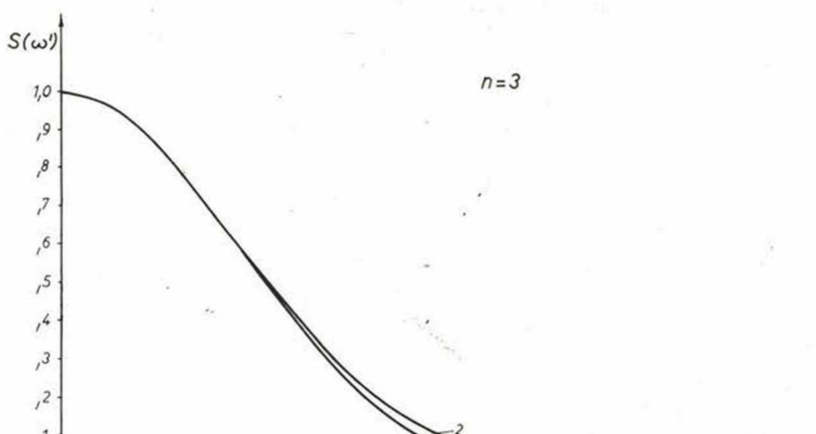

Fig. 5. Transfer functions of the binomial (1) and exponential (2) smoothing formula ($n = 3$).

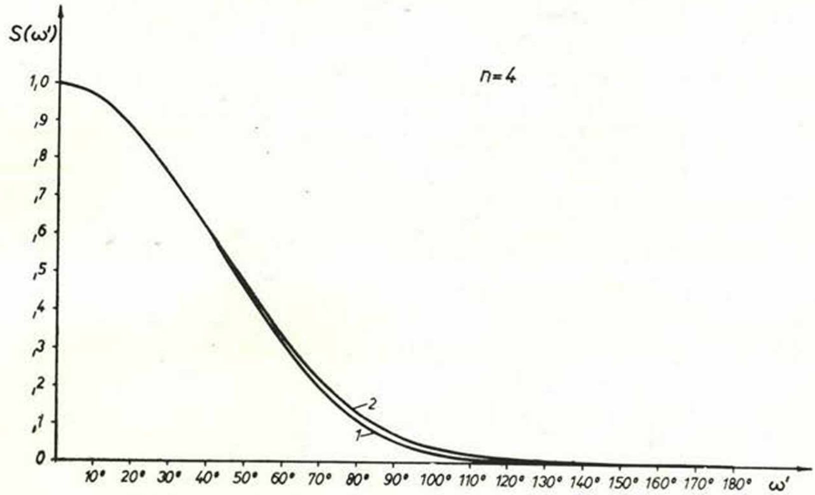

Fig. 6. Transfer functions of the binomial (1) and exponential (2) smoothing formula ($n=4$).

A continuous function, $f(x)$ can be smoothed by the formula

$$f_{sm}(x) = \sqrt{\frac{k}{\pi}} \int_{-\infty}^{+\infty} f(u) e^{-k(x-u)^2} \, du. \tag{11}$$

Applying the symbol of convolution, equation (11) can be written as

$$f_{sm}(x) = f(x) \quad * \quad \sqrt{\frac{k}{\pi}} e^{-kx^2}. \tag{12}$$

the weighting function then yields directly

$$s(x) = \sqrt{\frac{k}{\pi}} e^{-kx^2}.$$

Thus the transfer function (the Fourier transform of the weighting function) becomes

$$S(\omega) = F\left\{\sqrt{\frac{k}{\pi}} e^{-kx^2}\right\} = e^{-\frac{\omega^2}{4k}}. \tag{13}$$

Computation of regionals

There are several groups of methods for computation of regionals: various procedures of averaging, analytical continuations upward, statistical methods etc. We shall now treat some examples from the first and second groups.

One of the simplest methods is to consider the average of values observed on the circumference of a circle as the regional, i.e.

$$f_{reg}(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \varphi)\, d\varphi. \tag{14}$$

This integral expression may be approximated by the sum

$$f_{reg}(x_0, y_0) = \frac{1}{N} \sum_{k=1}^{N} f(P_k) \tag{15}$$

(P_k is a point on the circumference of a circle of radius r.) It has been shown in a previous publication (Meskó, 1966) that the transfer function of the operation (14) has the form

$$S(\varrho; r) = J_0(\varrho r), \tag{16}$$

or, using a relative (dimensionless) frequency variable

$$S(\varrho'; \mu) = J_0(\varrho' \mu), \tag{17}$$

where

$$r = \mu s$$

The transfer functions of operation (15) have been computed for several values of N in the special case of points regularly distributed on the circumference of a circle. The degree of approximation has also been considered. Therefore we now present the transfer functions (17) for a few values of the parameter μ only (Fig. 7).

