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Minimal surfaces

Maria Guadalupe Chaparro

Minimal Surfaces

A Thesis

Presented to the

Faculty of

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San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Approved by:

[redacted] 6/8/07

Wenxiang Wang, Committee Chair — Date

[redacted]

Rolland Trapp, Committee Member

[redacted]

Charles Stanton, Committee Member

[redacted]

Peter Williams, Chair,
Department of Mathematics

[redacted]

Joseph Chavez
Graduate Coordinator,
Department of Mathematics

ABSTRACT

The focus of this project consists of investigating when a ruled surface is a minimal surface. A minimal surface is a surface with zero mean curvature. In this project the basic terminology of differential geometry will be discussed including examples where the terminology will be applied to the different subjects of differential geometry. In addition to the basic terminology of differential geometry, we also focus on a classical theorem of minimal surfaces. It was referred as the Plateau's Problem. This theorem states that a surface with the minimal area is a minimal surface and the proof of the theorem will be provided. To investigate when a ruled surface is minimal, we need to solve a system of differential equations. In conclusion, we find that only ruled surfaces that are also minimal are helicoids. Some graphs of minimal surfaces will also be provided in this project, using MAPLE and other computer programs.

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Chapter 1

Introduction to Basic Terminology of Differential Equations

This chapter will be divided in five sections, where the terminology of differential geometry will be discussed. The first section begins with the definition of a surface in a space R^3. The other three sections will consist of the basic terminology of differential geometry, and the last section will applied the basic terminology of differential geometry to specific examples.

1.1 Surface in R^3

In this section the following definition consists of the properties that a surface in space R^3 must satisfy, where surfaces have no sharp point, edges, or self-intersection.

Definition 1.1.1. *A surface in R^3 is a subset $S \subset R^3$ such that for each point $p \in S$ there are a neighborhood V of p in R^3 and a mapping $\mathbf{x} : U \to V \cap S$ of an open set $U \subset R^2$ onto $V \cap S \subset R^3$ subject to the following three conditions. (Figure 1.1)*

1. $\mathbf{x}$ is of class C^3.

   This states that $\mathbf{x}$ is differentiable and the mapping $\mathbf{x}$ has continuous invariant partial derivatives. (first, second, and third derivatives)

2. $\mathbf{x}$ is a homeomorphism.

   This means that $\mathbf{x}$ is a bijection, $\mathbf{x}$ is continuous, and the inverse mapping $\mathbf{x}^{-1}$ is continuous.

3. $\mathbf{x}$ is regular at each point $(u, v) \in U$.

   This states that the differential mapping $d\mathbf{x} : R^2 \to R^3$ is injective, for every $(u, v) \in U$. In other words this is equivalent to the fact that the Jacobian matrix J_x has rank 2 at each point of $(u, v) \in U$. This implies that at each $(u, v) \in U$ the vector product of

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0,$$

   where $\mathbf{x}_u$ and $\mathbf{x}_v$ are denoted by

$$\mathbf{x}_u = \left(\frac{\partial \mathbf{x}_1}{\partial u}, \frac{\partial \mathbf{x}_2}{\partial u}, \frac{\partial \mathbf{x}_3}{\partial u}\right), \qquad \mathbf{x}_v = \left(\frac{\partial \mathbf{x}_1}{\partial v}, \frac{\partial \mathbf{x}_2}{\partial v}, \frac{\partial \mathbf{x}_3}{\partial v}\right).$$

   and $(u, v) \in U$. Thus, $\mathbf{x}$ is neither constant nor a function of u of v alone, so that the surface S is neither a point nor a curve.

The mapping $\mathbf{x}$ is called a parametrization or a local coordinate system either at a point p or in a neighborhood of the point p, and the neighborhood $V \cap S$ of p in S is known as the coordinate neighborhood. [Hsi97]

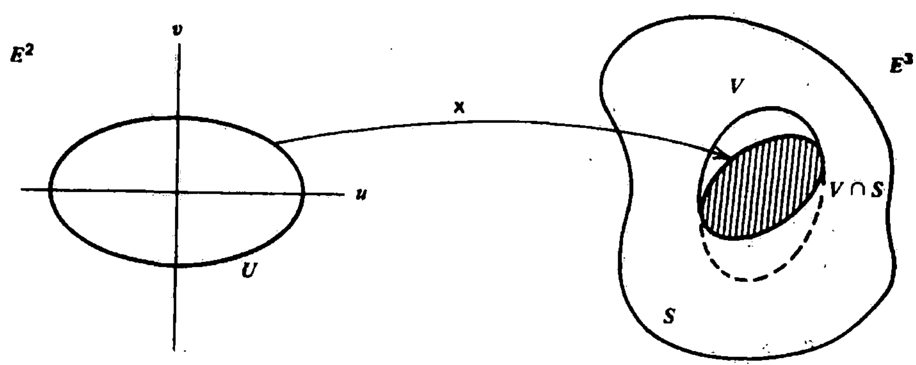


Figure 1.1: Surface

Lemma 1.1.2. *If $f : U \to R$ is of class C^3 function in an open set U of R^2, then the graph of f, that is, the subset of R^3 given by $(x_1, x_2, f(x_1, x_2))$ for $(x_1, x_2) \in U$ is a surface.*

The argument of Lemma (1.1.2) states that if the function f is any differential function on the open set U of R^2, then the function $\mathbf{x} : U \to R^3$ such that

$$\mathbf{x}(u, v) = (u, v, f(u, v))$$

is a proper patch. These patches are also known as Monge patches. The parametrization $\mathbf{x}$ is also called a Monge parametrization and the corresponding surface is a simple surface, where the general surface in R^3 can be constructed by gluing together simple surfaces. [O'N66]

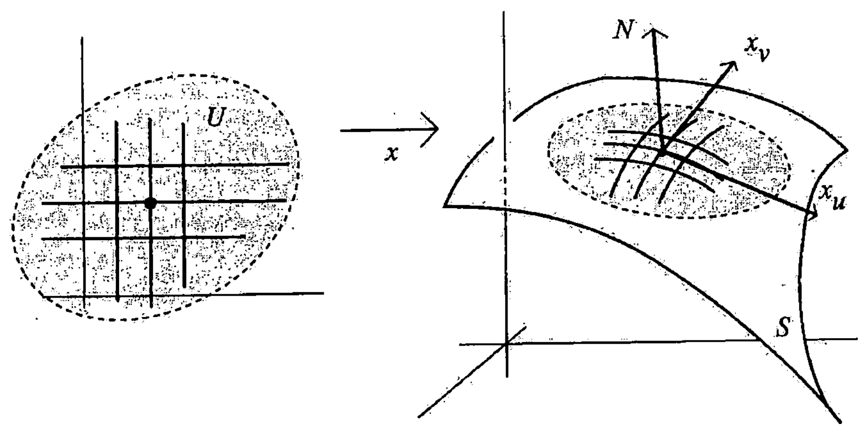


Figure 1.2: Tangent Vectors

Lemma 1.1.3. *All tangent vectors of a surface S at a point p form a plane that is called the tangent plane of S at p and is denoted by $T_p(S)$.(Figure 1.2)*

Proof. Let $\mathbf{x} : U \subset R^2 \to S$ be a parametrization of a surface S, and let $(u, v) \in U$ and $p = \mathbf{x}(u_0, v_0)$. If the coordinate function of any curve C on S through p are given by $u(t)$ and $v(t)$, where $t \in I$, then

$$u_0 = u(t_0), u_0 = u(t_0), \qquad \text{where } t_0 \in I$$

By the chain rule, the tangent vector of the curve C at p is

$$\left(\frac{d\mathbf{x}}{dt}\right)_0 = \mathbf{x}_u(u_0, v_0)u'(t_0) + \mathbf{x}_v(u_0, v_0)v'(t_0),$$

where the subscripts u, v denote the partial derivatives. Since $\mathbf{x}_u \times \mathbf{x}_v \neq 0$ and the vectors $\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)$ span a plane. Therefore, all tangent vectors of S at p are in the plane. [Hsi97]

$\square$

The tangent plane, which approximates a surface S near a point $p \in S$ is given by $T_p(S)$ and we have the following definition,

Definition 1.1.4. *The tangent plane of a surface S at a point p is given by*

$$\begin{aligned} T_p(S) &= \{v | v \text{ is tangent to } S \text{ at } p\} \\ T_p(S) &= Span\{\mathbf{x}_u, \mathbf{x}_v\} = \text{ Tangent Plane} \end{aligned}$$

Notice that the tangent plane $T_p(S)$ is a plane through the origin and it does not necessarily contain point p. If we consider this plane and by adding the vector determined by p, then we have the "plane tangent to the surface S at point p". In this case it is important to consider the tangent plane as a set of directions, where the operations of addition and multiplication by a scalar are satisfied. In the above definition $\{\mathbf{x}_u, \mathbf{x}_v\}$ is a basis for $T_p(S)$, where $T_p(S)$ contains the directions of the tangent vectors to any regular curve on the surface S through point p.

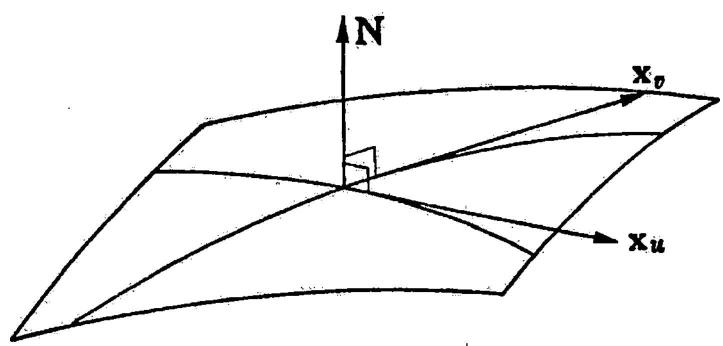


Figure 1.3: Unit Normal Vector

Definition 1.1.5. *The line orthogonal to the tangent plane $T_p(S)$ of a surface S at a point p is the normal to S at p. A normal vector field on S or on a region R of S is a function that assigns to each point p of S or R a normal vector of S at p.*

The normal to surface is the line at p perpendicular to the tangent plane. Thus, $\mathbf{N}$ is the unit normal vector to the tangent plane $T_p(S)$ given by *(Figure 1.2)*

$$\begin{aligned} \mathbf{N} &= \pm\frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{(\mathbf{x}_u \times \mathbf{x}_v)^2}} \\ &= \pm\frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{\mathbf{x}_u^2\mathbf{x}_v^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2}}, \end{aligned} \tag{1.1}$$

where $\mathbf{x}_u \times \mathbf{x}_v$ denotes the determinant of the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ in R^3 and is given by

$$\mathbf{x}_u \times \mathbf{x}_v = \left(\frac{\partial \mathbf{x}_2}{\partial u}\frac{\partial \mathbf{x}_3}{\partial v} - \frac{\partial \mathbf{x}_3}{\partial u}\frac{\partial \mathbf{x}_2}{\partial v}, \frac{\partial \mathbf{x}_1}{\partial u}\frac{\partial \mathbf{x}_3}{\partial v} - \frac{\partial \mathbf{x}_3}{\partial u}\frac{\partial \mathbf{x}_1}{\partial v}, \frac{\partial \mathbf{x}_1}{\partial u}\frac{\partial \mathbf{x}_2}{\partial v} - \frac{\partial \mathbf{x}_2}{\partial u}\frac{\partial \mathbf{x}_1}{\partial v} \right).$$

Let $\mathbf{x} : U \subset R^2 \to S$ be a parametrization of a surface S, and let $(u, v) \in U$. At a point $\mathbf{x} = \mathbf{x}(u, v)$ on S, we can define two tangent vector $\mathbf{w}_1, \mathbf{w}_2$ such that; $\mathbf{w}_i(u, v)$, $i = 1, 2$, are of class R^2 and the determinant of $|\mathbf{w}_1, \mathbf{w}_2, \mathbf{N}| > 0$. Notice that in this case we cannot assume that $\mathbf{w}_1, \mathbf{w}_2$ are orthogonal unit vectors since the inner products

$$\delta_{ij} = \mathbf{w}_i \cdot \mathbf{w}_j, \qquad i, j = 1, 2,$$

are not necessary the Kronecker deltas. Since $\mathbf{w}_1, \mathbf{w}_2$ are linearly independent, then any vector of S at a point $\mathbf{x}$ can be expresses as a linear combination, and it can be written as

$$\begin{aligned} \mathbf{x}_u &= a_1 \mathbf{w}_1 + a_2 \mathbf{w}_2, \\ \mathbf{x}_v &= b_1 \mathbf{w}_1 + b_2 \mathbf{w}_2, \end{aligned}$$

where a_i's and b_i's, $i = 1, 2$ are functions of u, v.
Now we can use the the following differential notation

$$d\mathbf{x} = \mathbf{x}_u du + \mathbf{x}_v dv. \tag{1.2}$$

1.2 First and Second Fundamental Form

The first fundamental form is basically a way of expressing geometrically the measurement of the distance on a curve, angles, and areas on a surface. The following is an expression of the first fundamental form in a basis for $T_p(S)$, which consist on the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ associated to a parametrization $\mathbf{x}(u, v)$ at $p \in S$. The vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are partial derivative with respect to u and v. Then the following equation can be expressed independently of the choice of parameter and is tangent to the surface S.

$$d\mathbf{x} = \mathbf{x}_u du + \mathbf{x}_v dv.$$

At a particular point $p = \mathbf{x}(u_0, v_0)$ the ratio dv/du determines the direction of the tangent to the surface S.

