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Tutte polynomial in knot theory

David Alan Petersen

TUTTE POLYNOMIAL IN KNOT THEORY

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

In this paper I will give a brief history of knot theory. Then I will give an introduction to knot theory with an emphasis on a diagrammatic approach to studying knots. I will also review basic concepts and notions from graph theory. Next, I will show how these two fields are related. Particularly, given a knot diagram I will show how to associate a graph. I will discuss ambiguities in the process and how certain diagrammatic properties translate into the associated graph. In particular, I will analyze the effect of flypes on associated graphs. After introducing the 2-variable Tutte polynomial of a graph I will show that this polynomial is flype invariant. This coupled with the Tait Flyping Theorem shows that the 2-variable Tutte polynomial is invariant for alternating knots. I will also show one aspect of the Tutte polynomial and its relationship to its associated knot diagram. Specifically, I will begin investigating how to determine the number of k twists in a knot diagram from the terms of the Tutte polynomial.

ACKNOWLEDGEMENTS

I would like to acknowledge the help of Dr. Rolland Trapp. I would not have been able to get through this process without his invaluable help. I would also like to thank my brother, Aaron, for the graphic conversion program he wrote for me. Without this program I would not have been able to create the necessary graphics for this paper. I would also like to thank all those who have given me their support and encouragement especially my family. I especially would like to thank my loving wife, Kiley, for her patience and understanding. Without her support I would be truly lost.

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Chapter 1

Introduction

This paper will apply graph theoretical techniques to the study of alternating knots and links. In 1984 V. Jones introduced a very powerful technique for distinguishing knots and links known as the Jones polynomial. The Jones polynomial is a 1-variable Laurent polynomial. This "new" polynomial inspired new research and generalizations including many applications to physics and real world situations. Thistlethwaite proved that it is possible to produce a 1-variable Tutte polynomial expansion for the Jones polynomial. [7] In the case of alternating knots, the Jones polynomial is a specialization of the 2-variable Tutte polynomial of an associated graph. This paper will first show that the 2-variable Tutte polynomial without specialization is an invariant of alternating knots. This will allow us to recognize that two equivalent reduced alternating knots are isomorphic regardless of diagram. Secondly, I will begin to investigate diagrammatic properties that are captured by the Tutte polynomial. Specifically, I will initiate investigations that determine the number of k twists in a knot diagram from the terms of the Tutte polynomial.

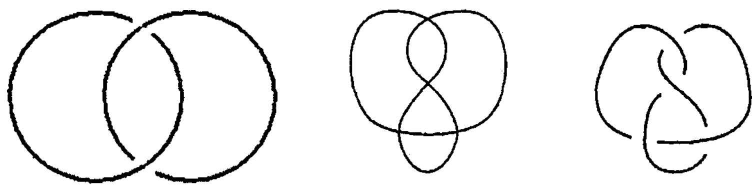
1.1 History of Knot Theory

Knot theory has been around since the late 1800's as scientists began to see the use of knots within nature. Lord Kelvin's theory of the atom stated the chemical properties of elements were related to knotting that occurs between atoms. This motivated P.G. Tait to begin to assemble a list of knots. Tait published the first set of papers describing the enumeration of knots in 1877. Tait viewed two knots as equivalent if one could be deformed, without breaking, to appear as the other. In 1928 J. Alexander described a method, the Alexander polynomial, which associated each knot with a polynomial. If one knot can be deformed into another knot they will both have the same associated polynomial. Unfortunately, the Alexander polynomial is not unique to a given knot. It is possible for more than one given knot to have the same Alexander polynomial. In 1932 K. Reidemeister developed tools that are sufficient, in theory, to distinguish almost any pair of distinct knots. Regrettably, Reidemeister's tools are too cumbersome and not practical however for large knots. The progress of topology as a mathematical field also helped in the advancements in knot theory. During the 1960's the investigation of higher dimensional knots became a very significant topic. In 1970 J. H. Conway introduced new combinatorial methods that began to lead to new invariants. In 1984, V. Jones introduced the Jones polynomial which is yet another way to distinguish knots from each other. The search for efficient ways to differentiate between knots continues today. [3]

1.2 Knot Theory

Intuitively speaking a knot is simply a knotted loop of rope. Mathematically speaking a *knot* is a simple closed curve in $\Re^3$. A *link* is the finite union of disjoint knots. The Hopf link is an example of a link that is not a knot since it is two separate loops that are hooked together (Figure 1.1 a). In particular, a knot is a link with only one component.

One common technique for studying a three dimensional knot is to look at a two dimensional projected image of the knot. The function from 3-space to the plane that takes a triple (x, y, z) to the pair (x, y) is called the *projection map.* A *projection* of a knot is its planar image resulting from the projection map (Figure 1.1 b). A *crossing* is when two strands in a projection of a knot cross each other. A *diagram* of a knot is a projection of a knot where gaps have been left to see crossings as underpasses and overpasses (Figure 1.1 c). An *underpass* of a diagram crossing is the strand that breaks while the *overpass* is the strand that remains unbroken or continuous.



(a) (b) (c)

Figure 1.1: (a)Hopf link (b)knot projection and (c) knot diagram

Knot diagrams are not unique since it is possible to draw many different diagrams of the same knot (Figures 1.2 a and b). Reidemeister proved that his three equivalence moves, known as Reidemeister moves, are an acceptable way to deform any diagram without changing the knot that it is associated with (Figure 1.3). In fact, Reidemeister proved that his three moves were the *only* moves necessary for any deformation between two diagrams of the same knot. [6] Unfortunately, there are practical limits based upon the size of the knot.

(a) (b)

Figure 1.2: (a)unknot and (b)unknot resembling trefoil

The trivial knot is known as the *unknot* and it has a diagram without any crossings (Figure 1.2 a). Any diagram of a nontrivial knot has at least three crossings. The most basic knot is the *trefoil* which has a diagram with exactly three crossings (Figure 1.4 a). A knot diagram is *alternating* if as the knot is traversed, crossings alternate between an overpass and an underpass (Figure 1.4 b).

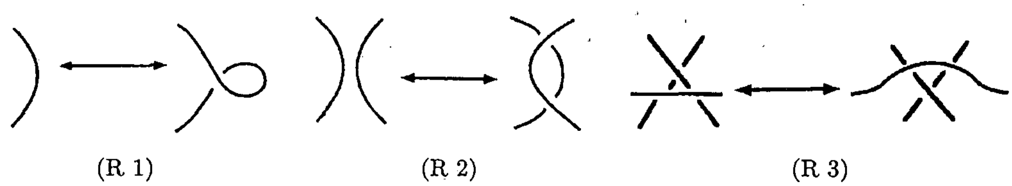


Figure 1.3: Reidemeister moves

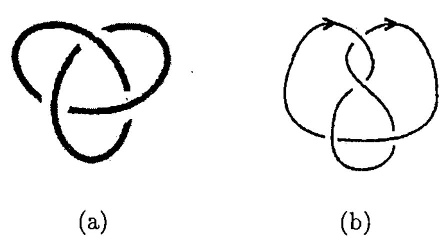


Figure 1.4: (a) trefoil and (b) alternating knot

When dealing with a knot diagram it can be useful to inspect limited portions of the diagram at a time. A *tangle* is a region of a knot diagram surrounded by a circle such that the diagram intersects with the circle exactly four times. From the intersections four arcs emerge pointing in the compass directions NW, NE, SW and SE (Figure 1.5). A diagram is considered *reduced* when there are no removable, also known as *nugatory*, crossings (Figure 1.6 a). A *flype* is a 180^o rotation of a tangle (Figure 1.6 b).