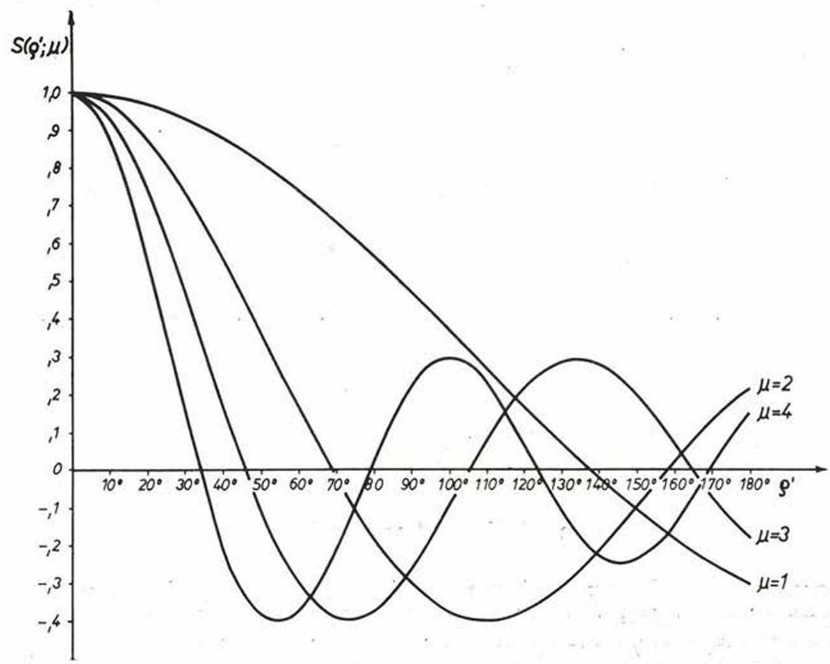

Fig. 7. Transfer functions of the averaging on the circumference of a circle for some values of the parameter $\mu = r/s$.

The transfer functions have relatively high values at high frequencies. Operation (14) does not sufficiently decrease the amplitudes of the high-frequency components.

It is more expedient to use the average of the observed values on the surface of a circular disk:

$$f_{reg}(x_0, y_0) = \frac{1}{R^2\pi} \int_{r=0}^{R} \int_{\varphi=0}^{2\pi} g(r, \varphi) r \, d\varphi \, dr. \tag{18}$$

Introducing in the frequency plane polar coordinates by the definitions

$$\omega = \varrho \cos \alpha; \quad \psi = \varrho \sin \alpha,$$

we have by equation (18)

$$S(\varrho\alpha) = \frac{1}{R^2\pi} \int_{r=0}^{R} \int_{\varphi=0}^{2\pi} r e^{i\varrho r \cos(\alpha-\varphi)} \, d\varphi \, dr =$$

$$= \frac{1}{R^2\pi} \cdot 2\pi \int_{r=0}^{R} r J_0(\varrho r) \, dr = \frac{2}{R^2} \left[\frac{r}{\varrho} J_1(\varrho r) \right]_{r=0}^{R} = \frac{2J_1(\varrho R)}{\varrho R}; \tag{19}$$

where J_1 is the first-order Bessel function.

Using a relative (dimensionless) frequency variable and introducing the parameter $\mu = R/s$, equation (19) yields

$$S(\varrho'; \mu) = \frac{2J_1(\mu\varrho')}{\mu\varrho'}. \tag{20}$$

The transfer functions are shown in Fig. 8 for a few values of the parameter μ. The "behaviour" of the functions is better, but further improvements are desirable.