The distance of two point p and q on a curve can be found by integrating

$$ds^2 = \sum_{i=1}^{3} dx_i dx_i = d\mathbf{x} \cdot d\mathbf{x} \tag{1.3}$$

along the curve.

Definition 1.2.1. *The quadratic form* $\mathbf{I}$ *is called the* ***first fundamental form*** *of the surface* S *at the point* $\mathbf{x}$ *defined by,*

$$\mathbf{I} = d\mathbf{x}^2. \tag{1.4}$$

Now we can express the first fundamental form in terms of the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ of the parametrization $\mathbf{x}(u, v)$ at $p \in S$.

$$\begin{aligned}
\mathbf{I} &= d\mathbf{x} \cdot d\mathbf{x} \\
&= (\mathbf{x}_u du + \mathbf{x}_v dv) \cdot (\mathbf{x}_u du + \mathbf{x}_v dv) \qquad \text{(by equation (1.2))} \\
&= (\mathbf{x}_u \cdot \mathbf{x}_u) du^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v) du dv + (\mathbf{x}_v \cdot \mathbf{x}_v) dv^2 \\
&= E du^2 + 2F du dv + G dv^2,
\end{aligned}$$

where

$$\begin{aligned}
E(u_0, v_0) &= \mathbf{x}_u \cdot \mathbf{x}_u, \\
F(u_0, v_0) &= \mathbf{x}_u \cdot \mathbf{x}_v, \\
G(u_0, v_0) &= \mathbf{x}_v \cdot \mathbf{x}_v.
\end{aligned}$$

The coefficients E, F, and G are functions of u and v computed at $t = 0$. Thus, E, F, and G are the coefficients for the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S)$.

The study of differential geometry of a surface leads to a number of quadratic forms on simple interpretations of geometry. The coefficients of the first fundamental form are important since they are used to calculate arc length, angles and areas of surfaces.

Let $c(t) = \mathbf{x}(u(t), v(t))$, where $a \leq t \leq b$, be a simple curve (arc) on $\mathbf{x}(u, v)$ since the arc length is given by

$$\begin{aligned}
S &= \int_a^b \left|c'(t)\right| dt \\
&= \int_a^b \sqrt{\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt}} dt \\
&= \int_a^b \sqrt{(\mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt}) \cdot (\mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt})} dt \\
&= \int \sqrt{E(\frac{du}{dt})^2 + 2F\frac{du}{dt} \cdot \frac{dv}{dt} + G(\frac{dv}{dt})^2} dt.
\end{aligned}$$

Now the distance between points p and q can be expressed as the following:

$$S = \int \sqrt{E(\frac{du}{dt})^2 + 2F\frac{du}{dt} \cdot \frac{dv}{dt} + G(\frac{dv}{dt})^2} dt. \tag{1.5}$$

Therefore, the arc length on a surface is the integral of the square root of $\mathbf{I}$, where $\mathbf{I}$ is called the element of the arc. Thus, the arc length is defined as the following,

Definition 1.2.2. *The first fundamental form* $\mathbf{I}$ *is called the* ***element of arc****, for the arc length of a curve on* S *is given by the formula*

$$S = \int \sqrt{\mathbf{I}}.$$

There are two important properties that apply to the first fundamental form.

1. $\mathbf{I}$ is invariant under a parameter transformation, where $\mathbf{I}$ depends on the surface and not on a particular representation for a surface.

2. $\mathbf{I}$ is positive definite implies that $\mathbf{I} \geq 0$ if and only if $du = 0$ and $dv = 0$. Since $\mathbf{I}$ is positive definite, it follows that the coefficients $E > 0$ and $G > 0$ must satisfy $EG - F^2 > 0$. Note that $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent and $\mathbf{x}_u \times \mathbf{x}_v \neq 0$. Since $E = \mathbf{x}_u \cdot \mathbf{x}_u > 0$ and $G = \mathbf{x}_u \cdot \mathbf{x}_v > 0$, then

$$\begin{aligned} EG - F^2 &= (\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)(\mathbf{x}_u \cdot \mathbf{x}_v) \\ &= (\mathbf{x}_u \times \mathbf{x}_v)(\mathbf{x}_u \times \mathbf{x}_v) \\ &= (\mathbf{x}_u \times \mathbf{x}_v)^2 > 0. \end{aligned}$$

Since $\mathbf{x}_u \times \mathbf{x}_v \neq 0$. [Str50]

The first fundamental form has been introduced and the following consist of the second fundamental form. Notice that both $\mathbf{x}$ and $\mathbf{N}$ are surface functions of u and v. Therefore,

$$\begin{aligned} d\mathbf{x} &= \mathbf{x}_u du + \mathbf{x}_v dv, \\ d\mathbf{N} &= \mathbf{N}_u du + \mathbf{N}_v dv. \end{aligned} \tag{1.6}$$

Definition 1.2.3. *The* ***second fundamental form*** *on S at point p is the quadratic form* **II**: *defined by,*

$$\mathbf{II} = -d\mathbf{x} \cdot d\mathbf{N}. \tag{1.7}$$

The second fundamental form can be expressed in terms of the vectors $\mathbf{x}_u$, $\mathbf{x}_v$ and the unit normal vector $\mathbf{N}$.

$$\begin{aligned} \mathbf{II} &= -d\mathbf{x} \cdot d\mathbf{N} \\ &= -(\mathbf{x}_u du + \mathbf{x}_v dv) \cdot (\mathbf{N}_u du + \mathbf{N}_v dv) \qquad \text{(by equations (1.2) and (1.6))} \\ &= -\big[(\mathbf{x}_u \cdot \mathbf{N}_u) du^2 + (\mathbf{x}_u \cdot \mathbf{N}_v) du dv + (\mathbf{x}_v \cdot \mathbf{N}_u) du dv + (\mathbf{x}_v \cdot \mathbf{N}_v) dv^2\big], \end{aligned}$$

Since $\mathbf{x}_u, \mathbf{x}_v$ are orthogonal to the unit normal vector $\mathbf{N}$, then

$$\mathbf{x}_u \cdot \mathbf{N} = \mathbf{x}_v \cdot \mathbf{N} = 0, \tag{1.8}$$

By differentiation with respect to u and v we generate

$$\mathbf{x}_{uu} \cdot \mathbf{N} + \mathbf{x}_u \cdot \mathbf{N}_u = 0, \tag{1.9}$$

$$\mathbf{x}_{uv} \cdot \mathbf{N} + \mathbf{x}_u \cdot \mathbf{N}_v = 0, \tag{1.10}$$

$$\mathbf{x}_{vv} \cdot \mathbf{N} + \mathbf{x}_v \cdot \mathbf{N}_v = 0. \tag{1.11}$$

Thus, we obtain

$$\mathbf{II} = (\mathbf{x}_{uu} \cdot \mathrm{N})du^2 + 2(\mathbf{x}_{uv} \cdot \mathrm{N})dudv + (\mathbf{x}_{vv} \cdot \mathrm{N})dv^2.$$

Thus,

$$\mathbf{II} = Ldu^2 + 2Mdudv + Ndv^2,$$

where the L, M and N are the coefficient of the second fundamental form defined by,

$$\begin{aligned} L &= \mathbf{x}_{uu} \cdot \mathbf{N}, \\ M &= \mathbf{x}_{uv} \cdot \mathbf{N}, \\ N &= \mathbf{x}_{vv} \cdot \mathbf{N}. \end{aligned}$$

$\mathbf{N}$ is the unit normal vector given by

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{\mathbf{x}_u^2 \mathbf{x}_v^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2}}.$$

Hence, by using the coefficients of the first fundamental form we obtain,

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{EG - F^2}},$$

where $EG - F^2 > 0$.

1.3 Normal, Gaussian, and Mean Curvature

In the previous section we defined the first and second fundamental form which consist on calculating important coefficients that are useful to determine the metric properties of a surface such as line element, area element, normal curvature, Gaussian curvature, and mean curvature. Therefore, it is important to consider the coefficients of the second fundamental form which will be efficient to calculate the curvatures of a surface. In this section normal curvature, Gaussian curvature, and mean curvature will be difined.

Definition 1.3.1. *The Gauss mapping g of the surface S is defined by $g : S \to S^2$, where S^2 is the unit sphere with center at point $0 \in S^3$, which sends each point p of S to the end point of the unit vector through 0 in the direction of the normal vector $\mathbf{N}$ of S at p.*

The geometrical interpretation of the normal curvature consist on the direction of the unit tangent vector $t = \dfrac{d\mathbf{x}}{ds}$. If any fixed point $p \in S$ is consider, then the values of the coefficient of the fundamental forms are fixed. Notice that in this case the choice of orientation on the curve passing through point p is independent. The meaning of this is that the curvature κ of a curve at p depends only on the direction of the principal normal vector and on the unit normal vector. Therefore, it is important to consider all curves that have tangents at point p going in the same direction since the curvature of these curves consist on the angle θ between the principal normal vector and the corresponding unit normal vector.

Let $\mathbf{x} : U \subset R^2 \to S$ be a parametrization of the surface S and $(u, v) \in U$. Let C be a curve passing through point $p = \mathbf{x}(u, v)$ with arc length s and $\mathbf{t}$ be the unit tangent vector (tangent to the surface S). Let $C \in S$ be a regular curve passing through a point p and consider the curvature of C at p, where κ is the curvature of C at p and $\cos\theta = \mathbf{n} \cdot \mathbf{N}$ where θ is the angle between the principal normal vector $\mathbf{n}$ to C and the corresponding unit normal vector $\mathbf{N}$ to S at p.

Let $\mathbf{x}(s) = c(s)$, where $c(s)$ is any curve on $S : \mathbf{x}(u, v)$. Then

$$\begin{aligned} c(s) &= \mathbf{x}(u(s), v(s)), \\ c(0) &= p, \\ \dot{c}(0) &= \mathbf{t}, \\ \ddot{c} &= \kappa\mathbf{n}, \end{aligned} \tag{1.12}$$

where κ is the curvature of the curve C at point p. By taking the inner product of equation (1.12) with the unit normal vector $\mathbf{N}$ we obtain

$$\begin{aligned} \kappa\mathbf{n} \cdot \mathbf{N} &= \ddot{c} \cdot \mathbf{N} \\ \kappa\cos\theta &= \ddot{c} \cdot \mathbf{N} \qquad (\text{since } \cos\theta = \mathbf{n} \cdot \mathbf{N}) \\ &= -\dot{c} \cdot \dot{\mathbf{N}} \qquad (\text{since } \dot{c} \cdot \mathbf{N} = 0 \Rightarrow \ddot{c} \cdot \mathbf{N} + \dot{c} \cdot \dot{\mathbf{N}} = 0) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{d\mathbf{x}}{ds}\cdot\frac{d\mathbf{N}}{ds} \quad \text{(since } \mathbf{x}(s) = c(s)\text{)}\\
&= \frac{-d\mathbf{x}\cdot d\mathbf{N}}{ds\cdot ds}\\
&= \frac{-d\mathbf{x}\cdot d\mathbf{N}}{ds^2}\\
&= \frac{II}{I}. \qquad \text{(by equations (1.4) and (1.7))}
\end{aligned}
$$

Therefore

$$\kappa\cos\theta = \frac{II}{I} = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}. \tag{1.13}$$

Lemma 1.3.2. *Curves on a surface S with the same tangent and osculating plane at a point p have the same curvature at p.*

By equation (1.13) we have

$$\kappa\cos\theta = \kappa_n, \tag{1.14}$$

where κ_n is called the ***normal curvature*** and is the curvature of the *normal section.* This is defined to be the curve of intersection of surface S by the plane through the unit tangent vector $\mathbf{t}$ and the unit normal vector. [Kre91]

Hence, the normal curvature in terms of the coefficient of the first and second fundamental form is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}. \tag{1.15}$$

