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On cubic difference equations with variable coefficients and fading stochastic perturbations

Ricardo Baccas, Cónall Kelly, and Alexandra Rodkina

Abstract We consider the stochastically perturbed cubic difference equation with variable coefficients

$$x_{n+1} = x_n(1 - h_n x_n^2) + \rho_{n+1}\xi_{n+1}, \quad n \in \mathbb{N}, \quad x_0 \in \mathbb{R}.$$

Here $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables, and $(\rho_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are sequences of nonnegative real numbers. We can stop the sequence $(h_n)_{n\in\mathbb{N}}$ after some random time $\mathscr{N}$ so it becomes a constant sequence, where the common value is an $\mathscr{F}_{\mathscr{N}}$-measurable random variable. We derive conditions on the sequences $(h_n)_{n\in\mathbb{N}}$, $(\rho_n)_{n\in\mathbb{N}}$ and $(\xi_n)_{n\in\mathbb{N}}$, which guarantee that $\lim_{n\to\infty} x_n$ exists almost surely (a.s.), and that the limit is equal to zero a.s. for any initial value $x_0 \in \mathbb{R}$.

1 Introduction

In this paper we analyse the global almost sure (a.s.) asymptotic behaviour of solutions of a cubic difference equation with variable coefficients and subject to stochastic perturbations

$$x_{n+1} = x_n(1 - h_n x_n^2) + \rho_{n+1}\xi_{n+1}, \quad n \in \mathbb{N}, \quad x_0 \in \mathbb{R}. \tag{1}$$

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Here $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent identically distributed random variables, $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative reals, and $(h_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative reals.

When $(\xi_n)_{n\in\mathbb{N}}$ is an independent sequence of standard Normal random variables, (1) can be interpreted as the Euler-Maruyama discretisation of the Itô-type stochastic differential equation

$$dX_t = -bX_t^3 dt + g(t)dW_t, \quad t \geq 0, \quad X_0 \in \mathbb{R}, \tag{2}$$

where $(W_t)_{t\geq 0}$ is a Wiener process, $b > 0$ is some constant, $g : [0,\infty) \to [0,\infty)$ is a continuous function. It was shown in [6] that when $\lim_{t\to\infty} g^2(t)\ln t = 0$, solutions of stochastic differential equation (2) are globally a.s. asymptotically stable, i.e. $\lim_{t\to\infty} X_t = 0$ a.s. for any initial value $X_0 \in \mathbb{R}$.

There is an extensive literature on the global a.s. asymptotic behaviour of solutions of nonlinear stochastic difference equations, and the most relevant publications for our purposes are: [1, 2, 3, 4, 5, 7, 14, 15]. However, if the timestep sequence in Eq. (1) is constant, so that $h_n \equiv h$, the global dynamics of (2) are not preserved and convergence of solutions to zero will only occur on a restricted subset of initial values. An early attempt to address local dynamics in an equation with bounded noise can be found in [8]; general results for equations with fading, state independent noise may be found in [2]. In [4] a complete description is given of these local dynamics (see also [2] and [5]). It was proved that the set of initial values can be partitioned into a "stability" region, within which solutions converge asymptotically to zero, an "instability" region, within which solutions rapidly grow without bound, and a region of unknown dynamics that is in some sense small. In the first two cases, the dynamic holds with probability at least $1-\gamma$ for $\gamma \in (0,1)$.

In the same article, it was shown that for any initial value $x_0 \in \mathbb{R}$, the behaviour of solution of the difference equation can be made consistent with the corresponding solution of the differential equation, with probability $1-\gamma$, by choosing the stepsize parameter h sufficiently small. This observation motivates the approach taken in this article, wherein the stepsize parameter is allowed to decrease over a random interval in order to capture trajectories within the basin of attraction of the point at zero long enough to ensure asymptotic convergence.

Several recent publications are devoted to the use of adaptive timestepping in a explicit Euler-Maruyama discretization of nonlinear equations: for example [3, 9, 12, 11]. In [9] (see also [7]) it was shown that suitably designed adaptive timestepping strategies could be used to ensure strong convergence of order $1/2$ for a class of equations with non-globally Lipschitz drift, and globally Lipschitz diffusion. These strategies work by controlling the extent of the nonlinear drift response in discrete time and required that the timesteps depend on solution values. In [11] an extension of that idea allows an explicit Euler-Maruyama discretisation to reproduce dynamical properties of a class of nonlinear stochastic differential equations with a unique equilibrium solution and non-negative drift and diffusion coefficients that are not globally Lipschitz continuous. The a.s. asymptotic stability and instability of the equilibrium at zero is closely reproduced, and positivity of solutions is preserved with arbitrarily high probability.

An element that these articles have in common is that the variable time-step depends upon the value of the solution. By contrast, in the present paper the sequence $(h_n)_{n\in\mathbb{N}}$ does not, and will be the same for any given initial value $x_0 \in \mathbb{R}$. However since the values of h_n can become arbitrarily small, it is not necessarily the case that x_n converges to zero: in fact if the stepsize sequence is summable we will show that the limit is nonzero a.s. So we freeze the sequence $(h_n)_{n\in\mathbb{N}}$ at an appropriate random moment $\mathscr{N}$, i.e. all step-sizes after $\mathscr{N}$ are the same: $h_n = h_{\mathscr{N}}$ for $n \ge \mathscr{N}$. The time at which this occurs depends on the initial value x_0, and is chosen to ensure that $(x_n)_{n\in\mathbb{N}}$ converges to zero a.s., as required.

The structure of the article is as follows. Some necessary technical results are stated in Section 2. In Section 3 we construct a timestep sequence $(h_n)_{n\in\mathbb{N}}$ that ensures solutions of the unperturbed cubic difference equation converge to a finite limit, and show that the summability of $(h_n)_{n\in\mathbb{N}}$ determines whether or not that limit is zero. In Section 4 we examine the convergence of solutions under the influence of a deterministic perturbation, and in Section 5 we consider two kinds of stochastic perturbation; one with bounded noise, and one with Gaussian noise. Illustrative numerical examples are provided in Section 6.

2 Mathematical preliminaries

Everywhere in this paper, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. A detailed discussion of probabilistic concepts and notation may be found, for example, in Shiryaev [14]. We will use the following elementary inequality: for each $a, b > 0$ and $\alpha \in (0,1)$

$$(a+b)^\alpha \le a^\alpha + b^\alpha. \tag{3}$$

The following lemmas also present additional useful technical results:

Lemma 1. *Let $f : [0,\infty) \to [0,\infty)$ be a decreasing continuous function, then*

$$\int_0^{n+1} f(x)dx > \sum_{i=1}^{n} f(i) > \int_1^{n+1} f(x)dx > \sum_{i=2}^{n+1} f(i).$$

Lemma 2. *(i) $\ln(1-x) < -x$ for $-\infty < x < 0$;*
(ii) For $0 < x < \frac{1}{2}$ the following estimate holds

$$\ln(1-x) > -2x. \tag{4}$$

Lemma 3. *Let $q_n \in [0,1)$ for all $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty}(1-q_n)$ converges to non zero limit if and only if $\sum_{n=1}^{\infty} q_n$ converges.*

We adopt the convention $\prod_{n=i}^{j} 1 = 1$ if $i > j$ from here forwards. The next result can be found in [14, Ch. 4.4, Ex. 1].

Lemma 4. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent $\mathscr{N}(0,1)$ distributed random variables. Then*

$$\mathbb{P}\left\{\limsup_{n\to\infty} \frac{\xi_n}{\sqrt{2\ln n}} = 1\right\} = 1. \tag{5}$$

We will use the following notation throughout the article:

Definition 1. Denote, for $k \in \mathbb{N}$,

$$\begin{aligned} e^a_{[k]} &= \exp\{\exp\{\ldots\{a\underbrace{\}\ldots\}}_{k\ times} \quad \text{for each} \quad a \in \mathbb{R}, \quad e^a_{[0]} = 1; \\ \ln_k b &= \ln[\ln[\ldots[\ln b\underbrace{]\ldots]]}_{k\ times} \quad \text{for each} \quad b \geq e^1_{[k]}, \quad \ln_0 b = b. \end{aligned} \tag{6}$$

Corollary 1. *For all $n, k \in \mathbb{N}$,*

$$\begin{aligned} \sum_{i=j}^{n} & \frac{1}{(i+1)\ln(i+1)\ldots\ln_k\left(i+e^1_{[k]}\right)} \\ &> \int_j^{n+1} \frac{dy}{(y+e^1_{[k]})\ln(y+e^1_{[k]})\ldots\ln_k\left(y+e^1_{[k]}\right)} \\ &= \ln_{k+1}(n+1+e^1_{[k]}) - \ln_{k+1}(j+e^1_{[k]}), \end{aligned} \tag{7}$$

and

$$\begin{aligned} \sum_{i=1}^{n+1} & \frac{1}{\left(i+e^1_{[k]}\right)\ln\left(i+e^1_{[k]}\right)\ldots\ln_k\left(i+e^1_{[k]}\right)} \\ &< \int_0^{n+2} \frac{dy}{(y+e^1_{[k]})\ln(y+e^1_{[k]})\ldots\ln_k\left(y+e^1_{[k]}\right)} \\ &= \ln_{k+1}(n+2+e^1_{[k]}) - \ln_{k+1}(e^1_{[k]}) = \ln_{k+1}(n+2+e^1_{[k]}). \end{aligned} \tag{8}$$

Proof. Applying Lemma 1 to the decreasing, continuous function

$$f(x) = \frac{1}{(x+1)\ln(x+1)\ldots\ln_k\left(x+e^1_{[k]}\right)}$$

yields the result.