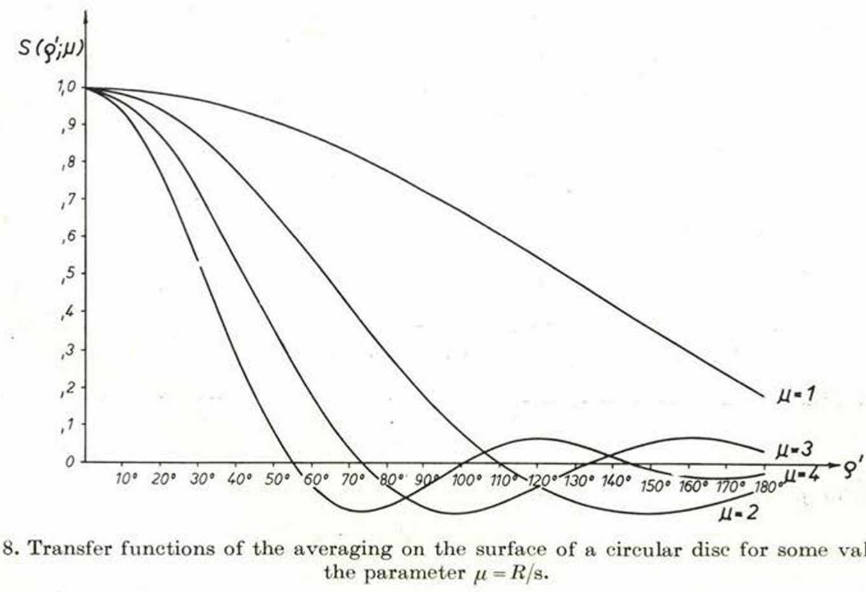

Fig. 8. Transfer functions of the averaging on the surface of a circular disc for some values of the parameter $\mu = R/s$.

An estimate of regionals can be obtained by analytical continuation upwards (e.g. K. Jung, 1961). The transfer function of the theoretical operation is of the form

$$S(\varrho; h) = e^{-h\varrho}, \tag{21}$$

where h means the height of the continuation (see e.g. Dean, 1958).

Introducing dimensionless $\chi = h/s$ and ϱ', we have

$$S(\varrho'; \chi) = e^{-\chi\varrho'}. \tag{22}$$

The transfer functions are shown in Fig. 9. It is seen that the operation has, indeed, the character of a low-pass filter.

The transfer functions of any linear method for determination of regionals can be similarly computed and illustrated. But instead of increasing the number of examples, let us inquire into the best possible way of smoothing and computation of regionals.

In the introduction, the aim of these operations has already been briefly outlined. It is essentially to remove the high-frequency components and preserve the others without distortion. The limit of the frequency band to be removed is, however, determined by the nature of the given problems to be solved by the survey. Therefore it would be improper to use any fixed formula (i. e. any fixed filter). We have to construct a set of filters. We can then choose from this set the particular filter (filters) appropriate to any situation that may arise.

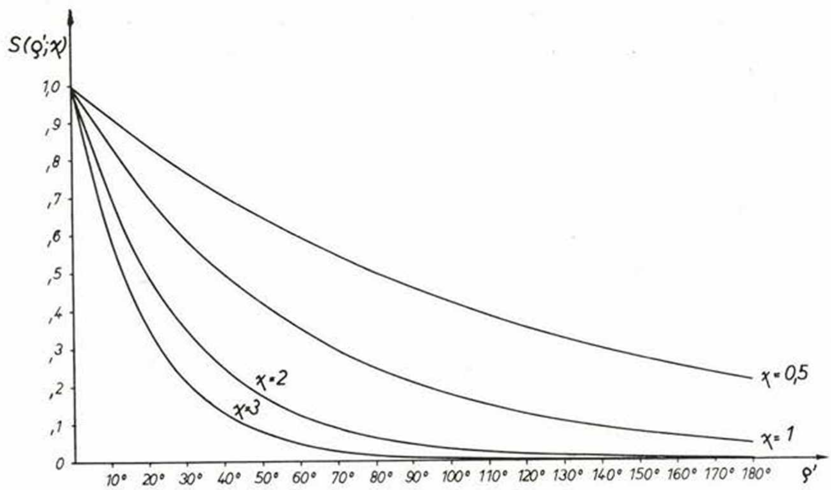

Fig. 9. Transfer functions of analytical continuation upwards for some values of the parameter $\chi = h/s$.

One possibility is to design low-pass filters of the form

$$\begin{aligned} S(\varrho) &= 1, \ \textit{for} \ |\varrho| > \varrho_0 \\ &= 0, \ \textit{for} \ |\varrho| \geqq \varrho_0. \end{aligned} \tag{23}$$

However, the corresponding weighting functions converge slowly to zero; hence, a long set of coefficients would be necessary. It is more convenient to use a generalization of the exponential formula, (13).

Sets of low-pass filters

Let us define a (relative) transmission frequency, ω_t by the property

$$S(\omega_t) = S(0)/e.$$

The transfer functions

$$S(\omega') = e^{-\left(\frac{18\,\omega'}{\varkappa'\pi}\right)^2}, \qquad (\varkappa' = 1, 2, \ldots, 9) \tag{24}$$

have, by the equations

$$\frac{18\omega_t'}{\varkappa'\pi} = 1,$$

the transmission frequencies

$$\omega' = \varkappa' \frac{\pi}{18}; \qquad (\varkappa' = 1, 2, \ldots, 9),$$

or

$$\omega_t' = \varkappa' \cdot 10^\circ; \qquad (\varkappa' = 1, 2, \ldots, 9). \tag{25}$$

The set of filters defined by equation (24) approximately preserve (i.e. transmit) the relative frequency band under 10°, 20°, ... 90° (Fig. 10).