Observing equation (1.14) let's consisder the fact that if the angle $\theta = 0$ then $\kappa = \kappa_n$, which represents the radius of the curvature of the curve with unit tangent vector t. On the other hand, if $\theta = \pi$ then $\kappa = -\kappa_n$, which means that $|\kappa_n|$ is the curvature of the intersection of the surface S with the plane passing through the tangent to the curve at point p and the normal vector to the surfade S at p. This curve is called the *normal section.* Notice that $\kappa_n \neq 0$ and $\kappa \neq 0$ because if it equal then the curvature of the normal section would vanish at p. [Kre91]

Now we can obtain the directions for each of which the normal curvature is maximum or minimum. Lets denote the normal curvature κ_n as κ. Recall the equation (1.15) of normal curvature in direction (du, dv) is given by

$$\begin{aligned}\kappa &= \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}. \\ \kappa(\lambda) &= \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2},\end{aligned} \tag{1.16}$$

where a function of $\lambda = \dfrac{dv}{du}$. Now we can find the extreme values of the normal curvature κ by calculating the critical point of the following:

$$\frac{d\kappa}{d\lambda} = 0.$$

Then we have

$$\frac{d\kappa}{d\lambda} = \frac{(2M + 2N\lambda)(E + 2F\lambda + G\lambda^2) - (L + 2M\lambda + N\lambda^2)(2F + 2G\lambda)}{(E + 2F\lambda + G\lambda^2)^2} = 0$$

since

$$(E + 2F\lambda + G\lambda^2)^2 \neq 0.$$

Thus,

$$(2M + 2N\lambda)(E + 2F\lambda + G\lambda^2) - (L + 2M\lambda + N\lambda^2)(2F + 2G\lambda) = 0$$

$$\begin{aligned}&\Rightarrow \quad (2M + 2N\lambda)(E + 2F\lambda + G\lambda^2) = (L + 2M\lambda + N\lambda^2)(2F + 2G\lambda) \\ &\Rightarrow \quad (E + F\lambda)(M + N\lambda) = (L + M\lambda)(F + G\lambda)\end{aligned}$$

or

$$\begin{vmatrix} E + F\lambda & L + M\lambda \\ F + G\lambda & M + N\lambda \end{vmatrix} = 0. \tag{1.17}$$

From equation (1.17) we obtain the following:

$$\kappa = \frac{\boldsymbol{II}}{\boldsymbol{I}} = \frac{M + N\lambda}{F + G\lambda}$$
$$= \frac{L + M\lambda}{E + F\lambda}. \tag{1.18}$$

Hence,

$$(FN - GM)\lambda^2 + (EN - GL)\lambda + (EM - FL) = 0 \tag{1.19}$$

or

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0. \tag{1.20}$$

is the quadratic equation in terms of λ with real roots and is obtain from equation (1.19).

The quadratic equation determines the two directions, where κ obtains an extreme value when the second fundamental form vanishes. On the other hand, if the second fundametal form and the first fundamental form are proportional, then one of the values must be a maximum and the other must be a minimum. Therefore, we have the following definition;

Definition 1.3.3. *The roots of equation (1.17) are the* ***principal directions*** *of the normal curvature of the Surface S at a point p and the normal curvatures of the curvature directions is callled the* ***principal curvatures*** *and is denoted by κ_1 and κ_2.*

Now we can find the principal curvatures using equation (1.18). We can calculate the following:

$$(F + G\lambda)\kappa = M + N\lambda,$$
$$(E + F\lambda)\kappa = L + M\lambda.$$

Solving for λ in terms of κ, we obtain the following values for λ respectively:

Therefore,

$$\begin{aligned}\lambda &= \frac{M - F\kappa}{G\kappa - N},\\ \lambda &= \frac{L - E\kappa}{F\kappa - M},\end{aligned}$$

where κ satisfies the following equations since $\lambda = \dfrac{dv}{du}$.

$$\begin{aligned}(L - E\kappa)du + (M - F\kappa)dv &= 0,\\ (M - F\kappa)du + (N - G\kappa)dv &= 0.\end{aligned} \tag{1.21}$$

Hence,

$$\begin{aligned}\frac{L - E\kappa}{F\kappa - M} &= \frac{M - F\kappa}{G\kappa - N}\\ \Rightarrow \quad (L - E\kappa)(G\kappa - N) &= (M - F\kappa)(F\kappa - M)\\ \Rightarrow \quad (E\kappa - L)(G\kappa - N) &= (F\kappa - M)(F\kappa - M)\end{aligned} \tag{1.22}$$

or

$$\begin{vmatrix} E\kappa - L & F\kappa - M \\ F\kappa - M & G\kappa - N \end{vmatrix} = 0. \tag{1.23}$$

From equation (1.23) we obtain the following quadratic equation:

$$(EG - F^2)\kappa^2 - (EN - 2FM + GL)\kappa + (NL - M^2) = 0$$

$$\Rightarrow \quad \kappa^2 - \frac{(EN - 2FM + GL)}{(EG - F^2)}\kappa + \frac{(LN - M^2)}{(EG - F^2)} = 0. \tag{1.24}$$

The quadratic equation (1.24) in terms of κ has κ_1 and κ_2 as roots.

Definition 1.3.4. *$\boldsymbol{K}$ is the **Gaussian curvature** and $\boldsymbol{H}$ is the **mean curvature** of a surface S at a point p defined by*

$$\boldsymbol{K} = \kappa_1\kappa_2, \tag{1.25}$$

$$\boldsymbol{H} = \frac{1}{2}(\kappa_1 + \kappa_2). \tag{1.26}$$

Note:

$$\begin{aligned} (\kappa - \kappa_1)(\kappa - \kappa_2) &= 0, \\ \Rightarrow \quad \kappa^2 - (\kappa_1 + \kappa_2)\kappa + \kappa_1\kappa_2 &= 0. \end{aligned} \tag{1.27}$$

By comparing equations (1.24) and (1.27) we can rewrite the ***Gaussian curvature*** and the ***mean curvature*** in terms of the coefficients of the first and second fundamental form.

Therefore, we obtain the following formulas

$$\boldsymbol{K} = \frac{LN - M^2}{EG - F^2}, \tag{1.28}$$

$$\boldsymbol{H} = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2}. \tag{1.29}$$

Notice that if we use equation (1.24) in terms of $\boldsymbol{H}$ and $\boldsymbol{K}$ we obtain the following quadratic equation

$$\kappa^2 - 2\boldsymbol{H}\kappa + \boldsymbol{K} = 0.$$

The quadratic equation generates solutions of the two principal curvatures expressed in terms of the Gaussian curvature and mean curvature,

$$\begin{aligned} \kappa_1 &= \boldsymbol{H} + \sqrt{\boldsymbol{H}^2 - \boldsymbol{K}}, \\ \kappa_2 &= \boldsymbol{H} - \sqrt{\boldsymbol{H}^2 - \boldsymbol{K}}. \end{aligned}$$

From equations (1.25) we have some cases where it follows that a point on a surface could be elliptic, parabolic, hyperbolic, or planar. The following cases represent the behavior of a point on the surface.

1. $LN - M^2 > 0$.

   In case one the Gaussian curvature is positive, therefore the point on the surface S is elliptic. Since the principal curvatures have the same sign, which means that all curves passing through the point have the normal vector in direction towards one side of the tangent plane. *(Figure 1.4)* [Alm66]

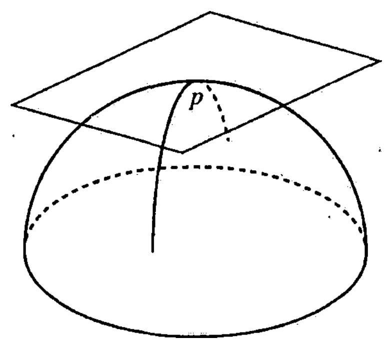


Figure 1.4: Elliptic

2. $LN - M^2 = 0$.

   In case two where the Gaussian curvature is zero, but the principal curvatures can be either $\kappa_1 > 0, \kappa_2 = 0$ or $\kappa_1 = 0, \kappa_2 < 0$. Therefore, the point on the surface S is parabolic and all points lie on the same side of the tangent plane. *(Figure 1.5)* [McC94]

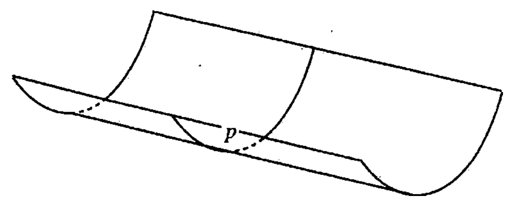


Figure 1.5: Parabolic

3. $LN - M^2 < 0$.

   This case three where the Gaussian curvature is negative, which means that the principal curvatures have opposite signs. Hence, the point on the surface is hyperbolic since the curves passing through point p have a normal vector at p in direction on either side of the tangent plane. (*Figure 1.6*) [McC94]

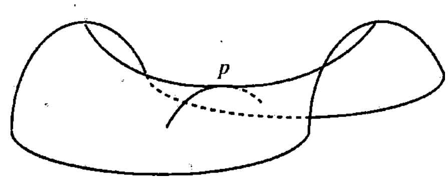


Figure 1.6: Hyperbolic

4. $L = M = N = 0$

   In this case the principal curvatures are zero, so $\kappa_1 = \kappa_2 = 0$. Thus, the point in the surface S is called planar and the behavior of S with respect to the tangent plane near a point p varies.[McC94]

If the principal curvatures are the same, but do not equal to zero ($\kappa_1 = \kappa_2 \neq 0$), then the point is called an ***Umbilic point***. This consists on taking any pair of orthogonal directions at such a point as principal directions. For instance, a plane or a Sphere consists entirely on umbilical points.

1.4 Fundamental Equations

In the previous section the coefficients of the first and second fundamental form were applied to the important field of curvature. Now these coefficients have a relationship on certain formulas that are derived from a moving trihedron, where this trihedron depends on two parameters (u, v). The coefficients of the first and second fundamental form have differential relations, where E, F and G depend on $\mathbf{x}_u$ and $\mathbf{x}_v$ and L, M and N also depends on $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$. The Theory of a surface is to express $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$, $\mathbf{N}_u$, and $\mathbf{N}_v$ as linear combinations of the basis $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{N}\}$, where $\mathbf{x} : U \subset R^2 \to S$ is a parametrization of a surface S and $\mathbf{N}$ is the unit normal vector. Consider the three linearly independent vectors $\mathbf{x}_u$, $\mathbf{x}_v$ and $\mathbf{N}$, where $\mathbf{x}_u$ and $\mathbf{x}_v$ lie on the tangent plane normal to $\mathbf{N}$, then the following equations are obtained for every vector that can be linearly expressed in terms of the basis $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{N}\}$:

$$
\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1_{1\,1}\mathbf{x}_u + \Gamma^2_{1\,1}\mathbf{x}_v + L\mathbf{N}, \\
\mathbf{x}_{uv} &= \Gamma^1_{1\,2}\mathbf{x}_u + \Gamma^2_{1\,2}\mathbf{x}_v + M\mathbf{N}, \\
\mathbf{x}_{vv} &= \Gamma^1_{2\,2}\mathbf{x}_u + \Gamma^2_{2\,2}\mathbf{x}_v + N\mathbf{N},
\end{aligned}
$$

where the functions of $\Gamma^j_{i\,k}$ are the ***Christoffel Symbols***. In this case [i, j, k] are called the Christoffel Symbols of the first kind while the functions of $\Gamma^j_{i\,k}$ are known as the Christoffel Symbols of the second kind. Notice that the Christoffel Symbols depends on the coefficient of the first fundamental form and their derivatives. The functions of $\Gamma^j_{i\,k}$ for $i, j, k = 1, 2$ are given by,

$$
\begin{aligned}
\Gamma^1_{1\,1} &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, & \Gamma^2_{1\,1} &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\
\Gamma^1_{1\,2} &= \frac{GE_v - FG_u}{2(EG - F^2)}, & \Gamma^2_{1\,2} &= \frac{EG_u - FE_v}{2(EG - F^2)}, \\
\Gamma^1_{2\,2} &= \frac{2GF_v - GGu - FG_v}{2(EG - F^2)}, & \Gamma^1_{2\,2} &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.\text{[Str50]}
\end{aligned}
$$

Now if we take the partial derivative of the unit normal vector $\mathbf{N}$ with respect to u and v we obtain the following;

$$\mathbf{N}_u = a\mathbf{x}_u + b\mathbf{x}_v, \tag{1.30}$$

$$\mathbf{N}_v = c\mathbf{x}_u + d\mathbf{x}_v. \tag{1.31}$$

The coefficients a, b, c, and d of these two equations can be obtained by doing some basic calculations. By taking the inner product of the first equation (1.30) with $\mathbf{x}_u$ and $\mathbf{x}_v$ we get

$$\begin{aligned} \mathbf{x}_u \cdot \mathbf{N}_u &= a(\mathbf{x}_u \cdot \mathbf{x}_u) + b(\mathbf{x}_u \cdot \mathbf{x}_v) \\ \Rightarrow \quad -L &= a(\mathbf{x}_u^2) + b(\mathbf{x}_u \cdot \mathbf{x}_v) \\ &= aE + bF. \\ \text{and} & \\ \mathbf{x}_v \cdot \mathbf{N}_u &= a(\mathbf{x}_v \cdot \mathbf{x}_u) + b(\mathbf{x}_v \cdot \mathbf{x}_v) \\ \Rightarrow \quad -M &= a(\mathbf{x}_v \cdot \mathbf{x}_u) + b(\mathbf{x}_v^2) \\ &= aF + bG \end{aligned}$$

Similarly, we take the inner product of the second equation (1.31) with $\mathbf{x}_u$ and $\mathbf{x}_v$.