3 The unperturbed deterministic cubic equation

Consider

$$x_{n+1} = x_n(1 - h_n x_n^2), \quad x_0 \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{9}$$

Everywhere in this paper we assume that $(h_n)_{n\in\mathbb{N}}$ is a non-increasing sequence of positive numbers. We derive an estimate on each $|x_n|$ and present a time-step sequence $(h_n)_{n\in\mathbb{N}}$ which provides convergence of the solution for any initial value $x_0 \in \mathbb{R}$.

3.1 Preliminary lemmata on solutions of Eq. (9)

Lemma 5. *Let x_n be a solution to Eq.* (9)*. Assume that*

$$\text{there exists } N \in \mathbb{N} \text{ such that } h_N x_N^2 < 2. \tag{10}$$

Then,

(a) the sequence $(|x_n|)_{n\in\mathbb{N}}$ is non-increasing and $h_n x_n^2 < 2$ for each $n \ge N$;
(b) the sequence $(|x_n|)_{n\in\mathbb{N}}$ converges to a finite limit.

Proof. (a) Since $h_N x_N^2 < 2$ implies that $1 - h_N x_N^2 \in (-1, 1)$ we have

$$|x_{N+1}| = |x_N||1 - h_N x_N^2| < |x_N|. \tag{11}$$

Since $(h_n)_{n\in\mathbb{N}}$ is a non-increasing sequence, we have $h_N \ge h_{N+1}$ and

$$h_{N+1}x_{N+1}^2 < h_N x_N^2 < 2.$$

The remainder of the proof of (a) follows by induction. To prove (b) we note that the sequence $(|x_n|)_{n\in\mathbb{N}}$ is non-increasing and bounded below by 0, and therefore it converges to a finite limit.

Lemma 6. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution to equation* (9)*. Assume that there exist $N \in \mathbb{N}$ such that*

$$2 > h_N x_N^2 > 1. \tag{12}$$

Then there exists $N_1 > N$ such that $h_{N_1} x_{N_1}^2 \le 1$.

Proof. By Lemma 5, the sequence $(|x_n|)_{n\in\mathbb{N}}$ is non-increasing. Furthermore, Lemma 5 part (b) implies that, for some $L \in \mathbb{R}$,

$$\lim_{n\to\infty} x_n^2 = L^2. \tag{13}$$

Proceed by contradiction and assume that $h_n x_n^2 > 1$ for all $n \ge N$. If either $L = 0$ or $\lim_{n\to\infty} h_n = 0$, it follows that $\lim_{n\to\infty} h_n x_n^2 = 0$. So $L \ne 0$ and $\lim_{n\to\infty} h_n = K \ne 0$. Since $h_n x_n^2$ is not increasing and by (12) we have

$$1 \le L^2 K < h_n x_n^2 < 2.$$

So it is only possible that either

(i) $2 > L^2 K > 1$ or
(ii) $L^2 K = 1$.

For case (i), $1 - L^2 K \in (-1, 0)$. Since $\lim_{n\to\infty} h_n x_n^2 = L^2 K$, there exists $\delta \in (0,1)$ and $N_1 \in \mathbb{N}$ such that $|1 - h_n x^2| < \delta$, for all $n \ge N_1$, implying

$$|x_{n+1}| < \delta |x_n|, \quad n \ge N_1. \tag{14}$$

Passing to the limit of both sides of (14) as $n \to \infty$, we get $L < \delta L$. Since $\delta \in (0,1)$, case (i) leads to a contradiction.

For case (ii), we have

$$\lim_{n\to\infty} |x_{n+1}| = \lim_{n\to\infty} |x_n| \lim_{n\to\infty} |1 - h_n x_n^2| = 0,$$

which implies that $\lim_{n\to\infty} |x_n| = 0$. Hence, case (ii) also leads to a contradiction. This completes the proof.

Lemma 7. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution to* (9) *with arbitrary initial condition $x_0 \neq 0$. If*

$$\textit{there exists } N \in \mathbb{N} \textit{ such that } h_N x_N^2 < 1, \tag{15}$$

then

(a) terms of the sequence $(x_n)_{n\ge N}$ do not change sign;
(b) the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a finite limit.

Proof. (a) Since (15) implies (10), we conclude that the sequence $(|x_n|)_{n\in\mathbb{N}}$ is non-increasing and therefore convergent, $1 - h_n x_n^2 \in (0,1)$ for all $n \ge N$ and then $x_n x_{n+1} > 0$ for all $n \ge N$. So the sign of x_n stops changing for $n \ge N$, which implies that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a finite limit.

Remark 1. From Lemma 6 we conclude that condition (10) implies (15). So without loss of generality we refer to (15) instead of (10) for the remainder of the article.

Remark 2. In the case where $h_N x_N^2 = 1$ for some $N \in \mathbb{N}$, we have $x_n = 0$ for all $n > N$, ensuring that $\lim_{n\to\infty} x_n = 0$. In the case when $h_N x_N^2 = 2$ for some $N \in \mathbb{N}$, we have

$$x_{N+1} = x_N (1 - h_N x_N^2) = -x_N,$$

which implies that $x_{N+k} = (-1)^k x_N$. In this case $\lim_{n\to\infty} |x_n| = |x_N|$ but $\lim_{n\to\infty} x_n$ does not exist.

3.2 Timestep summability and the limit of solutions

In this section we show that if (15) holds, then solutions converge to a nonzero limit if the stepsize sequence is summable. If not, solutions converge asymptotically to zero.

Lemma 8. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution of* (9) *with initial condition* $x_0 \neq 0$*. Suppose that* (15) *holds and that* $\sum_{j=1}^{\infty} h_j = S < \infty$*. Then,* $\lim_{n\to\infty} x_n = L \neq 0$*.*

Proof. Since (15) holds for some $N \in \mathbb{N}$, by Lemmata 5 and 7 we have, for all $k \in \mathbb{N}$,

$$x_{N+k}^2 < x_N^2, \quad 1 - h_{N+i}x_{N+i}^2 > 0. \tag{16}$$

Then, for all $k \in \mathbb{N}$,

$$\begin{aligned} x_{N+k} =& x_{N+k-1}\left(1 - h_{N+k-1}x_{N+k-1}^2\right) \\ &= x_{N+k-2}\left(1 - h_{N+k-2}x_{N+k-2}^2\right)\left(1 - h_{N+k-1}x_{N+k-1}^2\right) \\ &= x_N \prod_{i=0}^{k-1}\left(1 - h_{N+i}x_{N+i}^2\right). \end{aligned}$$

This implies

$$x_{N+k} = x_N e^{\sum_{i=0}^{k-1} \ln\left(1 - h_{N+i}x_{N+i}^2\right)}. \tag{17}$$

By Lemma 5, part (a),

$$\sum_{i=0}^{k-1} h_{N+i}x_{N+i}^2 < x_N^2 \sum_{i=0}^{k-1} h_{N+i} < x_N^2 S.$$

By Lemma 7, part (b), for some $L \in \mathbb{R}$ we have $\lim_{n\to\infty} x_n = L$. Also, $\lim_{j\to\infty} h_j = 0$, since $\sum_{j=1}^{\infty} h_j < \infty$. So there exists $N_1 \in \mathbb{N}$ such that $h_n x_n^2 < \frac{1}{2}$ for all $n \geq N_1$. Without loss of generality we may therefore suppose that $N_1 = N$. Part (ii) of Lemma 2 applies, and so for all $i \in \mathbb{N}$,

$$\ln\left(1 - h_{N+i}x_{N+i}^2\right) > -2h_{N+i}x_{N+i}^2. \tag{18}$$

Let $x_N > 0$. By applying (18) to (17), and by (16), we have

$$x_{N+k} > x_N e^{-2\sum_{i=0}^{k-1} h_{N+i}x_{N+i}^2} \geq x_N e^{-2x_N^2 \sum_{i=0}^{k-1} h_{N+i}} > x_N e^{-2x_N^2 S} > 0.$$

Passing to the limit for $k \to \infty$ in above inequality we get

$$L = \lim_{n\to\infty} x_n > x_N e^{-2x_N^2 S} > 0.$$

Similarly, for $x_N < 0$, we have

$$x_{N+k} < x_N e^{-2\sum_{i=0}^{k-1} h_{N+i}x_{N+i}^2} \leq x_N e^{-2x_N^2 \sum_{i=0}^{k-1} h_{N+i}} < x_N e^{-2x_N^2 S} < 0.$$

In both cases $\lim_{n\to\infty} x_n \neq 0$, proving the statement of the Lemma.