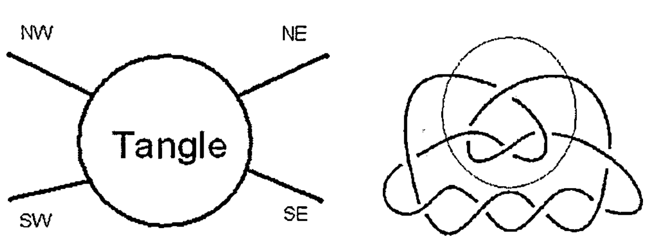


Figure 1.5: tangle

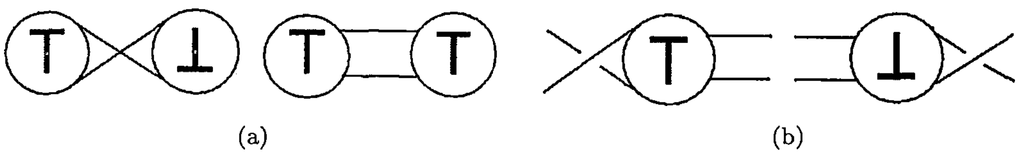


Figure 1.6: (a) nugatory crossing and (b) flype

A *knot invariant* is something that does not depend on the chosen diagram for a given knot. The *minimal crossing number* of a knot is the fewest number of crossings over all possible diagrams. Murasugi, and Thistlethwaite proved that if a knot has an alternating diagram, then all of its minimal crossing diagrams are alternating. [9] The minimal crossing number of a knot is an example of a knot invariant. Reducible crossings can be removed by twisting, and so cannot occur in a knot diagram of minimal crossing number. Other examples of knot invariants include knot colorings, the Jones polynomial and the Alexander polynomial; which will not be discussed at this time. Also, not all knots are alternating. A torus knot is one example of a knot that is non-alternating (Figure 1.7). In fact, it is conjectured that the proportion of knots which are alternating tends exponentially to zero with increasing crossing number. [9]

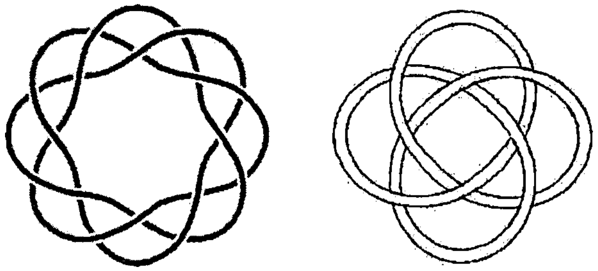


Figure 1.7: torus knots

P. Tait conjectured (known as the Tait Flype Theorem) that it was possible to transition from one diagram of a link to a different diagram of the same link by a finite number of flypes. This was later proven for alternating knots by Menasco and Thistlethwaite utilizing the Jones polynomial. [11] It is worth noting that the Tait Flype Theorem does not apply to non alternating knots such as torus knots.

Theorem 1.1. *(Tait Flype Theorem) Any two reduced alternating diagrams of a link are related by a finite number of flypes. [11]*

One last knot diagram detail necessary for this paper is the notion of a checkerboard shading. A *checkerboard shading* is when a region within the knot diagram is shaded and then each region that touches "across" at a crossing is also shaded (as in Figure 1.8). There are two choices for a checkerboard shading depending on the initial region shaded.

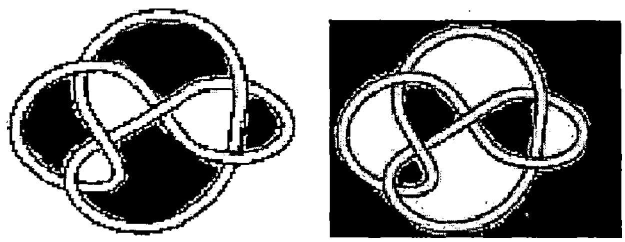
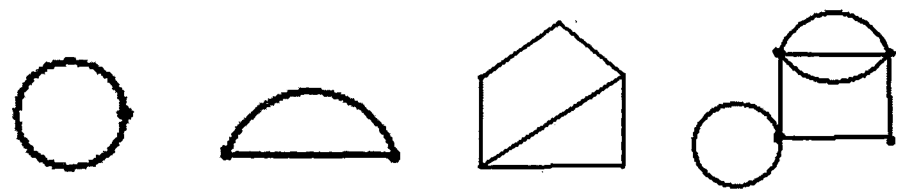


Figure 1.8: checkerboard shading

Many methods have arisen in the last one hundred years to help with the distinction of knots. This paper will focus on the Tutte polynomial as one description of how to differentiate between two alternating knots. This paper will discuss how the Tutte polynomial can be used to differentiate between knots and prove that the 2-variable Tutte polynomial is a knot invariant for any alternating knot. In particular it will be shown that all reduced alternating diagrams of a given knot will give the same Tutte polynomial as a result. Thus the 2-variable Tutte polynomial is an invariant of alternating knots.

1.3 Graph Theory

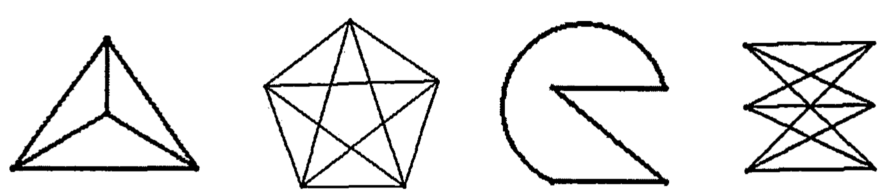
A *Graph* $G = (V, E)$ is a collection of vertices V and edges E. An edge is an unordered pair of vertices. Thus uv is the exact same edge as vu. Different aspects or properties of any graph will now be described. A *loop* is an edge that connects a single vertex to itself (Figure 1.9 a). A *multiedge* is when there exists multiple edges connecting two vertices (Figure 1.9 b). A *simple graph* is a graph that does not contain loops or multiedges (Figure 1.9 c). In a *multigraph* both loops and edges are allowed (Figure 1.9 d). In this paper the word "graph" will refer to a multigraph.



