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Classification of Finite Irreducible Modules over the Lie Conformal Superalgebra CK_6

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Abstract: We classify all continuous degenerate irreducible modules over the exceptional linearly compact Lie superalgebra $E(1, 6)$, and all finite degenerate irreducible modules over the exceptional Lie conformal superalgebra CK_6, for which $E(1, 6)$ is the annihilation superalgebra.

1. Introduction

Lie conformal superalgebras encode the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory [9].

A complete classification of finite simple Lie conformal superalgebras was obtained in [8]. The list consists of current Lie conformal superalgebras Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra, four series of "Virasoro like" Lie conformal superalgebras $W_n (n \geq 0)$, $S_{n,b}$ and $\tilde{S}_n (n \geq 2, b \in \mathbb{C})$, $K_n (n \geq 0, n \neq 4)$, K_4', and the exceptional Lie conformal superalgebra CK_6.

All finite irreducible representations of the simple Lie conformal superalgebras Cur $\mathfrak{g}$, $K_0 = \text{Vir}$ and K_1 were constructed in [4], and those of $S_{2,0}$, $W_1 = K_2$, K_3, and K_4 in [7]. More recently, the problem has been solved for all Lie conformal superalgebras of the three series W_n, $S_{n,b}$, and $\tilde{S}_n$ in [1], and for all Lie conformal superalgebras of the remaining series $K_n (n \geq 4)$ in [2]. The construction in all cases relies on the observation that the representation theory of a Lie conformal superalgebra R is controlled by the representation theory of the associated (extended) *annihilation algebra* $\mathfrak{g} = (\text{Lie}\, R)_+$ [4], thereby reducing the problem to the construction of continuous irreducible modules with discrete topology over the linearly compact Lie superalgebra $\mathfrak{g}$.

The construction of the latter modules consists of two parts. First one constructs a collection of continuous $\mathfrak{g}$-modules $\mathrm{Ind}(F)$, associated to all finite-dimensional irreducible $\mathfrak{g}_0$-modules F, where $\mathfrak{g}_0$ is a certain subalgebra of $\mathfrak{g}(= \mathfrak{gl}(1|n))$ or $\mathfrak{sl}(1|n)$ for the W and S series, and $= \mathfrak{cso}_n$ for the K_n series and CK_6).

The irreducible $\mathfrak{g}$-modules $\mathrm{Ind}(F)$ are called non-degenerate. The second part of the problem consists of two parts: (A) classify the $\mathfrak{g}_0$-modules F, for which the $\mathfrak{g}$-modules $\mathrm{Ind}(F)$ are non-degenerate, and (B) construct explicitly the irreducible quotients of $\mathrm{Ind}(F)$, called *degenerate* $\mathfrak{g}$-modules, for reducible $\mathrm{Ind}(F)$.

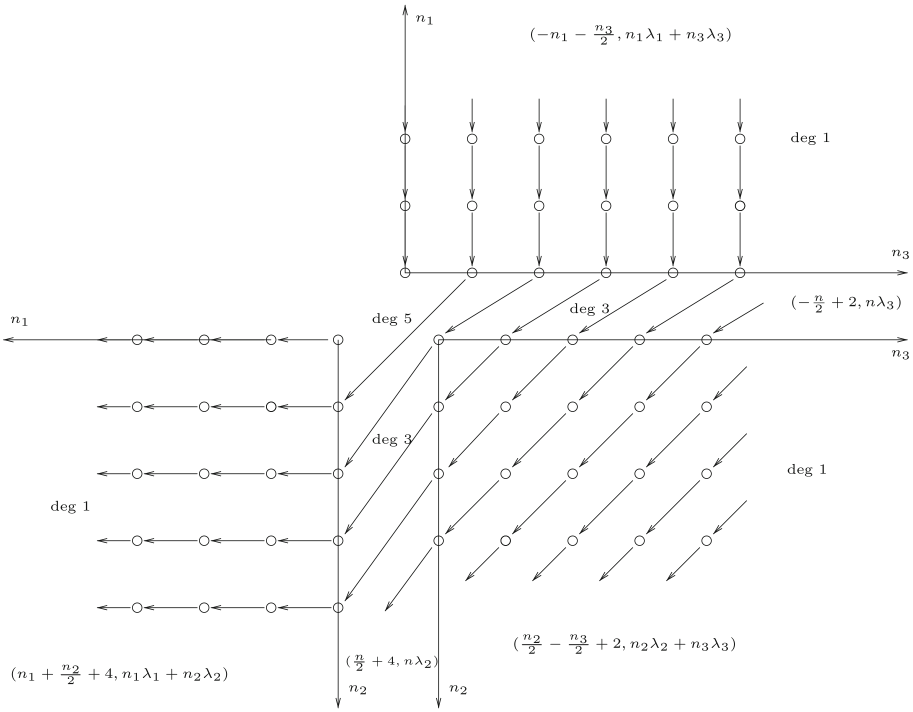
Both problems have been solved for types W and S in [1], and it turned out, remarkably, that all degenerate modules occur as cokernels of the super de Rham complex or their duals. More recently both problems have been solved for type K in [2], and it turned out that again, all degenerate modules occur as cokernels of a certain complex or their duals. This complex is a certain reduction of the super de Rham complex, called in [2] the super contact complex (since it is a "super" generalization of the contact complex of M. Rumin).

The present paper is the first in the series of three papers on construction of all finite irreducible representations of the Lie conformal superalgebra CK_6. In this paper we find all singular vectors of the $\mathfrak{g}$- modules $\mathrm{Ind}(F)$ for $\mathfrak{g} = E(1, 6)$, where F is a finite-dimensional irreducible representation of the Lie algebra $\mathfrak{cso}_6$. In particular, we find the list of all finite degenerate irreducible modules over CK_6. In our second paper [3] we give a proof of the key Lemma 4.4, and in the third paper construct the complexes, consisting of all degenerate $E(1, 6)$-modules $\mathrm{Ind}(F)$, providing thereby an explicit construction of all finite irreducible degenerate modules over CK_6.

All degenerate $E(1, 6)$-modules $\mathrm{Ind}(F)$ can be represented by the diagram at the end of the introduction (very similar to that for $E(5, 10)$ in [11]), with the point $(4, 0)$ excluded, where the nodes represent the highest weights of the modules $\mathrm{Ind}(F)$, and arrows represent the morphisms between these modules. Here $\lambda_2, \lambda_1, \lambda_3$ are the fundamental weights of $\mathfrak{so}_6 = A_3$ (where λ_1 is attached to the middle node of the Dynkin diagram).

In the subsequent publication we shall compute cohomology of these complexes, providing thereby an explicit construction of all degenerate continuous irreducible $E(1, 6)$-modules, hence of all degenerate finite irreducible CK_6-modules.

This work is organized as follows: In Sect. 2 we introduce notations and definitions of formal distributions, Lie conformal superalgebras and their modules. In Sect. 3, we introduce the Lie conformal superalgebra CK_6, the annihilation Lie superalgebra $E(1, 6)$ and the induced modules. In Sect. 4, we classify the singular vectors of the induced modules (Theorem 4.1), and in Theorem 4.3 we present the list of highest weights of degenerate irreducible modules. The last part of this section and Appendix A are devoted to their proofs through several lemmas. More precisely, we used the software Macaulay2 to simplify the computations, and Appendix A contains the notations in the complementary files that use Macaulay2 and the reduction procedure to find simplified conditions on singular vectors of small degrees. All these simplified and equivalent lists of equations, obtained with the software as explained in Appendix A, are analyzed in detail in the proofs of Lemmas 4.7-4.11 in Sect. 4.



2. Formal Distributions, Lie Conformal Superalgebras and their Modules

In this section we introduce the basic definitions and notations in order to have a more or less self-contained work, for details see [2,8,9] and references therein.

Definition 2.1. *A* **Lie conformal superalgebra R** *is a left* $\mathbb{Z}/2\mathbb{Z}$*-graded* $\mathbb{C}[\partial]$*-module endowed with a* $\mathbb{C}$*-linear map* $R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes R$, $a \otimes b \mapsto [a_\lambda b]$*, called the* λ*-bracket, satisfying the following axioms* $(a,\ b,\ c \in R)$*:*

Conformal sesquilinearity	$[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b],$
Skew-symmetry	$[a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial}\ a],$
Jacobi identity	$[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]].$

Here and further $p(a) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of a.

A Lie conformal superalgebra is called *finite* if it has finite rank as a $\mathbb{C}[\partial]$-module. The notions of homomorphism, ideal and subalgebras of a Lie conformal superalgebra are defined in the usual way. A Lie conformal superalgebra R is *simple* if $[R_\lambda R] \neq 0$ and contains no ideals except for zero and itself.

Definition 2.2. *A* **module** *M over a Lie conformal superalgebra R is a* $\mathbb{Z}/2\mathbb{Z}$*-graded* $\mathbb{C}[\partial]$*-module endowed with a* $\mathbb{C}$*-linear map* $R \otimes M \longrightarrow \mathbb{C}[\lambda] \otimes M$, $a \otimes v \mapsto a_\lambda^M v$*, satisfying the following axioms* $(a,\ b \in R),\ v \in M$*:*

$(M1)_\lambda\ (\partial a)_\lambda^M v = [\partial^M, a_\lambda^M]v = -\lambda a_\lambda^M v,$
$(M2)_\lambda\ [a_\lambda^M, b_\mu^M]v = [a_\lambda b]_{\lambda+\mu}^M v.$

An R-module M is called **finite** *if it is finitely generated over* $\mathbb{C}[\partial]$. *An R-module M is called* **irreducible** *if it contains no non-trivial submodule, where the notion of submodule is the usual one.*

Given a Lie conformal superalgebra R, let $\tilde{R} = R[t, t^{-1}]$ with $\tilde{\partial} = \partial + \partial_t$ and define the bracket [9]:

$$[at^n, bt^m] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} [a_{(j)}b] t^{m+n-j}. \tag{2.1}$$

Observe that $\tilde{\partial}\tilde{R}$ is an ideal of $\tilde{R}$ with respect to this bracket, and consider the Lie superalgebra $\mathrm{Alg}R = \tilde{R}/\tilde{\partial}\tilde{R}$ with this bracket.

An important tool for the study of Lie conformal superalgebras and their modules is the (extended) annihilation superalgebra. The *annihilation superalgebra* of a Lie conformal superalgebra R is the subalgebra $\mathcal{A}(R)$ (also denoted by $\mathrm{Alg}R_+$) of the Lie superalgebra $\mathrm{Alg}R$ spanned by all elements at^n, where $a \in R$, $n \in \mathbb{Z}_+$. It is clear from (2.1) that this is a subalgebra, which is invariant with respect to the derivation $\partial = -\partial_t$ of $\mathrm{Alg}R$. The *extended annihilation superalgebra* is defined as

$$\mathcal{A}(R)^e = (\mathrm{Alg}R)^+ := \mathbb{C}\partial \ltimes (\mathrm{Alg}R)_+.$$

Introducing the generating series

$$a_\lambda = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} (at^j), \quad a \in R, \tag{2.2}$$

we obtain from (2.1):

$$[a_\lambda, b_\mu] = [a_\lambda b]_{\lambda+\mu}, \quad \partial(a_\lambda) = (\partial a)_\lambda = -\lambda a_\lambda. \tag{2.3}$$

Formula (2.3) implies the following important proposition relating modules over a Lie conformal superalgebra R to certain modules over the corresponding extended annihilation superalgebra $(\mathrm{Alg}R)^+$.

Proposition 2.3 [4]. *A module over a Lie conformal superalgebra R is the same as a module over the Lie superalgebra* $(\mathrm{Alg}R)^+$ *satisfying the property*

$$a_\lambda m \in \mathbb{C}[\lambda] \otimes M \quad \text{for any } a \in R, m \in M. \tag{2.4}$$

(One just views the action of the generating series a_λ of $(\mathrm{Alg}R)^+$ *as the λ-action of $a \in R$.)*

The problem of classifying modules over a Lie conformal superalgebra R is thus reduced to the problem of classifying a class of modules over the Lie superalgebra $(\mathrm{Alg}R)^+$.

Let $\mathfrak{g}$ be a Lie superalgebra satisfying the following three conditions (cf. [7], p. 911):

(L1) $\mathfrak{g}$ is $\mathbb{Z}$-graded of finite depth $d \in \mathbb{N}$, i.e. $g = \oplus_{j \geq -d} \mathfrak{g}_j$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.
(L2) There exists a semisimple element $z \in \mathfrak{g}_0$ such that its centralizer in $\mathfrak{g}$ is contained in $\mathfrak{g}_0$.
(L3) There exists an element $\partial \in \mathfrak{g}_{-d}$ such that $[\partial, g_i] = g_{i-d}$, for $i \geq 0$.

Some examples of Lie superalgebras satisfying (L1)–(L3) are provided by annihilation superalgebras of Lie conformal superalgebras.

If $\mathfrak{g}$ is the annihilation superalgebra of a Lie conformal superalgebra, then the modules V over $\mathfrak{g}$ that correspond to finite modules over the corresponding Lie conformal superalgebra satisfy the following conditions:

(1) For all $v \in V$ there exists an integer $j_0 \geq -d$ such that $\mathfrak{g}_j v = 0$, for all $j \geq j_0$.
(2) V is finitely generated over $\mathbb{C}[\partial]$.

Motivated by this, the $\mathfrak{g}$-modules satisfying these two properties are called *finite conformal modules*.

We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_{<0} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{>0}, \qquad \text{with } \mathfrak{g}_{<0} = \oplus_{j<0}\mathfrak{g}_j, \mathfrak{g}_{>0} = \oplus_{j>0}\mathfrak{g}_j. \tag{2.5}$$

Let $\mathfrak{g}_{\geq 0} = \oplus_{j\geq 0}\mathfrak{g}_j$. Given a $\mathfrak{g}_{\geq 0}$-module F, we may consider the associated induced $\mathfrak{g}$-module

$$\mathrm{Ind}(F) = \mathrm{Ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} F = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F,$$

called the *generalized Verma module* associated to F. We shall identify $\mathrm{Ind}(F)$ with $U(\mathfrak{g}_{<0}) \otimes F$ via the PBW theorem.

Let V be a $\mathfrak{g}$-module. The elements of the subspace

$$\mathrm{Sing}(V) := \{v \in V | \mathfrak{g}_{>0} v = 0\}$$

are called *singular vectors*. For us the most important case is when $V = \mathrm{Ind}(F)$. The $\mathfrak{g}_{\geq 0}$-module F is canonically a $\mathfrak{g}_{\geq 0}$-submodule of $\mathrm{Ind}(F)$, and $\mathrm{Sing}(F)$ is a subspace of $\mathrm{Sing}(\mathrm{Ind}(F))$, called the *subspace of trivial singular vectors*. Observe that $\mathrm{Ind}(F) = F \oplus F_+$, where $F_+ = U_+(\mathfrak{g}_{<0}) \otimes F$ and $U_+(\mathfrak{g}_{<0})$ is the augmentation ideal of the algebra $U(\mathfrak{g}_{<0})$. Then non-zero elements of the space

$$\mathrm{Sing}_+(\mathrm{Ind}(F)) := \mathrm{Sing}(\mathrm{Ind}(F)) \cap F_+$$

are called *non-trivial singular vectors*. The following simple key result will be used in the rest of the paper, see [7, 10].

Theorem 2.4. *Let $\mathfrak{g}$ be a Lie superalgebra that satisfies (L1)–(L3).*

(a) If F is an irreducible finite-dimensional $\mathfrak{g}_{\geq 0}$-module, then the subalgebra $\mathfrak{g}_{>0}$ acts trivially on F and $\mathrm{Ind}(F)$ *has a unique maximal submodule.*
(b) Denote by $\mathrm{Ir}(F)$ *the quotient by the unique maximal submodule of* $\mathrm{Ind}(F)$. *Then the map $F \mapsto \mathrm{Ir}(F)$ defines a bijective correspondence between irreducible finite-dimensional $\mathfrak{g}_0$-modules and irreducible finite conformal $\mathfrak{g}$-modules.*
(c) A $\mathfrak{g}$-module $\mathrm{Ind}(F)$ *is irreducible if and only if the $\mathfrak{g}_0$-module F is irreducible and* $\mathrm{Ind}(F)$ *has no non-trivial singular vectors.*

In the following section we will describe the Lie conformal superalgebra CK_6 and its annihilation superalgebra $E(1,6)$. In the remaining sections we shall study the induced $E(1,6)$-modules and their singular vectors in order to apply Theorem 2.4 to get the classification of irreducible finite modules over the Lie conformal algebra CK_6.

3. Lie Conformal Superalgebra CK_6, Annihilation Lie Superalgebra $E(1,6)$ and the Induced Modules

Let $\Lambda(n)$ be the Grassmann superalgebra in the n odd indeterminates $\xi_1, \xi_2, \ldots, \xi_n$. Let t be an even indeterminate, and let $\Lambda(1,n)_+ = \mathbb{C}[t] \otimes \Lambda(n)$. The Lie conformal superalgebra K_n can be identified with

$$K_n = \mathbb{C}[\partial] \otimes \Lambda(n), \tag{3.1}$$

the λ-bracket for $f = \xi_{i_1} \ldots \xi_{i_r}$, $g = \xi_{j_1} \ldots \xi_{j_s}$ being as follows [8]:

$$[f_\lambda g] = \left((r-2)\partial(fg) + (-1)^r \sum_{i=1}^{n} (\partial_i f)(\partial_i g) \right) + \lambda(r+s-4) fg, \tag{3.2}$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$.

The annihilation Lie superalgebra of K_n can be identified with (see [2])

$$\mathcal{A}(K_n) = K(1,n)_+ = \Lambda(1,n)_+, \tag{3.3}$$

with the corresponding Lie bracket for elements $f, g \in \Lambda(1,n)$ being

$$[f,g] = \left(2f - \sum_{i=1}^{n} \xi_i \partial_i f\right)(\partial_t g) - (\partial_t f)\left(2g - \sum_{i=1}^{n} \xi_i \partial_i g\right) + (-1)^{p(f)} \sum_{i=1}^{n} (\partial_i f)(\partial_i g).$$

The extended annihilation superalgebra is

$$\mathcal{A}(K_n)^e = K(1,n)^+ = \mathbb{C}\,\partial \ltimes K(1,n)_+,$$

where ∂ acts on it as $-\mathrm{ad}\,\partial_t$. Note that $\mathcal{A}(K_n)^e$ is isomorphic to the direct sum of $\mathcal{A}(K_n)$ and the trivial 1-dimensional Lie algebra $\mathbb{C}(\partial + \frac{1}{2})$.

We define in $K(1,n)_+$ a gradation by putting

$$\deg(t^m \xi_{i_1} \ldots \xi_{i_k}) = 2m + k - 2,$$

making it a $\mathbb{Z}$-graded Lie superalgebra of depth 2: $K(1,n)_+ = \oplus_{j \geq -2} (K(1,n)_+)_j$. It is easy to check that $K(1,n)_+$ satisfies conditions (L1)–(L3).

We introduce the following notation:

$$\begin{aligned} \xi_I &:= \xi_{i_1} \ldots \xi_{i_k}, && \text{if} \quad I = \{i_1, \ldots, i_k\}, \\ |f| &:= k, && \text{if} \quad f = \xi_{i_1} \ldots \xi_{i_k}. \end{aligned}$$

For a monomial $\xi_I \in \Lambda(n)$, we let $\xi_I{}^*$ be its Hodge dual, i.e. the unique monomial in $\Lambda(n)$ such that $\xi_I \xi_I^* = \xi_1 \ldots \xi_n$.