Lemma 9. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution to* (9) *with the initial value* $x_0 \neq 0$*. Suppose that* (15) *holds and that* $\sum_{j=1}^{\infty} h_j = \infty$ *. Then* $\lim_{n\to\infty} x_n = 0$*.*

Proof. First, (15) implies (16). So, by Lemma 2 part (i), for each $k \in \mathbb{N}$,

$$\ln(1 - h_{N+k}x_{N+k}^2) < -h_{N+k}x_{N+k}^2. \tag{19}$$

Proceed by contradiction, and suppose that $\lim_{n\to\infty} x_n^2 = L^2$ for some $L > 0$. Since the sequence $(|x_n|)_{n\in\mathbb{N}}$ is non-increasing, we have $x_N^2 > x_{N+i}^2 \geq L^2$. Applying (17) and (19) we obtain

$$\begin{aligned} |x_{N+k}| &= |x_N| e^{\sum_{i=0}^{k-1} \ln\left(1 - h_{N+i}x_{N+i}^2\right)} < |x_N| e^{\sum_{i=0}^{k-1}\left(-h_{N+i}x_{N+i}^2\right)} \\ &< |x_N| e^{-L^2 \sum_{i=0}^{k-1} h_{N+i}}. \end{aligned} \tag{20}$$

Passing to the limit in (20) as $k \to \infty$, we arrive at

$$L^2 < |x_N| e^{-L^2 \sum_{j=1}^{\infty} h_j} = 0,$$

yielding the desired contradiction.

3.3 Estimation of $|x_n|$

In this section we establish a useful estimate for each $|x_n|$ when there exists $\bar{N} \in \mathbb{N}$ such that

$$\frac{1}{h_n x_n^2} \in (0,1), \text{ for all } n \leq \bar{N}. \tag{21}$$

Lemma 10. *If* (21) *holds for some* $\bar{N} \in \mathbb{N}$*, then for all* $n \leq \bar{N}$

$$|x_n| < |x_0|^{3^n} \prod_{i=0}^{n-1} h_{n-1-i}^{3^i}, \quad n \in \mathbb{N}. \tag{22}$$

Proof. For $n = 0$ we have,

$$x_1 = x_0(1 - h_0 x_0^2) = -h_0 x_0^3 \left(1 - \frac{1}{h_0 x_0^2}\right),$$

which, by (21), implies that

$$|x_1| = \left| h_0 x_0^3 \left(1 - \frac{1}{h_0 x_0^2}\right) \right| = h_0 |x_0|^3 \left| 1 - \frac{1}{h_0 x_0^2} \right| < h_0 |x_0|^3.$$

So (22) holds for $n = 1$. Assume that (22) holds for some $n = k < \bar{N}$. By (21), $|x_{k+1}| < h_k |x_k|^3$, which implies that

$$|x_{k+1}| < h_k|x_k|^3 < h_k|x_0|^{3^{k+1}}\prod_{i=0}^{k-1}h_{k-1-i}^{3^{i+1}} = |x_0|^{3^{k+1}}\prod_{i=-1}^{k-1}h_{k-1-i}^{3^{i+1}},$$

which demonstrates that (22) holds for $n = k+1$, and concludes the proof for all $n \le \bar{N}$ by induction.

Lemma 11. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution to* (9) *with arbitrary* $x_0 \in \mathbb{R}$ *and with* $(h_n)_{n\in\mathbb{N}}$ *satisfying the following condition*

$$\sum_{j=0}^{\infty} 3^{-j}\ln h_j^{-1} = \infty. \tag{23}$$

Then there exists $\bar{N} = \bar{N}(x_0)$ *such that* (15) *holds.*

Proof. Suppose that (15) fails to hold for any $\bar{N}$. Then $1/h_n x_n^2 \in (0,1)$, for all $n \in \mathbb{N}$. For an arbitrary $\bar{N}$, we can apply Lemma 10, making the change of variables

$$j = \bar{N}-1-i, \quad i = \bar{N}-1-j, \quad i = 0,\dots,\bar{N}-1, \quad j = \bar{N}-1,\dots,0,$$

to get

$$|x_{\bar{N}}| < |x_0|^{3^{\bar{N}}}\prod_{j=0}^{\bar{N}-1}h_j^{3^{\bar{N}-1-j}}. \tag{24}$$

Set

$$F(\bar{N}) := h_{\bar{N}}\left||x_0|^{3^{\bar{N}}}\prod_{j=0}^{\bar{N}-1}h_j^{3^{\bar{N}-1-j}}\right|^2. \tag{25}$$

Squaring both sides of (24) and multiplying throughout by $h_{\bar{N}}$, we obtain $h_{\bar{N}}|x_{\bar{N}}|^2 < F(\bar{N})$. Then

$$\begin{aligned}\ln[F(\bar{N})] &= \ln h_{\bar{N}} + 2\cdot 3^{\bar{N}}\ln|x_0| + \sum_{j=0}^{\bar{N}-1}\ln h_j^{2\cdot 3^{\bar{N}-1-j}}\\ &= \ln h_{\bar{N}} + 2\cdot 3^{\bar{N}}\ln|x_0| + \frac{2}{3}\cdot\sum_{j=0}^{\bar{N}-1}3^{\bar{N}-j}\ln h_j.\end{aligned} \tag{26}$$

Without loss of generality we can assume that $\ln h_{\bar{N}} < 0$, so $\ln h_{\bar{N}} < \frac{2}{3}\ln h_{\bar{N}}$ and, continuing from (26),

$$\begin{aligned}\ln[F(\bar{N})] &\le 2\cdot 3^{\bar{N}}\ln|x_0| + \frac{2}{3}\cdot 3^{\bar{N}}\sum_{j=0}^{\bar{N}}3^{-j}\ln h_j\\ &= \frac{2}{3}\cdot 3^{\bar{N}}\left[\ln|x_0|^3 + \sum_{j=0}^{\bar{N}}3^{-j}\ln h_j\right].\end{aligned} \tag{27}$$

The expression in the square brackets is negative for any $x_0 \in \mathbb{R}$ with $\bar{N}$ sufficiently large if condition (23) holds. In this case for each $x_0 \in \mathbb{R}$ we can find $\bar{N} = \bar{N}(x_0)$ s.t.

$$\sum_{j=0}^{\bar{N}} 3^{-j} \ln h_j^{-1} > \ln |x_0|^3.$$

Then $F(\bar{N}) < 1$ which means that $|x_{\bar{N}}| < 1$ as well as $h_{\bar{N}} x_{\bar{N}}^2 < 1$. So condition (15) holds for $\bar{N} = \bar{N}(x_0)$. The contradiction thus obtained proves the result.

Lemmata 8 and 11 imply the following corollary.

Corollary 2. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution to* (9) *with arbitrary $x_0 \in \mathbb{R}$ and with $(h_n)_{n\in\mathbb{N}}$ satisfying condition* (23)*. Then $\lim_{n\to\infty} x_n = L \neq 0$.*

Lemma 12. *Condition* (23) *holds if*

(i) $h_n \le e^{-3^n}$;
(ii) $h_n \le e^{-\frac{3^n}{n}}$;
(iii) $h_n \le e^{-\frac{3^n}{n \ln n}}$;
(iv) $h_n \le e^{-\frac{3^n}{n \ln n \ln_2 n \ldots \ln_k n}}$.

Proof. Case (i): we have $3^{-j} \ln h_j^{-1} \ge 1$. Case (ii): we have $3^{-j} \ln h_j^{-1} \ge \frac{1}{j}$. Cases (iii) and (iv): we have $3^{-j} \ln h_j^{-1} \ge \frac{1}{j \ln j}, \ldots$ *etc*. Note that the series

$$\sum^{\infty} 3^{-j} \ln h_j^{-1} = \infty,$$

for h_j defined by each of (i)-(iv). The lower limit of summation should be chosen according to the form of h_j in order to avoid zero denominators.

Remark 3. Applying Lemma 1 we conclude that for h_j defined by each of (i)-(iv), the corresponding $\bar{N}(x_0)$ can be estimated as

(i) $\bar{N}(x_0) > \ln |x_0|^3$;
(ii) $\ln \bar{N}(x_0) > \ln |x_0|^3$, so $\bar{N}(x_0) > |x_0|^3$;
(iii) $\ln[\ln[\bar{N}(x_0)]] > \ln |x_0|^3$, so $\bar{N}(x_0) > e^{|x_0|^3}$;
(iv) $\ln_{k-1}[\bar{N}(x_0)] > \ln |x_0|^3$, so $\bar{N}(x_0) > e^{e^{\cdots |x_0|^3}}$.

4 The perturbed deterministic cubic difference equation

Consider the perturbed difference equation

$$x_{n+1} = x_n(1 - h_n x_n^2) + u_{n+1}, \quad x_0 \in \mathbb{R}. \tag{28}$$

where $(u_n)_{n\in\mathbb{N}}$ is a real-valued sequence. We begin by providing an estimate for solutions of (28) under condition (21).

