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AN UPPERBOUND ON THE ROPELENGTH OF ARBORESCENT LINKS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Larry Andrew Mullins

September 2007

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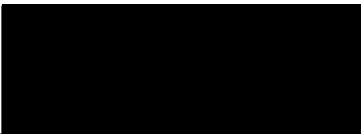
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Approved by:



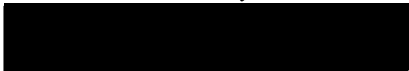
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8-16-2007

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ABSTRACT

In this paper we improve on the upperbounds for ropelength of a specific class of algebraic knots. Conway and Gabai use a diagrammatic approach to define algebraic knots and links. Here we realize these diagrammatic algebraic knots and links as three dimensional manifolds. This is done by ambient isotopy and issuing a proper parametrization of these knots and links. Then, the upperbounds on ropelength are shown to be dependent on the arclength of the parametrization. Finally, the arclength is counted and examples are given and shown to improve on the known bounds.

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Chapter 1

Introduction

If you were to take a piece of string, loop it around, physically tie some knots in it and connect the ends together, then you would have a string which is actually knotted. Mathematicians like to think of knots as a closed loop having no thickness and never intersecting itself. The most trivial example of a knot is the unknot which is a circle. Any knot which is not the unknot is considered “non-trivial”. The simplest non-trivial knot is the trefoil, as seen in Figure 1.1(a).

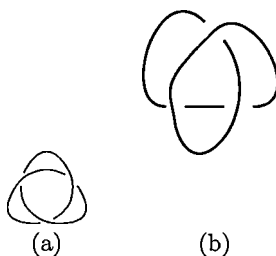


Figure 1.1: Projection of Trefoil and Unknot

The trefoil can be generated by taking an extension chord and tying one overhand, or underhand, loop and then connecting the ends together. Notice that you can lay the chord out in the same shape as Figure 1.1(a). Also, you can change the shape of that trefoil and not change the knot at all. After which you can lay the trefoil out again and get a different looking shape, possibly with new twists. Figure 1.1(b) is a projection of

an unknot having three new twists which can be untwisted to become the unknot. When the trefoil is laid out on a plane, then where the ropes overlap is called a crossing. This is the general idea behind knot projection.

Definition 1.1. *A knot projection is a projection of a (three dimensional) knot having no multiple crossings, where a broken line represents a crossing.*

For example, Figure 1.2 is another projection of the trefoil. Two knots are equivalent if there exists a isotopy between the two curves in \mathbb{R}^3 , i.e. if one knot can be continuously deformed into the other. Then, the two projections of the trefoil seen in Figure 1.1 are clearly equivalent. All knots equivalent to a given one are considered a knot type. Notice that the trefoil projections we gave still have three crossings. The fewest number

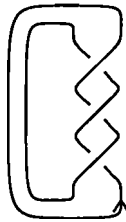


Figure 1.2: Another Projection of the Trefoil

of crossings in any projection of a knot is defined to be crossing number, and is denoted $C(K)$. Knots can be categorized by their crossings. The two projections above of the trefoil clearly can be transformed into one another. Any particular knot type, such as the trefoil, can have many crossing projections. Much of knot theory is dedicated to developing ways to distinguish between different knots and different knot types. These ways are often in the form of a number, and are called knot invariants. Also ambient isotopy does not change a knot invariant. Ropelength is considered to be the minimum ratio of the length of a rope to its radius over all knot configurations. Crossing number and ropelength are considered knot invariants. Although there are many more knot invariants, ropelength is the knot invariant considered in our research.

One interesting point to make, is that we often move between the projections of a knot, and its three dimensional counterpart. Crossing number is determined by considering projections of a knot, which is dependent on a two-dimensional image.

And ropelength is a topological invariant somewhat determined by the arclength of the parametrization of the projection knot embedded into three space. Combining these two ideas we are concerned with finding an upper bound on the ropelength of certain knot types based on crossing number. The knot types considered are a specific class of algebraic knots.

First, in Chapter 2 we take a close look at a paper by McCabe [McC05] where she describes how to obtain the particular class of knots we are concerned with. Here we will define algebraic knots and show how to obtain their projections from tree's. Then, in Chapter 3 we will define ropelength and state the theorem. Next, in Chapter 4 we will use methods similar to those found in Moran's paper [CFM04] to construct an upperbound on ropelength of these knots. Finally, in Chapter 5 provide examples of the upperbounds.

Chapter 2

Arborescent Links and Tree Diagrams

2.1 Introduction

In this chapter we introduce some definitions, notation, and knot constructions. First, the definitions of arborescent knots and links will be discussed, along with a few examples. Then, we describe how to construct the knots considered in this paper. Finally, we will construct a knot from its tree providing the reader with an example.

2.2 Arborescent Links and Conway's Notations

Arborescent knots and links have specific constructions. Given a knot or link projection, a tangle is a “region in the projection plane surrounded by by a circle such that the knot or link crosses the circle exactly four times,” [Ada04] (see Figure 2.1).

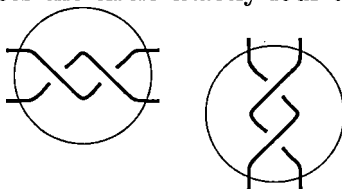


Figure 2.1: Horizontal and Vertical Integral Tangle

An integral tangle can be thought of as a series of crossings turned so that they are either horizontal or vertical. Also, these crossings are considered positive or negative as seen in Figure 2.2 where the arrows represent an orientation of the knot or link. Here the H_+ and the H_- , and V_+ and the V_- distinguish crossings for horizontal and vertical tangles respectively. Integral tangles have four arcs emanating from them, and they can

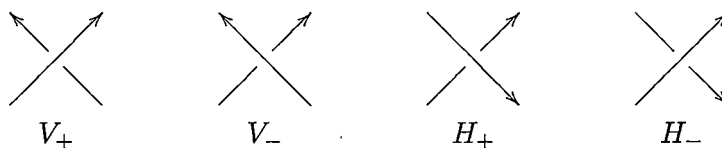


Figure 2.2: Positive and Negative Crossings

be connected “algebraically” if two arcs from one tangle are connected to two of another by unknotted arcs using the operations of tangle addition and multiplication, described below. The convention is to think of each arc as respectively labeled NW, NE, SW, and SE. Two tangles connected algebraically may be connected by two arcs from NE of one to NW of another, and SE to SW. This is often characterized as tangle addition. Tangle multiplication is the operation of reflecting “the first tangle across its NW to SE diagonal” [Ada04] and then connecting them the same way as addition. Tangles constructed using tangle addition and multiplication are called algebraic tangles, and new tangles may be formed by recursively adding or multiplying a number of algebraic tangles together. A non-algebraic knot is a knot that cannot be formed from tangle multiplication or addition, for example see Figure 2.3. An example of an algebraic tangle is found in Figure 2.4 where there is a negative two crossing tangle multiplied with the addition of a positive five tangle and a positive three tangle. When the ends of

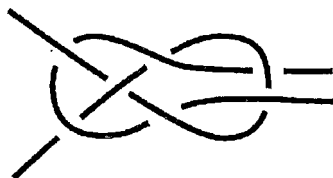


Figure 2.3: Nonalgebraic Tangle

the projection of an algebraic tangle are “closed-off”, either NW to NE and SW to SE,

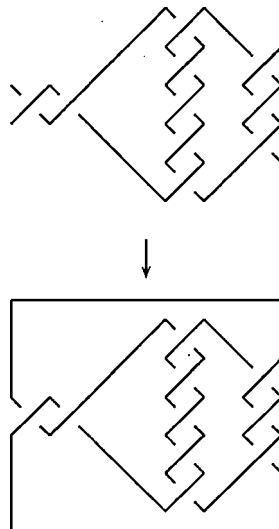


Figure 2.4: Arborescent Knot: $-2(5 + 3)$

or NW to SW and NE to SE, an *arborescent* knot or link is formed. Also, arborescent knots can be categorized by their *twist number*. The twist number, $T(K)$, of a knot, or link, K is the minimum number of integral tangles taken over all projections of K . See figure 2.7 for an example of an arborescent knot having $T(K) = 3$ and $T(K) = 4$. Next we will show how knots are formed by polyhedra in the plane.