Warning. This definition corresponds to the one in [4] or [7], p. 922, but in [2] Theorem 4.3, the Hodge dual was defined in a different way.

The Lie conformal superalgebra CK_6 is defined as the subalgebra of K_6 given by (cf. [5], Thm. 3.1)

$$CK_6 = \mathbb{C}[\partial]\text{-span}\,\{f - i(-1)^{\frac{|f|(|f|+1)}{2}} (-\partial)^{3-|f|} f^* \,:\, f \in \Lambda(6), 0 \leq |f| \leq 3\}.$$

Now, we define a linear operator $A : K(1,6)_+ \to K(1,6)_+$ by (cf. [6], p. 267)

$$A(f) = (-1)^{\frac{d(d+1)}{2}} \left(\frac{d}{dt}\right)^{3-d} (f^*), \tag{3.4}$$

where f is a monomial in $K(1,6)_+$, d is the number of odd indeterminates in f, the operator $(\frac{d}{dt})^{-1}$ indicates integration with respect to t (i.e. it sends t^n to $t^{n+1}/(n+1)$), and f^* is the Hodge dual of f. Then, the annihilation Lie superalgebra $E(1,6)$ of CK_6 is identified with the subalgebra of $K(1,6)_+$ given by the image of the operator $I - iA$. Since the linear map A preserves the $\mathbb{Z}$-gradation, the subalgebra $E(1,6)$ inherits the $\mathbb{Z}$-gradation.

Using Theorem 2.4, the classification of finite irreducible CK_6-modules can be reduced to the study of induced modules for $E(1,6)$. Observe that the graded subspaces of $E(1,6)$ and $K(1,6)_+$ with non-positive degree are the same. Namely,

$$\begin{aligned} E(1,6)_{-2} &= \langle\{\mathbf{1}\}\rangle, \\ E(1,6)_{-1} &= \langle\{\xi_i \ : \ 1 \le i \le 6\}\rangle \\ E(1,6)_0 &= \langle\{t\} \cup \{\xi_i\xi_j \ : \ 1 \le i < j \le 6\}\rangle. \end{aligned} \tag{3.5}$$

We shall use the following notation for the basis elements of $E(1,6)_0$ (cf. [2]):

$$E_{00} = t, \qquad F_{ij} = -\xi_i\xi_j. \tag{3.6}$$

Observe that $E(1,6)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(6) \simeq \mathfrak{cso}(6)$. Take

$$\partial := -\frac{1}{2}\mathbf{1} \tag{3.7}$$

as the element that satisfies (L3) in Sect. 2.

For the rest of this work, $\mathfrak{g}$ will be $E(1,6)$. Let F be a finite-dimensional irreducible $\mathfrak{g}_0$-module, which we extend to a $\mathfrak{g}_{\ge 0}$-module by letting $\mathfrak{g}_j$ with $j > 0$ acting trivially. Then we shall identify, as above:

$$\mathrm{Ind}(F) \simeq \Lambda(1,6)_+ \otimes F \simeq \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F \tag{3.8}$$

as $\mathbb{C}$-vector spaces.

Since the non-positive graded subspaces of $E(1,6)$ are the same as those of $K(1,6)_+$, the λ-action is given by restricting the λ-action in Theorem 4.1 in [2]. We describe the $\mathfrak{g}$-action of $K(1,6)_+$ on $\mathrm{Ind}(F)$ using the λ-action notation in (2.2), i.e.

$$f_\lambda(g \otimes v) = \sum_{j \ge 0} \frac{\lambda^j}{j!} (t^j f) \cdot (g \otimes v) \tag{3.9}$$

for $f, g \in \Lambda(6)$ and $v \in F$.

In the last part of this section we shall state an easier formula for the λ-action in the induced module (see Theorem 4.3, [2]). This is done by taking **the Hodge dual of the basis modified by a sign**, since we are using the definition of Hodge dual given in [4], instead of the one used in [2]. Namely, let T be the vector space automorphism of $\mathrm{Ind}(F)$ given by $T(g \otimes v) = (-1)^{|g|} g^* \otimes v$, then the following theorem gives the formula for the composition $T \circ (f_\lambda \cdot) \circ T^{-1}$.

Theorem 3.1. *Let F be a $\mathfrak{cso}(6)$-module. Then the λ-action of $K(1,6)_+$ in* $\mathrm{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F$*, is equivalent to the following one:*

$$f_\lambda(g \otimes v) = (-1)^{\frac{|f|(|f|+1)}{2}+|f||g|}$$
$$\times \Bigg\{ (|f|-2)\partial(fg) \otimes v - (-1)^{p(f)} \sum_{i=1}^{6} (\partial_i f)(\partial_i g) \otimes v - \sum_{r<s} (\partial_r \partial_s f) g \otimes F_{rs} v$$
$$+\lambda \Bigg[fg \otimes E_{00} v - (-1)^{p(f)} \sum_{i=1}^{6} \partial_i \big(f \xi_i g\big) \otimes v + (-1)^{p(f)} \sum_{i \neq j} (\partial_i f) \xi_j g \otimes F_{ij} v \Bigg]$$
$$-\lambda^2 \sum_{i<j} f \xi_i \xi_j g \otimes F_{ij} v \Bigg\}.$$

4. Singular Vectors

By Theorem 2.4, the classification of irreducible finite modules over the Lie conformal superalgebra CK_6 reduces to the study of singular vectors in the induced modules $\mathrm{Ind}(F)$, where F is an irreducible finite-dimensional $\mathfrak{cso}(6)$-module. This section will be devoted to the classification of singular vectors.

When we discuss the highest weights of vectors and singular vectors, we always mean with respect to the upper Borel subalgebra in $E(1,6)$ generated by $(E(1,6))_{>0}$ and the elements of the Borel subalgebra of $\mathfrak{so}(6)$ in $E(1,6)_0$. More precisely, recall (3.6), where we defined $F_{ij} = -\xi_i \xi_j \in E(1,6)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(6)$. Observe that F_{ij} corresponds to $E_{ij} - E_{ji} \in \mathfrak{so}(6)$, where E_{ij} are the elements of the standard basis of matrices. Consider the following (standard) notation for $\mathfrak{so}(6,\mathbb{C})$ (cf. [12], p. 83): We take

$$H_j = i\ F_{2j-1,2j}, \qquad 1 \leq j \leq 3, \tag{4.1}$$

a basis of a Cartan subalgebra $\mathfrak{h}_0$. Let $\varepsilon_j \in \mathfrak{h}_0^*$ be given by $\varepsilon_j(H_k) = \delta_{jk}$. Let

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j\}$$

be the set of roots. The root space decomposition is

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \text{with } \mathfrak{g}_\alpha = \mathbb{C}E_\alpha,$$

where, for $1 \leq l < j \leq 3$,

$$\begin{aligned} E_{\varepsilon_l - \varepsilon_j} &= F_{2l-1,2j-1} + F_{2l,2j} + i(F_{2l-1,2j} - F_{2l,2j-1}), \\ E_{\varepsilon_l + \varepsilon_j} &= F_{2l-1,2j-1} - F_{2l,2j} - i(F_{2l-1,2j} + F_{2l,2j-1}), \\ E_{-(\varepsilon_l - \varepsilon_j)} &= F_{2l-1,2j-1} + F_{2l,2j} - i(F_{2l-1,2j} - F_{2l,2j-1}), \\ E_{-(\varepsilon_l + \varepsilon_j)} &= F_{2l-1,2j-1} - F_{2l,2j} + i(F_{2l-1,2j} + F_{2l,2j-1}). \end{aligned} \tag{4.2}$$

Let $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_2 + \varepsilon_3\}$ and $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}$, be the simple and positive roots respectively. Consider

$$\begin{aligned} \alpha_{lj} &:= F_{2l-1,2j-1} - i F_{2l,2j-1} = \frac{1}{2}(E_{\varepsilon_l - \varepsilon_j} + E_{\varepsilon_l + \varepsilon_j}), \\ \beta_{lj} &:= F_{2l,2j} + i F_{2l-1,2j} = \frac{1}{2}(E_{\varepsilon_l - \varepsilon_j} - E_{\varepsilon_l + \varepsilon_j}). \end{aligned} \tag{4.3}$$

Then, the Borel subalgebra is

$$B_{\mathfrak{so}(6)} = \langle \{\alpha_{lj}, \beta_{lj} \mid 1 \le l < j \le 3\} \rangle. \tag{4.4}$$

Recall that the Cartan subalgebra $\mathfrak{h}$ in $(CK(1,6)_+)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(6) \simeq \mathfrak{cso}(6)$ is spanned by the elements E_{00}, H_1, H_2, H_3. Let $e_0 \in \mathfrak{h}^*$ be the linear functional given by $e_0(E_{00}) = 1$ and $e_0(H_i) = 0$ for all i. Let

$$\begin{aligned} \lambda_1 &= \varepsilon_1, \\ \lambda_2 &= \frac{\varepsilon_1 + \varepsilon_2 - \varepsilon_3}{2}, \\ \lambda_3 &= \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{2} \end{aligned} \tag{4.5}$$

be the fundamental weights of $\mathfrak{so}(6)$, extended to $\mathfrak{h}^*$ by letting $\lambda_i(E_{00}) = 0$. That is $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$, where $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_2 + \varepsilon_3$. Then the highest weight μ of the finite irreducible $\mathfrak{cso}(6)$-module F_μ can be written as

$$\mu = n_0 e_0 + n_1 \lambda_1 + n_2 \lambda_2 + n_3 \lambda_3, \tag{4.6}$$

where n_1, n_2 and n_3 are non-negative integers. In order to write explicitly weights for vectors in $CK(1,6)_+$-modules, we will consider the notation (4.6).

Consider a singular vector $\vec{m}$ in the $CK(1,6)_+$-module $\mathrm{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F$, where F is an irreducible $\mathfrak{cso}(6)$-module, and the λ-action of $CK(1,6)_+$ in $\mathrm{Ind}(F)$ is given by restricting the λ-action of $K(1,6)_+$ in Theorem 3.1.

Using (3.4), (3.9), and the description of the upper Borel subalgebra of $E(1,6)$, we obtain that a vector $\vec{m}$ in the $E(1,6)$-module $\mathrm{Ind}(F)$ is a singular vector if and only if the following conditions are satisfied:

(S1) For all $f \in \Lambda(6)$, with $0 \le |f| \le 3$,

$$\frac{d^2}{d\lambda^2} \left(f_\lambda \vec{m} - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|} (f^*{}_\lambda \vec{m}) \right) = 0.$$

(S2) For all $f \in \Lambda(6)$, with $1 \le |f| \le 3$,

$$\frac{d}{d\lambda} \left(f_\lambda \vec{m} - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|} (f^*{}_\lambda \vec{m}) \right) |_{\lambda=0} = 0.$$

(S3) For all f with $|f| = 3$ or $f \in B_{\mathfrak{so}(6)}$,

$$\left(f_\lambda \vec{m} - i(-1)^{\frac{|f|(|f|+1)}{2}} \lambda^{3-|f|} (f^*{}_\lambda \vec{m}) \right) |_{\lambda=0} = 0.$$

In order to classify the finite irreducible CK_6-modules we should solve Eqs. (S1–S3) to obtain the singular vectors.

Observe that the $\mathbb{Z}$-gradation in $E(1,6)$, translates into a $\mathbb{Z}_{\le 0}$-gradation in $\mathrm{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F$, where $\mathrm{wt}\, \partial = -2$ and $\mathrm{wt}\, \xi_i = -1$, but we shall work with the λ-action given by Theorem 3.1 where we considered the (modified) Hodge dual basis, therefore from now on the $\mathbb{Z}_{\le 0}$-gradation in $\mathrm{Ind}(F)$ is given by $\mathrm{wt}\, \partial = -2$ and $\mathrm{wt}\, \xi_I = |I| - 6$.

The next theorem is one of the main results of this section and gives us the complete classification of singular vectors:

Theorem 4.1. *Let F_μ be an irreducible finite-dimensional $\mathfrak{cso}(6)$-module with highest weight μ. Then, there exists a non-trivial singular vector $\vec{m}$ in the $E(1,6)$-module* $\mathrm{Ind}(F_\mu)$ *if and only if the highest weight μ is one of the following:*

(a) $\mu = \left(\frac{1}{2}+4\right)e_0+\lambda_2$, *where (up to scalar)* $\vec{m} = \partial^2\, g_5 + \partial\, g_3 + g_1$ *of degree -5, given by (4.40), with singular weight*

$$\left(-\frac{1}{2}\right)e_0+\lambda_3,$$

(b) $\mu = \left(\frac{n}{2}+4\right)e_0+n\lambda_2$, *with* $n \geq 2$, *where (up to scalar)* $\vec{m} = \sum_{|I|=3}\xi_I\otimes v_I$ *of degree*-3, *given by (4.181), with singular weight*

$$\left(\frac{n}{2}+1\right)e_0+(n-2)\lambda_2,$$

(c) $\mu = \left(-\frac{n}{2}+2\right)e_0+n\lambda_3$, *with* $n \geq 0$, *where (up to scalar)* $\vec{m} = \sum_{|I|=3}\xi_I\otimes v_I$ *of degree* -3, *given by (4.183), with singular weight*

$$\left(-\frac{n}{2}-1\right)e_0+(n+2)\lambda_3,$$

(d) $\mu = \left(n_1+\frac{n_2}{2}+4\right)e_0+n_1\lambda_1+n_2\lambda_2$, *with* $n_1 \geq 1$, $n_2 \geq 0$, *where (up to scalar)* $\vec{m} = \sum_{|I|=5}\xi_I\otimes v_I$ *of degree* -1, *given by (4.249), with singular weight*

$$\mu = \left(n_1+\frac{n_2}{2}+3\right)e_0+(n_1-1)\lambda_1+n_2\lambda_2,$$

(e) $\mu = \left(\frac{n_2}{2}-\frac{n_3}{2}+2\right)e_0+n_2\lambda_2+n_3\lambda_3$, *with* $n_2 \geq 1$, $n_3 \geq 0$, *where (up to scalar)* $\vec{m} = \sum_{|I|=5}\xi_I\otimes v_I$ *of degree* -1, *given by (4.254), with singular weight*

$$\mu = \left(\frac{n_2}{2}-\frac{n_3}{2}+1\right)e_0+(n_2-1)\lambda_2+(n_3+1)\lambda_3,$$

(f) $\mu = -\left(n_1+\frac{n_3}{2}\right)e_0+n_1\lambda_1+n_3\lambda_3$, *with* $n_1 \geq 0$, $n_3 \geq 0$. *where (up to scalar)* $\vec{m} = \left(\xi_{\{2\}^c}-i\xi_{\{1\}^c}\right)\otimes v_\mu$ *of degree* -1, *with singular weight*

$$\mu = -\left(n_1+\frac{n_3}{2}+1\right)e_0+(n_1+1)\lambda_1+n_3\lambda_3.$$

Remark 4.2. (a) We have the explicit expression of all singular vectors in terms of the highest weight vector v_μ in each case.
(b) The family (f) of singular vectors with $n_3 = 0$ corresponds to the first family of singular vectors in K_6, computed in Theorem 5.1 [2].
(c) The family (d) of singular vectors with $n_2 = 0$ corresponds to the second family of singular vectors in K_6, computed in Theorem 5.1 [2].
(d) The family (e) is new, with no analog in K_6.
(e) The highest weight for which we have a singular vector of degree -5 corresponds to the (not present) case $n = 1$ in the family (b), and the one parameter family (b) (resp. (c)) is the (not present) case $n_1 = 0$ (resp. $n_2 = 0$) in the family (d) (resp. (e)).

Using Theorem 4.1, we obtain the following theorem that is the main result of this work and gives us the complete list of highest weights of degenerate irreducible modules:

Theorem 4.3. *Let F_μ be an irreducible finite-dimensional $\mathfrak{cso}(6)$-module with highest weight μ. Then the $E(1,6)$-module $\mathrm{Ind}(F_\mu)$ is degenerate if and only if μ is one of the following:*

(a) $\mu = \left(n_1 + \frac{n_2}{2} + 4\right) e_0 + n_1\lambda_1 + n_2\lambda_2$, *with* $n_1 \geq 1,\ n_2 \geq 0$ *or* $n_1 = 0, n_2 \geq 1$,
(b) $\mu = \left(\frac{n_2}{2} - \dfrac{n_3}{2} + 2\right) e_0 + n_2\lambda_2 + n_3\lambda_3$, *with* $n_2 \geq 0,\ n_3 \geq 0$,
(c) $\mu = -\left(n_1 + \frac{n_3}{2}\right) e_0 + n_1\lambda_1 + n_3\lambda_3$, *with* $n_1 \geq 0,\ n_3 \geq 0$.

The rest of this section together with Appendix A is devoted to the proof of this theorem. The proof will be done through several lemmas.

Recall that the Cartan subalgebra $\mathfrak{h}$ in $(CK(1,6)_+)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(6) \simeq \mathfrak{cso}(6)$ is spanned by the elements

$$E_{00}, H_1, H_2, H_3,$$

and, for technical reasons as in our work [2], from now on we shall write the weights of an eigenvector for the Cartan subalgebra $\mathfrak{h}$ as an $1+3$-tuple for the corresponding eigenvalues of this basis:

$$\mu = (\mu_0; \mu_1, \mu_2, \mu_3). \tag{4.7}$$

Let $\mu = n_0 e_0 + n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3$ be the highest weight of the finite irreducible $\mathfrak{cso}(6)$-module F_μ, where n_1, n_2 and n_3 are non-negative integers, as in (4.6). Using the notation (4.7), this highest weight can be written as the $1+3$-tuple,

$$\mu = \left(n_0; n_1 + \frac{n_2}{2} + \frac{n_3}{2}\,,\ \frac{n_2}{2} + \frac{n_3}{2}\,,\ -\frac{n_2}{2} + \frac{n_3}{2}\right). \tag{4.8}$$

Let $\vec{m} \in \mathrm{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F$ be a singular vector, then

$$\vec{m} = \sum_{k=0}^{N} \sum_{I} \partial^k (\xi_I \otimes v_{I,k}), \quad \text{with } v_{I,k} \in F.$$

Lemma 4.4. *If $\vec{m} \in \mathrm{Ind}(F)$ is a singular vector, then the degree of $\vec{m}$ in ∂ is at most 2. Moreover, any singular vector has this form:*

$$\vec{m} = \partial^2 \sum_{|I|\geq 5} \xi_I \otimes v_{I,2} + \partial \sum_{|I|\geq 3} \xi_I \otimes v_{I,1} + \sum_{|I|\geq 1} \xi_I \otimes v_{I,0}.$$

Proof. The proof of this lemma will be published in a second paper [3] of the series of three papers. □

The $\mathbb{Z}$-gradation in $E(1,6)$, translates into a $\mathbb{Z}_{\leq 0}$-gradation in $\mathrm{Ind}(F)$:

$$\begin{aligned}\mathrm{Ind}(F) &\simeq \Lambda(1,6) \otimes F \simeq \mathbb{C}[\partial] \otimes \Lambda(6) \otimes F \\ &\simeq \underbrace{\mathbb{C}\,1 \otimes F}_{\text{deg } 0} \oplus \underbrace{\mathbb{C}^6 \otimes F}_{\text{deg } -1} \oplus \underbrace{((\mathbb{C}\,\partial \otimes F) \oplus (\Lambda^2(\mathbb{C}^6) \otimes F))}_{\text{deg } -2} \oplus \cdots .\end{aligned}$$

Therefore, in the previous lemma, we have proved that any singular vector must have degree at most -5.