Lemma 13. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution to equation* (28) *and let condition* (21) *hold. Then, for* $n \le \bar{N}$,

$$
\begin{aligned}
|x_{n+1}|^{\frac{1}{3^{n+1}}} &< |x_0|\prod_{i=0}^{n} h_i^{\frac{1}{3^{i+1}}} + \sum_{i=1}^{n+1}\prod_{j=i}^{n} h_j^{\frac{1}{3^{j+1}}}|u_i|^{\frac{1}{3^i}} \\
&= \prod_{i=0}^{n} h_i^{\frac{1}{3^{i+1}}}\left[|x_0| + \sum_{i=1}^{n+1}\prod_{j=0}^{i-1} h_j^{-\frac{1}{3^{j+1}}}|u_i|^{\frac{1}{3^i}}\right].
\end{aligned} \tag{29}
$$

Proof. By condition (21), for each $n \le \bar{N}$ we have

$$
\begin{aligned}
|x_{n+1}| &\le |x_n(1-h_n x_n^2)| + |u_{n+1}| \\
&\le h_n|x_n|^3\left|1-\frac{1}{h_n x_n^2}\right| + |u_{n+1}| \\
&\le h_n|x_n|^3 + |u_{n+1}|.
\end{aligned} \tag{30}
$$

Applying the inequality (3) with $\alpha_1 = \frac{1}{3}$, to (30) with $n = 0$, we get

$$
|x_1|^{\frac{1}{3}} \le h_0^{\frac{1}{3}}|x_0| + |u_1|^{\frac{1}{3}}. \tag{31}
$$

Applying the inequality (3) with $\alpha_2 = \frac{1}{3^2}$, to (30) with $n = 1$, and substituting (31), we get

$$
|x_2|^{\frac{1}{3^2}} \le h_1^{\frac{1}{3^2}}|x_1|^{\frac{1}{3}} + |u_2|^{\frac{1}{3^2}} \le h_1^{\frac{1}{3^2}} h_0^{\frac{1}{3}}|x_0| + h_1^{\frac{1}{3^2}}|u_1|^{\frac{1}{3}} + |u_2|^{\frac{1}{3^2}}. \tag{32}
$$

Continue this process inductively, and applying the inequality (3) with $\alpha_n = \frac{1}{3^{n+1}}$ we get

$$
\begin{aligned}
|x_{n+1}|^{\frac{1}{3^{n+1}}} &\le h_n^{\frac{1}{3^{n+1}}} h_{n-1}^{\frac{1}{3^n}} \dots h_1^{\frac{1}{3^2}} h_0^{\frac{1}{3}}|x_0| + h_n^{\frac{1}{3^{n+1}}} h_{n-1}^{\frac{1}{3^n}} \dots h_2^{\frac{1}{3^2}} h_1^{\frac{1}{3}}|u_1|^{\frac{1}{3}} \\
&\quad + h_n^{\frac{1}{3^{n+1}}} h_{n-1}^{\frac{1}{3^n}} \dots h_3^{\frac{1}{3^4}} h_2^{\frac{1}{3^3}}|u_2|^{\frac{1}{3^2}} + \dots + h_n^{\frac{1}{3^{n+1}}}|u_n|^{\frac{1}{3^n}} + |u_{n+1}|^{\frac{1}{3^{n+1}}},
\end{aligned}
$$

which completes the proof.

4.1 Boundedness of $(|x_n|)_{n\in\mathbb{N}}$ for particular $(h_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$

In this section we consider two special cases of $(h_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ each of which guarantees the boundedness of the sequence $(|x_n|)_{n\in\mathbb{N}}$. Both forms of $(h_n)_{n\in\mathbb{N}}$ were introduced in Lemma 12: the first corresponds to (ii)- (iv), the second corresponds to (i). Estimates for each $|u_n|$ are chosen relative to corresponding estimates for $|h_n|$.

4.1.1 Case 1

Let $e^1_{[k]}$ and $\ln_k(\cdot)$ be defined as in (6). Assume that, there exists $k \in \mathbb{N}$ and $\beta \in (0,1)$ such that

$$h_n \le \exp\left\{ -\frac{3^{n+1}}{\left(n+e^1_{[k]}\right)\ln\left(n+e^1_{[k]}\right)\dots\ln_k\left(n+e^1_{[k]}\right)} \right\}, \tag{33}$$
$$(h_n)_{n\in\mathbb{N}} \quad \text{is a decreasing sequence,}$$

and

$$|u_n| :\le \left(\frac{\beta}{\left(n+e^1_{[k]}\right)\ln\left(n+e^1_{[k]}\right)\dots\ln_k\left(n+e^1_{[k]}\right)} \right)^{3^n}. \tag{34}$$

Lemma 14. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution to equation* (28) *and let $(h_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ satisfy* (33) *and* (34)*, respectively. Then*

(i) there exists N_1 such that $|x_{N_1+1}| < 1$, and (15) *holds;*
(ii) $|x_{N_1+i}|$ is uniformly bounded for all $i \in \mathbb{N}$.

Proof. Suppose to the contrary that (21) holds for all n. Then, by Lemma 13, estimate (29) holds for all $n \in \mathbb{N}$.

Substituting the values of h_n from (33) and u_n from (34) into (29) we get

$$|x_{n+1}|^{\frac{1}{3^{n+1}}} \le \exp\left\{ -\sum_{i=0}^{n} \frac{1}{\left(i+e^1_{[k]}\right)\ln\left(i+e^1_{[k]}\right)\dots\ln_k\left(i+e^1_{[k]}\right)} \right\} |x_0|$$
$$+ \sum_{i=1}^{n+1} \exp\left\{ -\sum_{j=i}^{n} \frac{1}{(j+e^1_{[k]})\ln(j+e^1_{[k]})\dots\ln_k\left(n+2+e^1_{[k]}\right)} \right\} |u_j|^{\frac{1}{3^i}}.$$

Now we apply the inequalities from (7) and (8) and get

$$\exp\left\{ -\sum_{i=j}^{n} \frac{1}{\left(i+e^1_{[k]}\right)\ln\left(i+e^1_{[k]}\right)\dots\ln_k\left(i+e^1_{[k]}\right)} \right\} \le \frac{\ln_k(j+e^1_{[k]})}{\ln_k(n+1+e^1_{[k]})},$$

and

$$\exp\left\{ -\sum_{i=0}^{n} \frac{1}{\left(i+e^1_{[k]}\right)\ln\left(i+e^1_{[k]}\right)\dots\ln_k\left(i+e^1_{[k]}\right)} \right\} \le \frac{1}{\ln_k(n+1+e^1_{[k]})}.$$

Applying all the above we arrive at

$$
\begin{aligned}
|x_{n+1}|^{\frac{1}{3^{n+1}}} &\le \frac{|x_0|}{\ln_k(n+1+e^1_{[k]})} + \frac{\sum_{j=1}^{n+1} \ln_k(j+e^1_{[k]})|u_j|^{\frac{1}{3^j}}}{\ln_k(n+2+e^1_{[k]})} \\
&\le \frac{|x_0|\ln_k}{\ln_k(n+2+e^1_{[k]})} + \frac{\sum_{j=1}^{n+1} \frac{\beta}{(j+e^1_{[k]})\ln(j+e^1_{[k]})\ldots\ln_{k-1}(j+e^1_{[k]})}}{\ln_k(n+2+e^1_{[k]})} \\
&= \frac{|x_0|}{\ln_k(n+1+e^1_{[k]})} + \beta\left(\ln_k(n+2+e^1_{[k]})\right)^{-1}\ln_k(n+2+e^1_{[k]}) \\
&= \frac{|x_0|}{\ln_k(n+1+e^1_{[k]})} + \beta.
\end{aligned} \tag{35}
$$

So for each $\beta \in (0,1)$ we can find N_1 such that, for $n \ge N_1$,

$$\frac{|x_0|}{\ln_k(n+1+e^1_{[k]})} + \beta < 1,$$

which implies $|x_{N_1+1}| < 1$. Assume now that $N_2 > 2$ is such that, for $n \ge N_2$, we have

$$\left(n+e^1_{[k]}\right)\ln\left(n+e^1_{[k]}\right)\ldots\ln_k\left(n+e^1_{[k]}\right) \le 3^{\frac{n}{2}}.$$

Then, for $n \ge N_2$,

$$h_n \le e^{-\frac{3^{n+1}}{\left(n+e^1_{[k]}\right)\ln\left(n+e^1_{[k]}\right)\ldots\ln_k\left(n+e^1_{[k]}\right)}} < e^{-3^{\frac{n}{2}+1}} \le e^{-3^2} = e^{-9}. \tag{36}$$

Without loss of generality we can assume that $N_1 \ge N_2$. We have

$$0 < 1 - h_{N_1+1}x^2_{N_1+1} < 1, \quad |x_{N_1+2}| < |x_{N_1+1}| + |u_{N_1+2}|.$$

Also

$$x^2_{N_1+2} < 2x^2_{N_1+1} + 2u^2_{N_1+2}, \quad \text{and} \quad |u_n| < 1, \quad \forall n \in \mathbb{N},$$

so

$$
\begin{aligned}
&h_{N_1+2}x^2_{N_1+2} < 2h_{N_1+2}\left[x^2_{N_1+1} + u^2_{N_1+2}\right] = 2e^{-9}\left[x^2_{N_1+1} + 1\right] \\
&= 4e^{-9} \approx 0.00049 < 1.
\end{aligned} \tag{37}
$$