(a) (b) (c) (d)

Figure 1.9: (a) loop, (b) multiedge, (c) simple graph and (d) multigraph

A *complete graph*, K_n, is a graph with n vertices where all vertices are adjacent (as seen in figures 1.10 a and b). A *complete bipartite graph*, $K_{p,q}$ is a graph whose vertices are decomposed into two disjoint sets p and q such that no two vertices within the same set are adjacent but every pair of vertices in the two sets are adjacent (as seen in figures 1.10 c and d).[10]



(a) (b) (c) (d)

Figure 1.10: (a) K_4, (b) K_5, (c) $K_{2,2}$ and (d) $K_{3,3}$

Here are some more graph theoretical definitions. *Components* of a graph are the different disjoint pieces of the graph. A graph is *connected* if for every distinct pair of vertices $(u,\ v)$ there is a path from u to v. A connected graph consists of only one component. A *cutpoint* is a vertex that when deleted increases the number of components (Figure 1.11 a). An edge of a graph is a *bridge* if its deletion increases the number of components (Figure 1.11 b).

It is possible to study a graph by looking at subsets of it. A *subgraph* of graph G is $G' = (V',\ E')$ where $V' \subset V$ and $E' \subset E$. Every graph has at least one subgraph, which would be the graph itself. Every graph in figures 1.12 and 1.13 are subgraphs. A *spanning subgraph* is a subgraph that contains every vertex of a graph (as seen in Figures

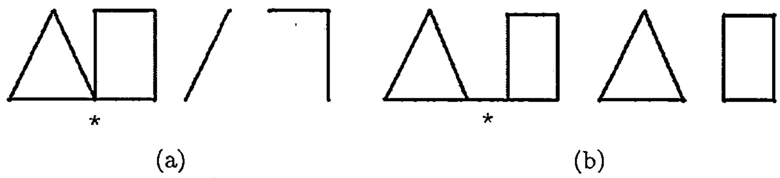


Figure 1.11: (a) cutpoint and (b) bridge

1.12 and 1.13 a, b, c and e). A *tree* is a graph in which any two vertices are connected by exactly one path. A tree has only one component and every edge is a bridge (as seen in Figures 1.12 and 1.13 c and d). A *spanning tree* is a tree that connects every vertex of the graph (as seen in Figure 1.12 c). Every spanning tree is a spanning subgraph. Notice figure 1.13 e has two components.

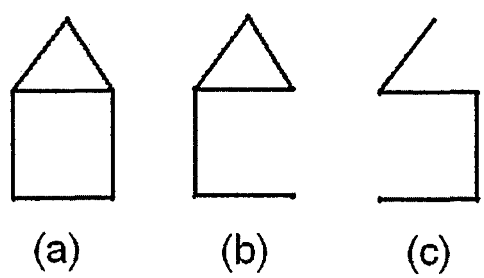


Figure 1.12: (a) graph, (b) spanning subgraph, (c) spanning tree

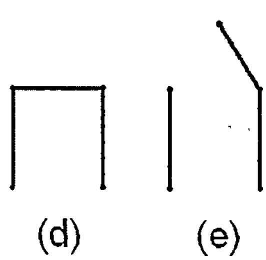


Figure 1.13: (d) tree and (e) two component subgraph

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency (Figure 1.14 a). An example of two non isomorphic graphs are K_4 and $K_{2,2}$ since not all vertices in $K_{2,2}$ are adjacent (Figure 1.14 b).

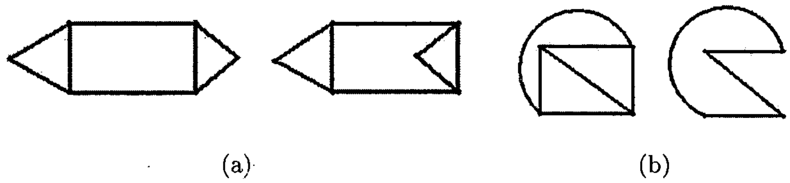


Figure 1.14: (a) different planar embeddings of the same graph (b) K_4 and $K_{2,2}$

A *planar graph* is a graph that can exist in a plane where all edges of the graph can be drawn between vertices without crossing any other edge. A planar graph can have multiple planar embeddings (Figure 1.14 a). Given a planar embedding of a graph G it's *dual graph* G^d is obtained by placing a vertex in every open region of the graph G and connecting the "new" vertices with "new" edges that cross each original edge (as seen in the Figure 1.15). Dual graphs are not unique. Different planar embeddings of the same graph can lead to different dual graphs that are not isomorphic (Figure 1.16).

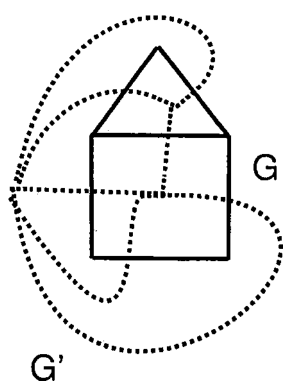


Figure 1.15: graph G and it's dual graph G'

The graph K_4 is an example of a graph with two common isomorphic planar embeddings (Figure 1.17 a). Not all graphs are planar. Both $K_{3,3}$ and K_5 are examples of non planar graphs (Figure 1.17 b and c). In fact, Kuratowski showed that any non planar graph essentially contains a copy of either $K_{3,3}$ or K_5. [12]

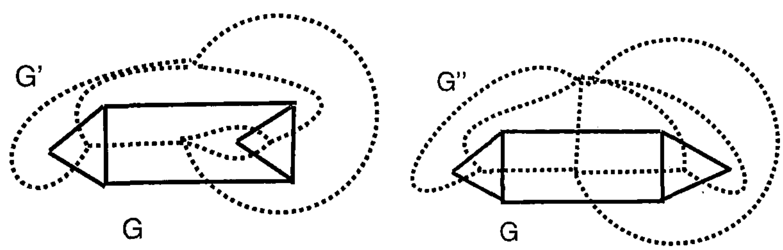


Figure 1.16: non isomorphic dual graphs associated to isomorphic graphs

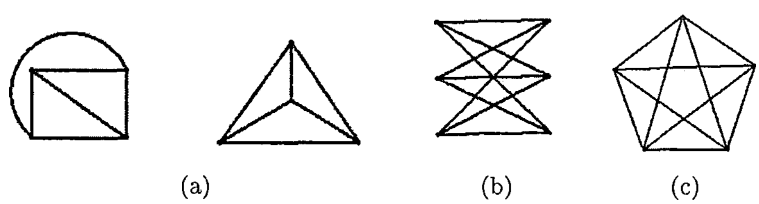


Figure 1.17: (a) different planar representations of K_4 (b) $K_{3,3}$, and (c) K_5

1.4 Knot Theory meets Graph Theory

It is possible to take any diagram of a knot and associate with it a planar embedding of a graph. This association of a knot diagram to a planar graph is accomplished in the following way. First, create a checkerboard shading of a knot diagram. Second, put a vertex in the center of each shaded region. Finally, connect vertices (regions) by edges through each crossing of the knot diagram so that the two shaded regions are connected via an edge (Figure 1.18).