Recall that in the theorem that gives us the λ-action, we considered the Hodge dual of the natural bases in order to simplify the formula of the action. Hence, any singular vector must have one of the following forms:

$$\begin{aligned}
\vec{m} &= \partial^2 \sum_{|I|=5} \xi_I \otimes v_{I,2} \ + \ \partial \sum_{|I|=3} \xi_I \otimes v_{I,1} + \sum_{|I|=1} \xi_I \otimes v_{I,0}, \ (\text{Degree} - 5),\\
\vec{m} &= \partial^2 \sum_{|I|=6} \xi_I \otimes v_{I,2} \ + \ \partial \sum_{|I|=4} \xi_I \otimes v_{I,1} + \sum_{|I|=2} \xi_I \otimes v_{I,0}, \ (\text{Degree} - 4),\\
\vec{m} &= \partial \sum_{|I|=5} \xi_I \otimes v_{I,1} + \sum_{|I|=3} \xi_I \otimes v_{I,0}, \ (\text{Degree} - 3), \qquad (4.9)\\
\vec{m} &= \partial \sum_{|I|=6} \xi_I \otimes v_{I,1} + \sum_{|I|=4} \xi_I \otimes v_{I,0}, \ (\text{Degree} - 2),\\
\vec{m} &= \sum_{|I|=5} \xi_I \otimes v_{I,0}, \ (\text{Degree} - 1).
\end{aligned}$$

Now, we shall introduce a very important notation. Observe that the formula for the action given by Theorem 3.1 has the form

$$f_\lambda(g \otimes v) = \partial\, a + b + \lambda\, B + \lambda^2\, C = (\lambda + \partial)\, a + b + \lambda\, (B - a) + \lambda^2\, C,$$

by taking the coefficients in ∂ and λ^j. Using it, we can write the λ-action on the singular vector $\vec{m} = \partial^2\, m_2 + \partial\, m_1 + m_0$ of degree 2 in ∂, as follows:

$$\begin{aligned}
f_\lambda \vec{m} = &\Big[(\lambda + \partial)\, a_0 + b_0 + \lambda\, (B_0 - a_0) + \lambda^2\, C_0\Big]\\
&+(\lambda + \partial) \Big[(\lambda + \partial)\, a_1 + b_1 + \lambda\, (B_1 - a_1) + \lambda^2\, C_1\Big]\\
&+(\lambda + \partial)^2 \Big[(\lambda + \partial)\, a_2 + b_2 + \lambda\, (B_2 - a_2) + \lambda^2\, C_2\Big]. \qquad (4.10)
\end{aligned}$$

Obviously, these coefficients depend also on f and m, and sometimes we shall write for example $a_2(f)$ or $a(f, m_2)$, instead of a_2, to emphasize the dependence, but we will keep it implicit in the notation if no confusion may arise. In a similar way, for the λ-action of f^* on $\vec{m} = \partial^2\, m_2 + \partial\, m_1 + m_0$ we use the notation

$$\begin{aligned}
f_\lambda^* \vec{m} = &\Big[(\lambda + \partial)\, ad_0 + bd_0 + \lambda\, (Bd_0 - ad_0) + \lambda^2\, Cd_0\Big]\\
&+(\lambda + \partial) \Big[(\lambda + \partial)\, ad_1 + bd_1 + \lambda\, (Bd_1 - ad_1) + \lambda^2\, Cd_1\Big]\\
&+(\lambda + \partial)^2 \Big[(\lambda + \partial)\, ad_2 + bd_2 + \lambda\, (Bd_2 - ad_2) + \lambda^2\, Cd_2\Big]. \qquad (4.11)
\end{aligned}$$

As before, these coefficients depend also in f^* and m, and sometimes we shall write for example $ad_2(f)$ or $ad(f, m_2)$, instead of ad_2, to emphasize the dependence, but we will keep it implicit in the notation if no confusion may arise.

Lemma 4.5. *Let $\vec{m} = \partial^2 m_2 + \partial m_1 + m_0$ be a vector of degree at most -5. The conditions (S1)–(S3) on $\vec{m}$ are equivalent to the following list of equations:*

- *For $|f| = 0$:*

$$C_0 = -B_1, \tag{4.12}$$
$$2\,B_2 = -a_2 - C_1, \tag{4.13}$$
$$2\,bd_0 = i\,a_2 - i\,C_1. \tag{4.14}$$

- *For $|f| = 1$:*

$$3\,B_2 = -2\,i\,bd_1 - 2\,i\,ad_0 - 2\,C_1, \tag{4.15}$$
$$2\,C_0 = a_1 - B_1 - 2\,bd_0\,i, \tag{4.16}$$
$$2a_2 = -B_2, \tag{4.17}$$
$$3\,Bd_0 = i\,C_1 - bd_1 + 2\,ad_0, \tag{4.18}$$
$$2\,b_2 = -a_1 - B_1, \tag{4.19}$$
$$b_1 = -B_0. \tag{4.20}$$

- *For $|f| = 2$:*

$$2\,C_0 = -2\,Bd_0\,i - B_1 + i\,ad_0 - i\,bd_1, \tag{4.21}$$
$$2\,b_2 = -i\,ad_0 - i\,bd_1 - B_1, \tag{4.22}$$
$$bd_0 = b_1\,i + B_0\,i. \tag{4.23}$$

- *For $|f| = 3$:*

$$C_0 = Cd_0\,i, \tag{4.24}$$
$$bd_0 = -i\,b_0, \tag{4.25}$$
$$B_1 = Bd_1\,i + a_1 - ad_1\,i, \tag{4.26}$$
$$b_2 = bd_2\,i - a_1 + ad_1\,i, \tag{4.27}$$
$$bd_1 = -Bd_0 - B_0\,i - b_1\,i, \tag{4.28}$$
$$ad_0 = -a_0\,i + Bd_0 + B_0\,i. \tag{4.29}$$

- *For $f \in B_{\mathfrak{so}(6)}$:*

$$b_2 = 0, \tag{4.30}$$
$$b_1 = 0, \tag{4.31}$$
$$b_0 = 0. \tag{4.32}$$

Remark 4.6. The equations in Lemma 4.5 are written using the previously introduced notation. For example, strictly speaking, if $\vec{m} = \partial^2 m_2 + \partial m_1 + m_0$ then Eq. (4.23) for an element $f = \xi_j \xi_k$ means

$$bd(\xi_j\xi_k, m_0) = b(\xi_j\xi_k, m_1)\,i + B(\xi_j\xi_k, m_0)\,i. \tag{4.33}$$

Proof. Using this notation, by taking coefficients in $\partial^i \lambda^j$, conditions (S1)-(S3) translate into the following list, for $f \in \Lambda(6)$ **(see file "equations.mws" where the computations were done using Maple, this file is located in the link written at the beginning of the Appendices)**:

$\bullet For|f| = 0$:

$$
\begin{aligned}
C_0 &= -B_1 - b_2,\\
B_2 &= -\frac{a_2}{2} - \frac{C_1}{2},\\
bd_0 &= \frac{1}{2}\, i\, a_2 - \frac{1}{2}\, i\, C_1,\\
ad_0 &= Bd_0,\\
C_2 &= 0,\ Cd_2 = 0,\ ad_2 = 0,\ Bd_2 = 0,\ Cd_1 = 0,\ Cd_0 = 0,\\
bd_2 &= -Bd_1,\\
bd_1 &= -Bd_0,\\
ad_1 &= Bd_1.
\end{aligned}
$$

$\bullet For|f| = 1$:

$$
\begin{aligned}
B_2 &= -\frac{2}{3}\, i\, bd_1 - \frac{2}{3}\, i\, ad_0 - \frac{2\, C_1}{3},\\
a_2 &= \frac{1}{3}\, i\, bd_1 + \frac{1}{3}\, i\, ad_0 + \frac{C_1}{3},\\
C_0 &= \frac{a_1}{2} - \frac{B_1}{2} - bd_0\, i,\\
C_2 &= -Bd_1\, i - bd_2\, i,\\
Bd_0 &= \frac{1}{3}\, i\, C_1 - \frac{bd_1}{3} + \frac{2\, ad_0}{3},\\
b_2 &= -\frac{a_1}{2} - \frac{B_1}{2},\\
b_1 &= -B_0,\\
Cd_2 &= 0, ad_2 = 0, Bd_2 = 0, Cd_1 = 0,\ Cd_0 = 0,\\
ad_1 &= Bd_1.
\end{aligned}
$$

$\bullet For|f| = 2$:

$$
\begin{aligned}
B_2 &= -bd_2\, i - \frac{1}{3}\, i\, ad_1 - \frac{2\, C_1}{3} - \frac{2}{3}\, i\, Bd_1,\\
Cd_0 &= \frac{1}{3}\, i\, C_1 - \frac{Bd_1}{3} + \frac{ad_1}{3},\\
C_0 &= -Bd_0\, i - \frac{B_1}{2} + \frac{1}{2}\, i\, ad_0 - \frac{1}{2}\, i\, bd_1 + \frac{a_1}{2},\\
a_2 &= \frac{C_1}{3} + \frac{1}{3}\, i\, Bd_1 - \frac{1}{3}\, i\, ad_1,\\
b_2 &= -\frac{1}{2}\, i\, ad_0 - \frac{1}{2}\, i\, bd_1 - \frac{a_1}{2} - \frac{B_1}{2},\\
bd_0 &= b_1\, i + B_0\, i,\\
C_2 &= -i\, Bd_2,\\
Cd_2 &= 0, ad_2 = 0,\ Cd_1 = 0.
\end{aligned}
$$

• $For|f| = 3$:

$$\begin{aligned}
C_0 &= Cd_0\, i,\\
C_1 &= Cd_1\, i,\\
C_2 &= Cd_2\, i,\\
bd_0 &= -i\, b_0,\\
a_2 &= ad_2\, i,\\
B_1 &= Bd_1\, i + a_1 - ad_1\, i,\\
b_2 &= bd_2\, i - a_1 + ad_1\, i,\\
bd_1 &= -Bd_0 - B_0\, i - b_1\, i,\\
ad_0 &= -a_0\, i + Bd_0 + B_0\, i,\\
B_2 &= Bd_2\, i.
\end{aligned}$$

• $For f \in B_{\mathfrak{so}(6)}$:

$$\begin{aligned}
0 &= b_0,\\
0 &= b_1 + a_0,\\
0 &= b_2 + a_1,\\
0 &= a_2.
\end{aligned}$$

Now, taking care of the length of the elements ξ_I involved in the expression of vectors in $\mathrm{Ind}(F)$ of degree at most -5, we observe that some equations are always zero, getting the list in the statement of the lemma. □

In order to classify the singular vectors we should impose Eqs. (4.12)–(4.32) to the 5 possible forms of singular vectors listed in (4.9), depending on the degree. The following lemmas describe the result in each case.

Lemma 4.7. *All the singular vectors of degree -5 are listed in the theorem.*

Proof. Using the softwares Macaulay2 and Maple, the conditions of Lemma 4.5 on the singular vector $\vec{m}_5$ were simplified in several steps. First, the conditions of Lemma 4.5 were reduced to a linear system of equations with a 992×544 matrix. After the reduction of this linear system, we obtained in the middle of the file "m5-macaulay-2" a simplified list of 542 equations (see Appendix A for the details of this reduction). In particular, we obtained the following identities:

$$\begin{array}{ll}
0 = v_1 + v_{1,3,4,5,6} & 0 = v_2 - i\ v_{1,3,4,5,6}\\
0 = v_3 + v_{1,2,3,5,6} & 0 = v_4 - i\ v_{1,2,3,5,6}\\
0 = v_5 + v_{1,2,3,4,5} & 0 = v_6 - i\ v_{1,2,3,4,5}\\
0 = v_{1,2,3} - i\ v_{1,2,3,5,6} & 0 = v_{1,2,4} - v_{1,2,3,5,6}\\
0 = v_{1,2,5} - i\ v_{1,2,3,4,5} & 0 = v_{1,2,6} - v_{1,2,3,4,5}\\
0 = v_{1,3,4} - i\ v_{1,3,4,5,6} & 0 = v_{1,3,5} + i\ v_{2,4,6}\\
0 = v_{1,3,6} + v_{2,4,6} & 0 = v_{1,4,5} + v_{2,4,6}\\
0 = v_{1,4,6} - i\ v_{2,4,6} & 0 = v_{1,5,6} - i\ v_{1,3,4,5,6}\\
0 = v_{2,3,4} - v_{1,3,4,5,6} & 0 = v_{2,3,5} + v_{2,4,6}\\
0 = v_{2,3,6} - i\ v_{2,4,6} & 0 = v_{2,4,5} - i\ v_{2,4,6}\\
0 = v_{2,5,6} - v_{1,3,4,5,6} & 0 = v_{3,4,5} - i\ v_{1,2,3,4,5}\\
0 = v_{3,4,6} - v_{1,2,3,4,5} & 0 = v_{3,5,6} - i\ v_{1,2,3,5,6}\\
0 = v_{4,5,6} - v_{1,2,3,5,6} & 0 = v_{2,3,4,5,6} + i\ v_{1,3,4,5,6}\\
0 = v_{1,2,4,5,6} + i\ v_{1,2,3,5,6} & 0 = v_{1,2,3,4,6} + i\ v_{1,2,3,4,5}
\end{array} \qquad (4.34)$$

In particular, all the vectors v_I can be written in terms of v_1, v_3, v_5 and $v_{1,3,5}$. By imposing these identities, we obtained at the end of the file “m5-macaulay-2” the following simplified list of 64 equations (see Appendix A for the details of this reduction):

$$\begin{array}{ll}
0 = H_1 v_1 + 1/2 v_1 & 0 = H_2 v_1 - 1/2 v_1 \\
0 = H_3 v_1 - 1/2 v_1 & 0 = H_1 v_3 - 1/2 v_3 \\
0 = H_2 v_3 + 1/2 v_3 & 0 = H_3 v_3 - 1/2 v_3 \\
0 = H_1 v_5 - 1/2 v_5 & 0 = H_2 v_5 - 1/2 v_5 \\
0 = H_3 v_5 + 1/2 v_5 & 0 = H_1 v_{1,3,5} + 1/2 v_{1,3,5} \\
0 = H_2 v_{1,3,5} + 1/2 v_{1,3,5} & 0 = H_3 v_{1,3,5} + 1/2 v_{1,3,5} \\
0 = E_{00} v_1 - 9/2 v_1 & 0 = E_{00} v_3 - 9/2 v_3 \\
0 = E_{00} v_5 - 9/2 v_5 & 0 = E_{00} v_{1,3,5} - 9/2 v_{1,3,5}
\end{array} \tag{4.35}$$

together with

$$\begin{array}{ll}
0 = E_{-(\varepsilon_1-\varepsilon_2)} v_1 & 0 = E_{-(\varepsilon_1-\varepsilon_2)} v_3 - 2v_1 \\
0 = E_{-(\varepsilon_1-\varepsilon_2)} v_5 & 0 = E_{-(\varepsilon_1-\varepsilon_2)} v_{1,3,5} \\
0 = E_{-(\varepsilon_1-\varepsilon_3)} v_1 & 0 = E_{-(\varepsilon_1-\varepsilon_3)} v_3 \\
0 = E_{-(\varepsilon_1-\varepsilon_3)} v_5 - 2v_1 & 0 = E_{-(\varepsilon_1-\varepsilon_3)} v_{1,3,5} \\
0 = E_{-(\varepsilon_2-\varepsilon_3)} v_1 & 0 = E_{-(\varepsilon_2-\varepsilon_3)} v_3 \\
0 = E_{-(\varepsilon_2-\varepsilon_3)} v_5 - 2v_3 & 0 = E_{-(\varepsilon_2-\varepsilon_3)} v_{1,3,5} \\
0 = E_{-(\varepsilon_1+\varepsilon_2)} v_1 & 0 = E_{-(\varepsilon_1+\varepsilon_2)} v_3 \\
0 = E_{-(\varepsilon_1+\varepsilon_2)} v_5 - 2v_{1,3,5} & 0 = E_{-(\varepsilon_1+\varepsilon_2)} v_{1,3,5} \\
0 = E_{-(\varepsilon_1+\varepsilon_3)} v_1 & 0 = E_{-(\varepsilon_1+\varepsilon_3)} v_3 + 2v_{1,3,5} \\
0 = E_{-(\varepsilon_1+\varepsilon_3)} v_5 & 0 = E_{-(\varepsilon_1+\varepsilon_3)} v_{1,3,5} \\
0 = E_{-(\varepsilon_2+\varepsilon_3)} v_1 - 2v_{1,3,5} & 0 = E_{-(\varepsilon_2+\varepsilon_3)} v_3 \\
0 = E_{-(\varepsilon_2+\varepsilon_3)} v_5 & 0 = E_{-(\varepsilon_2+\varepsilon_3)} v_{1,3,5}
\end{array} \tag{4.36}$$

and

$$\begin{array}{ll}
0 = E_{\varepsilon_1-\varepsilon_2} v_1 + 2v_3 & 0 = E_{\varepsilon_1-\varepsilon_2} v_3 \\
0 = E_{\varepsilon_1-\varepsilon_2} v_5 & 0 = E_{\varepsilon_1-\varepsilon_2} v_{1,3,5} \\
0 = E_{\varepsilon_1-\varepsilon_3} v_1 + 2v_5 & 0 = E_{\varepsilon_1-\varepsilon_3} v_3 \\
0 = E_{\varepsilon_1-\varepsilon_3} v_5 & 0 = E_{\varepsilon_1-\varepsilon_3} v_{1,3,5} \\
0 = E_{\varepsilon_2-\varepsilon_3} v_1 & 0 = E_{\varepsilon_2-\varepsilon_3} v_3 + 2v_5 \\
0 = E_{\varepsilon_2-\varepsilon_3} v_5 & 0 = E_{\varepsilon_2-\varepsilon_3} v_{1,3,5} \\
0 = E_{\varepsilon_1+\varepsilon_2} v_1 & 0 = E_{\varepsilon_1+\varepsilon_2} v_3 \\
0 = E_{\varepsilon_1+\varepsilon_2} v_5 & 0 = E_{\varepsilon_1+\varepsilon_2} v_{1,3,5} + 2v_5 \\
0 = E_{\varepsilon_1+\varepsilon_3} v_1 & 0 = E_{\varepsilon_1+\varepsilon_3} v_3 \\
0 = E_{\varepsilon_1+\varepsilon_3} v_5 & 0 = E_{\varepsilon_1+\varepsilon_3} v_{1,3,5} - 2v_3 \\
0 = E_{\varepsilon_2+\varepsilon_3} v_1 & 0 = E_{\varepsilon_2+\varepsilon_3} v_3 \\
0 = E_{\varepsilon_2+\varepsilon_3} v_5 & 0 = E_{\varepsilon_2+\varepsilon_3} v_{1,3,5} + 2v_1
\end{array} \tag{4.37}$$

Therefore, a vector

$$\vec{m}_5 = \partial^2 \sum_{|I|=5} \xi_I \otimes v_I \; + \; \partial \sum_{|I|=3} \xi_I \otimes v_I + \sum_{|I|=1} \xi_I \otimes v_I$$

satisfies conditions of Lemma 4.5 if and only if Eqs. (4.34–4.37) hold. We divided the final analysis of these equations in several cases:

• *Case* $v_5 \neq 0$. Using (4.37) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates v_5. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and by (4.35), the highest weight is

$$\mu = \left(\frac{9}{2}\,;\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right). \tag{4.38}$$

Using (4.36), the other vectors are completely determined by the highest weight vector v_5, namely

$$\begin{aligned} v_1 &= \frac{1}{2}E_{-(\varepsilon_1-\varepsilon_3)}\ v_5,\\ v_3 &= \frac{1}{2}E_{-(\varepsilon_2-\varepsilon_3)}\ v_5,\\ v_{1,3,5} &= \frac{1}{2}E_{-(\varepsilon_1+\varepsilon_2)}\ v_5. \end{aligned} \tag{4.39}$$

After lengthly computation, it is easy to see that (4.35–4.37) hold by using (4.38) and (4.39). From now on, we denote by A^c the complement of the subset A in $\{1,\ldots,6\}$. Therefore, the vector

$$\begin{aligned} \vec{m}_5 \ &= \partial^2\left[\sum_{l=1}^{3}(\xi_{\{2l\}^c} - i\ \xi_{\{2l-1\}^c})\otimes v_{2l-1}\right] + \ \partial\Big[\ (i\ \xi_{134}+\xi_{234}+i\xi_{156}+\xi_{256})\otimes v_1\\ &\quad +(i\ \xi_{123}+\xi_{124}+i\xi_{356}+\xi_{456})\otimes v_3 + \ (i\ \xi_{125}+\xi_{126}+i\xi_{345}+\xi_{346})\otimes v_5\\ &\quad + (i\ \xi_{136}+\xi_{236}+i\xi_{145}+\xi_{146}+i\ \xi_{235}+\xi_{245}-i\xi_{246}-\xi_{135})\otimes v_{1,3,5}\Big]\\ &\quad + \sum_{l=1}^{3}(\xi_{2l}+i\ \xi_{2l-1})\otimes v_{2l-1} \end{aligned} \tag{4.40}$$

is a singular vector of $\mathrm{Ind}(F_\mu)$, where v_5 is a highest weight vector in F_μ, $\mu = (9/2; 1/2, 1/2, -1/2)$ and $v_1, v_3, v_{1,3,5}$ are given by (4.39). By computing

$$\begin{aligned} E_{00}\cdot\vec{m}_5 &= \text{coefficient of } \lambda^1\ (1\ {}_\lambda\vec{m}_5)\,,\\ H_1\cdot\vec{m}_5 &= \text{coefficient of } \lambda^0\ (-i\ \xi_1\xi_2\ {}_\lambda\vec{m}_5)\,,\\ H_2\cdot\vec{m}_5 &= \text{coefficient of } \lambda^0\ (-i\ \xi_3\xi_4\ {}_\lambda\vec{m}_5)\,,\\ H_3\cdot\vec{m}_5 &= \text{coefficient of } \lambda^0\ (-i\ \xi_5\xi_6\ {}_\lambda\vec{m}_5)\,, \end{aligned} \tag{4.41}$$

one can prove that

$$wt\ \vec{m}_5 = \left(\frac{9}{2}\,;\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),$$

finishing this case.