In [Con70], Conway describes how to obtain arborescent links from a polyhedron in the plane by reducing integral tangles into vertices while keeping the arcs connected. First, reduce all of the integral tangles to vertices, resulting in a planer map. Next, recursively collect any two vertices connected to another by two arcs to form a single vertex, since they came from a larger tangle. Once all the vertices are collected, the *basic polyhedra* is formed for that projection (see Figure 2.5). In order to identify algebraic links McCabe[McC05] provides a useful definition.

Definition 2.1. *Any link which possesses a minimal-crossing projection whose basic polyhedron is the figure eight is defined to be a minimally and algebraically presentable (MAP) link.*

An example of a MAP link and how its associated polyhedra is the figure eight can

be found in Figure 2.5.

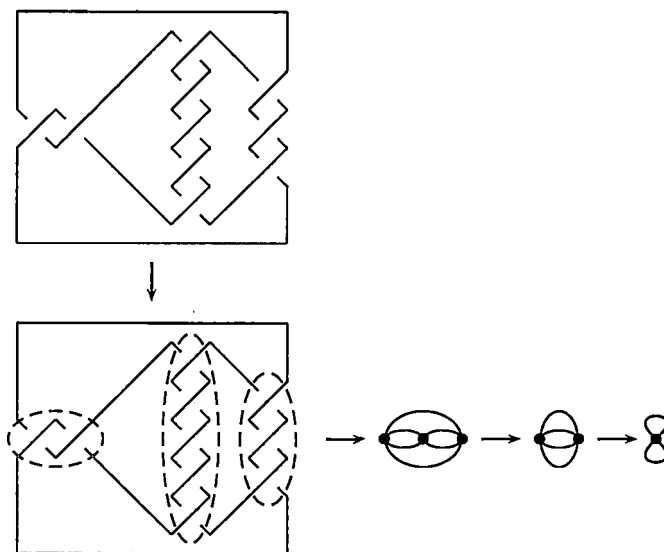


Figure 2.5: MAP Link

Because we are starting with links that possess a minimal crossing projection and the main theorem of this paper is based on the crossing numbers of the integral tangles, it is important to begin with a minimal crossing number projection having minimized twist number. For example, consider the bottom link in Figure 2.7. This link has the same number of crossings as the link in Figure 2.5. Moreover these links can be shown to be the same link by way of *flyping*. Flyping is an isotopy move that rotates a tangle of a knot or link by 180° , see Figure 2.6. Let the top link be A and the bottom

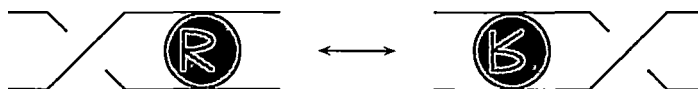


Figure 2.6: Flype

link B in Figure 2.7. Notice that the circled one crossing in B is an unnecessary twist because it causes more complexity than needed. This crossing can be combined with

the other one crossing in order to create a two tangle. This flyping move has reduced the connector strands from integral tangle to integral tangle, and tightened the knot up some. The MAP knots that McCabe has defined begin with links that are already in the form that has unnecessary crossings moved to the left of the projection. With a proper way to identify algebraic projections of links that possess a minimal crossing projection, we can move on to describing MAP links in terms of trees after Gabai and McCabe.

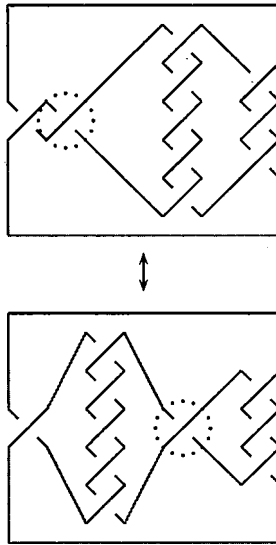


Figure 2.7: Same Link by Flyping

2.3 Tree Diagrams

In the previous section we described the general class of knots we are concerned with, arborescent knots. In this section we will see how to obtain a tree from an arborescent knot, and find out why that is important to the main theorem of this paper. First we will describe how the trees are related to the knots, and then show how they are constructed. Finally the general types of knots and its associated tree will be given.

McCabe's work on minimizing the crossing number of projections of MAP links by way of flyping to have a minimized representation of the knot projection is quite

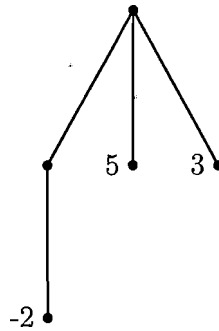


Figure 2.8: Tree Associated with Fig 2.4

useful here. This step is important so that we can begin minimizing the ropelength by minimizing the twist number, i.e. having tangles with unnecessary twists removed. Knots associated with trees have minimized twist number. One of the results of McCabes paper is that there corresponds a tree T in PL standard form for each MAP link K . [McC05] For the purpose of this paper we focus on a specific class of algebraic knots associated with a specific class of trees. We considered rooted trees as seen in see Figure 2.8. Gabai [Gab86] defines stumps and twigs as “a stump is a vertex which adjoins exactly one edge, and a twig is a vertex which adjoins exactly two edges.” [Gab86] A weight on a tree represents the crossing number of a particular integral tangle while its position in the tree is relative to the position of the integral tangle in the algebraic knot or link, see Figure 2.11.

One requirement for a tree to be in PL standard form is the tree must satisfy a general from associated with Gabai’s first theorem. [Gab86] The convention is that the first level contains only the root vertex and no weight. This corresponds to a horizontal tangle with zero crossings, a.k.a a primary band. Each horizontal level in the tree is considered to contain separate tangles where the root vertex is considered the primary band connecting the tangles, and each node is an associated integral tangle. The second level contains vertical tangles connected to the primary band. The third level contains horizontal tangles connected to the second level relative to the position of the weight in the tree. This alternating of the horizontal and vertical tangles continues for each level, always beginning with the first level as a horizontal primary band. See Figures 2.9, 2.10, and 2.11 for a nontrivial example of a PL tree and its knot, and refer to Figures 2.4 and

2.8 for a less complicated example.

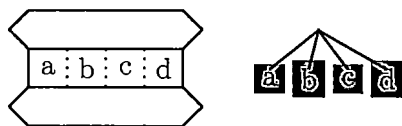


Figure 2.9: Primary Band and Associated Tree

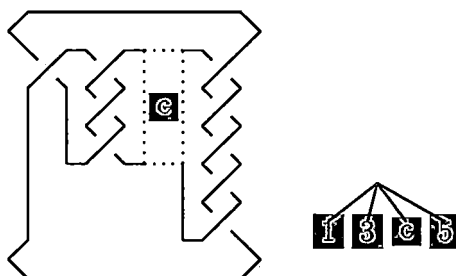


Figure 2.10: First Level

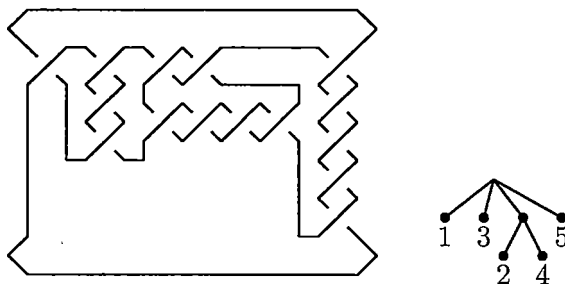


Figure 2.11: Link in PL Standard Form and its Tree

Notice that if a stump contains a ± 1 then the node can move up to the previous level with a sign change, because a positive horizontal one crossing tangle is the same as a negative vertical tangle, and vice-versa. The goal of the use of the PL standard form is to begin with arborescent knots that have minimized number of tangles in a projection so that ropelength is being minimized. With these conventions stated we now list the forms of arborescent knots that we worked with.