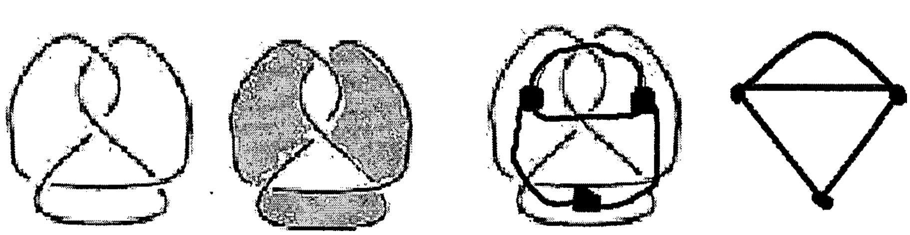


Figure 1.18: stages of knot diagram to planar graph association

Any given planar knot diagram will give two distinct planar graphs depending on the checkerboard shading chosen. These graphs are dual to each other (Figure 1.19).

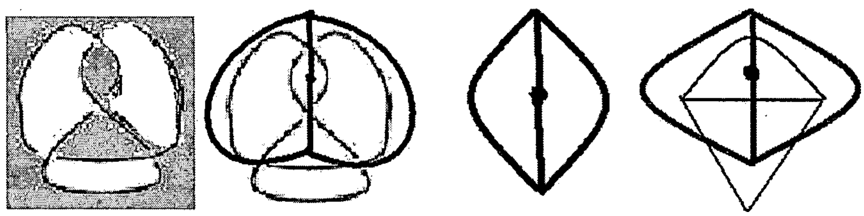


Figure 1.19: the other checkerboard shading results in the dual graph

Thus given any knot diagram, we can construct an associated planar graph. However, it is not possible to take any planar graph and find it's associated link diagram without some extra details. In a knot's associated graph each edge represents a crossing. Each crossing must be considered either a positive or negative crossing based on the type of crossing (as seen in Figure 1.20 a and b). Label each edge as positive or negative according to the sign of the associated crossing. This graph obtained is called a *signed graph* since each edge represents either a positive or negative crossing (Figure 1.21). [1]

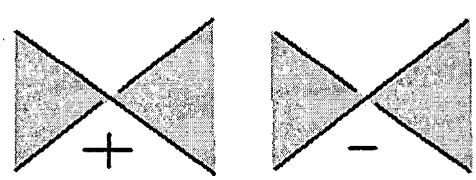
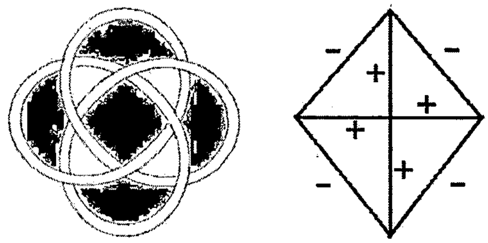


Figure 1.20: signed crossings

Any link diagram can be recreated from a signed planar graph. This is accomplished by marking the appropriate crossing on each signed edge and then connecting each strand as in the figure 1.22.

In an alternating diagram the crossings are either all positive or all negative based on the chosen shading (Figure 1.23). Based on this fact, it is not necessary to use signed graphs when dealing with alternating diagrams. It is worth noting that any reduced graph G will be loopless since loops in a graph correspond to reducible



(a) (b)

Figure 1.21: (a) torus knot shading and (b) torus knot signed graph

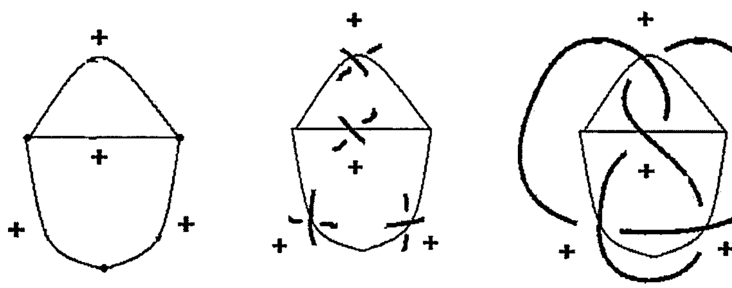


Figure 1.22: signed graph to link diagram

crossings, specifically Reidemeister 1 moves, in the knot diagram. Similarly, any reduced graph G will be bridgeless since bridges correspond to nugatory crossings in a diagram. We will restrict our attention to reduced and alternating knot diagrams in the next section consequently, all graphs will be unsigned, loopless, bridgeless and all edges will be assumed to be positive. Thus any planar embedding of a graph gives rise to a unique link diagram.

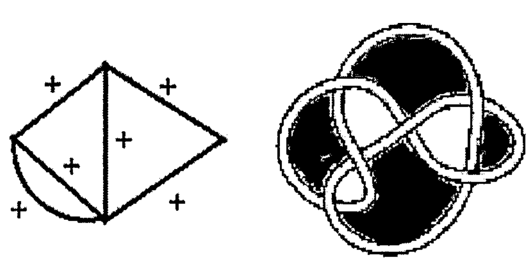


Figure 1.23: signed graph and it's associated alternating knot diagram

We now want to investigate how graphs associated with knot diagrams change under flypes. Given any knot K then there exists a knot diagram D that is associated with K and there exists a graph G that is associated with diagram D. Suppose that D has a tangle with a crossing. The results of a flype of the tangle are the following diagrams D and D_f (Figure 1.24).

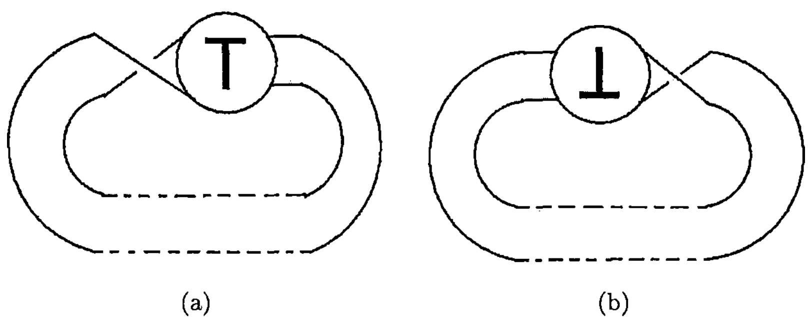


Figure 1.24: (a) diagram D and (b) diagram D_f

The associated graphs of D and D_f would look like the following graphs G and G_f (Figure 1.25). The graph of a tangle and it's rotated related tangle after a flype are isomorphic to each other since the adjacency of the edges is maintained (Figure 1.26). However, when inspecting the knot diagrams as a whole, the two associated graphs (pre-flype and post-flype) don't have to be isomorphic. In fact, the flype of the tangle results in a "surgery" where the associated graph piece is cut, flipped, and then reattached as seen in figures 1.27 and 1.28.