• *Case* $v_5 = 0$ *and* $v_3 \neq 0$. In this case, using (4.37) and (4.35), we have that v_3 is a highest weight vector in F_μ, with $\mu = (9/2; 1/2, -1/2, 1/2)$ that is not dominant integral, getting a contradiction.

• *Case* $v_5 = 0,\ v_3 = 0$ *and* $v_1 \neq 0$. In this case, using (4.37) and (4.35), we have that v_1 is a highest weight vector in F_μ, with $\mu = (9/2; -1/2, 1/2, 1/2)$ that is not dominant integral, getting a contradiction.

• *Case* $0 = v_5 = v_3 = v_1$ *and* $v_{1,3,5} \neq 0$. In this case, using (4.37) and (4.35), we have that $v_{1,3,5}$ is a highest weight vector in F_μ, with $\mu = (9/2; -1/2, -1/2, -1/2)$ that is not dominant integral, getting a contradiction and finishing the proof. □

Lemma 4.8. *There is no singular vector of degree* -4.

Proof. The proof of this lemma was done entirely with the softwares Macaulay2 and Maple. The conditions on the singular vector $\vec{m}_4$ were reduced to a linear system of equations with a 1104×527 matrix, whose rank is 527 (see Appendix A for details). Therefore, there is no non-trivial solution of this linear system, proving that there is no singular vector of degree -4, finishing the lemma. □

Lemma 4.9. *All the singular vectors of degree* -3 *are listed in the theorem.*

Proof. Using the softwares Macaulay2 and Maple, the conditions of Lemma 4.5 on the singular vector $\vec{m}_3$ were simplified in several steps. First, the conditions of Lemma 4.5 were reduced to a linear system of equations with a 694×442 matrix. After the reduction of this linear system, we obtained at the end of the file "m3-macaulay-1" a simplified list of 397 equations (see Appendix A for the details of this reduction). In particular, we obtained the following identities:

$$\begin{array}{ll} 0 = v_{1,2,3} - i\ v_{4,5,6} & 0 = v_{1,2,4} + i\ v_{3,5,6} \\ 0 = v_{1,2,5} - i\ v_{3,4,6} & 0 = v_{1,2,6} + i\ v_{3,4,5} \\ 0 = v_{1,3,4} - i\ v_{2,5,6} & 0 = v_{1,3,5} + i\ v_{2,4,6} \\ 0 = v_{1,3,6} - i\ v_{2,4,5} & 0 = v_{1,4,5} - i\ v_{2,3,6} \\ 0 = v_{1,4,6} + i\ v_{2,3,5} & 0 = v_{1,5,6} - i\ v_{2,3,4} \\ 0 = v_{2,3,4,5,6} & 0 = v_{1,3,4,5,6} \\ 0 = v_{1,2,4,5,6} & 0 = v_{1,2,3,5,6} \\ 0 = v_{1,2,3,4,6} & 0 = v_{1,2,3,4,5} \end{array} \tag{4.42}$$

Now, we have to impose the identities (4.42) to reduce the number of variables. Observe that everything can be written in terms of

$$v_{1,j,k} \quad \text{with } 2 \le j < k \le 6.$$

Unfortunately, the result is not enough to obtain in a clear way the possible highest weight vectors. For example, after the reduction and some extra computations it is possible to see that

$$(v_{1,3,6} - v_{1,4,5}) + i\ (v_{1,3,5} + v_{1,4,6}) \tag{4.43}$$

is annihilated by the Borel subalgebra. Hence, it is necessary to impose (4.42) and make a change of variables. We produced an auxiliary file where we imposed (4.42), and after the analysis of the results, we found that the following change of variable is convenient:

$$\begin{aligned} u_1 &= v_{1,2,3} - i\ v_{1,2,4} \\ u_2 &= v_{1,2,3} + i\ v_{1,2,4} \\ u_3 &= v_{1,2,5} - i\ v_{1,2,6} \\ u_4 &= v_{1,2,6} + i\ v_{1,2,6} \\ u_5 &= v_{1,3,4} -\ v_{1,5,6} \\ u_6 &= v_{1,3,4} +\ v_{1,5,6} \\ u_7 &= v_{1,3,5} - v_{1,4,6} + i\ (v_{1,3,6} + v_{1,4,5}) \\ u_8 &= v_{1,3,5} + v_{1,4,6} - i\ (v_{1,3,6} - v_{1,4,5}) \\ u_9 &= v_{1,3,5} - v_{1,4,6} - i\ (v_{1,3,6} + v_{1,4,5}) \\ u_{10} &= v_{1,3,5} + v_{1,4,6} + i\ (v_{1,3,6} - v_{1,4,5}) \end{aligned} \tag{4.44}$$

Observe that all the equations will be written in terms of u_i with $1 \leq i \leq 10$. By imposing these identities, we obtained at the end of the file "m3-macaulay-2" the following simplified list of 125 equations (see Appendix A for the details of this reduction):

$$0 = H_1u_1 - i/4E_{-(\varepsilon_1-\varepsilon_3)}u_{10} \ \ + u_1 \tag{4.45}$$
$$0 = -H_1u_1 + H_2u_1 - u_1 \tag{4.46}$$
$$0 = H_1u_1 + H_3u_1 \tag{4.47}$$
$$0 = H_1u_2 - 1/4E_{-(\varepsilon_1+\varepsilon_2)}u_5 - i/4E_{-(\varepsilon_1+\varepsilon_3)}u_8 \ \ + 3/2u_2 \tag{4.48}$$
$$0 = H_1u_2 + H_2u_2 - 1/2E_{-(\varepsilon_1+\varepsilon_2)}u_5 + 2u_2 \tag{4.49}$$
$$0 = -H_1u_2 + H_3u_2 + 1/2E_{-(\varepsilon_1+\varepsilon_2)}u_5 - u_2 \tag{4.50}$$
$$0 = H_1u_3 - 1/4E_{-(\varepsilon_1-\varepsilon_3)}u_5 + i/4E_{-(\varepsilon_1-\varepsilon_2)}u_8 \ \ + 3/2u_3 \tag{4.51}$$
$$0 = H_1u_3 + H_2u_3 - 1/2E_{-(\varepsilon_1-\varepsilon_3)}u_5 + u_3 \tag{4.52}$$
$$0 = -H_1u_3 + H_3u_3 + 1/2E_{-(\varepsilon_1-\varepsilon_3)}u_5 - 2u_3 \tag{4.53}$$
$$0 = H_1u_4 + H_3u_4 + u_4 \tag{4.54}$$
$$0 = H_2u_4 + H_3u_4 + u_4 \tag{4.55}$$
$$0 = H_3u_4 - i/4E_{-(\varepsilon_1+\varepsilon_2)}u_{10} \tag{4.56}$$
$$0 = H_1u_5 + i/4E_{-(\varepsilon_2-\varepsilon_3)}u_{10} \tag{4.57}$$
$$0 = -H_1u_5 + H_2u_5 + u_5 \tag{4.58}$$
$$0 = H_2u_5 + H_3u_5 \tag{4.59}$$
$$0 = 1/2E_{-(\varepsilon_1+\varepsilon_3)}u_3 + H_1u_6 + H_3u_6 + 2u_6 \tag{4.60}$$
$$0 = 1/2E_{-(\varepsilon_1+\varepsilon_2)}u_1 + H_1u_6 + H_2u_6 + 2u_6 \tag{4.61}$$
$$0 = H_2u_6 + H_3u_6 + i/2E_{-(\varepsilon_2+\varepsilon_3)}u_9 \ \ + 2u_6 \tag{4.62}$$
$$0 = -i\ E_{-(\varepsilon_1+\varepsilon_2)}u_4 \ \ + H_1u_7 + H_2u_7 - H_3u_7 + E_{00}u_7 - 2u_7 \tag{4.63}$$
$$0 = i\ E_{-(\varepsilon_1+\varepsilon_3)}u_2 \ \ + H_1u_7 - H_2u_7 + H_3u_7 + E_{00}u_7 - 2u_7 \tag{4.64}$$
$$0 = -i\ E_{-(\varepsilon_2+\varepsilon_3)}u_6 \ \ - H_1u_7 + H_2u_7 + H_3u_7 + E_{00}u_7 - 2u_7 \tag{4.65}$$
$$0 = H_1u_8 - E_{00}u_8 + 4u_8 \tag{4.66}$$
$$0 = -i/2E_{-(\varepsilon_2-\varepsilon_3)}u_5 \ \ + H_2u_8 + E_{00}u_8 - 3u_8 \tag{4.67}$$

$$0 = H_2 u_8 + H_3 u_8 \tag{4.68}$$
$$0 = H_1 u_9 + H_2 u_9 + H_3 u_9 - E_{00} u_9 + 4u_9 \tag{4.69}$$
$$0 = i/2 E_{-(\varepsilon_1-\varepsilon_2)} u_3 \ \ + H_2 u_9 - E_{00} u_9 + 3u_9 \tag{4.70}$$
$$0 = -i/2 E_{-(\varepsilon_1-\varepsilon_3)} u_1 \ \ + H_3 u_9 - E_{00} u_9 + 3u_9 \tag{4.71}$$
$$0 = H_1 u_{10} - E_{00} u_{10} + 4u_{10} \tag{4.72}$$
$$0 = H_2 u_{10} - E_{00} u_{10} + 4u_{10} \tag{4.73}$$
$$0 = H_3 u_{10} + E_{00} u_{10} - 4u_{10} \tag{4.74}$$
$$0 = -H_1 u_1 + E_{00} u_1 - 5u_1 \tag{4.75}$$
$$0 = -H_1 u_2 + E_{00} u_2 - 5u_2 \tag{4.76}$$
$$0 = -H_1 u_3 + E_{00} u_3 - 5u_3 \tag{4.77}$$
$$0 = H_3 u_4 + E_{00} u_4 - 4u_4 \tag{4.78}$$
$$0 = -H_3 u_5 + E_{00} u_5 + i/2 E_{-(\varepsilon_2-\varepsilon_3)} u_{10} \ \ - 3u_5 \tag{4.79}$$
$$0 = -H_1 u_6 + E_{00} u_6 + i/2 E_{-(\varepsilon_2+\varepsilon_3)} u_9 \ \ - 4u_6 \tag{4.80}$$

together with

$$0 = E_{-(\varepsilon_1-\varepsilon_2)} u_1 \tag{4.81}$$
$$0 = E_{-(\varepsilon_2-\varepsilon_3)} u_1 + E_{-(\varepsilon_1-\varepsilon_3)} u_5 \tag{4.82}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_2)} u_2 + E_{-(\varepsilon_1+\varepsilon_3)} u_3 \tag{4.83}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_3)} u_2 - E_{-(\varepsilon_1+\varepsilon_2)} u_3 \tag{4.84}$$
$$0 = E_{-(\varepsilon_2-\varepsilon_3)} u_2 + i\ E_{-(\varepsilon_1+\varepsilon_2)} u_8 \tag{4.85}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_3)} u_3 + i\ E_{-(\varepsilon_2-\varepsilon_3)} u_9 \tag{4.86}$$
$$0 = E_{-(\varepsilon_2-\varepsilon_3)} u_3 + i\ E_{-(\varepsilon_1-\varepsilon_3)} u_8 \tag{4.87}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_2)} u_4 \tag{4.88}$$
$$0 = E_{-(\varepsilon_1+\varepsilon_2)} u_1 + E_{-(\varepsilon_1-\varepsilon_3)} u_4 \tag{4.89}$$
$$0 = E_{-(\varepsilon_2-\varepsilon_3)} u_4 - E_{-(\varepsilon_1+\varepsilon_2)} u_5 \tag{4.90}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_2)} u_5 + 2u_1 \tag{4.91}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_2)} u_6 - i\ E_{-(\varepsilon_1+\varepsilon_3)} u_9 \tag{4.92}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_3)} u_6 + i\ E_{-(\varepsilon_1+\varepsilon_2)} u_9 \tag{4.93}$$
$$0 = -E_{-(\varepsilon_1+\varepsilon_2)} u_3 + E_{-(\varepsilon_2-\varepsilon_3)} u_6 \tag{4.94}$$
$$0 = i\ E_{-(\varepsilon_1+\varepsilon_3)} u_6 \ \ + E_{-(\varepsilon_1-\varepsilon_2)} u_7 \tag{4.95}$$
$$0 = -i\ E_{-(\varepsilon_1+\varepsilon_2)} u_6 \ \ + E_{-(\varepsilon_1-\varepsilon_3)} u_7 \tag{4.96}$$
$$0 = -i\ E_{-(\varepsilon_1+\varepsilon_2)} u_2 \ \ + E_{-(\varepsilon_2-\varepsilon_3)} u_7 \tag{4.97}$$
$$0 = E_{-(\varepsilon_1-\varepsilon_2)} u_{10} \tag{4.98}$$
$$0 = E_{-(\varepsilon_1+\varepsilon_3)} u_1 \tag{4.99}$$
$$0 = E_{-(\varepsilon_2+\varepsilon_3)} u_1 + 2u_4 \tag{4.100}$$
$$0 = E_{-(\varepsilon_2+\varepsilon_3)} u_2 \tag{4.101}$$
$$0 = E_{-(\varepsilon_2+\varepsilon_3)} u_3 - 2u_2 \tag{4.102}$$
$$0 = E_{-(\varepsilon_1+\varepsilon_3)} u_4 \tag{4.103}$$
$$0 = E_{-(\varepsilon_2+\varepsilon_3)} u_4 \tag{4.104}$$
$$0 = E_{-(\varepsilon_1+\varepsilon_3)} u_5 + 2u_4 \tag{4.105}$$

$$0 = E_{-(\varepsilon_2+\varepsilon_3)}u_5 \tag{4.106}$$
$$0 = E_{-(\varepsilon_2+\varepsilon_3)}u_8 \tag{4.107}$$
$$0 = E_{-(\varepsilon_1+\varepsilon_3)}u_{10} \tag{4.108}$$
$$0 = E_{-(\varepsilon_2+\varepsilon_3)}u_{10} \tag{4.109}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_1 - 2u_5 \tag{4.110}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_1 + 2\ i\ u_{10} \tag{4.111}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_1 \tag{4.112}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_2 \tag{4.113}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_2 \tag{4.114}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_2 + 2u_4 \tag{4.115}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_3 - 2\ i\ u_8 \tag{4.116}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_3 + 2u_5 \tag{4.117}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_3 - 2u_1 \tag{4.118}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_4 \tag{4.119}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_4 \tag{4.120}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_4 \tag{4.121}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_5 \tag{4.122}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_5 \tag{4.123}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_5 - 2\ i\ u_{10} \tag{4.124}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_6 + 2u_2 \tag{4.125}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_6 + 2u_4 \tag{4.126}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_6 \tag{4.127}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_7 \tag{4.128}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_7 \tag{4.129}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_7 \tag{4.130}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_8 \tag{4.131}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_8 \tag{4.132}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_8 + 4\ i\ u_5 \tag{4.133}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_9 + 4\ i\ u_3 \tag{4.134}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_9 - 4\ i\ u_1 \tag{4.135}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_9 \tag{4.136}$$
$$0 = E_{\varepsilon_1-\varepsilon_2}u_{10} \tag{4.137}$$
$$0 = E_{\varepsilon_1-\varepsilon_3}u_{10} \tag{4.138}$$
$$0 = E_{\varepsilon_2-\varepsilon_3}u_{10} \tag{4.139}$$
$$0 = E_{\varepsilon_1+\varepsilon_2}u_1 \tag{4.140}$$
$$0 = E_{\varepsilon_1+\varepsilon_3}u_1 \tag{4.141}$$
$$0 = E_{\varepsilon_2+\varepsilon_3}u_1 \tag{4.142}$$
$$0 = E_{\varepsilon_1+\varepsilon_2}u_2 + 2u_5 \tag{4.143}$$
$$0 = E_{\varepsilon_1+\varepsilon_3}u_2 + 2\ i\ u_8 \tag{4.144}$$

$$0 = E_{\varepsilon_2+\varepsilon_3} u_2 + 2u_3 \tag{4.145}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_3 \tag{4.146}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_3 \tag{4.147}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_3 \tag{4.148}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_4 - 2\, i\, u_{10} \tag{4.149}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_4 - 2u_5 \tag{4.150}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_4 - 2u_1 \tag{4.151}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_5 \tag{4.152}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_5 \tag{4.153}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_5 \tag{4.154}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_6 - 2u_1 \tag{4.155}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_6 - 2u_3 \tag{4.156}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_6 - 2\, i\, u_9 \tag{4.157}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_7 + 4\, i\, u_4 \tag{4.158}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_7 - 4\, i\, u_2 \tag{4.159}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_7 + 4\, i\, u_6 \tag{4.160}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_8 \tag{4.161}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_8 \tag{4.162}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_8 \tag{4.163}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_9 \tag{4.164}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_9 \tag{4.165}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_9 \tag{4.166}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} u_{10} \tag{4.167}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} u_{10} \tag{4.168}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} u_{10}. \tag{4.169}$$