The types of arborescent knots and links considered in this paper are very specific. The trees associated with the knots considered are made up of stumps and one twig. For example, Figure 2.8 has three stumps and one twig, which is the node with no weight other than the root. Having defined arborescent knots and their associated trees, we can now move on to their associated knot energies.

Chapter 3

Ropelength

3.1 Introduction

In 1986, Shinji Fukuhara proposed a canonical deformation of a knot in order to obtain its standard form. The original description is as follows:

make a knot of a non-elastic string and distribute electrons on it. Set the knot in viscous liquid which absorbs kinetic energy. Then the knot will move and gradually reduce its electric energy. If it reaches a critical but unstable point of the energy, then perturb it a little and let it move again. Repeat the process if necessary. Then finally its electric energy will become minimal and we will obtain a standard form of the previous knot.[Fuk88]

Fukuhara's paper continues to describe how to obtain minimal energies without changing the original isotopy class of the knot by way of studying polygonal knots. Also, Fukuhara is convinced that this method will save much time to make a knot table. Then, in 1991, Jun O'Hara published a topological paper on energies where he defines a real-valued energy functional.[O'H03] O'Hara's energy function is topologically motivated by curvature and defined below.

Definition 3.1. *Let $\alpha(t)$ be a parametric equation of a simple closed C^2 curve. Then*

$$E_M(\alpha) = \iint_{\alpha} \left[\frac{1}{\|\alpha(s) - \alpha(t)\|^2} - \frac{1}{D^2(\alpha(s), \alpha(t))} \right] \|\alpha(s)'\| \|\alpha(t)'\| ds dt \quad (3.1)$$

where $D(\alpha(s), \alpha(t))$ is the shortest distance along α between the points $\alpha(s)$ and $\alpha(t)$.

Call E_M the Möbius Energy.

Then, in 1997, Litherland et al define the energy *thickness*, and it's reciprocal, *ropelength*, also motivated by curvature. In this chapter we analyze ropelength and state the main theorem which provides upperbounds on ropelength for certain MAP links.

3.2 Thickness and Ropelength

Question 3.2. *What happens to a piece of rope when it is “tied” into a knot and then tightened, or cinched?*

The observation we make is that the amount of rope needed to make the knot is minimized, and this is the basic idea behind *ropelength*. In the above question we imagined a real piece of physical rope. Recall that mathematicians like to think of knots as a closed loop having no thickness and never intersecting itself. So, when we are talking about rope we must remember that we are still talking about one dimensional submanifolds in \mathbb{R}^3 . Since this rope has no real thickness, we must define thickness. Let $R(K)$ be the *injective radius* of a knot, which can be considered the maximum thickness of a knot for any smooth knot K in \mathbb{R}^3 . In order to properly describe the injective radius we take quote from [LSDR99], and provide a image in Figure 3.1.

For some radius $r > 0$, construct at each point x of K a standard disk of radius r centered at x in the plane normal to K at x . For small enough r , these disks are pairwise disjoint and form a solid tube around K . [LSDR99]

The injective radius is the supremum of all r such that the tube is embedded.

Definition 3.3.

$$Rop(K) = \frac{arclength(K)}{R(K)} \quad (3.2)$$

The arclength of a given knot conformation is the arclength of the parametrization of the knot in \mathbb{R}^3 . Ropelength can be thought of as a knot energy, or a geometric measure of complexity. As an energy, it is a scale invariant numerical measure of knot complication, similar to crossing or unknotting numbers.

In [LSDR99], Litherland et al prove that the thickness of a knot K is dependent on the curvature and doubly critical points of the knot. They considered a knot to be a “ C^2 submanifold of \mathbb{R}^3 homeomorphic to S^1 ,” [LSDR99] meaning that there exists a

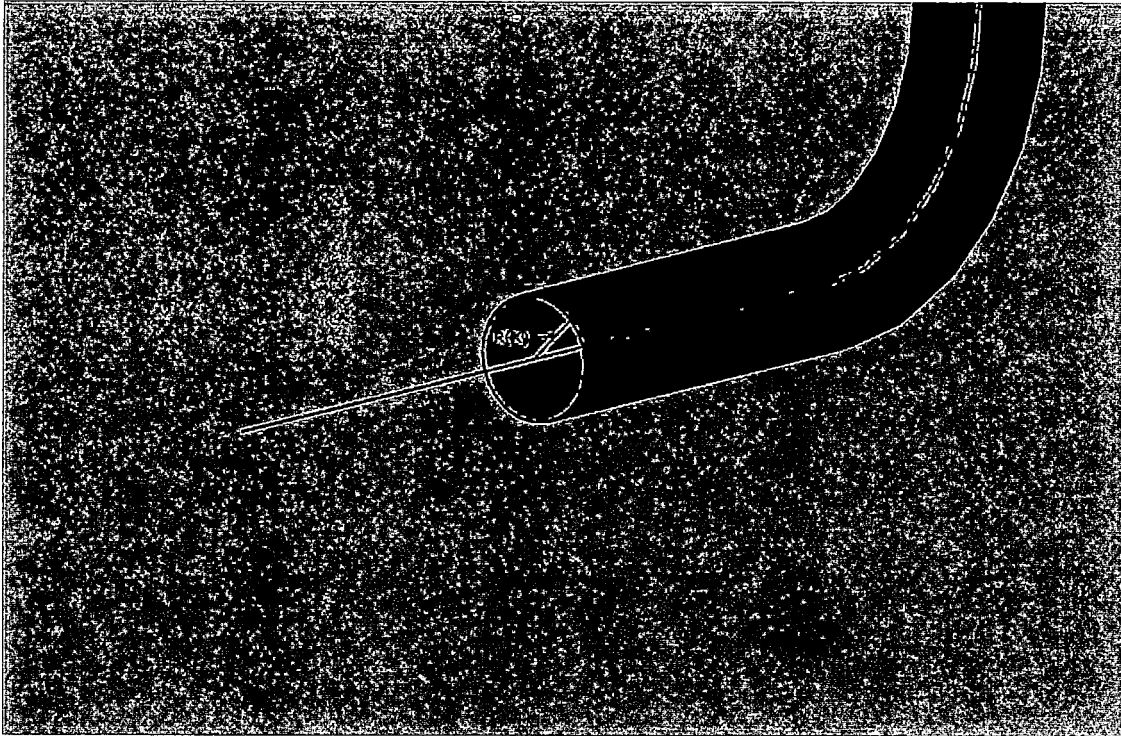


Figure 3.1: Injective Radius

C^2 parametrization of K . Let $\kappa(s)$ be the curvature of K at a point s , and let $\lambda(K)$ be the shortest doubly-critical self-distance of K , i.e. the length of the shortest chord perpendicular to the tangents of two points on K . According to the first theorem of [LSDR99]

$$R(K) = \min \left\{ \frac{1}{\max \kappa(K)}, \frac{1}{2} \min \lambda(K) \right\}. \quad (3.3)$$

For example, the curvature of the ellipse $E = (2\cos(t), \sin(t))$ is

$$\kappa(E) = \frac{2}{(4\sin^2 t + \cos^2 t)^{\frac{3}{2}}}.$$

And since the major and minor axis respectively yield

$$\begin{aligned} \kappa(0) &= \kappa(\pi) = 2 \\ \kappa\left(\frac{\pi}{2}\right) &= \kappa\left(\frac{3\pi}{2}\right) = \frac{1}{4} \\ \lambda(K) &= 2 \end{aligned}$$

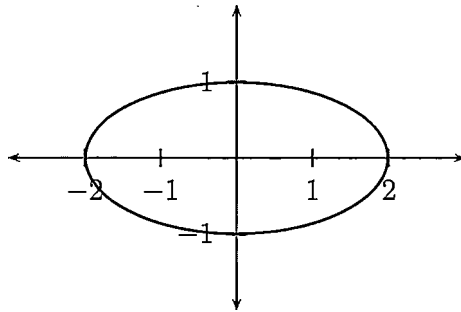


Figure 3.2: Graph of the Ellipse E

it follows that $Rop(E) = \frac{9.6884482}{\frac{1}{2}} = 19.3768964$. Figure 3.3 is a representation of the

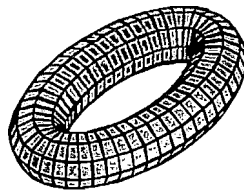


Figure 3.3: Graph of Thick Ellipse E

“thickened” ellipse.