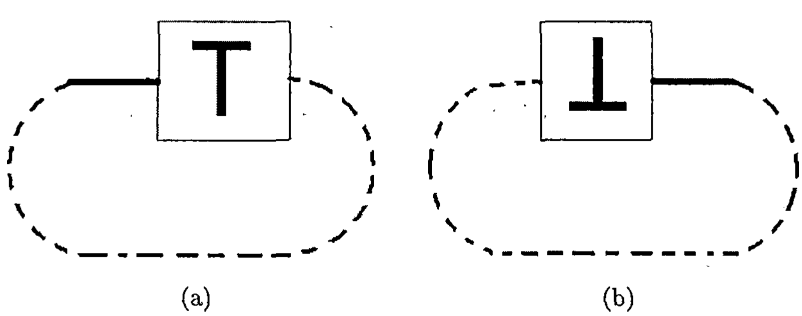


Figure 1.25: (a) graph of G and (b) graph of G_f

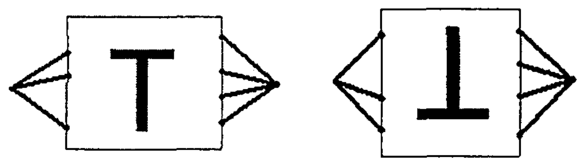


Figure 1.26: example of the graph (a) before the flype and (b) after the flype

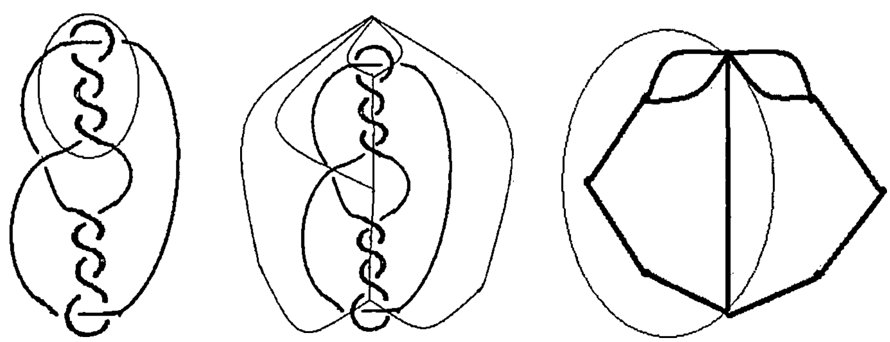


Figure 1.27: preflype knot diagram to graph

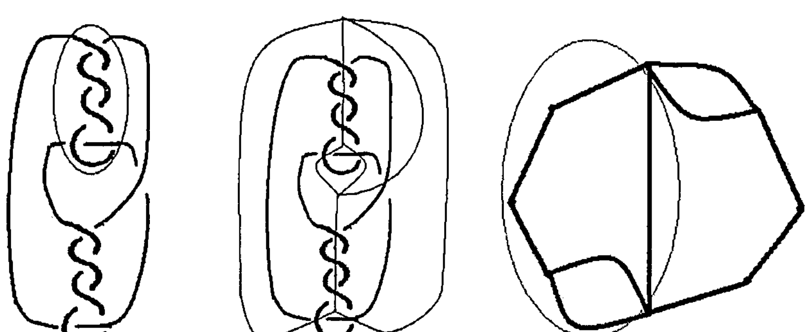


Figure 1.28: post flype knot diagram to graph

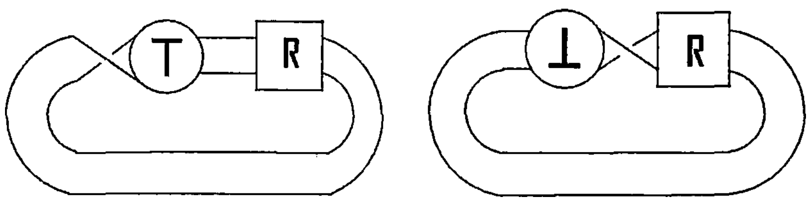


Figure 1.29: diagram of any alternating knot written as two tangles T and R

For this paper when a diagram is written as two tangles, tangle T will be the tangle to be flyped and tangle R will be for the rest of the knot (Figure 1.29).

Another relationship between knot diagrams and their associated graphs has to do with twists which are a major part of any knot diagram. It is easy to see that twists in a knot diagram will correspond to either a "chain" of edges or a multiedge with the number of edges or *multiplicity* of edges (in its associated graph) equal to the number of twists. Whether the graph has a chain or a multiedge depends on the initial checkerboard shading chosen. For example, a four twist tangle will result in a tree consisting of four edges or a multiedge with multiplicity of four (as seen in Figure 1.30 and 1.31). Later we shall see that the Tutte polynomial captures some of the twists in a diagram based on the checkerboard shading chosen.

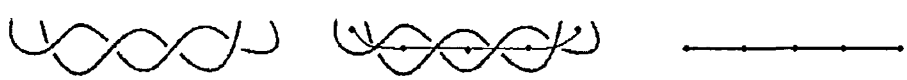


Figure 1.30: twists in a diagram correspond to bridges in a graph

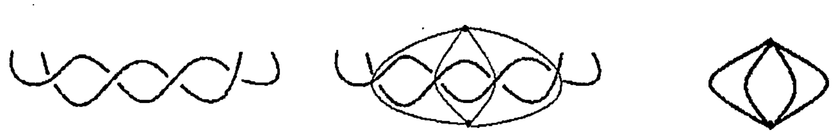


Figure 1.31: twists in a diagram correspond to multiedges in a graph

Chapter 2

Tutte Polynomial

In 1975 W. T. Tutte introduced what he called the Dichromatic polynomial (now known as the Tutte polynomial). The Tutte polynomial $T_G(x, y) = T_G$ of a graph can be defined inductively by deleting (cutting) and fusing (contracting) edges. Let $G = (V, E)$ be a multigraph where E is the set of all edges in G and $e \in E$. The deleting operation is given by taking $G - e = (V, E - e)$. Thus $G - e$ is obtained from G by cutting (deleting) the edge e. Similarly let G/e be the multigraph obtained from G by fusing (contracting) the edge e. Thus if $e \in E$ is incident with u and v (with $u = v$ if a loop) then in G/e the vertices u and v are replaced by a single vertex w. Each element $f \in E - e$ that is incident with either u or v is replaced by an edge or loop incident with w. (Figure 2.1)

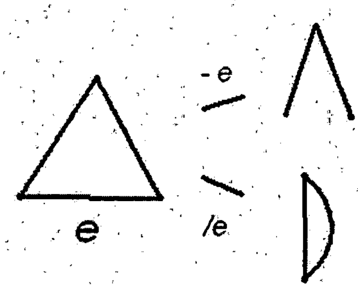


Figure 2.1: $-e$(cut or delete) and $/e$ (fuse or contraction)

Definition 2.1. *2-variable Tutte polynomial:*

$$T_G = \begin{cases} x(T_{G-e}) & \textit{if } e \textit{ is a bridge.} \\ y(T_{G-e}) & \textit{if } e \textit{ is a loop.} \\ T_{G-e} + T_{G/e} & \textit{if } e \textit{ is neither a bridge nor a loop.} \end{cases}$$

In the above definition if e is a loop or a bridge then $T_{G-e} = T_{G/e}$. The deletion and fusion of the edges continues until all that is left is a collection of subgraphs each made up solely of bridges and loops. In the decomposed graphs x represents the bridges and y represents the loops. The Tutte polynomial is found by summing all the polynomials associated with subgraphs together where each polynomial is the product of x's and y's for the corresponding bridges and loops. For example, the Tutte polynomial calculation of the graph associated with the figure eight knot is found in figure 2.2. (For ease of computation the up operation will be delete $(-e)$ and the down operation will be fuse $(/e)$ in the Tutte computational tree.)