Therefore, a singular vector of degree -3 must have the simplified form

$$\vec{m}_3 = \sum_{|I|=3} \xi_I \otimes v_I$$

and it satisfies conditions of Lemma 4.5 if and only if Eqs. (4.42), (4.44) and (4.45–4.169) hold. We divided the final analysis of these equations in several cases:

• *Case* $u_{10} \neq 0$. Using (4.137–4.139) and (4.167–4.169) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates u_{10}. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and by (4.72–4.74), the highest weight is

$$\mu = (k+4\,;\,k, k, -k)\,, \qquad \text{with } 2k \in \mathbb{Z}_{\geq 0}. \tag{4.170}$$

Then we shall prove that the cases $k = 0$ and $k = 1/2$ are not possible. Using (4.111), (4.127), (4.149) and other similar equations, we deduce that if $u_{10} \neq 0$ then $u_i \neq 0$

for all i. Now, we shall see that all u_i are completely determined by the highest weight vector u_{10}. If $k = 0$, we are working with the trivial $\mathfrak{so}(6)$ representation, and using (4.111) we obtain $u_{10} = 0$ getting a contradiction. Assume that $k \neq 0$. Now, applying $E_{\varepsilon_1-\varepsilon_3}$ to (4.45), we can prove that

$$u_1 = \frac{i}{4k} E_{-(\varepsilon_1-\varepsilon_3)} u_{10}. \tag{4.171}$$

Similarly, using (4.56) and (4.60), we have

$$u_4 = -\frac{i}{4k} E_{-(\varepsilon_1+\varepsilon_2)} u_{10}, \tag{4.172}$$

$$u_5 = -\frac{i}{4k} E_{-(\varepsilon_2-\varepsilon_3)} u_{10}. \tag{4.173}$$

Applying $E_{\varepsilon_1+\varepsilon_2}$ to (4.49) and using (4.143), we can prove that

$$2(2k-1)u_2 = E_{-(\varepsilon_1+\varepsilon_2)} u_5. \tag{4.174}$$

If $k \neq \frac{1}{2}$, we have

$$\begin{aligned} u_2 &= \frac{1}{2(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)}\, u_5 \\ &= \frac{-i}{8k(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)} E_{-(\varepsilon_2-\varepsilon_3)}\, u_{10} \\ &= \frac{1}{2(2k-1)} E_{-(\varepsilon_2-\varepsilon_3)}\, u_4. \end{aligned} \tag{4.175}$$

If $k = \frac{1}{2}$, we are working with a spin representation. Using (4.174) we get $0 = E_{-(\varepsilon_1+\varepsilon_2)} u_5$. In this case, by (4.49–4.50) and (4.76), we have wt $u_2 = (\frac{1}{2}+4; -\frac{1}{2}, -\frac{3}{2}, \frac{1}{2})$, which is impossible in a spin representation (see p. 288 [12]).

Similarly, using (4.52) and (4.117), we have

$$\begin{aligned} u_3 &= \frac{1}{2(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)}\, u_5 \\ &= \frac{-i}{8k(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)} E_{-(\varepsilon_2-\varepsilon_3)}\, u_{10} \\ &= \frac{-1}{2(2k-1)} E_{-(\varepsilon_2-\varepsilon_3)}\, u_1. \end{aligned} \tag{4.176}$$

Using (4.58) and (4.152), we have

$$\begin{aligned} u_6 &= \frac{-1}{2(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)}\, u_1 \\ &= \frac{-i}{8k(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)} E_{-(\varepsilon_1-\varepsilon_3)}\, u_{10} \\ &= \frac{1}{2(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)}\, u_4. \end{aligned} \tag{4.177}$$

By (4.63), (4.149), (4.158) and (4.172), we have

$$\begin{aligned} u_7 &= \frac{i}{2(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)}\, u_4 \\ &= \frac{1}{8k(2k-1)} E_{-(\varepsilon_1+\varepsilon_2)} E_{-(\varepsilon_1+\varepsilon_2)}\, u_{10}. \end{aligned} \tag{4.178}$$

Using (4.67), (4.127), (4.133) and (4.173), we have

$$\begin{aligned} u_8 &= \frac{i}{2(2k-1)} E_{-(\varepsilon_2-\varepsilon_3)}\, u_5 \\ &= \frac{1}{8k(2k-1)} E_{-(\varepsilon_2-\varepsilon_3)} E_{-(\varepsilon_2-\varepsilon_3)}\, u_{10}. \end{aligned} \tag{4.179}$$

By (4.71), (4.111), (4.135) and (4.171), we have

$$\begin{aligned} u_9 &= \frac{-i}{2(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)}\, u_1 \\ &= \frac{1}{8k(2k-1)} E_{-(\varepsilon_1-\varepsilon_3)} E_{-(\varepsilon_1-\varepsilon_3)}\, u_{10}. \end{aligned} \tag{4.180}$$

After a lengthly computation, it is possible to see that (4.45–4.169) hold by using (4.170) with $k \neq 1/2$, and the expressions of u_i obtained in (4.171–4.180). Therefore, using the expressions of $v_{k,l,j}$'s given in terms of u_i as in (A.14), the vector

$$\begin{aligned} \vec{m}_3 = 2\big[&(\xi_{\{1,2,3\}} - i\,\xi_{\{1,2,3\}^c}) - (\xi_{\{3,5,6\}} - i\,\xi_{\{3,5,6\}^c})\big] \otimes u_1, \\ +2\big[&(\xi_{\{1,2,3\}} - i\,\xi_{\{1,2,3\}^c}) + (\xi_{\{3,5,6\}} - i\,\xi_{\{3,5,6\}^c})\big] \otimes u_2, \\ +2\big[&(\xi_{\{1,2,5\}} - i\,\xi_{\{1,2,5\}^c}) - (\xi_{\{3,4,5\}} - i\,\xi_{\{3,4,5\}^c})\big] \otimes u_3, \\ +2\big[&(\xi_{\{1,2,5\}} - i\,\xi_{\{1,2,5\}^c}) + (\xi_{\{3,4,5\}} - i\,\xi_{\{3,4,5\}^c})\big] \otimes u_4, \qquad (4.181) \\ +2\big[&(\xi_{\{1,3,4\}} - i\,\xi_{\{1,3,4\}^c}) - (\xi_{\{1,5,6\}} - i\,\xi_{\{1,5,6\}^c})\big] \otimes u_5, \\ +2\big[&(\xi_{\{1,3,4\}} - i\,\xi_{\{1,3,4\}^c}) + (\xi_{\{1,5,6\}} - i\,\xi_{\{1,5,6\}^c})\big] \otimes u_6, \\ +\big[&(\xi_{\{1,3,5\}} + i\,\xi_{\{1,3,5\}^c}) - (\xi_{\{2,4,5\}} + i\,\xi_{\{2,4,5\}^c}) \\ &-(\xi_{\{2,3,6\}} + i\,\xi_{\{2,3,6\}^c}) - (\xi_{\{1,4,6\}} + i\,\xi_{\{1,4,6\}^c})\big] \otimes u_7, \\ +\big[&(\xi_{\{1,3,5\}} + i\,\xi_{\{1,3,5\}^c}) + (\xi_{\{2,4,5\}} + i\,\xi_{\{2,4,5\}^c}) \\ &-(\xi_{\{2,3,6\}} + i\,\xi_{\{2,3,6\}^c}) + (\xi_{\{1,4,6\}} + i\,\xi_{\{1,4,6\}^c})\big] \otimes u_8, \\ +\big[&(\xi_{\{1,3,5\}} + i\,\xi_{\{1,3,5\}^c}) + (\xi_{\{2,4,5\}} + i\,\xi_{\{2,4,5\}^c}) \\ &+(\xi_{\{2,3,6\}} + i\,\xi_{\{2,3,6\}^c}) - (\xi_{\{1,4,6\}} + i\,\xi_{\{1,4,6\}^c})\big] \otimes u_9, \\ +\big[&(\xi_{\{1,3,5\}} + i\,\xi_{\{1,3,5\}^c}) - (\xi_{\{2,4,5\}} + i\,\xi_{\{2,4,5\}^c}) \\ &+(\xi_{\{2,3,6\}} + i\,\xi_{\{2,3,6\}^c}) + (\xi_{\{1,4,6\}} + i\,\xi_{\{1,4,6\}^c})\big] \otimes u_{10}, \qquad (4.182) \end{aligned}$$

is a singular vector of $\mathrm{Ind}(F_\mu)$, where the u_i's are written in (4.171–4.180) in terms of u_{10}, where u_{10} is a highest weight vector in F_μ, and $\mu = (k+4; k, k, -k)$ with $2k \in \mathbb{Z}_{>0}$ and $k \neq \frac{1}{2}$. Now using (4.41), one can prove that

$$wt\ \vec{m}_3 = (k+1\,; k-1, k-1, -k+1)$$

finishing this case.

• *Case* $u_{10} = 0$ *and* $u_5 \neq 0$. In this case, using (4.122–4.124) and (4.152–4.154), we have that u_5 is a highest weight vector in F_μ. Considering (4.57–4.59) and (4.79), we have $\mu = (4; 0, -1, 1)$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = 0$ *and* $u_1 \neq 0$. In this case, using (4.110–4.112) and (4.140–4.142), we have that u_1 is a highest weight vector in F_μ. Considering (4.45–4.47) and (4.75), we have $\mu = (4; -1, 0, 1)$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = u_1 = 0$ *and* $u_8 \neq 0$. In this case, using (4.131–4.133) and (4.161–4.163), we have that u_8 is a highest weight vector in F_μ. Considering (4.66–4.68) we have $\mu = (k+4; k, -k-1, k+1)$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = u_1 = u_8 = 0$ *and* $u_4 \neq 0$. In this case, using (4.119–4.121) and (4.149–4.151), we have that u_4 is a highest weight vector in F_μ. Considering (4.54–4.56) and (4.78), we have $\mu = (4; -1, -1, 0)$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = u_1 = u_8 = u_4 = 0$ *and* $u_3 \neq 0$. In this case, using (4.116–4.118) and (4.146–4.148), we have that u_3 is a highest weight vector in F_μ. Considering (4.51–4.53) and (4.77), we have $\mu = (7/2; -3/2, 1/2, 1/2)$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = u_1 = u_8 = u_4 = u_3 = 0$ *and* $u_2 \neq 0$. In this case, using (4.113–4.115) and (4.143–4.145), we have that u_2 is a highest weight vector in F_μ. Considering (4.48–4.50) and (4.76), we have $\mu = (7/2; -3/2, -1/2, -1/2)$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = u_1 = u_8 = u_4 = u_3 = u_2 = 0$ *and* $u_9 \neq 0$. In this case, using (4.134–4.136) and (4.164–4.166), we have that u_9 is a highest weight vector in F_μ. Considering (4.69–4.71), we have $\mu = (-k+2; k, -(k+1), -(k+1))$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = u_1 = u_8 = u_4 = u_3 = u_2 = u_9 = 0$ *and* $u_6 \neq 0$. In this case, using (4.125–4.127) and (4.155–4.157), we have that u_6 is a highest weight vector in F_μ. Considering (4.60–4.62) and (4.80), we have $\mu = (3; -1, -1, -1)$ that is not dominant integral, getting a contradiction.

• *Case* $u_{10} = u_5 = u_1 = u_8 = u_4 = u_3 = u_2 = u_9 = u_6 = 0$ *and* $u_7 \neq 0$. In this case, using (4.128–4.130) and (4.158–4.160), we have that u_7 is a highest weight vector in F_μ. Considering (4.63–4.65), we have $\mu = (-k+2; k, k, k)$ which is a multiple of the spin representation with $2k \in \mathbb{Z}_{\geq 0}$. In this case $u_i = 0$ for all $i \neq 7$ and most of Eqs. (4.45–4.169) are trivial, and it is easy to check that the remaining equations hold in this case. Therefore, using (A.14), we have that the vector

$$\begin{aligned}\vec{m}_3 = &\left(\xi_{\{1,3,5\}} - \xi_{\{1,4,6\}} - i\ (\xi_{\{1,3,6\}} + \xi_{\{1,4,5\}})\right) \otimes u_7 \\ &+ \left(i\ (\xi_{\{1,3,5\}^c} - \xi_{\{1,4,6\}^c}) - (\xi_{\{1,3,6\}^c} + \xi_{\{1,4,5\}^c})\right) \otimes u_7 \end{aligned} \tag{4.183}$$

is a singular vector in $\mathrm{Ind}(F_\mu)$, where $\mu = (-k+2; k, k, k)$ with $2k \in \mathbb{Z}_{\geq 0}$ and u_7 is a highest weight vector in F_μ. Now using (4.41), one can prove that

$$wt\ \vec{m}_3 = (-k-1\ ; k+1, k+1, k+1)\,,$$

finishing the classification of singular vectors of degree -3. □

Lemma 4.10. *There is no singular vector of degree* -2.

Proof. Using the softwares Macaulay2 and Maple, the conditions of Lemma 4.5 on the singular vector $\vec{m}_2$ were reduced to a linear system of equations with a 268×272 matrix. After the reduction of this linear system, we obtained at the end of the file "m2-macaulay" or in file "m2-equations.pdf" a simplified list of 192 equations (see Appendix A for the details of this reduction). The 15th and 16th equations of this list are the following

$$0 = -i\ F_{1,2}\ v_{3,4,5,6} + E_{00}\ v_{3,4,5,6} - 5\ v_{3,4,5,6} + i\ v_{1,2,3,4,5,6}, \tag{4.184}$$
$$0 = E_{00}\ v_{1,2,3,4,5,6} - 3\ v_{1,2,3,4,5,6}, \tag{4.185}$$

and at the end of this list we have the conditions

$$\begin{aligned}
0 &= F_{1,2}\ v_{1,2,3,4,5,6} - v_{3,4,5,6}\\
0 &= F_{1,3}\ v_{1,2,3,4,5,6} + v_{2,4,5,6}\\
0 &= F_{1,4}\ v_{1,2,3,4,5,6} - v_{2,3,5,6}\\
0 &= F_{1,5}\ v_{1,2,3,4,5,6} + v_{2,3,4,6}\\
0 &= F_{1,6}\ v_{1,2,3,4,5,6} - v_{2,3,4,5}\\
0 &= F_{2,3}\ v_{1,2,3,4,5,6} - i\ v_{2,4,5,6}\\
0 &= F_{2,4}\ v_{1,2,3,4,5,6} + i\ v_{2,3,5,6}\\
0 &= F_{2,5}\ v_{1,2,3,4,5,6} - i\ v_{2,3,4,6}\\
0 &= F_{2,6}\ v_{1,2,3,4,5,6} + i\ v_{2,3,4,5}\\
0 &= F_{3,4}\ v_{1,2,3,4,5,6} - v_{1,2,5,6}\\
0 &= F_{3,5}\ v_{1,2,3,4,5,6} + v_{1,2,4,6}\\
0 &= F_{3,6}\ v_{1,2,3,4,5,6} - v_{1,2,4,5}\\
0 &= F_{4,5}\ v_{1,2,3,4,5,6} - i\ v_{1,2,4,6}\\
0 &= F_{4,6}\ v_{1,2,3,4,5,6} + i\ v_{1,2,4,5}\\
0 &= F_{5,6}\ v_{1,2,3,4,5,6} - v_{1,2,3,4}.
\end{aligned} \tag{4.186}$$

Therefore, if $\vec{m}_2 = \partial\ \sum_{|I|=6} \xi_I \otimes v_I + \sum_{|I|=4} \xi_I \otimes v_I$ is a singular vector in $\mathrm{Ind}(F_\mu)$, using Eqs. (4.186), we prove that $v_{1,2,3,4,5,6} \in F_\mu$ is annihilated by the Borel subalgebra of $\mathfrak{so}(6)$ (see (4.4)), and using that F_μ is irreducible, we get that $v_{1,2,3,4,5,6}$ is a highest weight vector. Now, we shall compute the corresponding weight μ. Recall (4.1) and observe that using (4.185) and (4.186), Eq. (4.184) is equivalent to the following:

$$(H_1^2 + 2H_1 + 1)v_{1,2,3,4,5,6} = 0, \tag{4.187}$$

obtaining that $H_1 v_{1,2,3,4,5,6} = -v_{1,2,3,4,5,6}$. Therefore, the weight μ is not dominant integral, getting a contradiction and finishing the proof. □

Lemma 4.11. *All the singular vectors of degree* -1 *are listed in the theorem.*

Proof. Since the singular vectors found in [2] for K_6 are also singular vectors for CK_6, using (B42–B43) in [2], we have that it is convenient to introduce the following notation:

$$\begin{aligned}
\vec{m}_1 &= \sum_{i=1}^{6} \xi_{\{i\}^c} \otimes v_{\{i\}^c}\\
&= \sum_{l=1}^{3} \Big[\big(\xi_{\{2l\}^c} + i\xi_{\{2l-1\}^c}\big) \otimes w_l + \big(\xi_{\{2l\}^c} - i\xi_{\{2l-1\}^c}\big) \otimes \overline{w}_l \Big],
\end{aligned} \tag{4.188}$$

that is, for $1 \le l \le 3$,

$$v_{\{2l\}^c} = w_l + \overline{w}_l, \qquad v_{\{2l-1\}^c} = i(w_l - \overline{w}_l), \tag{4.189}$$

or equivalently, for $1 \le l \le 3$,

$$w_l = \frac{1}{2}(v_{\{2l\}^c} - i\ v_{\{2l-1\}^c}), \qquad \overline{w}_l = \frac{1}{2}(v_{\{2l\}^c} + i\ v_{\{2l-1\}^c}). \tag{4.190}$$