There are three simple types of curves that have constant curvature and no doubly critical points, namely: helices, circles, and straight lines. These curves will be the centerline for the algebraic knots, and their parameterizations will be used to count arclength. In the next chapter we will show how to paste together these curves in such a way that we obtain a C^1 knot having $R(K) = 1$, and hence the ropelength is the arclength; but, first we will state the results and the main theorem.

3.3 Upperbounds

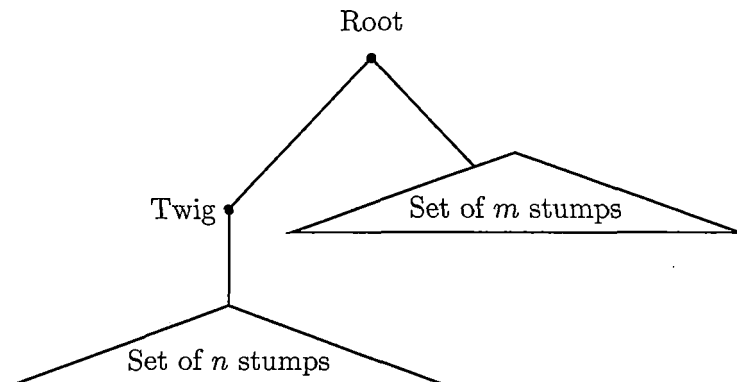


Figure 3.4: Tree with n and m Fans

The theorem that follows is based on a three dimensional representation of the knot associated with the tree in Figure 3.4. The reader can convince himself/herself that the three dimensional knot realized by a projection of an MAP knot of this form can be rotated along an axis of symmetry found vertically between the two sets of tangles as seen in Figure 5.1. These knot representations are realized using isotopy in \mathbb{R}^3 . Each integral tangle in the MAP knot K is represented by a double helix having crossing number associated with the number of twists in the double helix. Let p_1, \dots, p_n be the crossing number of the integral tangles associated with the n stumps, let q_1, \dots, q_m be the crossing number of the integral tangles associated with the m stumps. Let K be the knot associated with the above tree in Figure 3.4.

Theorem 3.4. *If K has $n = 1, 2, 3, \dots$, and $m = 2, 4, 6, \dots$, then*

$$\begin{aligned}
 \text{Rop}(K) \leq & \pi \left[2\sqrt{2} \left(\sum_{i=1}^n p_i + \sum_{i=1}^m q_i \right) + \sum_{i=2}^m |q_i - q_{i-1}| + \sum_{i=2}^n |p_i - p_{i-1}| + 3(n + m) + 5 \right] \\
 & + \left| \pi p_n + \sqrt{2} - 4m \right| + \left| \pi p_1 + \sqrt{2} - 4m \right| \\
 & + \left| \pi q_m + 1/\sqrt{2} - (2 + 4n) \right| + \left| \pi q_1 + 1/\sqrt{2} - (2 + 4n) \right| \\
 & + 0.168044(n + m - 2) + 16 + \sqrt{2}.
 \end{aligned}$$

(3.4)

Corollary 3.5. *If K has $n = 1, 2, 3, \dots$, and $m = 3, 5, 7, \dots$, then*

$$\begin{aligned}
 \text{Rop}(K) \leq & \pi \left[2\sqrt{2} \left(\sum_{i=1}^n p_i + \sum_{i=1}^m q_i \right) + \sum_{i=2}^m |q_i - q_{i-1}| + \sum_{i=2}^n |p_i - p_{i-1}| + 3(n+m) + 5 \right] \\
 & + \left| \pi p_n + \sqrt{2} - 4(m-1) \right| + \left| \pi p_1 + \sqrt{2} - 4(m-1) \right| \\
 & + \left| \pi q_m + 1/\sqrt{2} - (2+4n) \right| + \left| \pi q_1 + 1/\sqrt{2} - (2+4n) \right| \\
 & + 0.168044(n+m-2) + 16 + \sqrt{2}.
 \end{aligned}$$

Using Figure 5.1 as a reference, we can extract some useful bounds and explain why there needs to be two cases depending on whether m is even or odd. First, the even and odd cases must be considered due to the symmetry of the configuration of K . When m is even the plane that passes through the middle of the horizontal n stumps divides the vertical set of m stumps in half. And, when m is odd this plane will divide the the vertical integral tangles though the center of the middle tangle causing the horizontal pipes traveling along the horizontal n stumps to lose a distance of two in the process, since the helices cover a total distance of 4 in diameter. Also, it should be noted that the lower bound on p_1 and the q_i 's is 2. This is because if we have p_1, q_1 , and q_2 all having a crossing number of one, then we would have the trefoil.

Chapter 4

Proof of Theorem 3.4

4.1 Introduction

In this chapter we will prove Theorem 3.4 by construction. The constructions are based on one-dimensional submanifolds of \mathbb{R}^3 where the peices are the centerline of the tube with $R(K) = 1$ in the algebraic knot. The construction will be broken up into three main subsections, terminology, a series of lemmas, and total ropelength. Each subsection will address the issues of ropelength, embedding in three dimensions, and arclength. First we will look closely at the integral tangles, providing a parametrization of them by double helices. This idea originated in Moran's paper [Mor06]. Second, the strands needed to connect the integral tangles will be constructed by "piping" made of circular arcs and straight lines as one-dimensional submanifolds in \mathbb{R}^3 . Finally, we summarize with a counting of the total ropelngth.

4.2 Terminology

The purpose of this section is to make clear all of the terminology used in the proof, and to explain the positioning of the each piece in \mathbb{R}^3 . Using Figure 4.1 as a reference, we can explain all of the parts used in the construction of this knot. Recall that these thickened knots have an injective radius of one and that is in respect to a one dimensional parametrization of the knot in \mathbb{R}^3 . So, in order to explain what we mean by construction of the knot we first provide some parameterizations and name each type.

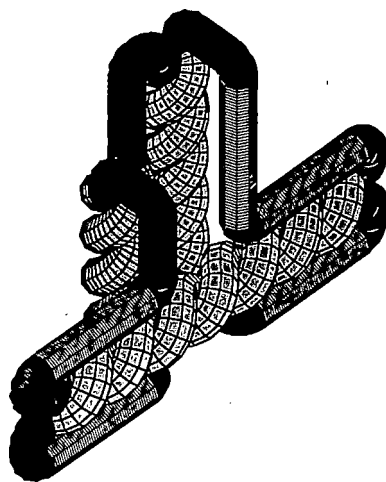


Figure 4.1: Example Knot

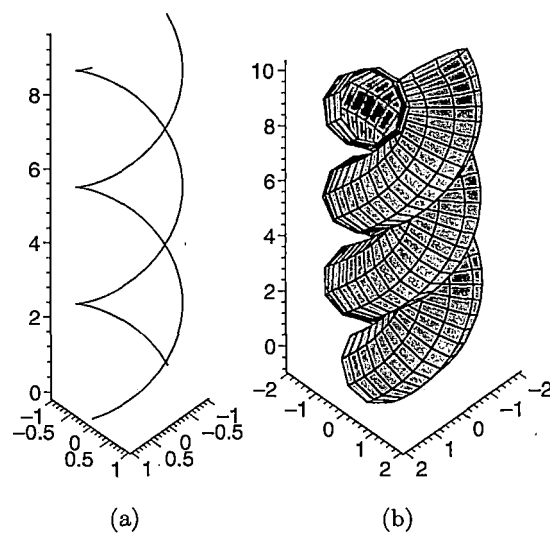


Figure 4.2: The Twists

First, consider the twists, which will be shown to represent the integral tangles. The core, or centerline, of one strand of the twists will be represented by helices having pitch 2π and radius one. This means that we will use double helices to represent the twists where one helix is rotated by π . For example the helices used in Figure 4.2 have

a parametrization of

$$H_1(t) = [\cos(3t), \sin(3t), 3t]$$

$$H_2(t) = [\cos(3t + \pi), \sin(3t + \pi), 3t]$$

where $t = 0 \dots \pi$.