Based on that we get

$$|x_{N_1+3}| < |x_{N_1+2}| + |u_{N_1+3}| < |x_{N_1+1}| + |u_{N_1+2}| + |u_{N_1+3}|.$$

Applying induction, assume that, for some $k \in \mathbb{N}$,

$$|x_{N_1+2+k}| \le |x_{N_1+1}| + \sum_{i=1}^{k} |u_{N_1+2+i}| \quad \text{and} \quad x^2_{N_1+2+k}h_{N_1+2+k} < 1, \tag{38}$$

and prove that relations in (38) hold for $k+1$. In order to do so we first get the estimate of $\sum_{i=1}^{k}|u_{N_1+2+i}|$. For all $n\in\mathbb{N}$, we have

$$|u_n| \le \left(\frac{\beta}{\left(n+e^1_{[k]}\right)\ln\left(n+e^1_{[k]}\right)\dots\ln_k\left(n+e^1_{[k]}\right)}\right)^{3^n} < \left(\frac{\beta}{n+e^1_{[k]}}\right)^{3^n}.$$

Then, for $n\ge N_1+2\ge 4$,

$$|u_n| \le \left(\frac{\beta}{n}\right)^{3^n} < \left(\frac{\beta}{n}\right)^{n} \le \left(\frac{\beta}{4}\right)^{n}$$

and

$$\sum_{i=1}^{k}|u_{N_1+2+i}| < \sum_{i=1}^{\infty}|u_{N_1+2+i}| \le \sum_{n=4}^{\infty}\left(\frac{\beta}{4}\right)^{n} = \frac{\left(\frac{\beta}{4}\right)^4}{1-\frac{\beta}{4}} < \frac{4\left(\frac{1}{4}\right)^4}{4-\beta} < \frac{1}{4^3\times 3} < 1. \quad (39)$$

Now,

$$|x_{N_1+2+k+1}| \le |x_{N_1+2+k}| + |u_{N_1+2+k+1}| \le |x_{N_1+1}| + \sum_{i=1}^{k+1}|u_{N_1+2+i}|,$$

proving the first part of (38) for each $k\in\mathbb{N}$, and

$$\begin{aligned} h_{N_1+2+k+1}x^2_{N_1+2+k+1} &\le 2h_{N_1+2+k+1}|x_{N_1+1}|^2 + 2h_{N_1+2+k+1}\left(\sum_{i=1}^{k+1}|u_{N_1+2+i}|\right)^2 \\ &\le 2e^{-9}\left[1+1\right] \le 4e^{-9} < 1, \end{aligned}$$

proving the second part of (38) for each $k\in\mathbb{N}$. This completes the proof of Part (i).

From (38) and (39) we have

$$|x_{N_1+2+k}| < |x_{N_1+1}| + 1,$$

for each $k\in\mathbb{N}$, which completes the proof of Part (ii).

4.1.2 Case 2

Assume that, for some $\beta\in(0,1)$,

$$h_n \le e^{-3^{n+1}}, \quad |u_n| \le \left[\frac{\beta(e-1)}{e}\right]^{3^n}. \quad (40)$$

Lemma 15. *The statement of Lemma 14 holds if, instead of conditions* (33)–(34)*, we assume that condition* (40) *holds.*

Proof. The proof is analogous to the proof of Lemma 14. Instead of (35) we obtain

$$\begin{aligned}|x_{n+1}|^{\frac{1}{3^{n+1}}} &\le e^{-(n+1)}|x_0| + \frac{\beta(e-1)}{e}[e^{-n}+e^{-n+1}+\cdots+e^{-1}+1] \\ &= e^{-(n+1)}|x_0| + \frac{\beta(e-1)}{e}\frac{1-e^{-n-1}}{1-e^{-1}} \le e^{-(n+1)}|x_0|+\beta.\end{aligned} \tag{41}$$

Taking $N_1 \ge \ln|x_0| - \ln[1-\beta]$ we get $|x_{n+1}| < 1$ for $n \ge N_1$. Instead of (37) we have

$$\begin{aligned}h_{N_1+2}x_{N_1+2}^2 &< 2h_{N_1+2}\left[x_{N_1+1}^2 + u_{N_1+2}^2\right] = 2e^{-3^{N_1+3}}\left[x_{N_1+1}^2 + \left[\frac{\beta(e-1)}{e}\right]^{2\cdot 3^{N_1+2}}\right] \\ &\le 2e^{-3^{N_1+3}}\left[1+1\right] \le 4e^{-3^{N_1+3}} < 4e^{-3^4} < 1,\end{aligned}$$

and instead of (39) we have

$$\begin{aligned}\sum_{i=1}^{k}|u_{N_1+2+i}| &= \sum_{i=1}^{k}\left[\frac{\beta(e-1)}{e}\right]^{3^{N_1+2+i}} \le \sum_{j=4}^{k}\left[\frac{\beta(e-1)}{e}\right]^{3^j} \\ &< \sum_{j=4}^{k}\left[\frac{\beta(e-1)}{e}\right]^{j} = \left[\frac{\beta(e-1)}{e}\right]^{3^4}\frac{1}{1-\frac{\beta(e-1)}{e}} \\ &< \left[\frac{\beta(e-1)}{e}\right]^{3^4} e < 1.\end{aligned}$$

The last inequality holds true since, in particular,

$$\left[\frac{(e-1)}{e}\right]^{3^4} \approx (0.6321)^8 1 < 0.3678 \approx e^{-1}$$

The rest of the proof is similar to the proof of Lemma 14.

4.2 Convergence of $(x_n)_{n\in\mathbb{N}}$ to a finite limit.

Theorem 1. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution to equation* (28) *and let* $(h_n)_{n\in\mathbb{N}}$ *and* $(u_n)_{n\in\mathbb{N}}$ *satisfy either conditions* (33)-(34) *or condition* (40). *Then the sequence* $(x_k)_{k\in\mathbb{N}}$ *converges to a finite limit as* $k\to\infty$.

Proof. It is sufficient to consider only the terms $\{x_{N_1+2+k}\}_{k\in\mathbb{N}}$. Since the sequence $\{x_{N_1+2+k}\}_{k\in\mathbb{N}}$ is bounded, it has a convergent subsequence $\{x_{N_1+2+k_l}\}_{l\in\mathbb{N}}$,

$$\lim_{l\to\infty} x_{N_1+2+k_l} = L.$$

We now show that

$$\lim_{m\to\infty} x_{N_1+2+m} = L$$

follows. For each $m \in \mathbb{N}$ denote $l_m \in \mathbb{N}$

$$l_m = \sup\{l : N_2 + 2 + k_l \le m\}.$$

Then

$$N_2 + 2 + k_{l_m} \le m \le N_2 + 2 + k_{l_m+1}$$

and

$$|x_{N_1+2+m}| \le |x_{N_1+2+m-1}| + |u_{N_1+2+m}| \le |x_{N_1+2+k_{l_m}}| + \sum_{i=k_{l_m}}^{m} |u_{N_1+2+i}|, \tag{42}$$

$$|x_{N_1+2+k_{l_m+1}}| \le |x_{N_1+2+m}| + \sum_{i=m}^{k_{l_m+1}} |u_{N_1+2+i}| \tag{43}$$

Passing to the limit in (42) and (43) we obtain, respectively,

$$\limsup_{m\to\infty} x_{N_1+2+m} \le L, \quad \text{and} \quad L \le \liminf_{m\to\infty} x_{N_1+2+m}.$$

This implies that $\lim_{m\to\infty} x_{N_1+2+m}$ exists and is equal to L.

When condition (40) holds it is possible that solutions of (28) converge to a nonzero limit. Example 1 below demonstrates that $\lim_{n\to\infty} x_n$ can be either zero or nonzero.

Example 1. We show that the limit of solutions of (28) can be positive, zero, or negative. For all three cases below, choose $h_n = e^{-3^{n+1}}$.

(i) **Zero limit** ($L = 0$)**.** Set

$$u_1 = -e^{-3} \approx -0.0498, \quad u_n = 0 \quad \text{for all} \quad n \ge 2.$$

Then (40) is satisfied for $\beta \in (1/(e-1), 1)$. The continuous function

$$f(x) = x - e^{-3}x^3.$$

takes its maximum $f_m = \frac{2}{3\sqrt{3e^{-3}}} \approx 1.724 > 0.0498 \approx -u_1$ at the point $x_m = \frac{1}{\sqrt{3e^{-3}}} \approx 2.586$, and $f(0) = 0$. So the equation

$$x - e^{-3}x^3 = e^{-1},$$

has a solution x_0 on the interval $\left(0, \frac{1}{\sqrt{3e^{-3}}}\right) \approx (0, 2.586)$. Consider now the equation (28) with this specific initial value. We get $x_1 = 0$ and since all $u_n = 0$ for $n \ge 2$, we have $x_n = 0$ for $n \ge 2$. Therefore $\lim_{n\to\infty} x_n = 0$.