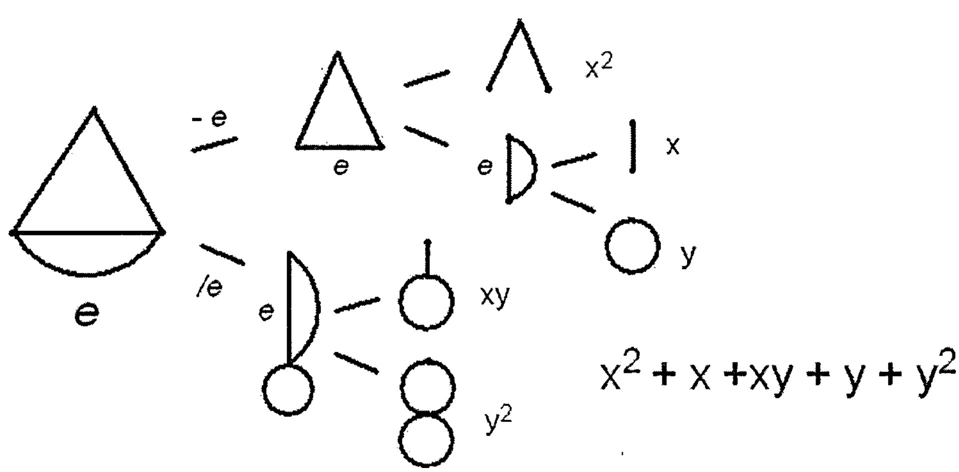


Figure 2.2: example calculation of Tutte polynomial for figure eight knot

This paper will utilize a few properties of the Tutte polynomial. The Tutte polynomial of a graph G and its dual graph H are related. In fact, $T_G(x,y) = T_H(y,x)$. [8] Another useful property of the Tutte polynomial is that it is multiplicative. Specifically, if a graph M consists of two graphs M_1 and M_2 with exactly one vertex in common (a cutpoint), then $T_M = (T_{M_1})(T_{M_2})$. [8]

2.1 Alternating Knots are 2-variable Tutte Polynomial Invariant

In this section it will be shown that the Tutte polynomial is an invariant of alternating knots. First, let two graphs G and G_f be graphs associated with reduced alternating diagrams of two knots related by a single flype. We will show the two graphs G and G_f have the same Tutte polynomial $T_G(x, y)$ or T_G. Next, we will apply the Tait Flyping Theorem to prove that reduced alternating knots are 2-variable Tutte polynomial invariant.

Theorem 2.2. *Let G and G_f be graphs associated with reduced alternating diagrams of two knots related by a single flype. The Tutte polynomial of a graph G is equal to the Tutte polynomial of G_f, or $T_G = T_{G_f}$.*

Proof. First, assume that in the checkerboard shading of the diagram the flype crossing is shaded (as in Figure 2.3). Let the corresponding single edge of the flype crossing in the graphs G and G_f be labeled e and e' respectively (Figure 2.4). Note that the definition of the Tutte polynomial gives that $T_G = T_{G-e} + T_{G/e}$ which also implies that $T_{G_f} = T_{G_f-e'} + T_{G_f/e'}$.

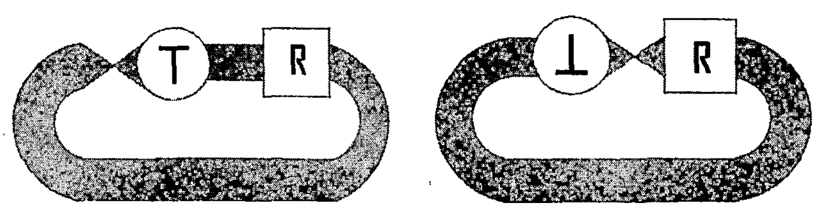


Figure 2.3: shading of flype crossing

By contraction G/e and G_f/e' are the same graph since all vertices maintain their adjacency as seen in figure 2.5.

Therefore $T_{G/e} = T_{G_f/e'}$.

The deletion operation is a different story. $G - e$ and $G_f - e'$ may not be the same graph (Figure 2.6). For example, let T have a vertex that has three edges on the left and a vertex with four edges on the right and let R have a vertex with two edges on the left and a vertex with single edge on the right. Then $G - e$ has a vertex with six

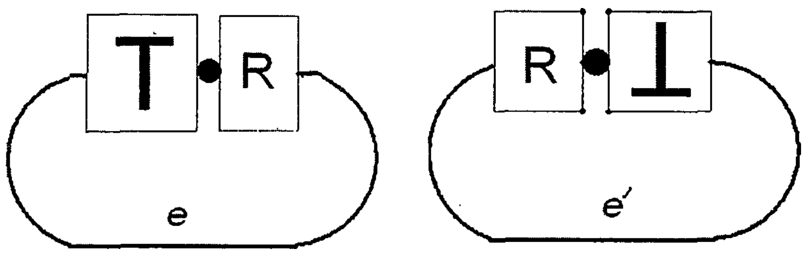


Figure 2.4: (a) graph of G and (b) graph of G_f with edge e and e' labeled

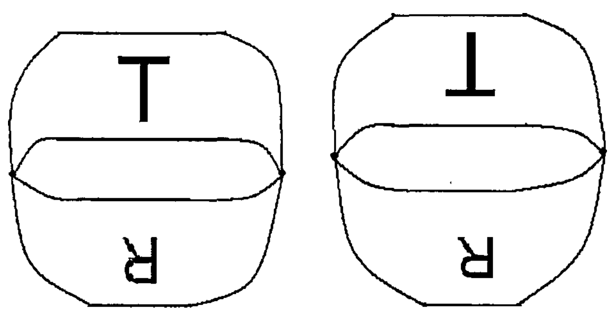
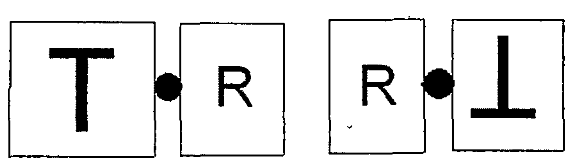


Figure 2.5: (a) graph of G and (b) graph of G_f with edge e and e' contracted

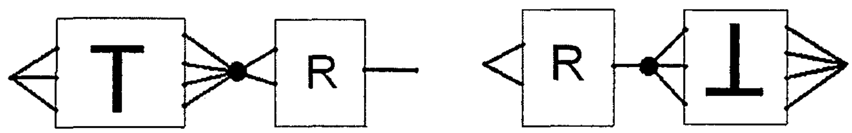
edges (4+2) but $G_f - e'$ would only have a vertex with only four edges (1+3). Clearly these two graphs are not isomorphic as seen in figure 2.7.