We applied the change of variables (4.189), and using the softwares Macaulay2 and Maple, the conditions of Lemma 4.5 on the singular vector $\vec{m}_1$ were simplified in several steps. First, the conditions of Lemma 4.5 were reduced to a linear system of equations with a 62×102 matrix. After the reduction of this linear system, we obtained at the end of the file "m1-macaulay" a simplified list of 51 equations (see Appendix A for the details of this reduction). More precisely, we obtained the following identities:

$$0 = H_1 w_1 - E_{00} w_1 + 4w_1 \tag{4.191}$$

$$0 = H_2 w_1 + H_3 w_1 \tag{4.192}$$

$$0 = \frac{1}{2}E_{-(\varepsilon_1-\varepsilon_2)} w_1 + H_1 w_2 + H_3 w_2 + w_2 \tag{4.193}$$

$$0 = -\frac{1}{2}E_{-(\varepsilon_1-\varepsilon_2)} w_1 + H_2 w_2 - E_{00} w_2 + 3w_2 \tag{4.194}$$

$$0 = \frac{1}{2}E_{-(\varepsilon_1-\varepsilon_3)} w_1 + \frac{1}{2}E_{-(\varepsilon_2-\varepsilon_3)} w_2 + H_1 w_3 + H_2 w_3 + 2w_3 \tag{4.195}$$

$$0 = -\frac{1}{2}E_{-(\varepsilon_1-\varepsilon_3)} w_1 - \frac{1}{2}E_{-(\varepsilon_2-\varepsilon_3)} w_2 + H_3 w_3 - E_{00} w_3 + 2w_3 \tag{4.196}$$

$$\begin{aligned} 0 = {} & \frac{1}{2}E_{-(\varepsilon_1+\varepsilon_2)} w_2 + \frac{1}{2}E_{-(\varepsilon_1+\varepsilon_3)} w_3 + H_1 \overline{w}_1 + E_{00} \overline{w}_1 \\ & -\frac{1}{2}E_{-(\varepsilon_1-\varepsilon_2)} \overline{w}_2 - \frac{1}{2}E_{-(\varepsilon_1-\varepsilon_3)} \overline{w}_3 \end{aligned} \tag{4.197}$$

$$\begin{aligned} 0 = {} & \frac{1}{2}E_{-(\varepsilon_1+\varepsilon_2)} w_2 - \frac{1}{2}E_{-(\varepsilon_1+\varepsilon_3)} w_3 + H_2 \overline{w}_1 - H_3 \overline{w}_1 \\ & +\frac{1}{2}E_{-(\varepsilon_1-\varepsilon_2)} \overline{w}_2 - \frac{1}{2}E_{-(\varepsilon_1-\varepsilon_3)} \overline{w}_3 \end{aligned} \tag{4.198}$$

$$\begin{aligned} 0 = {} & -\frac{1}{2}E_{-(\varepsilon_1+\varepsilon_2)} w_1 - \frac{1}{2}E_{-(\varepsilon_2+\varepsilon_3)} w_3 + H_1 \overline{w}_2 - H_3 \overline{w}_2 \\ & -\frac{1}{2}E_{-(\varepsilon_2-\varepsilon_3)} \overline{w}_3 + \overline{w}_2 \end{aligned} \tag{4.199}$$

$$\begin{aligned} 0 = {} & -\frac{1}{2}E_{-(\varepsilon_1+\varepsilon_2)} w_1 + \frac{1}{2}E_{-(\varepsilon_2+\varepsilon_3)} w_3 + H_2 \overline{w}_2 + E_{00} \overline{w}_2 \\ & -\frac{1}{2}E_{-(\varepsilon_2-\varepsilon_3)} \overline{w}_3 - \overline{w}_2 \end{aligned} \tag{4.200}$$

$$0 = -\frac{1}{2}E_{-(\varepsilon_1+\varepsilon_3)} w_1 + \frac{1}{2}E_{-(\varepsilon_2+\varepsilon_3)} w_2 + H_1 \overline{w}_3 - H_2 \overline{w}_3 \tag{4.201}$$

$$0 = -\frac{1}{2}E_{-(\varepsilon_1+\varepsilon_3)} w_1 - \frac{1}{2}E_{-(\varepsilon_2+\varepsilon_3)} w_2 + H_3 \overline{w}_3 + E_{00} \overline{w}_3 - 2\overline{w}_3 \tag{4.202}$$

$$0 = -E_{-(\varepsilon_1+\varepsilon_2)} w_3 + E_{-(\varepsilon_2-\varepsilon_3)} \overline{w}_1 - E_{-(\varepsilon_1-\varepsilon_3)} \overline{w}_2 \tag{4.203}$$

$$0 = E_{-(\varepsilon_1+\varepsilon_3)} w_2 + E_{-(\varepsilon_1-\varepsilon_2)} \overline{w}_3 \tag{4.204}$$

$$0 = E_{-(\varepsilon_2+\varepsilon_3)} w_1 \tag{4.205}$$

and

$$0 = E_{\varepsilon_1-\varepsilon_2} w_1 \tag{4.206}$$
$$0 = E_{\varepsilon_1-\varepsilon_3} w_1 \tag{4.207}$$
$$0 = E_{\varepsilon_2-\varepsilon_3} w_1 \tag{4.208}$$
$$0 = E_{\varepsilon_1-\varepsilon_2} w_2 - 2w_1 \tag{4.209}$$
$$0 = E_{\varepsilon_1-\varepsilon_3} w_2 \tag{4.210}$$
$$0 = E_{\varepsilon_2-\varepsilon_3} w_2 \tag{4.211}$$
$$0 = E_{\varepsilon_1-\varepsilon_2} w_3 \tag{4.212}$$
$$0 = E_{\varepsilon_1-\varepsilon_3} w_3 - 2w_1 \tag{4.213}$$
$$0 = E_{\varepsilon_2-\varepsilon_3} w_3 - 2w_2 \tag{4.214}$$
$$0 = E_{\varepsilon_1-\varepsilon_2} \overline{w}_1 + 2\overline{w}_2 \tag{4.215}$$
$$0 = E_{\varepsilon_1-\varepsilon_3} \overline{w}_1 + 2\overline{w}_3 \tag{4.216}$$
$$0 = E_{\varepsilon_2-\varepsilon_3} \overline{w}_1 \tag{4.217}$$
$$0 = E_{\varepsilon_1-\varepsilon_2} \overline{w}_2 \tag{4.218}$$
$$0 = E_{\varepsilon_1-\varepsilon_3} \overline{w}_2 \tag{4.219}$$
$$0 = E_{\varepsilon_2-\varepsilon_3} \overline{w}_2 + 2\overline{w}_3 \tag{4.220}$$
$$0 = E_{\varepsilon_1-\varepsilon_2} \overline{w}_3 \tag{4.221}$$
$$0 = E_{\varepsilon_1-\varepsilon_3} \overline{w}_3 \tag{4.222}$$
$$0 = E_{\varepsilon_2-\varepsilon_3} \overline{w}_3 \tag{4.223}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} w_1 \tag{4.224}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} w_1 \tag{4.225}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} w_1 \tag{4.226}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} w_2 \tag{4.227}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} w_2 \tag{4.228}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} w_2 \tag{4.229}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} w_3 \tag{4.230}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} w_3 \tag{4.231}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} w_3 \tag{4.232}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} \overline{w}_1 - 2w_2 \tag{4.233}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} \overline{w}_1 - 2w_3 \tag{4.234}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} \overline{w}_1 \tag{4.235}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} \overline{w}_2 + 2w_1 \tag{4.236}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} \overline{w}_2 \tag{4.237}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} \overline{w}_2 - 2w_3 \tag{4.238}$$
$$0 = E_{\varepsilon_1+\varepsilon_2} \overline{w}_3 \tag{4.239}$$
$$0 = E_{\varepsilon_1+\varepsilon_3} \overline{w}_3 + 2w_1 \tag{4.240}$$
$$0 = E_{\varepsilon_2+\varepsilon_3} \overline{w}_3 + 2w_2. \tag{4.241}$$

We divided the final analysis of these equations in several cases:

• *Case* $w_1 \neq 0$. Using (4.206–4.208) and (4.224–4.226) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates w_1. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and by (4.191–4.192), the (dominant integral) highest weight is

$$\mu = (k+4\,;k,l,-l)\,,\qquad \text{with } 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}_{\geq 0} \text{ and } k-l \in \mathbb{Z}_{\geq 0}. \tag{4.242}$$

Then we shall prove that the case $k = l$ is not possible. Using (4.209), (4.213), (4.233) and other similar equations, we deduce that if $w_1 \neq 0$ then $w_i \neq 0 \neq \overline{w}_i$ for all i. Now, we shall see that all w_i's and $\overline{w}_i$'s are completely determined by the highest weight vector w_1. More precisely, using (4.208), we have that $\text{wt}_{w_2} = (k+4; k-1, l+1, -l)$. Hence, from (4.193) we can prove that

$$2(l-k)w_2 = E_{-(\varepsilon_1-\varepsilon_2)}w_1. \tag{4.243}$$

If $k \neq l$, we have

$$w_2 = \frac{1}{2(l-k)}E_{-(\varepsilon_1-\varepsilon_2)}w_1. \tag{4.244}$$

If $k = l$, using (4.208), we have $w_2 \in [F_\mu]_{\mu-(\varepsilon_1-\varepsilon_2)}$ that has dimension 0 if $k = l$, which is a contradiction since $w_2 \neq 0$.

Therefore, from now on we shall assume that $k \neq l$. Similarly, using (4.213), we have that $\text{wt}_{w_3} = (k+4; k-1, l, -l+1)$. Hence, from (4.195) we can prove that

$$w_3 = \frac{-1}{2(k+l+1)}\left(E_{-(\varepsilon_1-\varepsilon_3)}w_1 + E_{-(\varepsilon_2-\varepsilon_3)}w_2\right). \tag{4.245}$$

Using (4.240), we have that $\text{wt}_{\overline{w}_3} = (k+4; k-1, l, -l-1)$. Hence, from the sum of (4.201) and (4.202) we can prove that

$$\overline{w}_3 = \frac{1}{2(k-l)}E_{-(\varepsilon_1+\varepsilon_3)}w_1. \tag{4.246}$$

Using (4.236), we have that $\text{wt}_{\overline{w}_2} = (k+4; k-1, l-1, -l)$. Hence, from the sum of (4.199) and (4.200) we can prove that

$$\overline{w}_2 = \frac{1}{2(k+l+1)}\left(E_{-(\varepsilon_1+\varepsilon_2)}w_1 + E_{-(\varepsilon_2-\varepsilon_3)}\overline{w}_3\right). \tag{4.247}$$

Using (4.233), we have that $\text{wt}_{\overline{w}_1} = (k+4; k-2, l, -l)$. Hence, from the sum of (4.197) and (4.198) we can prove that

$$\overline{w}_1 = \frac{1}{2(k+l+1)}\left(E_{-(\varepsilon_1-\varepsilon_3)}\overline{w}_3 - E_{-(\varepsilon_1+\varepsilon_2)}w_2\right). \tag{4.248}$$

Now, we have an explicit expression of all w_i's and $\overline{w}_j$'s in terms of w_1. After some lengthy computations it is possible to prove that Eqs. (4.191–4.241) hold. Hence, the vector

$$\vec{m}_1 = \sum_{l=1}^{3}\left[\left(\xi_{\{2l\}^c} + i\xi_{\{2l-1\}^c}\right)\otimes w_l + \left(\xi_{\{2l\}^c} - i\xi_{\{2l-1\}^c}\right)\otimes \overline{w}_l\right] \tag{4.249}$$

is a singular vector, where w_1 is a highest weight vector of F_μ, $\mu = (k+4\,;k,l,-l)$, with $2k \in \mathbb{Z}_{\geq 0}$, $2l \in \mathbb{Z}_{\geq 0}$, $k-l \in \mathbb{Z}_{>0}$, and all w_i's and $\overline{w}_j$'s are written in terms of w_1 in (4.244), (4.245), (4.248), (4.247) and (4.246). Now using (4.41), one can prove that

$$wt\ \vec{m}_1 = (k+3\,;k-1,l,-l)$$

finishing this case.

• *Case* $w_1 = 0$ *and* $w_2 \neq 0$. Using (4.209–4.211) and (4.227–4.229) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates w_2. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and considering (4.193–4.194), we have

$$\mu = (l+3\,;k,l,-k-1)\,, \qquad \text{with } 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}_{\geq 0},$$

that is not dominant integral, getting a contradiction.

• *Case* $w_1 = w_2 = 0$ *and* $w_3 \neq 0$. Using (4.212–4.214) and (4.230–4.232) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates w_3. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and considering (4.195–4.196), we have

$$\mu = (l+2\,;k,-k-2,l)\,,$$

that is not dominant integral, getting a contradiction.

• *Case* $w_1 = w_2 = w_3 = 0$ *and* $\overline{w}_3 \neq 0$. Using (4.221–4.223) and (4.239–4.241) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates $\overline{w}_3$. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and considering (4.201–4.202), we have

$$\mu = (-l+2\,;k,k,l)\,, \qquad \text{with } 2k \in \mathbb{Z}_{\geq 0}, 2l \in \mathbb{Z}, k+l \in \mathbb{Z}_{\geq 0}, k-l \in \mathbb{Z}_{\geq 0}. \quad (4.250)$$

Then we will see that the case $k = l$ is not possible. Using (4.216) and (4.220), we have $\overline{w}_1 \neq 0 \neq \overline{w}_2$.

Now, we shall see that all $\overline{w}_i$'s are completely determined by the highest weight vector $\overline{w}_3$. More precisely, applying $E_{\varepsilon_2-\varepsilon_3}$ to (4.199), we can prove that

$$2(k-l)\overline{w}_2 = E_{-(\varepsilon_2-\varepsilon_3)}\overline{w}_3. \quad (4.251)$$

If $k \neq l$, we have

$$\overline{w}_2 = \frac{1}{2(k-l)} E_{-(\varepsilon_2-\varepsilon_3)}\overline{w}_3. \quad (4.252)$$

If $k = l$, using (4.220), we have $0 \neq \overline{w}_2 \in [F_\mu]_{\mu-(\varepsilon_2-\varepsilon_3)}$, but $\dim\,[F_\mu]_{\mu-(\varepsilon_2-\varepsilon_3)} = 0$, getting a contradiction.

Therefore, from now on we shall assume that $k \neq l$. Similarly, applying $E_{\varepsilon_1-\varepsilon_3}$ to the sum of (4.197) and (4.198), we can prove that

$$\overline{w}_1 = \frac{1}{2(k-l)} E_{-(\varepsilon_1-\varepsilon_3)}\overline{w}_3. \quad (4.253)$$

In this case, Eqs. (4.191–4.241) collapse to a few ones and it is easy to see that all of them hold. Hence, the vector

$$\vec{m}_1 = \frac{1}{2(k-l)}\big(\xi_{\{2\}^c} - i\xi_{\{1\}^c}\big) \otimes E_{-(\varepsilon_1-\varepsilon_3)}\overline{w}_3 + \frac{1}{2(k-l)}\big(\xi_{\{4\}^c} - i\xi_{\{3\}^c}\big) \otimes E_{-(\varepsilon_2-\varepsilon_3)}\overline{w}_3 + \big(\xi_{\{6\}^c} - i\xi_{\{5\}^c}\big) \otimes \overline{w}_3 \quad (4.254)$$

is a singular vector, where $\overline{w}_3$ is a highest weight vector of F_μ, and $\mu = (-l+2\,;k,k,l)$, with $2k \in \mathbb{Z}_{\geq 0}$, $2l \in \mathbb{Z}$, $k+l \in \mathbb{Z}_{\geq 0}$, $k-l \in \mathbb{Z}_{>0}$. Now using (4.41), one can prove that

$$wt\ \vec{m}_1 = (-l+1\,;k,k,l+1)\,.$$

• *Case* $w_1 = w_2 = w_3 = \overline{w}_3 = 0$ *and* $\overline{w}_2 \neq 0$. Using (4.218–4.220) and (4.236–4.238) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates $\overline{w}_2$. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and considering (4.199–4.200), we have

$$\mu = (-k+1\,;l,k,l+1)\,,$$

that is not dominant integral, getting a contradiction.

• *Case* $w_1 = w_2 = w_3 = \overline{w}_3 = \overline{w}_2 = 0$ *and* $\overline{w}_1 \neq 0$. Using (4.215–4.217) and (4.233–4.235) we obtain that the Borel subalgebra of $\mathfrak{so}(6)$ annihilates $\overline{w}_1$. Hence, it is a highest weight vector in the irreducible $\mathfrak{cso}(6)$-module F_μ, and considering (4.197–4.198), we have

$$\mu = (-k\,;k,l,l)\,, \qquad \text{with } 2k \in \mathbb{Z}_{>0},\ 2l \in \mathbb{Z}_{\geq 0},\ \text{and } k-l \in \mathbb{Z}_{\geq 0}, \tag{4.255}$$

which is dominant integral. In this case, the conditions (4.191–4.241) reduce to the equation $E_{-(\varepsilon_2-\varepsilon_3)}\overline{w}_1 = 0$, that holds for the highest weight (4.255). Therefore, the vector

$$\vec{m}_1 = \left(\xi_{\{2\}^c} - i\xi_{\{1\}^c}\right) \otimes \overline{w}_1 \tag{4.256}$$

is a singular vector of $\mathrm{Ind}(F_\mu)$ with μ as in (4.255). Now using (4.41), one can prove that

$$wt\ \vec{m}_1 = (-k-1\,;k+1,l,l)$$

finishing the proof. □

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Appendices

A. Notations in the Files that use Macaulay2

This appendix contains the explanations of notations used in the files written for Macaulay2 in order to classify singular vectors in CK_6-induced modules of degree $-1, \ldots, -5$. These notations are the link between this paper and the files that use Macaulay2.

As we have seen in (4.9), the possible forms of the singular vectors are the following:

$$\begin{aligned}
\vec{m} &= \partial^2 \sum_{|I|=5} \xi_I \otimes v_{I,2} + \partial \sum_{|I|=3} \xi_I \otimes v_{I,1} + \sum_{|I|=1} \xi_I \otimes v_{I,0}, \text{(Degree } -5),\\
\vec{m} &= \partial^2 \sum_{|I|=6} \xi_I \otimes v_{I,2} + \partial \sum_{|I|=4} \xi_I \otimes v_{I,1} + \sum_{|I|=2} \xi_I \otimes v_{I,0}, \text{ (Degree } -4),\\
\vec{m} &= \partial \sum_{|I|=5} \xi_I \otimes v_{I,1} + \sum_{|I|=3} \xi_I \otimes v_{I,0}, \text{ (Degree } -3),\\
\vec{m} &= \partial \sum_{|I|=6} \xi_I \otimes v_{I,1} + \sum_{|I|=4} \xi_I \otimes v_{I,0}, \text{ (Degree } -2),\\
\vec{m} &= \sum_{|I|=5} \xi_I \otimes v_{I,0}, \text{ (Degree } -1).
\end{aligned} \tag{A.1}$$

In order to abbreviate and capture the length of the elements ξ_I in the summands of the possible singular vectors $\vec{m}$, and the degree of $\vec{m}$, we introduce the following notation that will be used in the software:

$$g_i = \sum_{|I|=i} \xi_I \otimes v_{I,-}, \tag{A.2}$$

so they can be rewritten as follows (that is $v_{I,-}$ will correspond to $v_{I,2},\ v_{I,1}$ or $v_{I,0}$):

$$\begin{aligned}
\vec{m}_5 &= \partial^2\, g_5 + \partial\, g_3 + g_1, && \text{(Degree } -5)\\
\vec{m}_4 &= \partial^2\, g_6 + \partial\, g_4 + g_2, && \text{(Degree } -4)\\
\vec{m}_3 &= \partial\, g_5 + g_3, && \text{(Degree } -3)\\
\vec{m}_2 &= \partial\, g_6 + g_4, && \text{(Degree } -2)\\
\vec{m}_1 &= g_5, && \text{(Degree } -1).
\end{aligned} \tag{A.3}$$

We have created a file (or a series of files in the case of $\vec{m}_5, \vec{m}_3$ and $\vec{m}_1$) for each possible singular vector of type (A.3). The first part of all the files have the same structure, and the idea is to impose the equations given in Lemma 4.5 to each $\vec{m}_i$. From these equations, we constructed a matrix by taking the coefficients of these equations in terms of a natural basis, getting in this way a homogeneous linear system that is solved in order to get a simplified list of conditions. Unfortunately we are not expert in Macaulay2 or Maple, therefore it is not done in the optimal or simplest way.

Description of the inputs:

• Input 1: We define $R0 = \mathbb{Q}[z]/(z^2+1) \simeq \mathbb{Q} + i\mathbb{Q}$. We defined $R0$ because the scalars involved in the equations of Lemma 4.5 and in the formula of the λ-action belong to this field.