(ii) **Positive limit ($L > 0$).** Set

$$u_1 = e^{-3} \approx 0.0498, \quad u_n > 0, \quad \text{for all} \quad n \geq 2,$$

so that (40) is satisfied. Suppose also that $x_0 > 0$ is chosen as in case (i). Then,

$$x_1 = 2u_1 + \underbrace{x_0(1 - h_0 x_0^2) - u_1}_{=0} = 2u_1 = 2e^{-3} > 0.$$

Moreover, note that $h_1 x_1^2 = 2e^{-12} < 1/2$. We can also write

$$x_{n+1} \geq x_n(1 - h_n x_n^2) \geq x_1 \prod_{i=1}^{n} (1 - h_i x_i^2).$$

The same approach as in Lemma 8 with $N = 1$ gives that $\lim_{n\to\infty} x_n > 0$.

(iii) **Negative limit ($L < 0$).** Set

$$u_1 = -2e^{-3} \approx -0.0996, \quad u_n < 0, \quad \text{for all} \quad n \geq 2,$$

so that (40) is satisfied, and choose $x_0 > 0$ as in Cases (i) and (ii). Then

$$x_1 = \underbrace{x_0(1 - h_0 x_0^2) + \frac{u_1}{2}}_{=0} + \frac{u_1}{2} < 0.$$

Again, we see that $h_1 x_1^2 = e^{-18} < 1/2$, and we can write for all $n \geq 1$

$$x_{n+1} \leq x_n(1 - h_n x_n^2) \leq x_1 \prod_{i=1}^{n} (1 - h_i x_i^2).$$

The same approach as in Lemma 8 with $N = 1$ gives that $\lim_{n\to\infty} x_n < 0$.

4.3 Modified process with a stopped timestep sequence

Based on Example 1 and Lemma 8 we cannot expect that, in general, the finite limit L will be zero. In order to obtain a sequence that converges to zero we modify the timestep sequence $(h_n)_{n\in\mathbb{N}}$ further by stopping it (preventing terms from varying further) after N_3 steps:

$$\hat{h}_n = \begin{cases} h_n, \; n < N_3, \\ h_N, \; n \geq N_3, \end{cases} \tag{44}$$

where N_3 is such that

$$|x_{N_3}| \leq 1. \tag{45}$$

Note that under the conditions of Lemmas 14 and 15 we would have $N_3 = N_1$. Note that N_3 is not necessarily the first moment where (45) holds; note also that (45) implies $x_{N_3}^2 h_{N_3} < 1$, but the converse does not necessarily hold.

Consider

$$x_{n+1} = x_n(1 - \hat{h}_n x_n^2) + u_{n+1}, \quad x_0 \in \mathbb{R}. \tag{46}$$

Theorem 2. *Let* $(h_n)_{n\in\mathbb{N}}$ *and* $(u_n)_{n\in\mathbb{N}}$ *satisfy either conditions* (33)-(34) *or condition* (40). *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution to equation* (46) *with* $(\hat{h}_n)_{n\in\mathbb{N}}$ *defined by* (44). *Then* $\lim_{n\to\infty} x_n = 0$ *for any initial value* $x_0 \in \mathbb{R}$.

Proof. Choose N_1 defined as in Lemmata 14 or 15 and set $N_3 = N_1$. To prove that

$$x_n^2 \hat{h}_n < 1, \quad \text{for all} \quad n > N_3,$$

we follow the approach taken in the proofs of Lemma 14, Part (i), and Lemma 15, Part (i).

Let assume first that conditions (33)–(34) hold, so we use N_1 from Lemma 14. We have $N_3 = N_1 > 2$, $|x_{N_3}| < 1$, $\hat{h}_{N_3+1} < e^{-3^{\frac{N_3}{2}+1}}$,

$$|x_{N_3+1}| \le |x_{N_3}| + |u_{N_3+1}|$$

and

$$\begin{aligned}\hat{h}_{N_3+1} x_{N_3+1}^2 &< 2\hat{h}_{N_3+1}\left[x_{N_3}^2 + u_{N_3+1}^2\right] = 2e^{-3^{\frac{N_3}{2}+1}}\left[x_{N_3}^2 + \left(\frac{\beta}{4}\right)^{N_3}\right] \\ &\le 2e^{-3^2}[1+1] = 4e^{-3^2} < 1.\end{aligned}$$

This gives us

$$|x_{N+2}| \le |x_{N+1}| + |u_{N+2}|,$$

which, as above, leads to

$$\begin{aligned}\hat{h}_{N+2} x_{N+2}^2 &< 2\hat{h}_{N_3}\left[x_{N_3}^2 + u_{N_3+1}^2\right] \le 2e^{-3^{\frac{N_3+1}{2}+1}}\left[1 + \left(\frac{\beta}{4}\right)^{N_3+1}\right] \\ &< 4e^{-3^2} < 1.\end{aligned}$$

Now we complete the proof by induction and arrive at

$$|x_{N+k}| \le |x_N| + \sum_{i=1}^{k} |u_{N+i}|, \tag{47}$$

which implies the boundedness of the sequence $(x_n)_{n\in\mathbb{N}}$. Note that Theorem 1 also holds when, instead of $(h_n)_{n\in\mathbb{N}}$ we have a stopped sequence $(\hat{h}_n)_{n\in\mathbb{N}}$, since its proof uses only (47) and convergence of the series $\sum_{i=1}^{\infty} u_i$. So we conclude that $\lim_{n\to\infty} x_n = L$. Passing to the limit in equation (46) we obtain the equality

$$L = L(1 - \hat{h}_N L),$$

which holds only for $L = 0$.

If condition (40) hold, we use N_1 from Lemma 15. The proof of this case is similar to that of the first, except that $\hat{h}_n \leq 3^{N_3+1}$.

Remark 4. Convergence of the solutions of equation (46) with stopped time-step sequence $(\hat{h}_n)_{n\in\mathbb{N}}$ may be slow, either if h_{N_3} is very small, or if N_3 is large. Alternative strategies for stopping the sequence $(\hat{h}_n)_{n\in\mathbb{N}}$ are as follows:

(i) Define

$$N_4 = \inf\{n \in \mathbb{N} : x_n^2 h_n < 1\}, \tag{48}$$

and assume that $x_{N_4} \neq 0$. Define

$$\hat{h}_n = \begin{cases} h_n, & n < N_4, \\ \frac{1}{x_{N_4}^2}, & n \geq N_4. \end{cases}$$

Then, $|x_{N_4+1}| = |u_{N_4+1}| < 1$, and the conditions of Theorem 2 hold. If $x_{N_4} = 0$, we also have $|x_{N_4+1}| = |u_{N_4+1}| < 1$.

(ii) Assume that $|u_{n+1}| \leq h_n$ for all $n \in \mathbb{N}$. Define again N_4 by (48). If $|x_{N_4}| \leq 1$ the conditions of Theorem 2 hold. If $|x_{N_4}| > 1$, we have

$$|x_{N_4}^3| > 1 \geq \frac{|u_{N_4+1}|}{h_{N_4}}, \text{ or } |x_{N_4}^3| h_{N_4} \geq |u_{N_4+1}|.$$

Then,

$$|x_{N_4+1}| \leq |x_{N_4}|(1 - x_{N_4}^2 h_{N_4}) + |u_{N_4+1}| = |x_{N_4}| - |x_{N_4}^3| h_{N_4} + |u_{N_4+1}| \leq |x_{N_4}|.$$

So

$$x_{N_4+1}^2 \hat{h}_{N_4+1} \leq x_{N_4}^2 h_{N_4} \leq 1.$$

By induction it can be shown that $x_{N_4+k}^2 \hat{h}_{N_4+k} \leq 1$ for all $k \in \mathbb{N}$. Now, applying the same reasoning as before we can prove that $(|x_{N_4+k}|)_{k\in\mathbb{N}}$ is uniformly bounded and converges to zero.

Theorem 3. *Let* $(h_n)_{n\in\mathbb{N}}$ *and* $(u_n)_{n\in\mathbb{N}}$ *satisfy either conditions* (33)-(34) *or condition* (40) *with* $\beta < \frac{1}{e-1}$, *in all cases with equality instead of inequality in the conditions placed upon each* h_n. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution to equation* (46) *with initial value* $x_0 \in \mathbb{R}$ *and* $(\hat{h}_n)_{n\in\mathbb{N}}$ *defined by* (44) *and* (48). *Then* $\lim_{n\to\infty} x_n = 0$.

Proof. By Lemmas 14, 15 and Remark (4), Part (ii), it is sufficient to show that $|u_{n+1}| \leq h_n$. Denote

$$Q(n) := \ln\beta - \sum_{i=1}^{k+1} \ln_i\left(n + e_{[k]}^1\right) + \left(\prod_{i=0}^{k} \ln_i\left(n + e_{[k]}^1\right)\right)^{-1}$$

Note that, for $n \geq 1$,

$$
\begin{aligned}
Q(n) &< \ln\beta - \ln\left(n + e^1_{[k]}\right) + \frac{1}{\left(n + e^1_{[k]}\right)} \\
&\leq \ln\beta - \ln 2 + \frac{1}{2} \\
&\approx \ln\beta - 0.1931 < 0.
\end{aligned}
$$

When conditions (33)–(34) hold we have, for $n \geq 1$,

$$
\frac{|u_{n+1}|}{h_n} \leq \exp\left\{3^{n+1} Q(n)\right\} \leq 1.
$$

When condition (40) holds with $\beta(e-1) \leq 1$, we have, for $n \geq 1$,

$$
\frac{|u_{n+1}|}{h_n} \leq (\beta(e-1))^{3^{n+1}} \leq 1.
$$

5 The stochastically perturbed cubic difference equation

In this section we consider a stochastic difference equation

$$
x_{n+1} = x_n(1 - h_n x_n^2) + \rho_{n+1}\xi_{n+1}, \quad n \in \mathbb{N}, \quad x_0 \in \mathbb{R}, \tag{49}
$$

where $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent identically distributed random variables. We discuss only two cases: $|\xi_n| \leq 1$ and $\xi_n \sim \mathcal{N}(0,1)$. Denoting

$$
u_n := \rho_n \xi_n,
$$

we can apply the results of Section 4 pathwise to solutions of (49) for almost all $\omega \in \Omega$.