(a) (b)

Figure 2.6: (a) graph of G and (b) graph of G_f with edge e and e' deleted

However the edges e and e' when deleted create a cutpoints between the subgraphs of T and R of G and of R and T_f of G_f respectively. (In this situation the order matters as seen in the previous example.) Recall, if a graph M consists of two graphs M_1 and M_2 with exactly one vertex in common (a cutpoint), then $T_M = (T_{M_1})(T_{M_2})$. That is to say the Tutte polynomial is multiplicative if the subgraphs share only one



(a) (b)

Figure 2.7: example of (a) G and (b) G_f after deletion of edges e and e'

point in common. Keep in mind the graphs T and T_f are isomorphic.

Therefore $T_{G-e} = T_{G_f-e'}$.

Since $T_{G/e} = T_{G_f/e'}$ and $T_{G-e} = T_{G_f-e'}$ then $T_G = T_{G_f}$

In the previous argument we decided on a specific shading for the "flype" edge e and e'. It is possible to show that the Tutte polynomial is invariant even if the other checkerboard shading is chosen by using the properties of dual graphs. Recall, the Tutte polynomial of a graph G and its dual graph H are related. In fact, $T_G(x,y) = T_H(y,x)$. Therefore if the "other" (non flype) shading is chosen we first find H the dual of G, then using the previous argument it follows that $T_H = T_{H_f}$. Therefore by applying the dual relationship and the previous argument the following is true:

$$T_G(x,y) = T_H(y,x) = T_{H_f}(y,x) = T_{G_f}(x,y)$$

Therefore for all cases the Tutte polynomial is flype invariant. □

Theorem 2.3. *The 2-variable Tutte polynomial $T_G(x,y)$ is an invariant of alternating knots.*

Proof. The Tait Flype Theorem states that any two reduced alternating diagrams of a link are related by a finite number of flypes. [11] Coupling this with Theorem 2.1 shows that all reduced alternating projections of the same knot have the same Tutte polynomial. □

2.2 Multiplicity of Edges

Now that we know the Tutte polynomial is an invariant of alternating knots it is natural to ask what aspects of the polynomial are related to the diagram of the knot. Recall, given a knot K and its associated graph G then if there are twists in K they show up as either a multiedge or as a "chain" of edges in G (Figure 2.8 a and b). Interestingly, the number of twists or the multiplicity of a multiedge can be found be inspecting the Tutte polynomial of the associated graph. For example, if there is a tangle containing five twists in K then it will show up as a multiedge with multiplicity five within the graph G and therefore within the terms of the Tutte polynomial. It is possible to count exact multiplicities of an edge. For example all tangles with five twists can be found explicitly by using the terms of the Tutte polynomial. Consequently, the Tutte polynomial counts the exact number of twists in a given tangle depending on the shading chosen.

(a) (b)

Figure 2.8: (a) twist as a multiedge and (b) twist as a "chain" of bridges

Let $G = (V,\ E)$ be multigraph with vertex set V and edge set E. For every subset $A \subseteq E$, its *rank* is $r(A) = n - k(G|A)$ where $n = |V|$ and $k(G|A)$ is the number of connected components of the spanning subgraph $(V,\ A)$. [4] Therefore, the rank of a graph is equal to the number of vertices less components. Notice that the number components of a graph associated to a knot will always be one since a knot by definition is a link with a single component.

Now, let G be the associated graph of a reduced alternating knot diagram. Remember that G will be loopless and bridgeless since the knot diagram is reduced and has only one component. The following Lemma is true. Note that the coefficient mentioned in the Lemma is a polynomial in terms of y, in fact it is exactly the polynomial y^l.

Lemma 2.4. *The highest exponent of x in $T_G(x,y)$ is $r(E)$. Moreover, the coefficient of $x^{r(E)}$ is y^l, where the graph has l loops. (follows from an argument found in [1])*

First, here is some notation to go along with the following arguments. Let $T_G^{a,b}$ = the coefficient of $x^a y^b \in T_G(x,y)$. Therefore Lemma 2.4 implies:

$$T_G^{r(E),k} = \begin{cases} 1 & \text{if there are } k \text{ loops} \\ 0 & \text{otherwise} \end{cases}$$

Using this lemma it is possible to find the number of different multiedges with different multiplicities. Before proving Theorem 2.5 and Corollary 2.6 lets illustrate them with example 2.9. This knot diagram has a multiedge with multiplicity three and a multiedge with multiplicity six. Its Tutte polynomial is

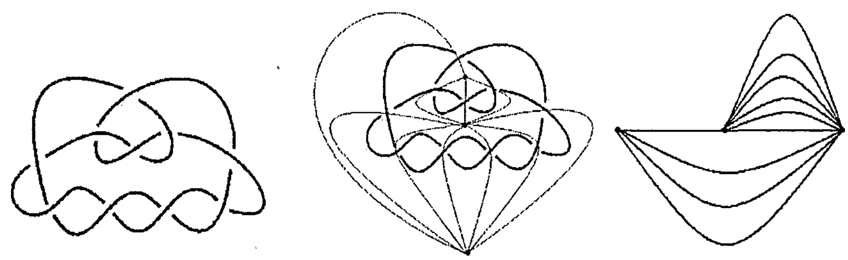


Figure 2.9: example of finding multiplicities in a graph

$$x^2 + x + 2xy + 2xy^2 + xy^3 + xy^4 + xy^5 + y + 2y^2 + 3y^3 + 3y^4 + 3y^5 + 3y^6 + 2y^7 + y^8.$$

The graph has three vertices and one component therefore the rank of the graph is two $(3 - 1)$. Therefore, in the Tutte computational tree we are looking for graphs that are multiedges of k multiplicity that turn into graphs with $k-1$ loops after the contraction operation. These graphs correspond to terms of the Tutte polynomial. Consequently, we need to inspect terms of the Tutte polynomial with x^1y^k terms. We want terms of x^1 since the rank of the graph is two. In this case there are the following terms $xy^5, xy^4, xy^3, 2xy^2, 2xy$. Since xy^5 is the first such term and it's coefficient is 1 then there is one multiedge with multiplicity six. Notice that the term xy^6 has a coefficient of zero and $1 - 0 = 1$. Also notice that a multiedge with multiplicity six appears as a term of xy^{6-1}. By continuing this process the next the next term without a coefficient of 1 is $2xy^2$ which implies that there is one $(2-1)$ multiedge of multiplicity three, since $2xy^{3-1}$. Notice that by subtracting the coefficients we get the exact number of multiedges $(2-1)$ and that it has multiplicity three since y has a power of two $(3 - 1)$. We now prove the following theorem.

Theorem 2.5. *Let G be a loopless graph then for $k \geq 1$ the coefficient of $x^{r(E)-1}y^k$ is equal to the number of multiedges with multiplicity $> k$.*

Proof. We will proceed by induction on $|E|$.

Let $|E| = 2$ then there exists two cases. Case 1, is when there is a multiedge with multiplicity two that has x + y as it's Tutte polynomial. Case 2, is when there is a tree of two bridges which has a Tutte polynomial of x^2 as seen in figure 2.10.