Equation (24) yields the theoretical transfer functions. Because of digital realization, the actual functions deviate from the theoretical ones. The deviation increases with increasing $\varkappa'$, but even for $\varkappa' = 9$ it remains negligibly small.

The sets of coefficients corresponding to the weighting functions may be obtained by means of the inverse Fourier transform

$$S(l;\, \varkappa') = F\{S(\omega';\, \varkappa')\} = \frac{\varkappa'\sqrt{\pi}}{36} e^{-\frac{l^2\varkappa'^2\pi^2}{36^2}}. \tag{26}$$

The actual transfer functions have also been computed, using the coefficient sets (26) and formula (7). The deviations between the theoretical (intended) and actual functions are so small that a separate representation of the latter has not been necessary: Fig. 10 may be taken to show the actual transfer functions as well.

In the two-dimensional case we may use a suitable generalization of the transfer function (24).

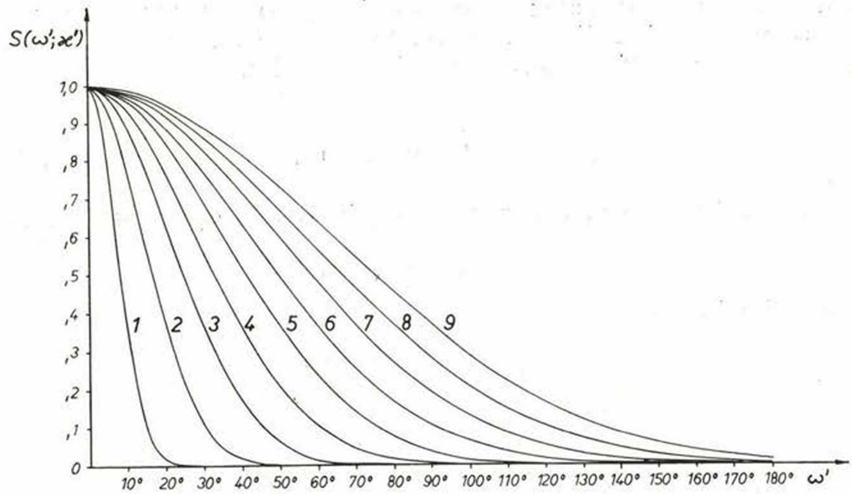

Fig. 10. Sets of low-pass filters (theoretical and actual), designed to perform the operations of smoothing and computation of regionals.

Whenever the transfer function and weighting function possess circular symmetry, their relation may be expressed as a zero-order Hankel transform

$$S(\varrho) = 2\pi \int_0^\infty r s(r)\, J_0(\varrho r)\, dr, \tag{27}$$

$$S(r) = \frac{1}{2\pi} \int_0^\infty \varrho S(\varrho)\, J_0(\varrho r)\, d\varrho. \tag{28}$$

(Dean, 1958.)

Now, the required generalization of the transfer function (24) becomes

$$S(\varrho;\, \varkappa) = e^{-\varkappa\varrho^2}, \tag{29}$$

and the weighting functions may be obtained by means of the inverse Hankel transform

$$s(r;\, \varkappa) = \frac{1}{2\pi} \int_0^\infty \varrho\, e^{-\varkappa\varrho^2} J_0(\varrho r)\, d\varrho = \frac{1}{4\pi\varkappa} \cdot e^{-\frac{r^2}{4\varkappa}}, \tag{30}$$

(For evaluation of the integral see e.g. Gradstein, Rusik, 1963).

Introducing the relative frequency variable ϱ' and the parameter $\varkappa'$,

$$\varkappa = \left(\frac{18s}{\varkappa'\pi}\right)^2; \tag{31}$$

equation (29) becomes

$$S(\varrho'; \varkappa') = e^{-\frac{18\varrho'}{\varkappa'\pi}}. \tag{32}$$

The coefficient set corresponding to the weighting function yields, after substitution of $\varkappa$ from eq. (31)

$$s(\mu; \varkappa') = \pi\left(\frac{\varkappa'}{36}\right)^2 \cdot e^{-\frac{\mu^2\varkappa'^2\pi^2}{36^2}}, \tag{33}$$

where $\mu = r/s$ and r is the distance from the point of reference.

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