We also consider a stochastically perturbed equation with stopped timestep sequence $(\hat{h}_n)_{n\in\mathbb{N}}$

$$
x_{n+1} = x_n(1 - \hat{h}_n x_n^2) + \rho_{n+1}\xi_{n+1}, \quad n \in \mathbb{N}, \quad x_0 \in \mathbb{R}, \tag{50}
$$

where $\hat{h}_n$ is defined by (44) with N_3 selected as equal to N_1 from Lemmas 14, 15 or as equal to N_4 from Remark 4. Note that since solutions of (49) are stochastic processes, N_1 and N_4 are a.s. finite $\mathbb{N}$-valued random variables, which we therefore denote by $\mathcal{N}_1$ and $\mathcal{N}_4$, respectively.

5.1 Case 1: bounded noise ($|\xi_n| \le 1$)

In this case, for all $\omega \in \Omega$ and all $n \in \mathbb{N}$, we have

$$|u_n| = |\rho_n \xi_n| \le |\rho_n|.$$

So we may apply the results of Section 4 to each trajectory, arriving at:

Theorem 4. *Let $(h_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ satisfy either conditions* (33)-(34) *or condition* (40) *(ρ_n satisfying the constraint for u_n). Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of random variables s.t. $|\xi_n| \le 1$ for all $n \in \mathbb{N}$. Let $(x_n)_{n\in\mathbb{N}}$ be a solution to* (49)*, $(\hat{h}_n)_{n\in\mathbb{N}}$ defined as in* (44)*, and $(\hat{x}_n)_{n\in\mathbb{N}}$ a solution to* (50)*. Then, a.s.,*

(i) $\lim_{n\to\infty} x_n = L$*, where L is an a.s. finite random variable;*
(ii) $\lim_{n\to\infty} \hat{x}_n = 0$.

5.2 Case 2: unbounded noise ($\xi_n \sim \mathcal{N}(0,1)$).

Theorem 5. *Let $(h_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ satisfy either conditions* (33)-(34) *or condition* (40) *(ρ_n satisfying the constraint for u_n). Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of mutually independent $\mathcal{N}(0,1)$ random variables. Let $(x_n)_{n\in\mathbb{N}}$ be a solution to* (49)*, $(\hat{h}_n)_{n\in\mathbb{N}}$ as defined in* (44)*, and $(\hat{x}_n)_{n\in\mathbb{N}}$ a solution to* (50)*. Then, a.s.,*

(i) $\lim_{n\to\infty} x_n = L$*, where L is an a.s. finite random variable;*
(ii) $\lim_{n\to\infty} \hat{x}_n = 0$.

Proof. If (40) holds for $\beta \in (0,1)$, then for some $\beta_1 \in (\beta,1)$ we have

$$\left[\frac{\beta(e-1)}{e}\right]^{3^n} = \left[\frac{\beta_1(e-1)}{e}\right]^{3^n} \times \left[\frac{\beta}{\beta_1}\right]^{3^n},$$

and, for each $\varsigma > 0$,

$$\lim_{n\to 0} \left[\frac{\beta}{\beta_1}\right]^{3^n} \ln^{\frac{1}{2}+\zeta} n = 0,$$

Applying Lemma 4 we conclude that there exists $\mathcal{N}$ such that for all $n \ge \mathcal{N}$,

$$\left|\frac{1}{(\ln n)^{1/2+\varsigma}}\xi_n\right| < 1.$$

Then, for all $n \ge \mathcal{N}$,

$$|u_{n+1}| = \left|\left[\frac{\beta_1(e-1)}{e}\right]^{3^n} \times \left[\frac{\beta}{\beta_1}\right]^{3^n} \xi_n\right| \le \left[\frac{\beta_1(e-1)}{e}\right]^{3^n}.$$

If (34) hold holds for $\beta \in (0,1)$, then for some $\beta_1 \in (\beta,1)$ we use the estimate

$$|u_{n+1}| \le \left(\frac{\beta_1}{\left(n+e^1_{[k]}\right) \ln\left(n+e^1_{[k]}\right) \dots \ln_k\left(n+e^1_{[k]}\right)} \right)^{3^n} \left[\frac{\beta}{\beta_1}\right]^{3^n} |\xi_{n+1}|,$$

and apply the same reasoning as above.

Define for a.a. $\omega \in \Omega$

$$y_m := x_{m+\mathcal{N}(\omega)}, \quad u_{m+1} := \rho_{m+\mathcal{N}(\omega)} \xi_{m+\mathcal{N}(\omega)}(\omega), \quad \mathrm{h}_m := h_{m+\mathcal{N}(\omega)},$$

and consider the deterministic stochastic equation

$$y_{m+1} = y_m(1-\mathrm{h}_m y_m^2) + u_{m+1}, \quad m \in \mathbb{N}, \quad y_0 = x_{\mathcal{N}(\omega)}. \tag{51}$$

Equation (51) satisfies the conditions of either Lemma 14 or Lemma 15. So there exists N_1 (which depends on ω) such that $h_{N_1} x_{N_1}^2 < 1$. The remainder of the proof follows by the same argument as that in Section 4.

6 Illustrative numerical examples

In this section we illustrate the asymptotic behaviour of solutions of the unperturbed equation (9) with summable and non-summable timestep sequences, as described in Lemmas 8 & 9, and the stochastically perturbed equation (49) with unbounded Gaussian noise as described in Theorem 5.

Figure 1, parts (a) and (b) show three solutions of the unperturbed deterministic equation (9) corresponding to the initial values $x_0 = 1.1, 0.5, -1.1$, with timestep sequence $h_n = 1/n^{10}$, so that $\sum_{i=1}^{\infty} h_i < \infty$. We observe that all three solutions appear to converge to different finite limits, as predicted by Lemma 8.

Parts (c) and (d) show three solutions of (9) with the same initial values and with timestep sequence $h_n = 1/n^{0.1}$, so that $\sum_{i=1}^{\infty} h_i = \infty$. Note that we have selected values of x_0 that are sufficiently small for (15) to hold with this choice of h_n, hence avoiding the possibility of blow-up. All three solutions appear to converge to a zero limit, as predicted by Lemma 9.

Figure 2, parts (a) and (b) show three solution trajectories of the stochastic equation (49) each corresponding to initial values given by $x_0 = 2.5, 0.5, -2.5$ with timestep sequence $h_n = e^{-\frac{3^{n+1}}{n+e}}$, satisfying (33) for $k = 1$, $(\xi_n)_{n\in\mathbb{N}}$ a sequence of i.i.d. $N(0,1)$ random variables, and

$$\rho_n = \left(\frac{\beta}{n+e}\right)^{3^n}, \tag{52}$$

with $\beta = 0.5$ satisfying (34) with $k = 1$. We observe that all three solutions approach different nonzero limits, as predicted by Theorem 5.

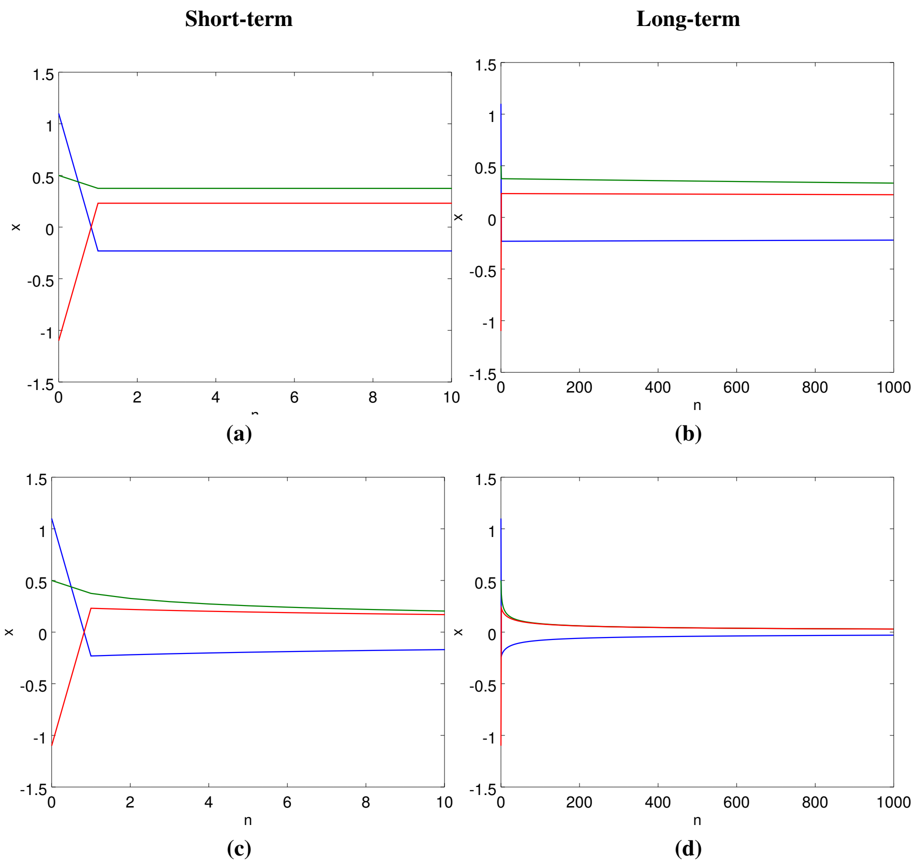

Fig. 1 Solutions of (9) with summable (Parts (a) and (b)) and non-summable (Parts (c) and (d)) timestep sequences. Short term and long term dynamics are given in the first and second columns, respectively.