• Input 2: We define the polynomial ring R with coefficients in $R0$, in the skew-commutative variables $x_1, \ldots, x_6$ and the commutative variables $F_{(i,j)}$ $(1 \le i < j \le 6)$, E , v_I $(1 \le |I| \le 6)$. Observe that the variables x_i correspond to the variables ξ_i in the paper and E corresponds to the operator E_{00}. All the other variables are the same as in the paper. Note that in this case the software considers the term $F_{(1,2)} v_3$ as a monomial in the polynomial ring, not as the element $F_{(1,2)} \in \mathfrak{so}(6)$ acting in $v_3 \in F$.

Remark A.1. Observe that the command "diff(x, f)" in Macaulay2 is the derivative of f **on the right** with respect to x. Since we work with skew-commutative variables x_i and we need to compute the left derivative ∂_{x_i} (see the formula of the λ-action on induced modules). In our case, we have

$$\partial_{x_i}(f) = (-1)^{|f|-1}(\text{diff}(x_i, f)), \tag{A.4}$$

and

$$\partial_{x_i}\partial_{x_j}(f) = -(\text{diff}(x_i, \text{diff}(x_j, f))). \tag{A.5}$$

• Input 3: We define $f\_(0) = 1_R$, $f\_(I) = x_I = \xi_I$ for $1 \leq |I| \leq 3$, and $fd\_(I) = (f\_(I))^* = \xi_I^*$ for $0 \leq |I| \leq 3$. Observe that we used "diff" in the definition of fd.

• Input 4: For $1 \leq i \leq 6$, we define $g\_(i) = \sum_{|I|=i} x_I * v\_(I)$ as in (A.2) and (A.3).

• Inputs 5–10: Now we write the terms used in the notation introduced in (4.10) and (4.11). Namely, we define the terms

$$\begin{aligned} a\_((I), k) &:= a(\xi_I, g_k), \quad ad\_((I), k) := ad(\xi_I, g_k),\\ b\_((I), k) &:= b(\xi_I, g_k), \quad bd\_((I), k) := bd(\xi_I, g_k),\\ B\_((I), k) &:= B(\xi_I, g_k), \quad Bd\_((I), k) := Bd(\xi_I, g_k),\\ C\_((I), k) &:= C(\xi_I, g_k), \quad Cd\_((I), k) := Cd(\xi_I, g_k), \end{aligned}$$

for all $1 \leq k \leq 6$ and $0 \leq |I| \leq 3$. Observe that in order to write the terms that appear in the λ-action, we have to take care of the sign in the derivative by using (A.4) and (A.5).

• Inputs 11–17: According to Lemma 4.5, the conditions (S1)–(S3) on a vector $\vec{m}$, of degree at most -5, are equivalent to the following list of equations:

$*For|f| = 0:$

$$\begin{aligned} 0 &= C_0 + B_1, && ec\_((0), 1)\\ 0 &= 2\,B_2 + a_2 + C_1, && ec\_((0), 2)\\ 0 &= 2\,bd_0 - i\,a_2 + i\,C_1. && ec\_((0), 3) \end{aligned}$$

$*For f = \xi_i:$

$$\begin{aligned} 0 &= 3\,B_2 + 2\,i\,bd_1 + 2\,i\,ad_0 + 2\,C_1, && ec\_((i), 1)\\ 0 &= 2\,C_0 - a_1 + B_1 + 2\,bd_0\,i, && ec\_((i), 2)\\ 0 &= 2a_2 + B_2, && ec\_((i), 3)\\ 0 &= 3\,Bd_0 - i\,C_1 + bd_1 - 2\,ad_0, && ec\_((i), 4)\\ 0 &= 2\,b_2 + a_1 + B_1, && ec\_((i), 5)\\ 0 &= b_1 + B_0. && ec\_((i), 6) \end{aligned}$$

$*For f = \xi_i\xi_j \ \ (i < j):$

$$0 = 2\,C_0 + 2\,Bd_0\,i + B_1 - i\,ad_0 + i\,bd_1, \qquad ec\_((i, j), 1)$$

$$0 = 2\,b_2 + i\,ad_0 + i\,bd_1 + B_1, \qquad ec\_((i,j),2)$$
$$0 = bd_0 + b_1\,i - B_0\,i. \qquad ec\_((i,j),3)$$

$*For f = \xi_i \xi_j \xi_k \ (i < j < k)$:

$$0 = C_0 - Cd_0\,i, \qquad ec\_((i,j,k),1)$$
$$0 = bd_0 + i\,b_0, \qquad ec\_((i,j,k),2)$$
$$0 = B_1 - Bd_1\,i - a_1 + ad_1\,i, \qquad ec\_((i,j,k),3)$$
$$0 = b_2 - bd_2\,i + a_1 - ad_1\,i, \qquad ec\_((i,j,k),4)$$
$$0 = bd_1 + Bd_0 + B_0\,i + b_1\,i, \qquad ec\_((i,j,k),5)$$
$$0 = ad_0 + a_0\,i - Bd_0 - B_0\,i. \qquad ec\_((i,j,k),6)$$

$*For f = \alpha_{ij} or \beta_{ij} \in B_{\mathfrak{so}(6)} \ (1 \le i < j \le 3)$:

$$b_1(\alpha_{ij}) = 0, \qquad ecborel\_((i,j),1)$$
$$b_1(\beta_{ij}) = 0, \qquad ecborel\_((i,j),2)$$
$$b_2(\alpha_{ij}) = 0, \qquad ecborel\_((i,j),3)$$
$$b_2(\beta_{ij}) = 0, \qquad ecborel\_((i,j),4)$$
$$b_0(\alpha_{ij}) = 0, \qquad ecborel\_((i,j),5)$$
$$b_0(\beta_{ij}) = 0, \qquad ecborel\_((i,j),6)$$

The right column of the previous list of conditions contains the name that is used in the Macaulay file of $\vec{m}_5$ for each equation. Observe that for each vector $\vec{m}_i$ the equations are implemented in a different way, taking care of the elements g_k. Namely, if we work with $\vec{m}_4 = \partial^2 g_6 + \partial g_4 + g_2$, then equation $ec\_((0),3)$ is written in the Macaulay file as

$$2\,bd\_((0),2) - z\,a\_((0),6) + z\,C\_((0),4) = 0, \tag{A.6}$$

where z corresponds to the complex number i. And for $\vec{m}_5 = \partial^2 g_5 + \partial g_3 + g_1$, then equation $ec\_((0),3)$ is written in the corresponding Macaulay file as

$$ec\_((0),3) = 2\,bd\_((0),1) - z\,a\_((0),5) + z\,C\_((0),3) = 0. \tag{A.7}$$

Not all the equations are non-trivial for the different $\vec{m}_i$, since the length of the monomial ξ_I may be greater than 6. For example, in (A.7), the length of ξ_I is 6, but this equation is trivial when it is implemented for $\vec{m}_3$ since the length of ξ_I is 7. In the following table we indicate which equations appear for the different $\vec{m}_i$ and we give the length of ξ_I that is present in each case. Therefore, the name and number of the equations is modified for the file of each $\vec{m}_i$.

	$\vec{m}_5$	$\vec{m}_4$	$\vec{m}_3$	$\vec{m}_2$	$\vec{m}_1$
• $\|f\| = 0$:					
$C_0 + B_1$	3	4	5	6	–
$2\, B_2 + a_2 + C_1$	5	6	–	–	–
$2\, bd_0 - i\, a_2 + i\, C_1$	5	6	–	–	–
• $\|f\| = 1$:					
$3\, B_2 + 2\, i\, bd_1 + 2\, i\, ad_0 + 2\, C_1$	6	–	–	–	–
$2\, C_0 - a_1 + B_1 + 2\, bd_0\, i$	4	5	6	–	–
$2a_2 + B_2$	6	–	–	–	–
$3\, Bd_0 - i\, C_1 + bd_1 - 2\, ad_0$	6	–	–	–	–
$2\, b_2 + a_1 + B_1$	4	5	6	–	–
$b_1 + B_0$	2	3	4	5	6
• $\|f\| = 2$:					
$2\, C_0 + 2\, Bd_0\, i + B_1 - i\, ad_0 + i\, bd_1$	5	6	–	–	–
$2\, b_2 + i\, ad_0 + i\, bd_1 + B_1$	5	6	–	–	–
$bd_0 + b_1\, i - B_0\, i$	3	4	5	6	–
• $\|f\| = 3$:					
$C_0 - Cd_0\, i$	6	–	–	–	–
$bd_0 + i\, b_0$	2	3	4	5	6
$B_1 - Bd_1\, i - a_1 + ad_1\, i$	6	–	–	–	–
$b_2 - bd_2\, i + a_1 - ad_1\, i$	6	–	–	–	–
$bd_1 + Bd_0 + B_0\, i + b_1\, i$	4	5	6	–	–
$ad_0 + a_0\, i - Bd_0 - B_0\, i$	4	5	6	–	–
• $f \in$ Borel:					
b_2	5	6	–	–	–
b_1	3	4	5	6	–
b_0	1	2	3	4	5

• Input 18–21: We denote by $A\_((I), k)$ a one column matrix whose entries are the coefficients in the monomials x_J of the equation $ec\_((I), k)$. We should impose that the equation of each entry must be zero.

• Input 22: We denote by $M\_((I), k)$ a one column matrix whose entries are the coefficients in the monomials x_J of the equation $ecborel\_((I), k)$. We should impose that the equation of each entry must be zero.

• Input 23: The previously defined matrices $A\_((I), k)$ and $M\_((I), k)$ are one column matrices whose entries are R0-linear combinations of the monomials $v\_(I)$, $F\_(i, j) * v\_(I)$ and $E * v\_(I)$. Each entry must be zero, for that reason we define the lists zvari=list$\{v_(I),\ F_(i, j),\ E\}$ and wvari=list$\{v_(I),\ F_(i, j) * v_(I),\ E * v_(I)\}$, in this order, with the auxiliary lists avari1, avari2 and avari3.

• Input 28–31: We take the transpose of $A\_((I), k)$ getting a one row matrix. Then for each entry in this one row matrix, we produce a column formed by the coefficients of this entry with respect to the variables in wvari, obtaining in this way a matrix with coefficients in R0 whose transpose is called $D\_((I), k)$. If we consider wvari as a one column matrix, then we have $A_((I), k) = D_((I), k)*$wvari and it must be zero. Therefore we obtained a homogeneous linear system that must be solved.

• Input 32: With the same procedure, using the matrices $M\_((I), k)$, we define the matrices $N\_((I), k)$ that complete the linear system.

• Input 33: The matrices $D\_((I), k)$ and $N\_((I), k)$ are put together into one matrix that is called X whose coefficients are in R0. So, we need to solve the homogeneous linear system associated to X.

Observe that with this procedure, we consider the elements $F\_(i, j) * v\_(I)$ as a monomial in the ring R, not as an element in $\mathfrak{so}(6)$ acting in $v\_(I)$. Since the software (at least from our knowledge) does not work with Lie theory, we first solve the linear system, and then we impose the Lie setting by hand. The description of the inputs that we gave is essentially the structure of all the Macaulay files associated to the vectors $\vec{m}_5$, $\vec{m}_4$ and $\vec{m}_2$. The files associated to $\vec{m}_3$ and $\vec{m}_1$ have a modification: before the definition of the matrices "D" and "M" all the variables $F\_(i, j)$ are written as linear combinations of the more natural basis of $\mathfrak{so}(6)$ given by the H_i and E_α. Therefore the list of monomials in wvari is written in terms of them.

Now, we describe in details the list of files associated to each $\vec{m}_i$.

Files associated to $\vec{m}_5$

• *File "m5-macaulay-1"*

With the list of inputs previously described, we get a 1952×544 matrix X of rank 540 (see inputs 33–40). This matrix X is constructed by joining together the list of matrices $l0, l1, \ldots, l4$. In order to reduce the size of the matrix, we study the rank of these matrices and we found that the 992×544 matrix, called $Y25$, formed with the matices $l0, l1, l2, l4$ also has rank 540. Unfortunately, the software Macaulay2 can solve a linear system if the matrix is over $\mathbb{Z}_p$, $\mathbb{R}$ or $\mathbb{C}$, and it must be a non-singular square matrix in the cases $\mathbb{R}$ or $\mathbb{C}$. Therefore, we exported the matrix $Y25$ and we used Maple, see the file "m5-maple-1", to find the row-reduced echelon matrix of $Y25$, that is called C in that file.

• *File "m5-macaulay-2"*

If we try to copy the matrix C in the file "m5-macaulay-1" the software runs out of memory. Therefore, we continue the work in this NEW Macaulay file "m5-macaulay-2". Now, we describe the inputs in details:

$*$ Input 1–7: The rings $R0$ and R, and the list of variables $wvari$ are copied from the file "m5-macaulay-1". We need $wvari$ because, in input 19, we reconstruct the (reduced) equations as linear combinations of the monomials $v\_(I)$, $F\_(i, j) * v\_(I)$ and $E * v\_(I)$.

$*$ Input 8–15: The matrix C that is produced in the file "m5-maple-1", which is the row-reduced echelon matrix of $Y25$, is introduced in this NEW Macaulay file "m5-macaulay-2" divided in several parts, called $X1, \ldots, X7$. These parts are put together to reconstruct the matrix C and it is called $X11$ (input 15). Observe that $Y25$ was a 992×544 matrix of rank 540. For this reason, we copied the first 542 rows of C (the row-reduced echelon matrix of $Y25$). Therefore $X11$ is a 542×544 matrix with zero in the last two rows.

$*$ Input 16–19: We obtain a reduced (and equivalent) list of equations in a one column matrix $X28 = X27 * wvari$ (whose size is 542×1), where $X27$ is $X11$ viewed with entries in the ring R. Each entry must be zero.

At the end of this list of equations, we observe the following conditions:

$$\begin{array}{ll} 0 = v_1 + v_{1,3,4,5,6} & 0 = v_2 - i\ v_{1,3,4,5,6} \\ 0 = v_3 + v_{1,2,3,5,6} & 0 = v_4 - i\ v_{1,2,3,5,6} \\ 0 = v_5 + v_{1,2,3,4,5} & 0 = v_6 - i\ v_{1,2,3,4,5} \\ 0 = v_{1,2,3} - i\ v_{1,2,3,5,6} & 0 = v_{1,2,4} - v_{1,2,3,5,6} \\ 0 = v_{1,2,5} - i\ v_{1,2,3,4,5} & 0 = v_{1,2,6} - v_{1,2,3,4,5} \\ 0 = v_{1,3,4} - i\ v_{1,3,4,5,6} & 0 = v_{1,3,5} + i\ v_{2,4,6} \\ 0 = v_{1,3,6} + v_{2,4,6} & 0 = v_{1,4,5} + v_{2,4,6} \\ 0 = v_{1,4,6} - i\ v_{2,4,6} & 0 = v_{1,5,6} - i\ v_{1,3,4,5,6} \\ 0 = v_{2,3,4} - v_{1,3,4,5,6} & 0 = v_{2,3,5} + v_{2,4,6} \\ 0 = v_{2,3,6} - i\ v_{2,4,6} & 0 = v_{2,4,5} - i\ v_{2,4,6} \\ 0 = v_{2,5,6} - v_{1,3,4,5,6} & 0 = v_{3,4,5} - i\ v_{1,2,3,4,5} \\ 0 = v_{3,4,6} - v_{1,2,3,4,5} & 0 = v_{3,5,6} - i\ v_{1,2,3,5,6} \\ 0 = v_{4,5,6} - v_{1,2,3,5,6} & 0 = v_{2,3,4,5,6} + i\ v_{1,3,4,5,6} \\ 0 = v_{1,2,4,5,6} + i\ v_{1,2,3,5,6} & 0 = v_{1,2,3,4,6} + i\ v_{1,2,3,4,5} \end{array} \tag{A.8}$$

In order to simplify the 540 equations, we need to impose conditions (A.8). Observe that all the vectors v_I can be written in terms of the set $\{v\_1, v\_3, v\_5, v\_(1,3,5)\}$. This is done in the following inputs.

$*$ Input 20–21: We define a ring P that is isomorphic to R. In this case, P is the polynomial ring with coefficients in $R0$, in the skew-commutative variables $t\_1, \ldots, t\_6$ and the commutative variables $h\_i, e_{(i,j)}, em_{(i,j)}, me_{(i,j)}, mem_{(i,j)}$ $(1 \leq i < j \leq 3)$, $E0$, u_I $(1 \leq |I| \leq 6)$. The idea is to replace the basis $F_{(i,j)} \in \mathfrak{so}(6)$ by the basis given by H_i and E_α. We are using the following notation, for $1 \leq i < j \leq 3$:

$$\begin{aligned} e\_(i,j) &= E_{\varepsilon_i - \varepsilon_j}, \\ em\_(i,j) &= E_{\varepsilon_i + \varepsilon_j}, \\ me\_(i,j) &= E_{-(\varepsilon_i - \varepsilon_j)}, \\ mem\_(i,j) &= E_{-(\varepsilon_i + \varepsilon_j)}. \end{aligned} \tag{A.9}$$

$*$ Input 22: We define a map $Q : R \to P$, that impose conditions (A.8) and change the basis in $\mathfrak{so}(6)$ using the notation (A.9). The definition of Q is the following:

Input 22: $Qvari = \{x\_i => t\_i,$
$F\_(1,2) => -z * h\_1, F\_(3,4) => -z * h\_2, F\_(5,6) => -z * h\_3,$
$F\_(2*i-1, 2*j-1) => (e\_(i,j) + em\_(i,j) + me\_(i,j) + mem\_(i,j))/4),$
$F\_(2*i, 2*j) => (e\_(i,j) - em\_(i,j) + me\_(i,j) - mem\_(i,j))/4),$
$F\_(2*i-1, 2*j) => -z * (e\_(i,j) - em\_(i,j) - me\_(i,j) + mem\_(i,j))/4),$
$F\_(2*i, 2*j-1) => -z * (-e\_(i,j) - em\_(i,j) + me\_(i,j) + mem\_(i,j))/4),$
$E => E0,$
$v\_1 => u\_1, v\_2 => -z * u\_1, v\_3 => u\_3,$
$v\_4 => -z * u\_3, v\_5 => u\_5, v\_6 => -z * u\_5,$
$v\_(i,j) => u\_(i,j)),$
$v\_(1,2,3) => -z * u\_3, v\_(1,2,4) => -u\_3, v\_(1,2,5) => -z * u\_5,$
$v\_(1,2,6) => -u\_5, v\_(1,3,4) => -z * u\_1, v\_(1,3,5) => u\_(1,3,5),$

$v\_(1,3,6) => -z*u\_(1,3,5), v\_(1,4,5) => -z*u\_(1,3,5), v\_(1,4,6)$
$=> -u\_(1,3,5),$
$v\_(1,5,6) => -z*u\_1, v\_(2,3,4) => -u\_1, v\_(2,3,5) => -z*u\_(1,3,5),$
$v\_(2,3,6) => -u\_(1,3,5), v\_(2,4,5) => -u\_(1,3,5), v\_(2,4,6)$
$=> z*u\_(1,3,5),$
$v\_(2,5,6) => -u\_1, v\_(3,4,5) => -z*u\_5, v\_(3,4,6) => -u\_5,$
$v\_(3,5,6) => -z*u\_3, v\_(4,5,6) => -u\_3,$
$v\_(i,j,k,l) => u\_(i,j,k,l)))),$
$v\_(2,3,4,5,6) => z*u\_1, v\_(1,3,4,5,6) => -u\_1, v\_(1,2,4,5,6) => z*u\_3,$
$v\_(1,2,3,5,6) => -u\_3, v\_(1,2,3,4,6) => z*u\_5, v\_(1,2,3,4,5) => -u\_5,$
$v\_(1,2,3,4,5,6) => u\_(1,2,3,4,5,6)\},$
$Q = map(P, R, Qvari).$

$*$ Input 23–27: We define a new 'wvari', which is the list of variables that will appear in the equations. More precisely, wvari=list$\{u_(I),\ h_k*u_(I),\ e_(i,j)*u_(I),\ em_(i,j)*u_(I),\ me_(i,j)*u_(I),\ mem_(i,j)*u_(I),\ E0*u_(I)\}$, where $u\_(I)$ is restricted in this case to the set $\{u\_1, u\_3, u\_5, u\_(1,3,5)\}$.