Now we can explain the placement in \mathbb{R}^3 . Let the set of integral tangles associated with the set of n stumps be called P , and the other set of m stumps Q . An example of such a placement is seen in figure 4.3 where Q is the set of 4 vertical twists and P is the set of 3 horizontal twists. Then place the integral tangles in Q such that the cross

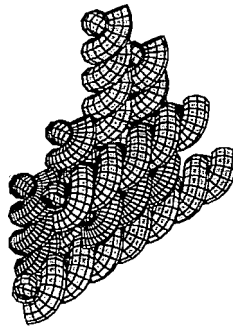


Figure 4.3: Placement of P, Q Helices

section of the helices which intersects the xy -plane will have this image seen in Figure 4.4. Each Q_i is a double helix with q_i twists and the dotted line represents the starting

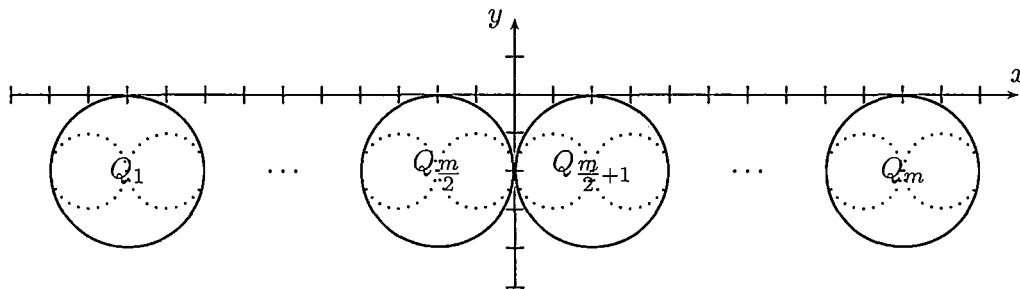


Figure 4.4: Placement of Q Helices

position of a given helix having radius one and pitch 2π . So, one can imagine that each integral tangle is surrounded by a tube of radius 2 which is a cross-section represented

by solid circles. It is important to note that each integral tangle is placed so that these tubes do not intersect, i.e. the integral tangles do not intersect. The integral tangles associated with the n stumps will be stacked horizontally along the yz -plane next to the tangles just described. The middle of each tangle will intersect the yz -plane, meaning that the tangles have a symmetry through the yz -plane. A cross section of the yz -plane would yield the image seen in Figure 4.5. Each P_i is a double helix with P_i twists and

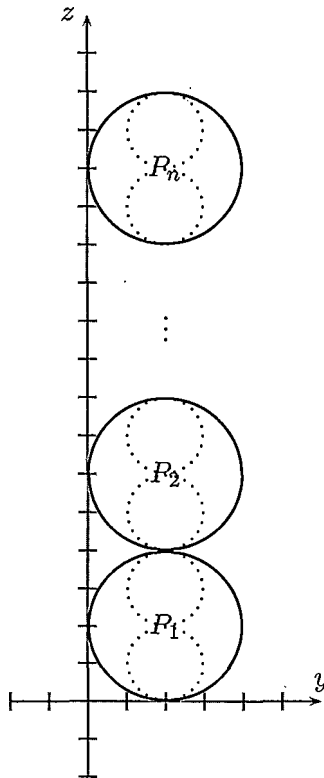


Figure 4.5: Placement of P Helices

the dotted line represents the starting position of a given helix having radius one and pitch 2π . After obtaining the arclength of each helix, each helix must be connected in such a way that respects the structure of the arborescent MAP knot or link. This will be done by way of *piping*.

Piping is the structure that “connects” P and Q together. This consists of one-eighth circles, quarter circles and straight lines all still being one-dimensional sub-manifolds in \mathbb{R}^3 , where they will all be connected piecewise in such a way that the knot,

i.e. the centerline, is continuous. The one-eighth circles are needed to flatten off each of the helices because each helix begins and ends at a 45° with its starting plane. An example of such a flattening of a helix can be found in Figure 4.6. For example, consider the helix with parametrization $H(t) = (\cos t, \sin t, t)$ having an injective radius of one. Then, $H'(t) = (-\sin t, \cos t, 1)$. Then the unit tangent plane can be found by

$$\begin{aligned}(x, y, z) \cdot (-\sin t, \cos t, 1) &= 0 \\ -x\sin t + y\cos t + z &= 0.\end{aligned}$$

So, at $t = 0$ we have $y = -z$ and at $t = \pi$ we have $y = z$. Thus, the helix meets the plane at 45° , and adding an eighth circle flattens out the tangent plane making it easier to connect the needed quarter circles.

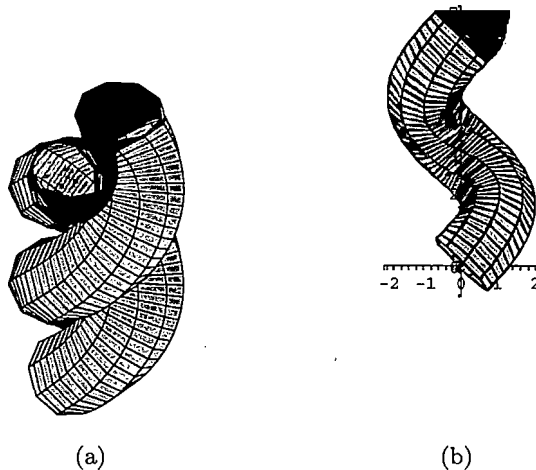


Figure 4.6: One-Eighth Circle

After flattening the helices, all adjacent helices will need to be of the same height in order to connect them together. This will require *height connectors*. These height connectors will simply be straight lines having an injective radius of one extending from one flattened helices to the height of another. An example of such a construction is found in subfigure 4.7(a). Once the adjacent helices are the same height they need to be

connected by quarter circles with radius one and small *connector pipes*. The subfigure 4.7(b) is a close-up of the quarter circles connecting the two adjacent helices in subfigure 4.7(a) where the connector pipe has been removed in order to emphasize where the connector pipes are needed. The quarter circles considered here are used to connect

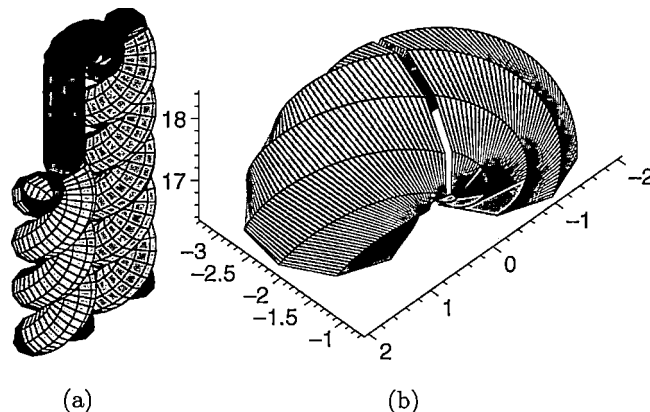


Figure 4.7: Quarter Circles

each adjacent helix. Also, it will be shown that there are ten more needed to connect P and Q together. It should be noted that the connector pipes are straight line segments connecting the two quarter circles together. This is small yet significant distance. Also, the centerline images do not emphasize the distance nearly as well as the tubular images do, and remember the knot is the centerline. Having described how the adjacent helices are connected, we now need to connect P and Q together.