Parts (c) and (d) repeat the computation, but with the timestep sequence stopped so that its values become fixed when $h_n x_n^2 < 1$ is satisfied for the first time. Solutions demonstrate behaviour consistent with asymptotic convergence to zero, also as predicted by Theorem 5.

Note that $\beta \in (0,1)$ in Condition (34), but that in Figure 2 the effect of the stochastic perturbation decays too rapidly for differences between trajectories to be visible. Therefore in each part of Figure 3 we choose larger values of β and generate fifteen trajectories of (49) with $(\xi_n)_{n\in\mathbb{N}}$ a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables, timestep sequence $h_n = e^{-\frac{3^{n+1}}{n+e}}$ stopped when $h_n x_n^2 < 1$ is satisfied for the first time, $x_0 = 2.5$, and each ρ_n chosen to satisfy (52). Parts (a) and (b) show that, when $\beta = 3/2$, trajectories appear to converge to zero. However, Parts (c)–(f) indi-

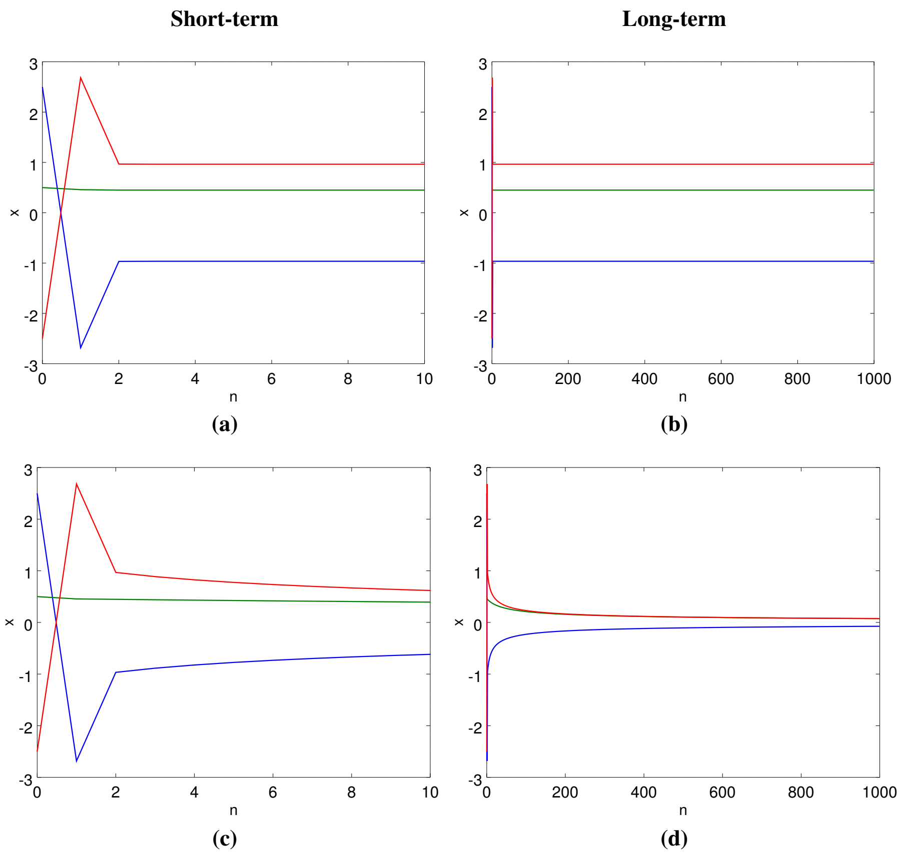

Fig. 2 Solutions of (49) with Gaussian perturbation and non-stopped (Parts (a) and (b)) and stopped (Parts (c) and (d)) timestep sequences. Short term and long term dynamics are given in the first and second columns, respectively.

cate that, when β is increased, first to $\beta = 3$ and then to $\beta = 5$, trajectories may no longer converge to zero a.s., but instead to a random limit.

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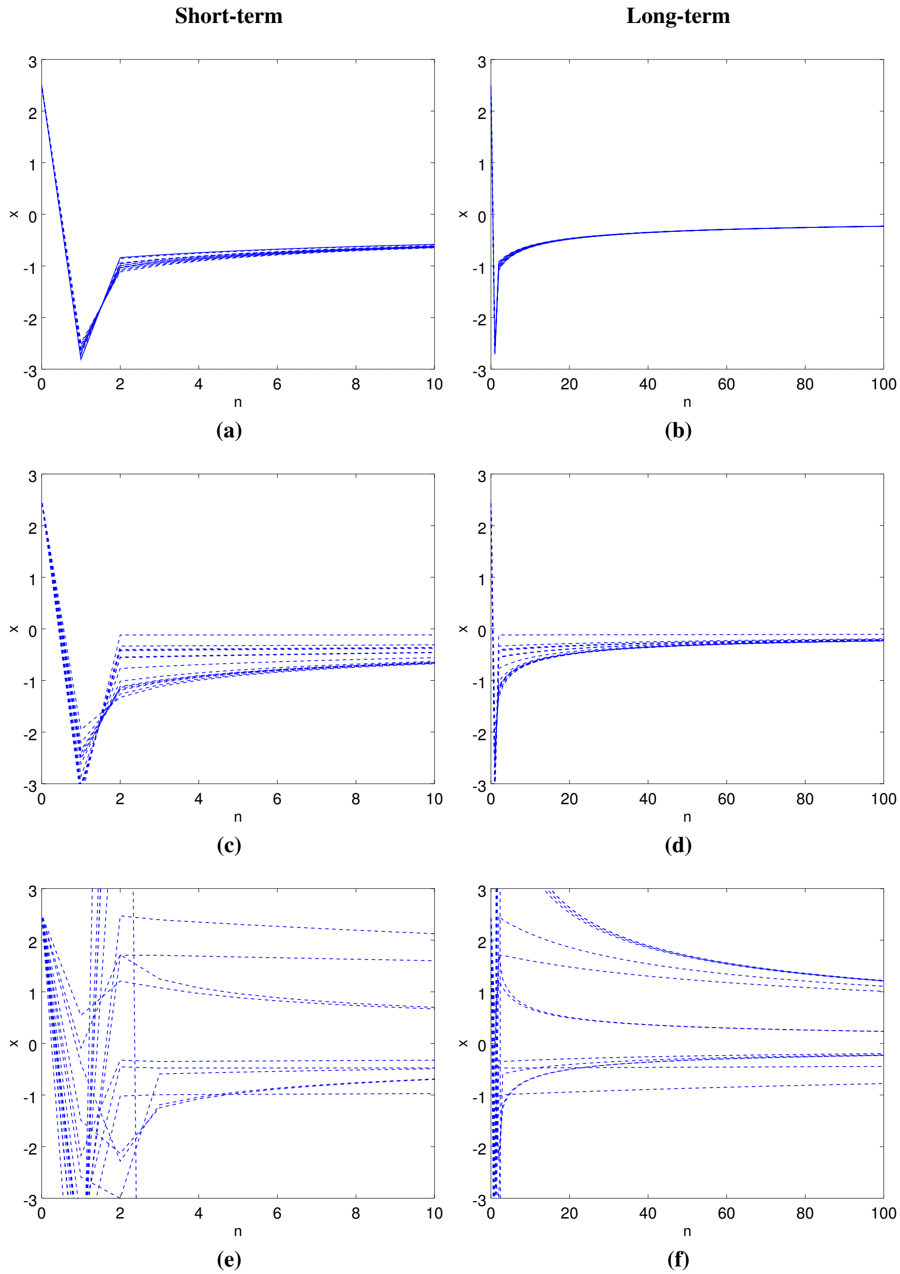

Fig. 3 Multiple trajectories of (49) with Gaussian perturbation and stopped timestep sequence. Here, $x_0 = 2.5$, $\beta = 3/2$ (Part (a) short-term and Part (b) long-term behaviour), $\beta = 3$ (Part (c) short-term and Part (d) long-term behaviour), and $\beta = 5$ (Part (e) short-term and Part (f) long-term behaviour).

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