(a) case 1 (b) case 2

Figure 2.10: (a) x + y and (b) x^2

In the first case $|V| = 2$ and there is one component. Consequently, the rank of the graph is one $(2 - 1)$. Hence, we are looking for the term of the Tutte polynomial with x^0 since $1 - 1 = 0$. Notice the only term that meets this criteria is y^1. Therefore there is a single multiedge of multiplicity two since the power of y is one less than the multiplicity of the multiedge. Consequently, the theorem holds in this case. Similarly, in the second case $|V| = 3$ and there is one component. Thus the rank of the graph is 2 $(3 - 1)$. As a result, we are looking for the term of the Tutte polynomial with x^1 since $2 - 1 = 1$. Notice there are no terms of the Tutte polynomial with this criteria. Therefore there are no multiedges in the graph and the theorem also holds in this case.

Clearly the statement is true for both cases.

Assume that the statement is is true for $|E| = n$.

Recall the notation, let $T_G^{a,b}$ = the coefficient of $x^a y^b \in T_G(x, y)$. Additional notation is still necessary, let $m_k(G)$ = the number of multiedges of multiplicity more than $k \in G$. Now let G be the graph such that $|E| = n + 1$ and let e be an edge of G (not a bridge) with multiplicity of j. Also let $G - e = G'$ and let $G/e = G''$ then $T_G = T_{G'} + T_{G''}$ as seen in figure 2.11. Notice that $r(G') = r(G)$ since e was not a bridge so the graph G' has the same number of vertices and components as G. This is true even if

$j = 1$ since j is not a bridge. Also notice that $r(G'') = r(G) - 1$ since there is one less vertex but the same number of components in G'' when compared to G.

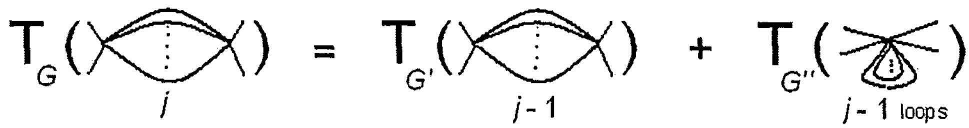


Figure 2.11: contraction and deletion on multiedge with multiplicity j

So $T_G^{r(G)-1,k} = T_{G'}^{r(G')-1,k} + T_{G''}^{r(G''),k}$

where $T_{G''}^{r(G''),k} = \begin{cases} 1 & \text{if } j = k + 1 \text{ and contraction creates } k \text{ loops} \\ 0 & \text{otherwise} \end{cases}$

by Lemma 2.4 $T_{G'}^{r(G')-1,k} = m_k(G')$ (by the induction hypothesis)

In a loopless graph using the contraction operation is the only way to make loops. These loops correspond to the powers of y within the Tutte polynomial. Therefore we only find loops in the G'' graph. Consequently, the only way to get k loops or a term of y^k in the Tutte polynomial is to contract G'' that contains a multiedge with multiplicity $k + 1 \in G$.

$$\text{Therefore we need to show that } m_k(G) = m_k(G') + \begin{cases} 1 & \text{if } j = k + 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

The process of showing the above argument consists of three cases. Case 1 is when $j \leq k$, case 2 is for $j > k + 1$ and case 3 is when $j = k + 1$.

Case $j \leq k$

Then the edge e does not contribute to $m_k(G)$, nor will it in $m_k(G')$. Note, $T_{G''}^{r(G''),k}$ will be zero since $j \neq k + 1$. Therefore we get equality in equation 2.5.

Case $j > k + 1$

The multiedge containing e contributes to $m_k(G)$. After deletion, the multiedge remaining contributes to $m_k(G')$. As seen previously, $m_k(G) = m_k(G')$. Note that $T_{G''}^{r(G''),k}$ will be zero since $j \neq k + 1$. Therefore we get equality in equation 2.5.

Case $j = k + 1$

The multiedge containing e contributes to $m_k(G)$, but after deletion the remaining multiedge does not contribute to $m_k(G')$, so $m_k(G) = m_k(G') + 1$. Remember from above $T_{G''}^{r(G''),k} = 1$ when $j = k+1$. Therefore equation 2.5 is verified.

□

Corollary 2.6. *If G is a loopless graph and $k > 1$, then $T_G^{r(E)-1,k-1} - T_G^{r(E)-1,k}$ is the number of multiedges of multiplicity exactly $k \in G$.*

Proof. By theorem 2.5, $T_G^{r(E)-1,k-1} - T_G^{r(E)-1,k} = m_{k-1}(G) - m_k(G)$. This is equal to the number of multiedges $\in G$ with multiplicity exactly k. □

Using the duality properties of graphs it should also be similarly possible to locate all "chains" of bridges that represent tangles of multiple twists within a knot and it's associated graph. This conjecture is not addressed in this paper and will be considered in future work.

Chapter 3

Conclusion

Knot theory has been around for more than two hundred years. It has seen many approaches and advancements on the basic question of how to efficiently differentiate between knots that look different but are actually the same. In 1984 V. Jones and his single variable Jones polynomial reinvigorated this search. However, this paper focused on one approach, the Tutte polynomial, which is without specialization, unlike the Jones polynomial, when dealing with alternating knots.

This paper began with a brief history of knot theory followed by an introduction to knot theory. Useful graph properties were also introduced which showed how these two fields, graph theory and knot theory, are related when using a diagrammatic approach. Then it was shown how a flype of a tangle effects the associated graph of a given knot diagram. Next the Tutte polynomial was introduced inductively. Some of the properties of the Tutte polynomial where also introduced. In particular, that the Tutte polynomial is multiplicative when the graph has a cutpoint. In addition, the relationship between the Tutte polynomials of dual graphs was described. Utilizing graph theoretical techniques it was shown that the Tutte polynomial is invariant for flypes. Combining this fact with the Tait Flype Theorem it was proven that the Tutte polynomial is invariant for all alternating knots. Next, using a given Tutte polynomial it was shown that certain features of a knot diagram could be discovered. Consequently, given a Tutte polynomial it is possible to discover some of the number of twists that exist in a given tangle of a knot diagram.

This paper, however, leaves some unanswered questions and/or conjectures. The first obvious conjecture would be that it is possible to locate chains of k edges within the terms of the Tutte polynomial. Depending on the original checkerboard shading chosen in a diagram D a twist of k multiplicity would show up as either a multiedge or a "chain" of edges. We have shown that any multiedges of an associated graph would appear within the terms of the corresponding Tutte polynomial. We have also discussed the duality relationship of the Tutte polynomial where $T_G(x,y) = T_H(y,x)$. Therefore it seems that the terms of the Tutte polynomial that correspond to a chain of edges would simply be $x^k y^{r(E)}$. However, the rank of the graph G and it's dual G^d are not always equal. Therefore more work is needed to show where the "chain" terms show up in an associated Tutte polynomial. It would be necessary to establish how the rank of G is related to the rank of G^d. Then using the dual property of the Tutte polynomial mentioned above it should be possible to show how the "chain" terms show up with in the polynomial. This would then allow for us to find all tangles of twists of any multiplicities of any diagram regardless of initial checkerboard shading. This and other questions will have to wait for a later work.

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