$*$ Input 28–31: We apply the map Q to the equations in the matrix $X28$, and then we obtain a 542×68 matrix, called $X29$, given by the coefficients in the monomials of 'wvari' that appear in the equations of $Q(X28)$. The rank of X29 is 64. The matrix $X29$ is exported in order to reduce the linear system.

$*$ Input 33: We use Maple to reduce the matrix $X29$. The matrix C that is produced in the file "m5-maple-2", which is the row-reduced echelon matrix of $X29$, is introduced in this input and it is called $X30$. Observe that $X29$ was a 542×68 matrix of rank 64. For this reason, we copied the first 70 rows of C (the row-reduced echelon matrix of $X29$). Therefore $X30$ is a 70×68 matrix with zero in the last six rows.

$*$ Input 34–38: We obtain a reduced (and equivalent) list of equations in a one column matrix $X34 = X33 * wvari$ (whose size is 70×1), where $X33$ is $X30$ viewed with entries in the ring P. Each entry must be zero. This final list of 64 simplified and equivalent equations is copied in the proof of Lemma 4.7.

Files associated to $\vec{m}_4$

• *File "m4-macaulay"*

With the list of inputs previously described, except that we do not need to impose Borel equations (hence the matrices "M" and "N" are not needed), we get a 1104×527 matrix X of rank 527. Therefore, there is no non-trivial solution of this linear system, proving that there is no singular vector of degree -4.

Files associated to $\vec{m}_3$

• *File "m3-macaulay-1"*

With the list of inputs previously described, we get a 694×442 matrix X of rank 397. This matrix X is exported to a file and using Maple, see the file "m3-maple-1", we obtain the row-reduced echelon matrix of X, that is called C. This matrix C is introduced as the matrix $X11$ in the Macaulay file (see input 43-44), to reconstruct the (reduced) equations as linear combinations of the monomials $v\_(I)$, $F\_(i,j) * v\_(I)$ and $E * v\_(I)$. In fact, the matrix $X11$ is 400×442 because we removed the last zero rows of the row-reduced echelon matrix, therefore it has zero in the last three rows.

$*$ Input 45–48: We obtain a reduced (and equivalent) list of equations in a one column matrix $X14 = X12 * wvari$ (whose size is 400×1), where $X12$ is $X11$ viewed with entries in the ring R. Each entry must be zero.

At the end of this list of equations, we observe the following conditions:

$$\begin{array}{ll} 0 = v_{1,2,3} - i\ v_{4,5,6} & 0 = v_{1,2,4} + i\ v_{3,5,6} \\ 0 = v_{1,2,5} - i\ v_{3,4,6} & 0 = v_{1,2,6} + i\ v_{3,4,5} \\ 0 = v_{1,3,4} - i\ v_{2,5,6} & 0 = v_{1,3,5} + i\ v_{2,4,6} \\ 0 = v_{1,3,6} - i\ v_{2,4,5} & 0 = v_{1,4,5} - i\ v_{2,3,6} \\ 0 = v_{1,4,6} + i\ v_{2,3,5} & 0 = v_{1,5,6} - i\ v_{2,3,4} \\ 0 = v_{2,3,4,5,6} & 0 = v_{1,3,4,5,6} \\ 0 = v_{1,2,4,5,6} & 0 = v_{1,2,3,5,6} \\ 0 = v_{1,2,3,4,6} & 0 = v_{1,2,3,4,5} \end{array} \tag{A.10}$$

Observe that (A.10) can be written as

$$v_I = 0 \quad \text{if } |I| = 5, \quad \text{and} \quad v_{\{a,b,c\}} = (-1)^{a+b+c}\, i\ v_{\{a,b,c\}^c} \quad \text{for } a < b < c. \tag{A.11}$$

• *File "m3-macaulay-2"*

Now, we have to impose the identities (A.10) to reduce the number of variables. Using (A.11), everything can be written in terms of

$$v_{1,j,k} \quad \text{with} \quad 2 \leq j < k \leq 6.$$

Unfortunately, the result is not enough to obtain in a clear way the possible highest weight vectors. For example, after the reduction and some extra computations it is possible to see that

$$(v_{1,3,6} - v_{1,4,5}) + i\ (v_{1,3,5} + v_{1,4,6}) \tag{A.12}$$

is annihilated by the Borel subalgebra. Hence, it is necessary to impose (A.10) and make a change of variables. We produced an auxiliary file where we imposed (A.10), and after the analysis of the results, we found that the following change of variable is convenient:

$$\begin{aligned} u_1 &= v_{1,2,3} - i\ v_{1,2,4} \\ u_2 &= v_{1,2,3} + i\ v_{1,2,4} \\ u_3 &= v_{1,2,5} - i\ v_{1,2,6} \\ u_4 &= v_{1,2,6} + i\ v_{1,2,6} \\ u_5 &= v_{1,3,4} - \ v_{1,5,6} \\ u_6 &= v_{1,3,4} + \ v_{1,5,6} \\ u_7 &= v_{1,3,5} - v_{1,4,6} + i\ (v_{1,3,6} + v_{1,4,5}) \\ u_8 &= v_{1,3,5} + v_{1,4,6} - i\ (v_{1,3,6} - v_{1,4,5}) \\ u_9 &= v_{1,3,5} - v_{1,4,6} - i\ (v_{1,3,6} + v_{1,4,5}) \\ u_{10} &= v_{1,3,5} + v_{1,4,6} + i\ (v_{1,3,6} - v_{1,4,5}) \end{aligned} \tag{A.13}$$

or equivalently

$$
\begin{aligned}
v_{1,2,3} &= \frac{1}{2}(u_1 + u_2) = i\ v_{4,5,6}\\
v_{1,2,4} &= \frac{i}{2}(u_1 - u_2) = -i\ v_{3,5,6}\\
v_{1,2,5} &= \frac{1}{2}(u_3 + u_4) = i\ v_{3,4,6}\\
v_{1,2,6} &= \frac{i}{2}(u_3 - u_4) = -i\ v_{3,4,5}\\
v_{1,3,4} &= \frac{1}{2}(u_5 + u_6) = i\ v_{2,5,6}\\
v_{1,5,6} &= \frac{-1}{2}(u_5 - u_6) = i\ v_{2,3,4}\\
v_{1,3,5} &= \frac{1}{4}(u_7 + u_8 + u_9 + u_{10}) = -i\ v_{2,4,6}\\
v_{1,4,6} &= \frac{1}{4}(-u_7 + u_8 - u_9 + u_{10}) = -i\ v_{2,3,5}\\
v_{1,3,6} &= \frac{i}{4}(-u_7 + u_8 + u_9 - u_{10}) = i\ v_{2,4,5}\\
v_{1,4,5} &= \frac{i}{4}(-u_7 - u_8 + u_9 + u_{10}) = i\ v_{2,3,6}
\end{aligned}
\tag{A.14}
$$

We also need to replace the basis $F_{(i,j)} \in \mathfrak{so}(6)$ by the basis given by H_i and E_α. We are using the notation introduced in (A.9). The identities (A.10), the change of variables (A.14) and the new basis of $\mathfrak{so}(6)$ are implemented with the definition of a ring P that is isomorphic to R and a map $Q : R \to P$. More precisely,
$*$ Input 1–22: They are the same inputs in the file “m3-macaulay-1”.

$*$ Input 23: We define a ring P as the polynomial ring with coefficients in $R0$, in the skew-commutative variables $t\_1, \ldots, t\_6$ and the commutative variables u_i $(1 \le i \le 10)$, h_i, $e_{(i,j)}$, $em_{(i,j)}$, $me_{(i,j)}$, $mem_{(i,j)}$ $(1 \le i < j \le 3)$, $E0$.

$*$ Input 24: We define a map $Q : R \to P$, that impose conditions (A.10), the change of variables (A.14) and change the basis in $\mathfrak{so}(6)$ using the notation (A.9). The definition of Q is the following:

$Qvari = \{x_i => t_i,$

$F\_(1,2) => -z * h_1,\ F\_(3,4) => -z * h_2,\ F\_(5,6) => -z * h_3,$

$F\_(2*i-1, 2*j-1) => (e\_(i,j) + em\_(i,j) + me\_(i,j) + mem\_(i,j))/4,$

$F\_(2*i, 2*j) => (e\_(i,j) - em\_(i,j) + me\_(i,j) - mem\_(i,j))/4,$

$F\_(2*i-1, 2*j) => -z * (e\_(i,j) - em\_(i,j) - me\_(i,j) + mem\_(i,j))/4,$

$F\_(2*i, 2*j-1) => -z * (-e\_(i,j) - em\_(i,j) + me\_(i,j) + mem\_(i,j))/4,$

$E => E0,$

$v_i => u_i,$

$v\_(i,j) => u\_(i,j),$

$v\_(1,2,3) => (u_1 + u_2)/2,\ v\_(1,2,4) => z * (u_1 - u_2)/2,$

$v\_(1,2,5) => (u_3+u_4)/2, v\_(1,2,6) => z*(u_3-u_4)/2,$
$v\_(1,3,4) => (u_5+u_6)/2, v\_(1,5,6) => -(u_5-u_6)/2,$
$v\_(1,3,5) => (u_7+u_8+u_9+u_{10})/4,$
$v\_(1,3,6) => z*(-u_7+u_8+u_9-u_{10})/4,$
$v\_(1,4,5) => z*(-u_7-u_8+u_9+u_{10})/4,$
$v\_(1,4,6) => (-u_7+u_8-u_9+u_{10})/4,$
$v\_(2,3,4) => z*(u_5-u_6)/2,$
$v\_(2,3,5) => z*(-u_7+u_8-u_9+u_{10})/4,$
$v\_(2,3,6) => (-u_7-u_8+u_9+u_{10})/4,$
$v\_(2,4,5) => (-u_7+u_8+u_9-u_{10})/4,$
$v\_(2,4,6) => z*(u_7+u_8+u_9+u_{10})/4,$
$v\_(2,5,6) => -z*(u_5+u_6)/2, v\_(3,4,5) => -(u_3-u_4)/2,$
$v\_(3,4,6) => -z*(u_3+u_4)/2, v\_(3,5,6) => -(u_1-u_2)/2,$
$v\_(4,5,6) => -z*(u_1+u_2)/2,$
$v\_(i,j,k,l) => u\_(i,j,k,l),$
$v\_(1..i-1|i+1..6) => 0_P,$
$v\_(1,2,3,4,5,6) => u\_(1,2,3,4,5,6)\},$
$Q = map(P, R, Qvari).$

$*$ Input 25–41: With the same list of inputs as in the file "m3-macaulay-1", but applying the map Q, we get a 694×170 matrix X of rank 125. This matrix X is constructed by joining together the list of matrices $l0, l1, \ldots, l4$.
$*$ Input 42–47: In order to reduce the size of the matrix, we studied the rank of these matrices and we found that the 354×170 matrix, called $X2$, formed with the matices $l0, l1, l2, l4$ also has rank 125. We exported the matrix $X2$ and we used Maple, see the file "m3-maple-2", to find the row-reduced echelon matrix of $X2$, that is called C in that file.

$*$ Input 48–49: The matrix C that is produced in the file "m3-maple-2", which is the row-reduced echelon matrix of $X2$, is introduced in this file and it is called $X11$. Observe that $X2$ was a 354×170 matrix of rank 125. For this reason, we copied the first 127 rows of C (the row-reduced echelon matrix of $X2$). Therefore $X11$ is a 127×170 matrix with zero in the last two rows.

$*$ Input 50–55: We obtain a reduced (and equivalent) list of equations in a one column matrix $X14 = X12 * wvari$ (whose size is 127×1), where $X12$ is $X11$ viewed with entries in the ring P. Each entry must be zero.

$*$ Input 56–60: In order to simplify the list of equations, we define Z_i =row i of $X14$, and then we consider the following list of linear combinations of these rows:

for i in 0..124 do $Z\_i = X14\_(i,0)$;
$i57:$
$X25 = \{Z\_0, Z\_1-Z\_0, Z\_2+Z\_0, Z\_3, Z\_4+Z\_3, Z\_5-Z\_3, Z\_6, Z\_7+Z\_6,$
$Z\_8-Z\_6, Z\_9+Z\_11, Z\_10+Z\_11, Z\_11, Z\_12, Z\_13-Z\_12, Z\_14+Z\_13,$

Z_15+Z_17, Z_16+Z_15, Z_17+Z_16, $Z\_18+Z\_19-Z\_20$, $Z\_18-Z\_19+Z\_20$, $-Z\_18+Z\_19+Z\_20$, Z_21, Z_22, Z_23+Z_22, $Z\_24+Z\_25+Z\_26$,
for i in 25..29 list Z_i,
Z_30-Z_0, Z_31-Z_3, Z_32-Z_6, Z_33+Z_11, Z_34-Z_14, Z_35-Z_15,
for i in 36..124 list $Z\_i\}$;

obtaining an equivalent list of equations given by the rows of the 127×1 matrix $X29$ (all of them must be zero). These equations are copied in the proof of Lemma 4.9 together with the final analysis of them, see (4.45–4.169).

Files associated to $\vec{m}_2$

• *File "m2-macaulay"*

With the list of inputs previously described, we get a 268×272 matrix X of rank 192. This matrix X is exported to a file and using Maple (see the file "m2-maple.mws") we obtain the row-reduced echelon matrix of X, that is called $X11$. In fact, the matrix $X11$ is 195×272 because we removed the last zero rows of the row-reduced echelon matrix. This matrix $X11$ is introduced in the Macaulay file "m2-macaulay" as the input 43. In order to reconstruct the reduced system of equations as linear combinations of the monomials $v\_(I)$, $F\_(i, j) * v\_(I)$ and $E * v\_(I)$ we multiply X11*wvari, obtaining a one column matrix, called X14, with the list of equations that must be zero (see inputs 45–52). This matrix $X14$ is exported into a latex-pdf file "m2-equations.pdf", and the analysis of these equations is done in the paper (see the proof of the lemma 4.10 corresponding to $\vec{m}_2$).

Files associated to $\vec{m}_1$

• *File "m1-macaulay"*

Since the singular vectors found in [2] for K_6 are also singular vectors for CK_6, using (B42–B43) in [2], we have that it is convenient to introduce the following notation:

$$\begin{aligned} \vec{m}_1 &= \sum_{i=1}^{6} \xi_{\{i\}^c} \otimes v_{\{i\}^c} \\ &= \sum_{l=1}^{3} \Big[\big(\xi_{\{2l\}^c} + i\xi_{\{2l-1\}^c}\big) \otimes w_l + \big(\xi_{\{2l\}^c} - i\xi_{\{2l-1\}^c}\big) \otimes \overline{w}_l \Big], \end{aligned} \tag{A.15}$$

that is, for $1 \le l \le 3$,

$$v_{\{2l\}^c} = w_l + \overline{w}_l, \qquad v_{\{2l-1\}^c} = i(w_l - \overline{w}_l), \tag{A.16}$$

or equivalently, for $1 \le l \le 3$,

$$w_l = \frac{1}{2}(v_{\{2l\}^c} - i\ v_{\{2l-1\}^c}), \qquad \overline{w}_l = \frac{1}{2}(v_{\{2l\}^c} + i\ v_{\{2l-1\}^c}). \tag{A.17}$$

Now, with the usual list of inputs previously described, we have to impose the identities (A.16) to change the variables. We also need to replace the basis $F_{(i,j)} \in \mathfrak{so}(6)$ by the basis given by H_i and E_α. We are using the notation introduced in (A.9). The change of variables (A.16) and the new basis of $\mathfrak{so}(6)$ are implemented with the definition of a ring P that is isomorphic to R and a map $Q : R \to P$. More precisely,

$*$ Input 1–19: They are the usual inputs, for example as in the file "m3-macaulay-1".

$*$ Input 20–21: We define a ring P as the polynomial ring with coefficients in $R0$, in the skew-commutative variables $t\_1, \dots, t\_6$ and the commutative variables u_i $(1 \le i \le 10)$, $omega_1, omega_2, omega_3, domega_1, domega_2, domega_3, h\_i, e_{(i,j)}, em_{(i,j)}, me_{(i,j)}, mem_{(i,j)}$ $(1 \le i < j \le 3)$, $E0$.

$*$ Input 22: We define a map $Q : R \to P$, that change of variables (A.16) and change the basis in $\mathfrak{so}(6)$ using the notation (A.9). The definition of Q is the following:

$$
\begin{aligned}
&Qvari = \{x\_i => t\_i,\\
&F\_(1,2) => -z * h\_1,\ F\_(3,4) => -z * h\_2,\ F\_(5,6) => -z * h\_3,\\
&F\_(2*i-1, 2*j-1) => (e\_(i,j) + em\_(i,j) + me\_(i,j) + mem\_(i,j))/4),\\
&F\_(2*i, 2*j) => (e\_(i,j) - em\_(i,j) + me\_(i,j) - mem\_(i,j))/4),\\
&F\_(2*i-1, 2*j) => -z * (e\_(i,j) - em\_(i,j) - me\_(i,j) + mem\_(i,j))/4),\\
&F\_(2*i, 2*j-1) => -z * (-e\_(i,j) - em\_(i,j) + me\_(i,j) + mem\_(i,j))/4),\\
&E => E0,\\
&v\_i => u\_i,\\
&v\_(i,j) => u\_(i,j),\\
&v\_(i,j,k) => u\_(i,j,k),\\
&v\_(i,j,k,l) => u\_(i,j,k,l),\\
&v\_(2,3,4,5,6) => z * (omega\_1 - domega\_1),\\
&v\_(1,2,4,5,6) => z * (omega\_2 - domega\_2),\\
&v\_(1,2,3,4,6) => z * (omega\_3 - domega\_3),\\
&v\_(1,3,4,5,6) => (omega\_1 + domega\_1),\\
&v\_(1,2,3,5,6) => (omega\_2 + domega\_2),\\
&v\_(1,2,3,4,5) => (omega\_3 + domega\_3),\\
&v\_(1,2,3,4,5,6) => u\_(1,2,3,4,5,6)\},\\
&Q = map(P, R, Qvari).
\end{aligned}
$$

$*$ Input 23–37: With the usual list of inputs, but applying the map Q to the equations, we get a 62×102 matrix X of rank 51. We exported the matrix X and we used Maple, see the file "m1-maple", to find the row-reduced echelon matrix of X, that is called C in that file.

$*$ Input 39: The matrix C that is produced in the file "m1-maple", which is the row-reduced echelon matrix of X, is introduced in this file and it is called $X11$. Observe that X was a 62×102 matrix of rank 51. Therefore $X11$ is a 62×102 matrix with zero in the last 11 rows.

$*$ Input 40–45: We obtain a reduced (and equivalent) list of equations in a one column matrix $X14 = X12 * wvari$ (whose size is 62×1), where $X12$ is $X11$ viewed with entries in the ring P. Each entry must be zero.

These equations are copied in the proof of Lemma 4.11, in Eqs. (4.191–4.241), and the final analysis of them is done in that proof.

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