Then

$$\begin{aligned} \mathbf{x}_u \cdot \mathbf{N}_v &= c(\mathbf{x}_u \cdot \mathbf{x}_u) + d(\mathbf{x}_u \cdot \mathbf{x}_v) \\ \Rightarrow \quad -M &= c(\mathbf{x}_u^2) + d(\mathbf{x}_u \cdot \mathbf{x}_v) \\ &= cE + dF. \\ \text{and} & \\ \mathbf{x}_v \cdot \mathbf{N}_v &= c(\mathbf{x}_v \cdot \mathbf{x}_u) + d(\mathbf{x}_v \cdot \mathbf{x}_v) \\ \Rightarrow \quad -N &= c(\mathbf{x}_v \cdot \mathbf{x}_u) + d(\mathbf{x}_v^2) \\ &= cF + dG. \end{aligned}$$

Therefore

$$-L = aE + bF \tag{1.32}$$

$$-M = aF + bG \tag{1.33}$$

and

$$-M = cE + dF \tag{1.34}$$

$$-N = cF + dG \tag{1.35}$$

Using systems of differential equations, we can focus on equation (1.32) and (1.33) to solve for coefficients a and b.

Therefore

$$-L = aE + bF$$

$$-M = aF + bG$$

$$\begin{aligned} \Rightarrow \qquad GL &= -aEG - bFG \\ -FM &= aF^2 + bFG \\ \Rightarrow \qquad GL - FM &= -a(EG - F^2) \\ \Rightarrow \qquad a &= \frac{FM - GL}{EG - F^2}. \end{aligned}$$

By substitution we can replace our solution for coefficient a in equation (1.32) which gives the following solution for coefficient b.

$$b = \frac{FL - ME}{EG - F^2}.$$

Similarly, using equation (1.34) and (1.35) and system of equations we can calculate the solutions for the coefficients c and d we obtain

$$-M = cE + dF$$

$$-N = cF + dG$$

$$
\begin{aligned}
\Rightarrow \qquad GM &= -cEG - dFG \\
-FN &= cF^2 + dFG \\
\Rightarrow \qquad GM - FN &= -c(EG - F^2) \\
\Rightarrow \qquad c &= \frac{FN - GM}{EG - F^2}, \\
d &= \frac{FM - EN}{EG - F^2}.
\end{aligned}
$$

Now that we have calculated the coefficients a, b, c and d lets substitute these results in equations (1.30) and (1.31). We obtain the following equations which are called the ***Weingarten formulas.***

$$
\mathbf{N}_u = \frac{(FM - GL)\mathbf{x}_u + (FL - EM)\mathbf{x}_v}{EG - F^2}, \tag{1.36}
$$

$$
\mathbf{N}_v = \frac{(NF - MG)\mathbf{x}_u + (FM - NE)\mathbf{x}_v}{EG - F^2}. \tag{1.37}
$$

1.5 Examples

The following examples consists of calculating the coefficients for the first and second fundamental form, Gaussian Curvature and Mean curvature of different surfaces. The first example is on a sphere and the second is a Torus.

Example 1.5.1. *Find the coefficients of first and second fundamental form, the Gaussian curvature and the mean curvature of a sphere given by: (figure 1.7)*

$$
\begin{aligned}
x_1 &= a \sin\theta \cos\varphi, \\
x_2 &= a \sin\theta \sin\varphi, \\
x_3 &= a \cos\theta.
\end{aligned}
$$

The parametrization of a sphere is given by

$$
\mathbf{x}(\theta, \varphi) = (a \sin\theta \cos\varphi, a \sin\theta \sin\varphi, a \cos\theta).
$$

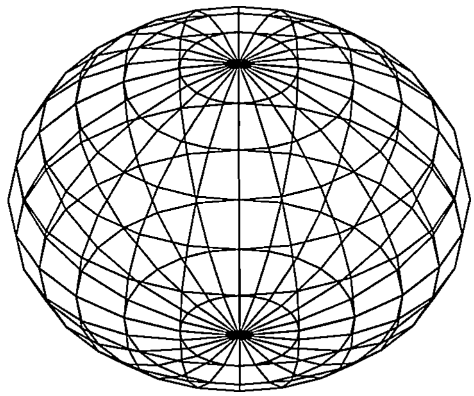


Figure 1.7: Sphere

The partial derivative with respect to θ and φ are given by

$$\begin{aligned} \mathbf{x}_\theta &= (a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, -a\sin\theta), \\ \mathbf{x}_\varphi &= (-a\sin\theta\sin\varphi, a\sin\theta\cos\varphi, 0). \end{aligned}$$

$$\mathbf{x}_\theta \times \mathbf{x}_\varphi = \begin{vmatrix} i & j & k \\ a\cos\theta\cos\varphi & a\cos\theta\sin\varphi & -a\sin\theta \\ -a\sin\theta\sin\varphi & a\sin\theta\cos\varphi & 0 \end{vmatrix},$$

$$\mathbf{x}_\theta \times \mathbf{x}_\varphi = (a^2\sin^2\theta\cos\varphi, a^2\sin^2\theta\sin\varphi, a^2\sin\theta\cos\theta).$$

The unit normal vector $\mathbf{N}$ of S at point p is given by

$$\mathbf{N} = \frac{\mathbf{x}_\theta \times \mathbf{x}_\varphi}{\sqrt{(\mathbf{x}_\theta \times \mathbf{x}_\varphi)^2}}.$$

Thus,

$$\begin{aligned} \mathbf{N} &= \frac{(a^2\sin^2\theta\cos\varphi, a^2\sin^2\theta\sin\varphi, a^2\sin\theta\cos\theta)}{\sqrt{(a^2\sin^2\theta\cos\varphi)^2 + (a^2\sin^2\theta\sin\varphi)^2 + (a^2\sin\theta\cos\theta)^2}} \\ &= \frac{(a^2\sin^2\theta\cos\varphi, a^2\sin^2\theta\sin\varphi, a^2\sin\theta\cos\theta)}{a^2\sin\theta}. \end{aligned}$$

Now lets find the coefficients of the first fundamental form of a sphere.

$$\begin{aligned}
E &= \mathbf{x}_\theta \cdot \mathbf{x}_\theta \\
&= (a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, -a\sin\theta)\cdot(a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, -a\sin\theta) \\
&= a^2\cos^2\theta\cos^2\varphi + a^2\cos^2\theta\sin^2\varphi + a^2\sin^2\theta \\
&= a^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + a^2\sin^2\theta \\
&= a^2\cos^2\theta + a^2\sin^2\theta \\
&= a^2(\sin^2\theta + \cos^2\theta) \\
&= a^2.
\end{aligned}$$

$$\begin{aligned}
F &= \mathbf{x}_\theta \cdot \mathbf{x}_\varphi \\
&= (a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, -a\sin\theta)\cdot(-a\sin\theta\sin\varphi, a\sin\theta\cos\varphi, 0) \\
&= -a^2\sin\theta\cos\theta\sin\varphi\cos\varphi + a^2\sin\theta\cos\theta\sin\varphi\cos\varphi + 0 \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
G &= \mathbf{x}_\varphi \cdot \mathbf{x}_\varphi \\
&= (-a\sin\theta\sin\varphi, a\sin\theta\cos\varphi, 0)\cdot(-a\sin\theta\sin\varphi, a\sin\theta\cos\varphi, 0) \\
&= a^2\sin^2\theta\sin^2\varphi + a^2\sin^2\theta\cos^2\varphi \\
&= a^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi) \\
&= a^2\sin^2\theta.
\end{aligned}$$

The coefficients of the first fundamental form are the following:

$$\begin{aligned}
E &= a^2, \\
F &= 0, \\
G &= a^2\sin^2\theta.
\end{aligned}$$

and the first fundamental form is given by

$$\boldsymbol{I} = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2.$$

Now lets find the coefficients for the second fundamental form by using the second partial derivatives with respect to θ and φ.

$$\begin{aligned}
\mathbf{x}_{\theta\theta} &= (-a\sin\theta\cos\varphi, -a\sin\theta\cos\varphi, -a\cos\theta), \\
\mathbf{x}_{\theta\varphi} &= (-a\cos\theta\sin\varphi, a\cos\theta\cos\varphi, 0), \\
\mathbf{x}_{\varphi\varphi} &= (-a\sin\theta\cos\varphi, -a\sin\theta\sin\varphi, 0).
\end{aligned}$$

$$\begin{aligned}
L &= \mathbf{x}_{\theta\theta}\cdot\mathbf{N} \\
&= (-a\sin\theta\cos\varphi, -a\sin\theta\cos\varphi, -a\cos\theta)\cdot\frac{(a^2\sin^2\theta\cos\varphi, a^2\sin^2\theta\sin\varphi, a^2\sin\theta\cos\theta)}{a^2\sin\theta} \\
&= \frac{-a^3\sin^3\theta(\sin^2\varphi+\cos^2\varphi) - a^3\sin\theta\cos^2\theta}{a^2\sin\theta} \\
&= \frac{-a^3\sin^3\theta - a^3\sin\theta\cos^2\theta}{a^2\sin\theta} \\
&= \frac{-a^3\sin\theta(\sin^2\theta+\cos^2\theta)}{a^2\sin\theta} \\
&= \frac{-a^3\sin\theta}{a^2\sin\theta} \\
&= -a.
\end{aligned}$$

$$\begin{aligned}
M &= \mathbf{x}_{\theta\varphi}\cdot\mathbf{N} \\
&= (-a\cos\theta\sin\varphi, a\cos\theta\cos\varphi, 0)\cdot\frac{(a^2\sin^2\theta\cos\varphi, a^2\sin^2\theta\sin\varphi, a^2\sin\theta\cos\theta)}{a^2\sin\theta} \\
&= \frac{-a^3\sin^2\theta\cos\theta\sin\varphi\cos\varphi + a^3\sin^2\theta\cos\theta\sin\varphi\cos\varphi + 0}{a^2\sin\theta} \\
&= 0.
\end{aligned}$$

$$
\begin{aligned}
N &= \mathbf{x}_{\varphi\varphi} \cdot \mathbf{N} \\
&= (-a\sin\theta\cos\varphi, -a\sin\theta\sin\varphi, 0) \cdot \frac{(a^2\sin^2\theta\cos\varphi, a^2\sin^2\theta\sin\varphi, a^2\sin\theta\cos\theta)}{a^2\sin\theta} \\
&= \frac{-a^3\sin^3\theta(\sin^2\varphi + \cos^2\varphi)}{a^2\sin\theta} \\
&= \frac{-a^3\sin^3\theta}{a^2\sin\theta} \\
&= -a\sin^2\theta.
\end{aligned}
$$

Thus, the coefficients for the second fundamental form are:

$$
\begin{aligned}
L &= -a, \\
M &= 0, \\
N &= -a\sin^2\theta.
\end{aligned}
$$

and the second fundamental form is given by

$$
\mathit{II} = -ad\theta^2 - a\sin^2\theta d\varphi^2.
$$

Using the coefficients of the first and second fundamental form we can calculate the Gaussian curvature of the sphere. Recall the coefficients for the first and the second fundamental form. Coefficients of the first fundamental form:

$$
\begin{aligned}
E &= a^2, \\
F &= 0, \\
G &= a^2\sin^2\theta.
\end{aligned}
$$

and the coefficients of the second fundamental form:

$$
\begin{aligned}
L &= -a, \\
M &= 0, \\
N &= -a\sin^2\theta.
\end{aligned}
$$

Using equation (1.28) and the results of the coefficients of the first and second fundamental form for a sphere we can calculate the *Gaussian curvature.*

$$\begin{aligned} \boldsymbol{K} &= \frac{(-a)(-a\sin^2\theta - 0^2)}{(a^2)(a^2\sin^2\theta) - 0^2} \\ &= \frac{a^2\sin^2\theta}{a^4\sin^2\theta} \\ &= \frac{1}{a^2}. \end{aligned}$$

Therefore, the *Gaussian curvature* for the sphere is constant, where $a > 0$.