Using Figure 4.8 as a reference, we can now describe how P and Q connect together. First notice that the bottom helix in P needs a small piece of straight pipe of length $1/\sqrt{2}$ because the helices in Q that originally began at $z = 0$ now extend down to $-1/\sqrt{2}$ after they were flattened. This is because a quarter circle must travel $1/\sqrt{2}$ in the z direction. Hence we need two small *extenders* to extend down from the quarter circles coming off of the bottom of P_1 to meet the lower parts of Q_1 and Q_m . After these extenders we need quarter circles to come around to the bottom of P_1 , and then straight pipes so that the length of P_1 is extended to the width of Q . Also, these straight pipes will be used on top of P for the same reason. We will call these pipes *horizontal straight pipes*. These pipes will be parallel to P . Then, two quarter circles are needed to send

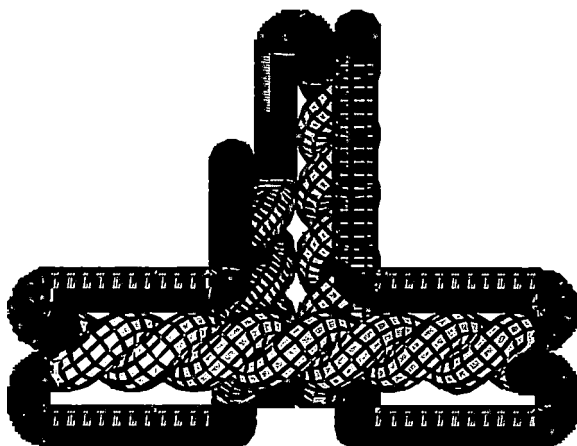


Figure 4.8: Frontview Example

the pipe up or over to Q , depending on whether the width of P exceeds the height of Q . At this point it is necessary to use *vertical straight pipes* to extend to the height of Q . Finally we need *pipes passing through the plane* that separates P and Q in order to connect them together. Now that we have a basis on terminology we can proceed to prove that this parametrization satisfies the properties of ropelength.

4.3 Lemmas

Here we will show through a series of lemmas that each piece in this parametrization satisfies the following three main properties that are a result of the theorem on ropelength.

- I. Distances between any two pieces must be at least twice the radius.
- II. Distances between doubly critical points must be at least twice the radius.
- III. The radius of curvature must be at least the radius of the rope.

Recall that we are always working in reference to the one-dimensional submanifolds in \mathbb{R}^3 . Condition I is needed to show that the “tubes” having an injective radius one will never cross each other and will be tailored to each given piece, while II and III will always need to be shown. First we will prove the length of the small connectors. Then we will prove each of the conditions for the helices, then the one-eighth circles, the quarter circles, and finally all of the straight pipes.

Lemma 4.1. *The distance between the connecting quarter circles is*

$$2(\sqrt{\frac{5}{2} - \sqrt{2}} - 1) \approx 0.084022$$

Proof. A cross section passing through the center of a helix being flattened can be seen in Figure 4.9 where $\alpha = 1 - 1/\sqrt{2}$. Also, a top down projection of the perpendicular

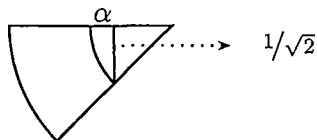


Figure 4.9: Cross Section of Flattening

plane, taken from subfigure 4.7(b), can be seen in Figure 4.10 where this image contains

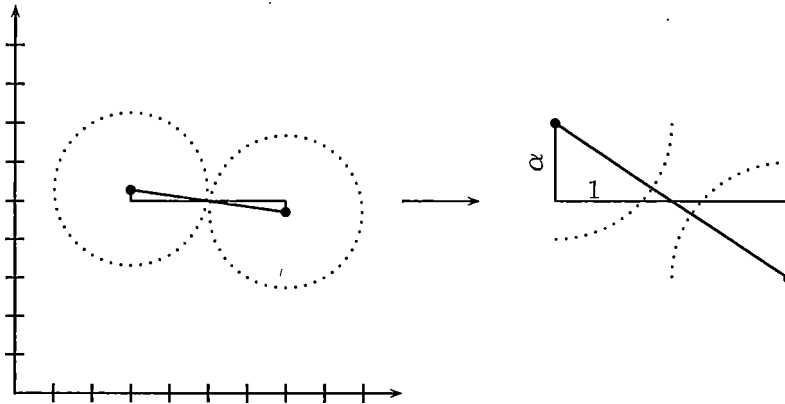


Figure 4.10: Projection of Connectors with Close-up

a close up of the distance to be calculated, which is the distance between the two dotted circles. Hence this distance is

$$2(\sqrt{1^2 - \alpha^2} - 1) = 2(\sqrt{\frac{5}{2} - \sqrt{2}} - 1).$$

□

Lemma 4.2. *The conditions I, II, and III are satisfied by a double helix with helices having radius one and pitch one.*

Proof. Let $H_1(t) = (\cos(t), \sin(t), t)$ be one helix and $H_2(t) = (\cos(t + \pi), \sin(t + \pi), t)$ be the second helix of the double helix. First, it is clear that the minimal distance between any two points on the helix is 2, because if $H_2(0) = (-1, 0, 0)$ we have that the minimal

distance between $H_1(t)$ and $H_2(0)$ is $d(t) = \sqrt{(\cos t + 1)^2 + (\sin t)^2 + t^2}$. Then,

$$\begin{aligned}
 (d(t)^2)' &= 2(\cos t + 1)(-\sin t) + 2\sin t \cos t + 2t \\
 &= -2\sin t + 2t \\
 &= 2(t - \sin t) \\
 &= 0 \\
 &\Rightarrow \\
 t &= \sin t \\
 &\Rightarrow \\
 t &= 0
 \end{aligned}$$

Second, there are no doubly critical points on the helix. Let $t = 0$. Then $H_1'(0) = (0, 1, 1)$ and we must find a chord orthogonal to both $H_1'(0)$ and some other tangent vector $H_1'(t)$. Let $\vec{v} = H_1(0)$ and $\vec{w} = H_1(t)$, and $\vec{v}' = H_1'(0)$ and $\vec{w}' = H_1'(t)$. Then,

$$(\vec{v} - \vec{w}) \cdot \vec{v}' = 0 \Rightarrow \sin t + t = 0 \Rightarrow t = 0.$$

Hence the only vector tangent to the chord is the original vector, and there are no doubly critical points. The reader can easily show that the curvature of a helix having radius r and pitch α is given by

$$\kappa(t) = \frac{r}{r^2 + \alpha^2}.$$

Finally, note that the curvature of a helix having radius one and pitch one is given by

$$\kappa(t) = \frac{1}{1^2 + 1^2} = \frac{1}{2}.$$

Thus $\frac{1}{\kappa(t)} = 2$ and the final condition is satisfied. Therefore conditions I, II, and III are satisfied. \square

Lemma 4.3. *The conditions I, II, and III are satisfied by the one-eighth circles.*

Proof. Conditions II and III still hold since circles have no doubly critical points and their radius of curvature is one. Condition I holds because the circles will be placed on their respective helices so that they match up as seen in Figure 4.6, and the helices have already been shown to be a distance of at least twice the radius away. Therefore the one-eighth circles satisfy the conditions on ropelength. \square