By using equation (1.29) and the coefficients of the first and second fundamental form we can calculate the mean curvature of the sphere.

$$\begin{aligned} \boldsymbol{H} &= \frac{1}{2}\frac{a^2(-a\sin^2\theta) - a(a^2\sin^2\theta)}{a^2(a^2\sin^2\theta)} \\ &= -\frac{1}{2}\frac{2a^3\sin^2\theta}{a^4\sin^2\theta} \\ &= -\frac{1}{a}. \end{aligned}$$

Thus, the mean curvature of a sphere is constant, where $a > 0$.

Example 1.5.2. *Find the coefficient of the first and second fundamental form, the Gaussian curvature and the mean curvature of the Torus given by: (Figure 1.8)*

$$\begin{aligned} x_1 &= u\cos v, \\ x_2 &= u\sin v, \\ x_3 &= \sqrt{b^2 - (u-a)^2}, \ \textit{where } a > b. \end{aligned}$$

The parametrization of a Torus is given by

$$\mathbf{x}(u,v) = \left(u\cos v, u\sin v, \sqrt{b^2 - (u-a)^2}\right).$$

To find the coefficients for the first fundamental form, we want to calculate the firt partial derivate with respect to u and v. $\mathbf{x}_u$ and $\mathbf{x}_v$ are the partial derivatives with respect to u and v denoted as,

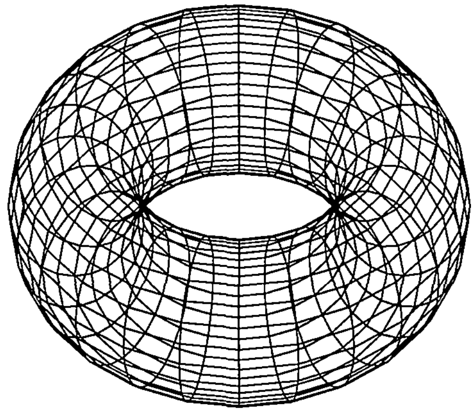


Figure 1.8: Torus

$$\mathbf{x}_u = \left(\cos v, \sin v, -\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}\right),$$

$$\mathbf{x}_v = (-u\sin v, u\cos v, 0).$$

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & -\frac{(u-a)}{\sqrt{b^2-(u-a)^2}} \\ -u\sin v & u\cos v & 0 \end{vmatrix},$$

$$\mathbf{x}_u \times \mathbf{x}_v = \left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\cos v, \frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\sin v, u\right).$$

So, the normal unit vector is given by

$$\mathbf{N} = \frac{\left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\cos v, \frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\sin v, u\right)}{\sqrt{\left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}\right)^2 + \left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\sin v\right)^2 + u^2}}$$

$$= \frac{\left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\cos v, \frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\sin v, u\right)}{\sqrt{\frac{u^2b^2}{b^2-(u-a)^2}}}$$

$$= \frac{\left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\cos v, \frac{(u-a)}{\sqrt{b^2-(u-a)^2}}u\sin v, u\right)}{\frac{ub}{\sqrt{b^2-(u-a)^2}}}.$$

The following consists of calculating the coefficients for the first fundamental form.

$$\begin{aligned}
E &= \mathbf{x}_u \cdot \mathbf{x}_u \\
&= \left(\cos v, \sin v, -\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}\right) \cdot \left(\cos v, \sin v, -\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}\right) \\
&= \cos^2 v + \sin^2 v + \frac{(u-a)^2}{b^2-(u-a)^2} \\
&= 1 + \frac{(u-a)^2}{b^2-(u-a)^2} \\
&= \frac{b^2}{b^2-(u-a)^2}. \\
F &= \mathbf{x}_u \cdot \mathbf{x}_v \\
&= \left(\cos v, \sin v, -\frac{(u-a)}{\sqrt{b^2-(u-a)^2}}\right) \cdot (-u\sin v, u\cos v, 0) \\
&= -u\sin v\cos v + u\sin v\cos v + 0 \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
G &= \mathbf{x}_v \cdot \mathbf{x}_v \\
&= (-u\sin v, u\cos v, 0) \cdot (-u\sin v, u\cos v, 0) \\
&= u^2 \sin^2 v + u^2 \cos^2 v + 0 \\
&= u^2(\sin^2 v + \cos^2 v) \\
&= u^2.
\end{aligned}$$

Thus, the coefficients for the first fundamental form are the following:

$$\begin{aligned}
E &= \frac{b^2}{b^2 - (u-a)^2}, \\
F &= 0, \\
G &= u^2,
\end{aligned}$$

and the first fundamental form is given by,

$$\mathbf{I} = \frac{b^2}{b^2 - (u-a)^2} du^2 + u^2 dv^2.$$

Now to calculate the coefficients for the second fundamental form we must calculate the second partial derivatives with respect to u and v.

$$\begin{aligned}
\mathbf{x}_{uu} &= \left(0, 0, \frac{-b^2}{(b^2 - (u-a)^2)^{\frac{3}{2}}}\right), \\
\mathbf{x}_{uv} &= (-\sin v, \cos v, 0), \\
\mathbf{x}_{vv} &= (-u\cos v, -u\sin v, 0).
\end{aligned}$$

Therefore, the following shows the calculation for the coefficients of the second fundamental form.

$$\begin{aligned}
L &= \mathbf{x}_{uu} \cdot \mathbf{N} \\
&= \left(0, 0, \frac{-b^2}{(b^2-(u-a)^2)^{\frac{3}{2}}}\right) \cdot \frac{\left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u\cos v, \frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u\sin v, u\right)}{\frac{ub}{\sqrt{b^2-(u-a)^2}}} \\
&= \frac{-b^2 u}{(b^2-(u-a)^2)^{\frac{3}{2}}} \cdot \frac{\sqrt{b^2-(u-a)^2}}{ub}. \\
&= \frac{-b}{b^2-(u-a)^2}. \\
M &= \mathbf{x}_{uv} \cdot \mathbf{N} \\
&= (-\sin v, \cos v, 0) \cdot \frac{\left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u\cos v, \frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u\sin v, u\right)}{\frac{ub}{\sqrt{b^2-(u-a)^2}}} \\
&= \frac{-\frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u \sin v \cos v + \frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u \sin v \cos v + 0}{\frac{ub}{\sqrt{b^2-(u-a)^2}}} \\
&= 0. \\
N &= \mathbf{x}_{vv} \cdot \mathbf{N} \\
&= (-u\cos v, -u\sin v, 0) \cdot \frac{\left(\frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u\cos v, \frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u\sin v, u\right)}{\frac{ub}{\sqrt{b^2-(u-a)^2}}} \\
&= \frac{-\frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u^2 \cos^2 v - \frac{(u-a)}{\sqrt{b^2-(u-a)^2}} u^2 \sin^2 v + 0}{\sqrt{\frac{u^2 b^2}{b^2-(u-a)^2}}}.
\end{aligned}$$

$$= \frac{\frac{-u^2(u-a)}{\sqrt{b^2-(u-a)^2}}(\cos^2 v + \sin^2 v)}{\frac{ub}{\sqrt{b^2-(u-a)^2}}}$$

$$= \frac{-u(u-a)}{b}.$$

Thus, the coefficients for the second fundamental form are the following:

$$\begin{aligned} L &= \frac{-b}{b^2-(u-a)^2}, \\ M &= 0, \\ N &= \frac{-u(u-a)}{b}, \end{aligned}$$

and the second fundamental form of a Torus is given by,

$$\mathbf{II} = \frac{-b}{b^2-(u-a)^2}du^2 + \frac{-u(u-a)}{b}dv^2.$$

Using the results of the coefficients of the first and second fundamental form we can calculate the Gaussian curvature using equation (1.28).

$$\begin{aligned} \boldsymbol{K} &= \frac{\left(\frac{-b}{b^2-(u-a)^2}\right)\left(\frac{-u(u-a)}{b}\right) - 0^2}{\frac{u^2b^2}{b^2-(u-a)^2} - 0^2} \\ &= \frac{u(u-a)}{b^2-(u-a)^2}\frac{b^2-(u-a)^2}{u^2b^2} \\ &= \frac{u-a}{ub^2}. \end{aligned}$$

Hence, the Gaussian curvature depends on the parameter u.

Now we can calculate the mean curvature of a Torus using equation (1.29).

$$
\begin{aligned}
\boldsymbol{H} &= \frac{1}{2}\frac{\left(\dfrac{b^2}{b^2-(u-a)^2}\right)\left(\dfrac{-u(u-a)}{b}\right)-\dfrac{u^2b}{b^2-(u-a)^2}}{\dfrac{u^2b^2}{b^2-(u-a)^2}} \\
&= \frac{1}{2}\frac{ub(a-2u)}{b^2-(u-a)^2}\frac{b^2-(u-a)^2}{u^2b^2} \\
&= \frac{a-2u}{2ub}.
\end{aligned}
$$

Thus, the mean curvature of a Torus depends on the parameter u.

Chapter 2

Ruled and Minimal Surfaces

This chapter introduces the definition of a minimal surface and a complete proof for *The Plateau's Problem*, then this information will be applied on investigating when a ruled surface is a minimal surface.

2.1 The Plateau's Problem

Definition 2.1.1. *A surface S of class $r \geq 2$ whose mean curvature $\boldsymbol{H}$ is zero at every point of S is called a* ***minimal surface.***

The Plateau's Problem states: Let C be a simple closed curve in R^3, there exists a surface S_0 bounded by C with the smallest area (i.e. S_0 has the minimal area among all the surfaces baunded by C).

Theorem 2.1.2. *The surface S_0 in Plateau's Problem has zero mean curvature, therefore is a minimal surface.*

Proof. Let $\mathbf{x} : U \subset R^2 \to R^3$ be a paremetrization of the surface S_0, where $(u, v) \in U$. Then a small normal variation of the surface S_0 with respect to a differentiable function λ on S_0, which vanishes on C, is a mapping $\mathbf{x}^* : U \times (-\epsilon, \epsilon) \to R^3$ defined by

$$\mathbf{x}^*(u, v, t) = \mathbf{x}(u, v) + t\lambda(u, v)\mathbf{N}(u, v),$$

where $\mathbf{N}$ is the unit normal vector of S_0 and ϵ is small for each $t \in (-\epsilon, \epsilon)$, the mapping $\mathbf{x}^t : U \to R^3$ is defined by

$$\begin{aligned}\mathbf{x}^t(u,v) &= \mathbf{x}^*(u,v,t)\\ &= \mathbf{x}(u,v) + t\lambda(u,v)\mathbf{N}(u,v).\end{aligned}$$

Then, we have the following partial derivatives with respect to u and v.

$$\mathbf{x}_u^t = \mathbf{x}_u + t\lambda_u\mathbf{N} + t\lambda\mathbf{N}_u, \tag{2.1}$$

$$\mathbf{x}_v^t = \mathbf{x}_v + t\lambda_v\mathbf{N} + t\lambda\mathbf{N}_v. \tag{2.2}$$

Assume that the given surface S_0 has minimal area. Recall that we denote the area of a surface over the region D by

$$A = \text{Surface area} \quad = \quad \int_D\int |\mathbf{x}_u \times \mathbf{x}_v| dudv.$$

Then,

$$\begin{aligned}A &= \int_D\int |\mathbf{x}_u \times \mathbf{x}_v| dudv\\
&= \int_D\int |\mathbf{x}_u|\,|\mathbf{x}_v| \sin\theta dudv\\
&\quad (\text{since } |\mathbf{x}_u \times \mathbf{x}_v| = |\mathbf{x}_u|\,|\mathbf{x}_v|\sin\theta)\\
&= \int_D\int |\mathbf{x}_u|\,|\mathbf{x}_v| \sqrt{1-\cos^2\theta} dudv\\
&\quad (\text{since } \sin^2\theta + cos^2\theta = 1)\\
&= \int_D\int |\mathbf{x}_u|\,|\mathbf{x}_v| \sqrt{1-\left(\frac{|\mathbf{x}_u\cdot\mathbf{x}_v|}{|\mathbf{x}_u|\,|\mathbf{x}_v|}\right)^2} dudv\\
&\quad (\text{by the dot product definition: } \mathbf{x}_u\cdot\mathbf{x}_v = |\mathbf{x}_u||\mathbf{x}_v|\cos\theta\ )\\
&= \int_D\int |\mathbf{x}_u|\,|\mathbf{x}_v| \sqrt{\frac{|\mathbf{x}_u|^2\,|\mathbf{x}_v|^2 - |\mathbf{x}_u\cdot\mathbf{x}_v|^2}{|\mathbf{x}_u|^2\,|\mathbf{x}_v|^2}} dudv\\
&= \int_D\int \sqrt{|\mathbf{x}_u|^2\,|\mathbf{x}_v|^2 - |\mathbf{x}_u\cdot\mathbf{x}_v|^2} dudv.\end{aligned}$$

Recall from section 1.2. the coefficients of the first fundamental form;

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, F = \mathbf{x}_u \cdot \mathbf{x}_v \text{ and } G = \mathbf{x}_v \cdot \mathbf{x}_v.$$

Thus,

$$A = \int_D \int \sqrt{EG - F^2} dudv. \tag{2.3}$$

By using the ***Weingarten formulas*** we can obtain the following equations:

$$\mathbf{x}_u^t = \mathbf{x}_u + t\lambda_u \mathbf{N} + \frac{t\lambda}{EG - F^2}[(FM - GL)\mathbf{x}_u + (FL - EM)\mathbf{x}_v],$$

$$\mathbf{x}_v^t = \mathbf{x}_v + t\lambda_v \mathbf{N} + \frac{t\lambda}{EG - F^2}[(FN - GM)\mathbf{x}_u + (FM - EN)\mathbf{x}_v],$$

where $\mathbf{N}_u$ and $\mathbf{N}_v$ are the ***Weingarten formulas*** see equations (1.36) and (1.37).

Let E^t, G^t and F^t be the coefficients of the first fundamental form of the surface $\mathbf{x}^t(u, v)$, then

$$\begin{aligned}
E^t &= \mathbf{x}_u^t \cdot \mathbf{x}_u^t \\
&= (\mathbf{x}_u + t\lambda_u \mathbf{N} + t\lambda \mathbf{N}_u) \cdot (\mathbf{x}_u + t\lambda_u \mathbf{N} + t\lambda \mathbf{N}_u) \qquad \text{(by equation (2.1))} \\
&= (\mathbf{x}_u \cdot \mathbf{x}_u) + 2t\lambda(\mathbf{x}_u \cdot \mathbf{N}) + 2t\lambda(\mathbf{x}_u \cdot \mathbf{N}_u) + 2t^2\lambda_u\lambda(\mathbf{N} \cdot \mathbf{N}_u) \\
&\quad + t^2\lambda_u^2(\mathbf{N} \cdot \mathbf{N}) + t^2\lambda^2(\mathbf{N}_u \cdot \mathbf{N}_u) \\
&= (\mathbf{x}_u \cdot \mathbf{x}_u) + 2t\lambda(\mathbf{x}_u \cdot \mathbf{N}_u) + O(t^2) \\
&\quad \text{(by equation (1.8) and since } \mathbf{N} \cdot \mathbf{N} = 1 \text{ and } \mathbf{N}_u \cdot \mathbf{N}_u = 1\text{)} \\
&= (\mathbf{x}_u \cdot \mathbf{x}_u) - 2t\lambda(\mathbf{x}_{uu} \cdot \mathbf{N}) + O(t^2) \qquad \text{(by equation(1.9))} \\
&= E - 2t\lambda L + O(t^2). \qquad (\text{since } L = \mathbf{x}_{uu} \cdot \mathbf{N})
\end{aligned}$$

where $O(t^2)$ is in terms of degree ≥ 2.

By interchanging u and v we obtain the following:

$$\begin{aligned}
F^t &= \mathbf{x}_u^t \cdot \mathbf{x}_v^t \\
&= (\mathbf{x}_u + t\lambda_u \mathbf{N} + t\lambda \mathbf{N}_u) \cdot (\mathbf{x}_v + t\lambda_v \mathbf{N} + t\lambda \mathbf{N}_v) \qquad \text{(by equation (2.1) and (2.2))} \\
&= (\mathbf{x}_u \cdot \mathbf{x}_v) + t\lambda(\mathbf{x}_u \cdot \mathbf{N}_v) + t\lambda(\mathbf{x}_v \cdot \mathbf{N}_u) + O(t^2) \qquad \text{(by equation (1.8))} \\
&= (\mathbf{x}_u \cdot \mathbf{x}_v) - t\lambda(\mathbf{x}_{uv} \cdot \mathbf{N}) - t\lambda(\mathbf{x}_{vu} \cdot \mathbf{N}) + O(t^2) \qquad \text{(by equation (1.10))} \\
&= (\mathbf{x}_u \cdot \mathbf{x}_v) - 2t\lambda(\mathbf{x}_{uv} \cdot \mathbf{N}) + O(t^2) \\
&= F - 2t\lambda M + O(t^2). \qquad \text{(since } M = \mathbf{x}_{uv} \cdot \mathbf{x}_N\text{)}
\end{aligned}$$

Similarly we can obtain G^t

$$\begin{aligned}
G^t &= \mathbf{x}_v^t \cdot \mathbf{x}_v^{t'} \\
&= (\mathbf{x}_v + t\lambda_v \mathbf{N} + t\lambda \mathbf{N}_v) \cdot (\mathbf{x}_v + t\lambda_v \mathbf{N} + t\lambda \mathbf{N}_v) \qquad \text{(by equation (2.2))} \\
&= (\mathbf{x}_v \cdot \mathbf{x}_v) + 2t\lambda(\mathbf{x}_v \cdot \mathbf{N}) + 2t\lambda(\mathbf{x}_v \cdot \mathbf{N}_v) + 2t^2\lambda_v\lambda(\mathbf{N} \cdot \mathbf{N}_v) \\
&\quad + t^2\lambda_v^2(\mathbf{N} \cdot \mathbf{N}) + t^2\lambda^2(\mathbf{N}_v \cdot \mathbf{N}_v) \\
&= (\mathbf{x}_v \cdot \mathbf{x}_v) + 2t\lambda(\mathbf{x}_v \cdot \mathbf{N}_v) + O(t^2) \qquad \text{(by equation (1.8))} \\
&= (\mathbf{x}_v \cdot \mathbf{x}_v) - 2t\lambda(\mathbf{x}_{vv} \cdot \mathbf{N}) + O(t^2) \qquad \text{(by equation (1.11))} \\
&= G - 2t\lambda N + O(t^2). \qquad \text{(since } N = \mathbf{x}_{vv} \cdot \mathbf{N}\text{)}
\end{aligned}$$

Thus, we have the following coefficients for the first fundamental form of the surface $\mathbf{x}^t(u, v)$:

$$\begin{aligned}
E^t &= E - 2t\lambda L + O(t^2), \\
F^t &= F - 2t\lambda M + O(t^2),
\end{aligned}$$

and

$$G^t = G - 2t\lambda N + O(t^2).$$

Now using the above coefficient we can compute the following:

$$\begin{aligned} E^tG^t - (F^t)^2 &= (E - 2t\lambda L + O(t^2))(G - 2t\lambda N + O(t^2)) - (F - 2t\lambda M + O(t^2))^2 \\ &= EG - F^2 - 2t\lambda(EN - 2FM + GL) + O(t^2) \end{aligned}$$

$$\Big(\text{Since} \qquad \boldsymbol{H} = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2}$$
$$\Rightarrow 2\boldsymbol{H}(EG - F^2) = EN - 2FM + GL\Big)$$

Then,

$$\begin{aligned} &= EG - F^2 - 2t\lambda(2\boldsymbol{H}(EG - F^2)) + O(t^2) \\ &= (EG - F^2)(1 - 4t\lambda\boldsymbol{H} + O(t^2)). \end{aligned}$$

Therefore,

$$E^tG^t - (F^t)^2 = (EG - F^2)(1 - 4t\lambda\boldsymbol{H} + O(t^2)).$$

By taking the square root to both side we have the following:

$$\begin{aligned} \sqrt{E^tG^t - (F^t)^2} &= \sqrt{(EG - F^2)(1 - 4t\lambda\boldsymbol{H} + O(t^2))} \\ &= \sqrt{EG - F^2}(1 - 4t\lambda\boldsymbol{H} + O(t^2))^{\frac{1}{2}} \end{aligned}$$

(by using the binomial expansion for a square root we obtain,)

$$= \sqrt{EG - F^2}(1 - 2t\lambda\boldsymbol{H} + O(t^2)).$$

Hence,

$$\sqrt{E^tG^t - (F^t)^2} = \sqrt{EG - F^2}(1 - 2t\lambda\boldsymbol{H} + O(t^2)). \tag{2.4}$$

Then the area of the surface $\mathbf{x}^t(u,v)$ over the region D for (u,v) is obtain by

$$\begin{aligned}
A^t &= \int_D\int \sqrt{E^tG^t-(F^t)^2}dudv \\
&= \int_D\int \sqrt{EG-F^2}(1-2t\lambda\boldsymbol{H}+O(t^2))dudv \quad \text{(by equation (2.4))} \\
&= \int_D\int \sqrt{EG-F^2}-2t\lambda\boldsymbol{H}\sqrt{EG-F^2}+O(t^2)\sqrt{EG-F^2}dudv \\
&= \int_D\int \sqrt{EG-F^2}dudv-\int_D\int 2t\lambda\boldsymbol{H}\sqrt{EG-F^2}dudv \\
&\quad +\int_D\int O(t^2)\sqrt{EG-F^2}dudv \\
&= A-\int_D\int 2t\lambda\boldsymbol{H}\sqrt{EG-F^2}dudv+\int_D\int O(t^2)\sqrt{EG-F^2}dudv \\
&\quad \text{(By equation (1.24))} \\
&= A-2t\int_D\int \lambda\boldsymbol{H}\sqrt{EG-F^2}dudv+O(t^2).
\end{aligned}$$

Thus,

$$A^t = A-2t\int_D\int \lambda\boldsymbol{H}\sqrt{EG-F^2}dudv+O(t^2). \tag{2.5}$$

Surface S_0 to have minimal area means A^0 is the minimal value for A^t and the derivative of A^t at t must be zero.

$$\begin{aligned}
& \left(\frac{dA^t}{dt}\right)\Big|_{t=0} = -2\int_D\int \lambda\boldsymbol{H}\sqrt{EG-F^2}dudv \\
\Rightarrow\quad & \left(\frac{dA^t}{dt}\right)\Big|_{t=0} = 0 \\
\Rightarrow\quad & \boldsymbol{H}\sqrt{EG-F^2} = 0 \\
\Rightarrow\quad & \boldsymbol{H} = 0. \qquad \text{(since } EG-F^2>0\text{)}
\end{aligned}$$

$\square$

2.2 Ruled Minimal Surfaces

Definition 2.2.1. *A **ruled surface** is a surface with a parametrization given by*

$$\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{z}(u), \tag{2.6}$$

which consist on a surface formed by lines with direction vector $\mathbf{z}(u)$ *along the curve* $\mathbf{y}(u)$*, where* $\mathbf{y}(u)$ *is called the **directrix** of the **ruled surface** and the line with direction* $\mathbf{z}(u)$ *is called a **ruling** on the surface.*

The proof of the following theorem consist on the main result of this project.

Theorem 2.2.2. *The right helicoid* $\mathbf{x}(u,v) = (v\cos u, v\sin u, au)$*, where* $0 < u < 2\pi$*,* $-\infty < v < \infty$*,* $a =$ **constant** *is the only minimal surface, other than the plane, that is also a ruled surface.*

Proof. Let S be a ruled surface with the parametrization given by

$$\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{z}(u).$$

We are going to show that: If S is a minimal surface then S is either a plane or a right helicoid. Since $\mathbf{z}(u)$ represents the direction we may assume

$$\begin{aligned} & \mathbf{z}(u)\cdot\mathbf{z}(u) = 1 \\ \Rightarrow \quad & (\mathbf{z}(u)\cdot\mathbf{z}(u))' = 0 \\ \Rightarrow \quad & \mathbf{z}'(u)\cdot\mathbf{z}(u) + \mathbf{z}(u)\cdot\mathbf{z}'(u) = 0 \\ \Rightarrow \quad & 2\mathbf{z}'(u)\cdot\mathbf{z}(u) = 0. \end{aligned}$$

Therefore,

$$\mathbf{z}'(u)\cdot\mathbf{z}(u) = 0. \tag{2.7}$$

We may also assume that the curve $\mathbf{z}(u)$ is parametrized by the arc length.