Lemma 4.4. *The conditions I, II, and III are satisfied by the quarter circles.*

Proof. Again conditions II and III still hold since circles have no doubly critical points and their radius of curvature is one. Also, locally the quarter circles are going to satisfy condition I. First consider the quarter circles used to connect adjacent helices. They are placed on top of pieces that already a distance of 2 away from each other, namely the one-eighth circles or the height connectors. Clearly the 10 quarter circles needed to finish the connections will always satisfy condition I. Therefore the quarter circles satisfy the conditions on ropelength. \square

Lemma 4.5. *The conditions I, II, and III are satisfied by the connectors.*

Proof. Conditions II and III still hold since straight lines have no doubly critical points and no radius of curvature. By subfigure 4.7(b) the connectors will be at least twice the radius away from any other local piece. Therefore the connectors satisfy the conditions on ropelength. \square

Lemma 4.6. *The conditions I, II, and III are satisfied by the both the straight horizontal and vertical pipes, and pipe passing through the plane.*

Proof. Conditions II and III still hold since straight lines have no doubly critical points and infinite radius of curvature. Because P and Q do not intersect, the pipe passing through the plane will always be at least twice the radius away from any other local piece. Therefore the pipe passing through the plane satisfy the conditions on ropelength. \square

Due to these lemmas, each piece of the parametrization satisfies the conditions on ropelength we have proved that the algebraic knot described in Theorem 3.4 satisfies these conditions as well. Recall that the sum of the arclength in this case is the ropelength. Hence, we will next count the arclength of each piece and sum them up in order to prove the upperbounds.

4.4 Total Ropelength

Now that we have all of the necessary pieces, we can add up the total arclength. Recall that the arclength is the ropelength since the $R(K) = 1$. The table in Figure 4.11

Arclength	Piece
$2\sqrt{2}\pi(\sum_{i=1}^n p_i + \sum_{i=1}^m q_i)$	Helices
$\pi(n + m)$	1/8 Circles
$\pi(2(n + m) + 5)$	1/4 Circles
$0.168044(n + m - 2)$	Connectors
$\pi[\sum_{i=2}^m q_i - q_{i-1} + \sum_{i=2}^n p_i - p_{i-1}]$	Height Connectors
$ \pi p_n + \sqrt{2} - 4m + \pi p_n + \sqrt{2} - 4m $	Straight Horizontal Pipe
$ \pi q_m + 1/\sqrt{2} - (2 + 4n) + \pi q_1 + 1/\sqrt{2} - (2 + 4n) $	Straight Vertical Pipe
$\sqrt{2}$	Extenders
16	Pipe Passing through the Plane

Figure 4.11: Total Ropelength

is a summary of the arclengths of each piece. The following are arguments for counting the arclength of each piece.

First the arclength of one twist of a single helix is

$$l = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} dt = \sqrt{2}\pi.$$

This implies the length of a double helix having p twist is $l_p = 2\sqrt{2}\pi p$. Thus the arclength of the set of helices in both P and Q is

$$l_{PQ} = 2\sqrt{2}\pi\left(\sum_{i=1}^n p_i + \sum_{i=1}^m q_i\right).$$

The arclength of a one-eighth circle is $1/8(2\pi) = \pi/4$. Since there are a total of $4n + 4m$ one-eighth circles needed, two for each end of a double helix, we must have a total arclength of $\pi(n + m)$ for all. There are a total of $4(n + m) + 10$ quarter circles needed, 4 for each double helix and 10 more for the fan connections. The total arclength of the quarter circles is

$$1/4(2\pi)(4(n + m) + 10) = \pi(2(n + m) + 5).$$

The total arclength of the Height Connectors in Q is

$$\sum_{i=2}^m \pi |q_i - q_{i-1}|$$

and P is

$$\sum_{i=2}^n \pi |p_i - p_{i-1}|.$$

Hence, the total arclength of the height connectors in both P and Q is

$$\pi \left[\sum_{i=2}^m |q_i - q_{i-1}| + \sum_{i=2}^n |p_i - p_{i-1}| \right].$$

The total arclength for the small connector pipes is

$$2(1 - \sqrt{\frac{5}{2} - \sqrt{2}})(2(n + m) - 4) = 4(1 - \sqrt{\frac{5}{2} - \sqrt{2}})(n + m - 2) \approx 0.168044(n + m - 2).$$

To verify the length of the horizontal and vertical pipes we shall count each of them individually in reference to Figure 4.8. The total arclength of the Straight Horizontal Pipe along the bottom is

$$2 |(\pi p_1/2 + 1/\sqrt{2}) - 4m/2| = |\pi p_1 + \sqrt{2} - 4m|.$$

And The total arclength of the Straight Horizontal Pipe along the top is

$$2 |(\pi p_n/2 + 1/\sqrt{2}) - 4m/2| = |\pi p_n + \sqrt{2} - 4m|.$$

This is based on the fact that the plane passing through the middle of P splits Q in half. So, the length of half of P_1 is $\pi p_1/2$. And we have $+1/\sqrt{2}$ because of the one-eighth circle needed to flatten off P_1 . And there is a $-4m/2$ since the pipe needs to only go as far as Q_1 . Similarly for the top. Hence the total arclength contributed from top and bottom is

$$|\pi p_n + \sqrt{2} - 4m| + |\pi p_1 + \sqrt{2} - 4m|.$$

The total arclength of the Straight Vertical Pipe is

$$|(\pi q_1 + 1/\sqrt{2}) - (2 + 4n)| + |(\pi q_n + 1/\sqrt{2}) - (2 + 4n)|.$$

First the $\pi q_1 + 1/\sqrt{2}$ comes from the height of Q_1 and the one-eighth circle, and the $-(2 + 4n)$ is a result of the height of P . Similarly for the pipe opposite this one.

Finally the two extenders have a length of $1/\sqrt{2} + 1/\sqrt{2} = \sqrt{2}$. And the 4 pipes that pass through the plane separating P and Q must have a total length of

$$4[(1 + 1/\sqrt{2}) + (3 - 1/\sqrt{2})] = 4(4) = 16.$$

Now add it all up. Let T be the total arclength of this knot configuration. Then

$$\begin{aligned}
T &= \pi \left[2\sqrt{2} \left(\sum_{i=1}^n p_i + \sum_{i=1}^m q_i \right) + \sum_{i=2}^m |q_i - q_{i-1}| + \sum_{i=2}^n |p_i - p_{i-1}| + 3(n+m) + 5 \right] \\
&\quad + \left| \pi p_n + \sqrt{2} - 4m \right| + \left| \pi p_1 + \sqrt{2} - 4m \right| \\
&\quad + \left| \pi q_m + 1/\sqrt{2} - (2+4n) \right| + \left| \pi q_1 + 1/\sqrt{2} - (2+4n) \right| \\
&\quad + 0.168044(n+m-2) + 16 + \sqrt{2}
\end{aligned}$$

Therefore there exists an upper bound on K which is dependent on the number of tangles and there corresponding crossing numbers, and it is represented as

$$Rop(K) \leq T \quad \square$$

Corollary 4.7. *The ropelength of one of these links grows linearly with crossing number.*

Proof. In order to prove this, we must write the inequality 3.4 in the form

$$Rop(K) \leq \lambda C(K) + \mu,$$

since crossing number is linear. First notice that the twist number is less than the crossing number.

$$m + n = T(K) \leq C(K). \quad (4.1)$$

Also, the crossing number of an individual twist is less than that of the crossing number of the knot, and

$$\sum_{i=2}^n |p_i - p_{i-1}| \leq C(K) \quad (4.2)$$

$$\sum_{i=2}^m |q_i - q_{i-1}| \leq C(K). \quad (4.3)$$

Consequently,

$$\begin{aligned}
\left| \pi p_n + \sqrt{2} - 4(m-1) \right| &\leq \pi p_n + \sqrt{2} + 4(m-1) \\
&\leq \pi C(K) + \sqrt{2} + 4C(K) \\
&\leq (\pi + 4)C(K) + \sqrt{2}.
\end{aligned} \quad (4.4)$$