Therefore,

$$\begin{aligned} & \mathbf{z}'(u)\cdot\mathbf{z}'(u) = 1 \\ \Rightarrow \quad & \mathbf{z}''(u)\cdot\mathbf{z}'(u) = 0. \end{aligned}$$

Since the partial derivatives of the parametrization $\mathbf{x}(u,v)$ with respect to u and v are given by

$$\begin{aligned}\mathbf{x}_u &= \mathbf{y}'(u) + v\mathbf{z}'(u),\\ \mathbf{x}_v &= \mathbf{z}(u).\end{aligned}$$

Then the dot product of $\mathbf{x}_u$ and $\mathbf{x}_v$ is given by

$$\begin{aligned}\mathbf{x}_u \cdot \mathbf{x}_v &= (\mathbf{y}'(u) + v\mathbf{z}'(u)) \cdot \mathbf{z}(u)\\ &= \mathbf{y}'(u) \cdot \mathbf{z}(u) + v\mathbf{z}'(u) \cdot \mathbf{z}(u)\\ &= \mathbf{y}'(u) \cdot \mathbf{z}(u) \qquad (\text{since } \mathbf{z}'(u) \cdot \mathbf{z}(u) = 0).\end{aligned}$$

If we choose the orthogonal parametrization for the surface $\mathbf{x}(u,v))$, then

$$\begin{aligned}\mathbf{x}_u \cdot \mathbf{x}_v &= 0\\ \Rightarrow \qquad \mathbf{y}'(u) \cdot \mathbf{z}(u) &= 0.\end{aligned}$$

Now we consider a ruled surface with the following conditions.

$$\begin{aligned}\mathbf{z}(u) \cdot \mathbf{z}(u) = 1 \quad &\Rightarrow \quad \mathbf{z}'(u) \cdot \mathbf{z}(u) = 0,\\ \mathbf{z}'(u) \cdot \mathbf{z}'(u) = 1 \quad &\Rightarrow \quad \mathbf{z}''(u) \cdot \mathbf{z}'(u) = 0,\end{aligned}$$

and

$$\mathbf{y}'(u) \perp \mathbf{z}(u) \quad \Rightarrow \quad \mathbf{y}'(u) \cdot \mathbf{z}(u) = 0.$$

Recall the first partial derivatives of equation (2.6) with respect to u and v. Recall that the equation (2.6) is given by

$$\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{z}(u),$$

where the following are the first partial derivative for equation (2.6) with respect to u and v:

$$\begin{aligned}\mathbf{x}_u &= \mathbf{y}'(u) + v\mathbf{z}'(u),\\ \mathbf{x}_v &= \mathbf{z}(u).\end{aligned}$$

and the second partial derivative with respect to u and v are generate

$$\begin{aligned}\mathbf{x}_{uu} &= \mathbf{y}''(u) + v\mathbf{z}''(u),\\ \mathbf{x}_{uv} &= \mathbf{z}'(u),\\ \mathbf{x}_{vv} &= 0.\end{aligned}$$

The normal unit vector $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{(\mathbf{x}_u \times \mathbf{x}_v)^2}}.$$

Now we can calculate the coefficients of the first fundamental form of the parametrization $\mathbf{x}(u,v)$:

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u \\ &= \big(\mathbf{y}'(u) + v\mathbf{z}'(u)\big) \cdot \big(\mathbf{y}'(u) + v\mathbf{z}'(u)\big). \end{aligned}$$

$$\begin{aligned} F &= \mathbf{x}_u \cdot \mathbf{x}_v \\ &= \big(\mathbf{y}'(u) + v\mathbf{z}'(u)\big) \cdot \mathbf{z}(u) \\ &= \mathbf{y}'(u) \cdot \mathbf{z}(u) + v\mathbf{z}'(u) \cdot \mathbf{z}(u) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

notice that the coefficient F for the first fundamental form is zero, since $\mathbf{z}'$ is orthogonal to $\mathbf{z}$ and $\mathbf{y}'$ is orthogonal to $\mathbf{z}$. (by assumptions)
And the coefficient G for the first fundamental form is

$$\begin{aligned} G &= \mathbf{x}_v \cdot \mathbf{x}_v \\ &= \mathbf{z}(u) \cdot \mathbf{z}(u) \\ &= 1. \end{aligned}$$

Since $\mathbf{z}(u)$ is the unit vector.

The following results are the calculations for the coefficients of second fundamental form.

$$\begin{aligned}
L &= \mathbf{x}_{uu} \cdot \mathbf{N} \\
&= \Big(\mathbf{y}''(u) + v\mathbf{z}''(u)\Big) \cdot \Big(\big(\mathbf{y}'(u) + v\mathbf{z}'(u)\big) \times \mathbf{z}(u)\Big). \\
M &= \mathbf{x}_{uv} \cdot \mathbf{N} \\
&= \mathbf{z}'(u) \cdot \Big(\big(\mathbf{y}'(u) + v\mathbf{z}'(u)\big) \times \mathbf{z}(u)\Big). \\
N &= \mathbf{x}_{vv} \cdot \mathbf{N} \\
&= 0 \cdot \mathbf{N} \\
&= 0.
\end{aligned}$$

The coefficient N is zero since $\mathbf{x}_{vv} = 0$.

Now we can calculate the mean curvature using the coefficients of the first and second fundamental form. Recall the mean curvature

$$\boldsymbol{H} = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2}.$$

S is a minimal surface, then $\boldsymbol{H} = 0$
Then,

$$EN - 2FM + GL = 0 \tag{2.8}$$

Since the coefficient F of the first fundamental form and the coefficient N of the second fundamental form are equal to zero, then equation (2.8) becomes

$$
\begin{aligned}
L &= 0 \\
&= \Big(\mathbf{y}''(u) + v\mathbf{z}''(u)\Big) \cdot \Big(\big(\mathbf{y}'(u) + v\mathbf{z}'(u)\big) \times \mathbf{z}(u)\Big) \\
&= \Big(\mathbf{y}''(u) + v\mathbf{z}''(u)\Big) \cdot \Big(\mathbf{y}'(u) \times \mathbf{z}(u) + v\mathbf{z}'(u) \times \mathbf{z}(u)\Big) \\
&= v^2\mathbf{z}''(u) \cdot \big(\mathbf{z}'(u) \times \mathbf{z}(u)\big) + v\Big(\mathbf{y}''(u) \cdot \big(\mathbf{z}'(u) \times \mathbf{z}(u)\big) + \mathbf{z}''(u) \cdot \big(\mathbf{y}'(u) \times \mathbf{z}(u)\big)\Big) \\
&\qquad + \ \mathbf{y}''(u) \cdot \big(\mathbf{y}'(u) \times \mathbf{z}(u)\big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0 &= v^2\mathbf{z}''(u) \cdot \big(\mathbf{z}'(u) \times \mathbf{z}(u)\big) + v\Big(\mathbf{y}''(u) \cdot \big(\mathbf{z}'(u) \times \mathbf{z}(u)\big) + \mathbf{z}''(u) \cdot \big(\mathbf{y}'(u) \times \mathbf{z}(u)\big)\Big) \\
&\qquad + \ \mathbf{y}''(u) \cdot \big(\mathbf{y}'(u) \times \mathbf{z}(u)\big).
\end{aligned}
$$

We can observed the coefficients of the quadratic equation in terms of v, where the coefficients must be zero.

Therefore, we have the following three differential equations:

$$\mathbf{z}''(u) \cdot \big(\mathbf{z}'(u) \times \mathbf{z}(u)\big) = 0, \tag{2.9}$$

$$\mathbf{y}''(u) \cdot \big(\mathbf{z}'(u) \times \mathbf{z}(u)\big) + \mathbf{z}''(u) \cdot \big(\mathbf{y}'(u) \times \mathbf{z}(u)\big) = 0, \tag{2.10}$$

$$\mathbf{y}''(u) \cdot \big(\mathbf{y}'(u) \times \mathbf{z}(u)\big) = 0. \tag{2.11}$$

From equation (2.9) we obtain

$\Rightarrow$ $\mathbf{z}''(u)$ is a linear combination of $\{\mathbf{z}(u), \mathbf{z}'(u)\}$.

$\Rightarrow$ $\mathbf{z}''(u) = a\mathbf{z}(u) + b\mathbf{z}'(u)$

By taking the dot product with $\mathbf{z}'(u)$, then

$\Rightarrow$ $\mathbf{z}''(u) \cdot \mathbf{z}'(u) = a\mathbf{z}(u) \cdot \mathbf{z}'(u) + b\mathbf{z}'(u) \cdot \mathbf{z}'(u)$

$\Rightarrow$ $\mathbf{z}''(u) \cdot \mathbf{z}'(u) = b\mathbf{z}'(u) \cdot \mathbf{z}'(u)$

(Since $\mathbf{z}(u) \cdot \mathbf{z}'(u) = 0$

and by assumption $\mathbf{z}''(u) \cdot \mathbf{z}'(u) = 0$)

$$\Rightarrow \quad b = 0 \quad (\text{since } \mathbf{z}'(u) \cdot \mathbf{z}'(u) = 1 \text{ by assumption})$$

Since $b = 0$ then,

$$\mathbf{z}''(u) = a\mathbf{z}(u)$$

$$\Rightarrow \quad \mathbf{z}''(u) \| \mathbf{z}(u).$$

So, $\{\mathbf{z}(u), \mathbf{z}'(u), \mathbf{y}'(u)\}$ is linearly independent.

If not, then $\mathbf{y}'(u)$ is a linear combination of $\{\mathbf{z}(u), \mathbf{z}'(u)\}$.

$$\Rightarrow \quad \mathbf{y}'(u) = a\mathbf{z}(u) + b\mathbf{z}'(u)$$

by taking the dot product with $\mathbf{z}(u)$, then

$$\Rightarrow \quad \mathbf{y}'(u) \cdot \mathbf{z} = a\mathbf{z}(u) \cdot \mathbf{z}(u) + b\mathbf{z}'(u) \cdot \mathbf{z}(u)$$

$$\Rightarrow \quad \mathbf{y}'(u) \cdot \mathbf{z} = a\mathbf{z}(u) \cdot \mathbf{z}(u)$$

(Since $\mathbf{z}'(u) \cdot \mathbf{z}(u) = 0$)

$$\Rightarrow \quad a = 0 \qquad (\text{since } \mathbf{z}(u) \cdot \mathbf{z}(u) = 1)$$

Since $a = 0$,

$$\Rightarrow \quad \mathbf{y}'(u) = b\mathbf{z}'(u)$$

$$\Rightarrow \quad \mathbf{y}'(u) \| \mathbf{z}'(u)$$

$$\begin{aligned} \Rightarrow \quad \mathbf{N} &= \mathbf{x}_u \times \mathbf{x}_v \\ &= (\mathbf{y}'(u) + v\mathbf{z}'(u)) \cdot \mathbf{z}(u) \\ &= (b\mathbf{z}'(u) + v\mathbf{z}'(u)) \cdot \mathbf{z}(u) \quad (\text{since } \mathbf{y}'(u) = b\mathbf{z}'(u)) \end{aligned}$$

$$= (b + v)\mathbf{z}'(u) \times \mathbf{z}(u)$$

$$\left(\text{since} \quad \mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\right)$$

Therefore,

$$\mathbf{N} = \mathbf{z}'(u) \times \mathbf{z}$$

$$
\begin{aligned}
\Rightarrow \quad \mathbf{N}' &= \left(\mathbf{z}'(u) \times \mathbf{z}\right)' \\
&= \mathbf{z}''(u) \times \mathbf{z}(u) + \mathbf{z}'(u) \times \mathbf{z}'(u) \\
&= \mathbf{z}''(u) \times \mathbf{z}(u) \quad (\text{since } \mathbf{z}'(u) \times \mathbf{z}'(u) = 0) \\
&= a\mathbf{z}(u) \times \mathbf{z}(u) \quad (\text{since } \mathbf{z}''(u) = a\mathbf{z}(u)) \\
&= 0 \qquad (\text{since } \mathbf{z}(u) \times \mathbf{z}(u) = 0) \\
\Rightarrow \quad & \mathbf{N} \text{ is a constant.} \\
\Rightarrow \quad & \textbf{Surface is a plane.}
\end{aligned}
$$

From equation (2.10):

$$
\begin{aligned}
\Rightarrow \quad & \mathbf{y}''(u) \cdot \left(\mathbf{z}'(u) \times \mathbf{z}(u)\right) = 0 \\
\Rightarrow \quad & \mathbf{y}''(u) \text{ is a linear combination of } \{\mathbf{z}(u), \mathbf{z}'(u)\}. \\
\Rightarrow \quad & \mathbf{y}''(u) = a\mathbf{z}(u) + b\mathbf{z}'(u).
\end{aligned}
$$

From equation (2.11):

$$
\begin{aligned}
\Rightarrow \quad & \mathbf{y}''(u) \text{ is a linear combination of } \{\mathbf{z}(u), \mathbf{y}'(u)\}. \\
\Rightarrow \quad & \mathbf{y}''(u) = c\mathbf{z}(u) + d\mathbf{y}'(u).
\end{aligned}
$$

From equation (2.10) and (2.11) we have the following differential equations:

$$
\begin{aligned}
\mathbf{y}''(u) &= a\mathbf{z}(u) + b\mathbf{z}'(u), \\
\mathbf{y}''(u) &= c\mathbf{z}(u) + d\mathbf{y}'(u).
\end{aligned}
$$

Then,

$$
\begin{aligned}
& a\mathbf{z}(u) + b\mathbf{z}'(u) = c\mathbf{z}(u) + d\mathbf{y}'(u) \\
\Rightarrow \quad & (a-c)\mathbf{z}(u) + b\mathbf{z}'(u) - d\mathbf{y}'(u) = 0. \qquad (2.12)
\end{aligned}
$$

Since $\{\mathbf{z}(u), \mathbf{z}'(u), \mathbf{y}'(u)\}$ is linearly independent, then the coefficients $(a-c), b$ and d of equation (2.8) are equal to zero.
Therefore,

$$
\mathbf{y}''(u) = a\mathbf{z}(u) \quad \text{and} \quad \mathbf{y}''(u) = c\mathbf{z}(u).
$$

Since $a = c, b = 0$ and $d = 0$.