Similarly,

$$\left| \pi p_n + \sqrt{2} - 4(m-1) \right| \leq (\pi + 4)C(K) + \sqrt{2} \quad (4.5)$$

$$\left| \pi p_1 + \sqrt{2} - 4(m-1) \right| \leq (\pi + 4)C(K) + \sqrt{2} \quad (4.6)$$

$$|\pi q_1 + 1/\sqrt{2} - (2 + 4n)| \leq (\pi + 1)C(K) + 1/\sqrt{2} \quad (4.7)$$

$$|\pi q_m + 1/\sqrt{2} - (2 + 4n)| \leq (\pi + 1)C(K) + 1/\sqrt{2} \quad (4.8)$$

The inequality 3.4 can be written in terms of crossing number and twist number.

$$\begin{aligned} Rop(K) &\leq 2\sqrt{2}\pi C(K) + 3.168044T(K) + 32.78608883 \\ &\quad + \pi \left[\sum_{i=2}^m |q_i - q_{i-1}| + \sum_{i=2}^n |p_i - p_{i-1}| \right] \\ &\quad + \left| \pi p_n + \sqrt{2} - 4(m-1) \right| + \left| \pi p_1 + \sqrt{2} - 4(m-1) \right| \\ &\quad + |\pi q_m + 1/\sqrt{2} - (2 + 4n)| + |\pi q_1 + 1/\sqrt{2} - (2 + 4n)|. \end{aligned}$$

Then using the above inequalities

$$\begin{aligned} Rop(K) &\leq 2\sqrt{2}\pi C(K) + 3.168044C(K) + 32.78608883 \\ &\quad + \pi [2C(K)] \\ &\quad + (\pi + 4)C(K) + \sqrt{2} + (\pi + 4)C(K) + \sqrt{2} \\ &\quad + (\pi + 1)C(K) + 1/\sqrt{2} + (\pi + 1)C(K) + 1/\sqrt{2}. \end{aligned}$$

Thus

$$Rop(K) \leq (2\sqrt{2}\pi + 5\pi + 13.168044)C(K) + 32.78608883 + 3\sqrt{2}.$$

Therefore we have written the inequality 3.4 of the form $Rop(K) \leq \lambda C(K) + \mu$, and the ropelength of one of these links grows linearly with crossing number \square

Chapter 5

Examples

5.1 Example 1

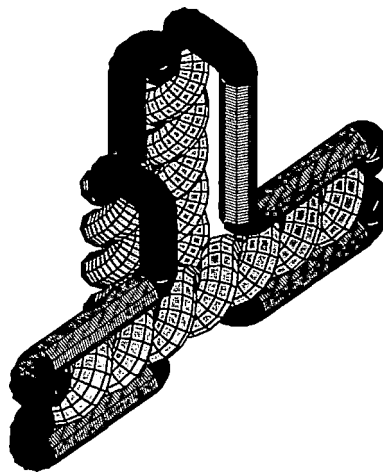


Figure 5.1: Example 1

Here is an example of a configuration based on Figure 5.1. Let $|\bullet|$ represent the crossing number of a tangle. Let the knot K be the knot associated with $Q = \{q_1, q_2\}$ and $P = \{p_1\}$ where $|p_1| = 7$, $|q_1| = 3$, and $|q_2| = 5$, and figure 5.1 is the three dimensional representation of that knot. Then,

$$\begin{aligned}
\text{Rop}(K) &\leq \pi \left[2\sqrt{2}(7 + 3 + 5) + 2 + 0 + 3(2 + 1) + 5 \right] \\
&\quad + \left| 7\pi + \sqrt{2} - 4 \times 2 \right| + \left| 7\pi + \sqrt{2} - 4 \times 2 \right| \\
&\quad + \left| 5\pi + 1/\sqrt{2} - (2 + 4 \times 1) \right| + \left| \pi 3 + 1/\sqrt{2} - (2 + 4 \times 1) \right| \\
&\quad + 0.168044(1 + 2 - 2) + 16 + \sqrt{2} \\
&= \pi \left[30\pi\sqrt{2} + 16 \right] + 2 \times \left| 7\pi + \sqrt{2} - 8 \right| \\
&\quad + \left| 5\pi + 1/\sqrt{2} - 6 \right| + \left| \pi 3 + 1/\sqrt{2} - 6 \right| + 0.168044 + 16 + \sqrt{2} \\
&\approx 252.6238529
\end{aligned}$$

$$\Rightarrow \text{Rop}(K) \leq 252.6238529$$

5.2 Example 2

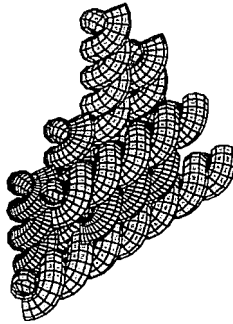


Figure 5.2: Example 2

The next example will be based in the tangle placement found in Figure 5.2. This example has 7 tangles, 4 vertical and 3 horizontal. This example helps illustrate what information is needed to calculate an upper bound on ropelength. The image shown has no piping to make it easier to visualize. We only need to know the tangle placement and their values.

$$\begin{array}{rcl}
|p_1| & = & 7 \\
|p_2| & = & 3 \\
|p_3| & = & 5
\end{array}
\quad
\begin{array}{rcl}
|q_1| & = & 3 \\
|q_2| & = & 4 \\
|q_3| & = & 7 \\
|q_4| & = & 5
\end{array}$$

We have the needed information and hence the calculation is as follows:

$$\begin{aligned}
\text{Rop}(K) &\leq \pi \left[2\sqrt{2}(7 + 3 + 5 + 3 + 4 + 7 + 5) + 1 + 3 + 2 + 4 + 2 + 21 + 5 \right] \\
&\quad + \left| 5\pi + \sqrt{2} - 4 \times 4 \right| + \left| 7\pi + \sqrt{2} - 4 \times 4 \right| \\
&\quad + \left| 5\pi + 1/\sqrt{2} - (2 + 4 \times 3) \right| + \left| \pi 3 + 1/\sqrt{2} - (2 + 4 \times 3) \right| \\
&\quad + 0.168044(3 + 4 - 2) + 16 + \sqrt{2} \\
&\approx 453.8055205
\end{aligned}$$

$$\Rightarrow \text{Rop}(K) \leq 453.8055205$$

Notice that Example 5.1 has a lower bound than Example 5.2 as it should since there are more tangles in Example 5.2. Finally in the next section we will compare these examples to known upper bounds.

Chapter 6

Conclusion

The upper bounds described in this paper are by no means the lowest; however, they do improve on Cantarella et al found in [CFM04]. Their upperbounds on the examples above are $Rop(K) \leq 491.0900000$ and $Rop(K) \leq 2164.040000$. Whereas these new upper bounds, $Rop(K) \leq 252.6238529$ and $Rop(K) \leq 453.8055205$, greatly improve on these numbers. The new upperbounds may also be improved on and the reader is encouraged to consider other configurations of this knot class to help minimize ropelength. One way to minimize ropelength is to have the piping connect directly from helix to helix by pipes of constant curvature, such as pieces of circles, or straight pipe that goes from P_n to Q_1 so that it does not ride along the top of the horizontal fan. Another very interesting question is what happens to the curvature of the horizontal helices as they wrap around the other vertical helices. In other words, what happens to the curvature of a double helix, having helices with radius 1 and pitch 1, as you wrap it around a tube of radius r . This is similar to the supercoiling idea by Rawdon in [ref]. Finally, what pitch will, smaller than 1, will minimize the arclength of a double helix. This will surely help minimize the upper bounds. Although or techniques are not optimal, we have constructed an arborescent knot with minimized ropelength in respect to known upper bounds.

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