Thus,

$$\mathbf{y}''(u) \| \mathbf{z}(u).$$

Recall that a ruled surface is defined as a surface generated by the motion of a straight line, which we refered to ruling. By observing the three coefficients of the quadratic equation in terms of v we concluded the following:

$$\mathbf{z}''(u) \| \mathbf{z}(u), \tag{2.13}$$

and

$$\mathbf{y}''(u) \| \mathbf{z}(u). \tag{2.14}$$

By equations (2.9) and (2.10) implies the following:

$$\begin{aligned}
(\mathbf{y}'(u) \cdot \mathbf{z}'(u))' &= \mathbf{y}''(u) \cdot \mathbf{z}'(u) + \mathbf{y}'(u) \cdot \mathbf{z}''(u) \\
&= 0 \qquad (\text{since } \mathbf{z} \perp \mathbf{z}' \text{ and } \mathbf{z} \perp \mathbf{y}') \\
\Rightarrow \qquad \mathbf{y}'(u) \cdot \mathbf{z}'(u) &= a. \qquad (2.15)
\end{aligned}$$

where $a =$ **constant**.

Now we reparametrize the surface S as the following;

$$\begin{aligned}
\mathbf{x}^*(u, v) &= \mathbf{y}(u) - a\mathbf{z}(u) + (v + a)\mathbf{z}(u) \\
&= \mathbf{y}_1(u^*) + v^*\mathbf{z}_1(u^*),
\end{aligned}$$

where

$$\begin{aligned}
u^* &= u, \\
v^* &= (v + a), \\
\mathbf{y}_1(u^*) &= \mathbf{y}(u) - a\mathbf{z}(u), \\
&\text{and} \\
\mathbf{z}_1(u^*) &= \mathbf{z}(u).
\end{aligned}$$

Then, the following conditions still satiesfied for this reparametrization:

$$\begin{aligned}
\mathbf{z}_1(u^*) \cdot \mathbf{z}_1(u^*) &= 1, \\
\mathbf{z}_1'(u^*) \cdot \mathbf{z}_1'(u^*) &= 1, \\
&\text{and} \\
\mathbf{y}_1'(u^*) \cdot \mathbf{z}_1(u^*) &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{y}_1'(u^*) &= \mathbf{y}'(u) - a\mathbf{z}'(u), \\
&\text{and} \\
\mathbf{z}_1'(u^*) &= \mathbf{z}'(u).
\end{aligned}$$

Now consider the orthagonal reparametrization of the surface $\mathbf{x}^*(u, v)$. Then,

$$\begin{aligned}
\mathbf{y}_1'(u^*) \cdot \mathbf{z}_1'(u^*) &= \big(\mathbf{y}'(u) - a\mathbf{z}'(u)\big) \cdot \big(\mathbf{z}'(u)\big) \\
&= (\mathbf{y}'(u) \cdot \mathbf{z}'(u)) - a(\mathbf{z}'(u) \cdot \mathbf{z}'(u)) \\
&= a - a \qquad \text{(by equation (2.15) and } \mathbf{z}'(u) \cdot \mathbf{z}'(u) = 1).
\end{aligned}$$

Therefore,

$$\mathbf{y}_1'(u^*) \cdot \mathbf{z}_1'(u^*) = 0. \tag{2.16}$$

Now if we take the first derivative of $\mathbf{y}_1'(u^*) \cdot \mathbf{z}_1(u^*) = 0$ using the product rule we obtain

$$\begin{aligned}
& & \big(\mathbf{y}_1'(u^*) \cdot \mathbf{z}_1(u^*)\big)' &= 0 \\
&\Rightarrow & \mathbf{y}_1''(u^*) \cdot \mathbf{z}_1(u^*) + \mathbf{y}_1'(u^*) \cdot \mathbf{z}_1'(u^*) &= 0 \\
&\Rightarrow & \mathbf{y}_1''(u^*) \cdot \mathbf{z}_1(u^*) &= 0 \qquad \text{(by equation (2.16))} \\
&\Rightarrow & \mathbf{y}_1''(u^*) &= 0 \\
&\Rightarrow & \mathbf{y}_1(u^*) &\text{ is linear.}
\end{aligned}$$

Hence,

$$\mathbf{y}_1(u^*) \text{ is a straight line.}$$

We may assume

$$\mathbf{y_1}(u^*) = (0, 0, bu^*)$$

Then,

$$\begin{aligned} \mathbf{z_1}(u^*) &= (\cos u^*, \sin u^*, 0) \\ \Rightarrow \qquad \mathbf{x}^*(u, v) &= (0, 0, bu^*) + v^*(\cos u^*, \sin u^*, 0) \end{aligned}$$

Thus, we have a **right helicoid**. *(Figure 2.1)*

$\square$

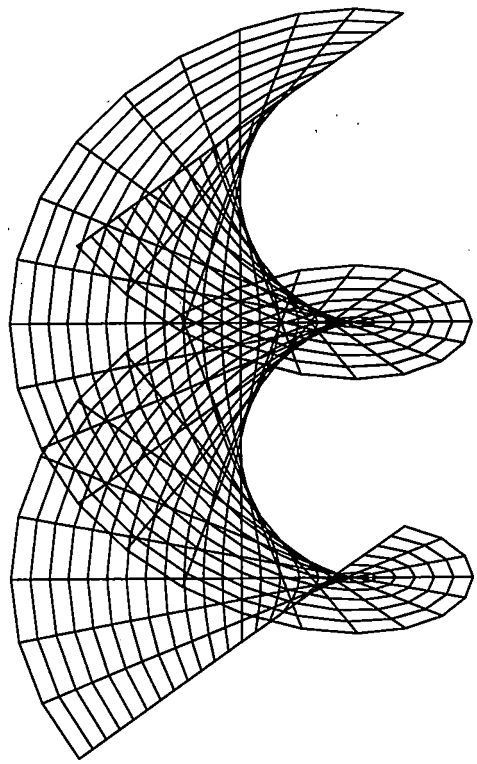


Figure 2.1: Right Helicoid

Appendix A

Maple

The majority of the geometrical figures provided in this project are created using MAPLE. The following information consist on the comands that provides the geometrical figures on MAPLE. Moreover, other figures are provided that represents minimal surfaces. The figure of a Sphere (1.7) can be executed using the following comands on MAPLE.

Figure 1.7: Sphere

> with(plots):

> sphereplot(1,theta=0..2*Pi,phi=0..2*Pi,color=black, style=wireframe, axes=none);

Figure 1.8: Torus

> with(plots): > setoptions3d(style=wireframe,color=black, axes=none, scaling=CONSTRAINED);

> plot3d([cos(y) * (10.0 + 4.0 * cos(x)), sin(y) * (10.0 + 4.0 * cos(x)), -6.0*sin(x)], x=-Pi..Pi,y=-Pi..Pi);

Figure 2.1: Right Helicoid

> with(plots):

> setoptions3d(scaling=constrained):

> a:=2;

> Helicat := (x,y,t) $\rightarrow$

$[((1/a) * (\text{sqrt}(1 + (a * x)^2)) * t + (1 - t) * x) * \cos(a * y),$

$((1/a) * (\text{sqrt}(1 + (a * x)^2)) * t + (1 - t) * x) * \sin(a * y),$

$((1/a) * \text{arcsinh}(a * x)) * t - (1 - t) * y];$

> Surface := plot3d(Helicat(x,y,0.0), x=-1..1, y=0..2*Pi/a,color=black,style=wireframe):

> display(Surface);

Catenoid.

> with(plots):

> setoptions3d(scaling=constrained):

> a:=2;

> Helicat := (x,y,t) $\rightarrow$

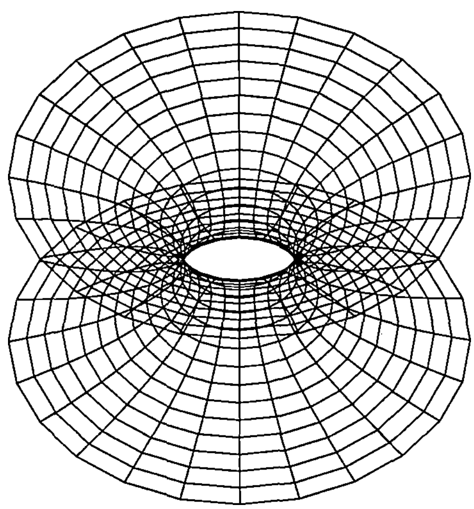
$[((1/a) * (\mathrm{sqrt}(1 + (a * x)^2)) * t - (1 - t) * x) * \cos(a * y),$

$((1/a) * (\mathrm{sqrt}(1 + (a * x)^2)) * t + (1 - t) * x) * \sin(a * y),$

$((1/a) * \mathrm{arcsinh}(a * x)) * t + (1 - t) * y];$

> Surface := plot3d(Helicat(x,y,1), x=-2..2, y=Pi..2*Pi,style=wireframe, color=black, axes=none):

>display(Surface);

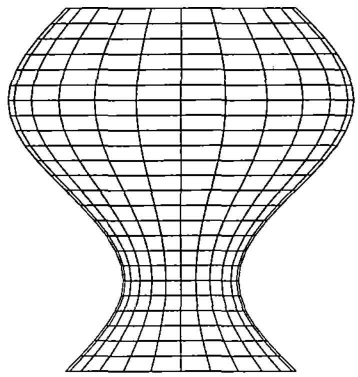


Surface of Revolution

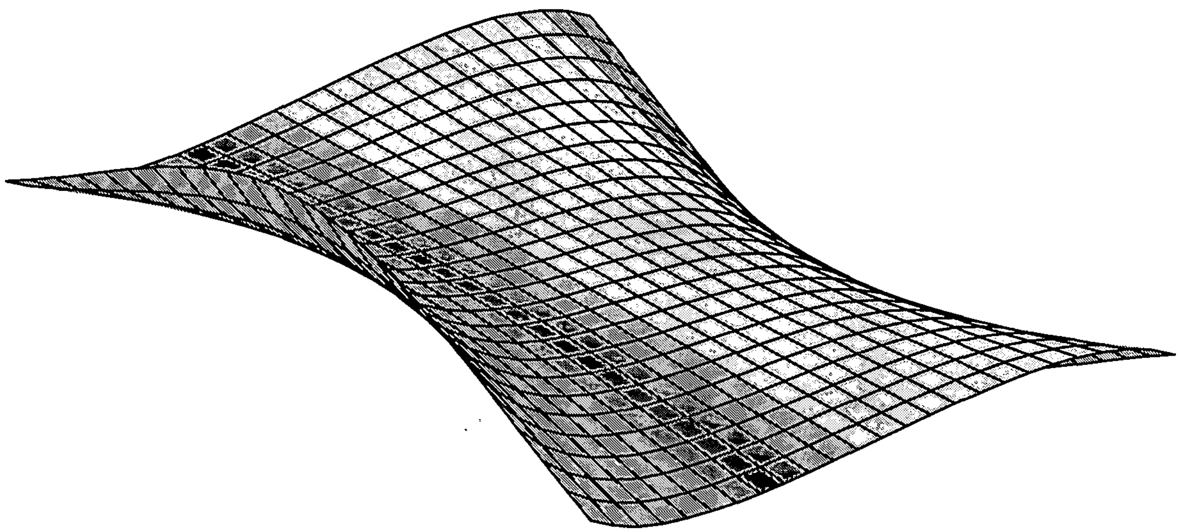
> with(plots):

> setoptions3d(scaling=constrained); f:= (x) → sin(x) + 2;

> f := x → sin(x) + 2; plot3d([u, f(u)*cos(v), f(u)*sin(v)], u=0..2*Pi, v=0..2*Pi, style=wireframe, axes=none, orientation=[180,-180], color=black);



> $\mathrm{plot3d}(\mathrm{x} * \exp(\mathrm{x}^2 - \mathrm{y}^2), \mathrm{x} = -1..1, \mathrm{y} = -1..1, \mathrm{color} = \mathrm{x});$



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