Bounds on the entanglement of two-qutrit systems from fixed marginals

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We discuss the problem of characterizing upper bounds on entanglement in a bipartite quantum system when only the reduced density matrices (marginals) are known. In particular, starting from the known two-qubit case, we propose a family of candidates for maximally entangled mixed states with respect to fixed marginals for two qutrits. These states are extremal in the convex set of two-qutrit states with fixed marginals. Moreover, it is shown that they are always quasidistillable. As a by-product we prove that any maximally correlated state that is quasidistillable must be pure. Our observations for two qutrits are supported by numerical analysis.

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I. INTRODUCTION

The preparation of a quantum system in a certain state is regarded as a central target in several contexts, and if the system is multipartite, the possible *entanglement* among subsystems is a useful resource for quantum information processing and quantum communication [1]. Suitable criteria to characterize or quantify entanglement are then of primary importance [2]. For pure bipartite states $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ with $|\Psi_{AB}\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$, the von Neumann entropy of any of the two reduced density matrices or *marginals* reads

$$E(\Psi_{AB}) = S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A), \tag{1}$$

where $\rho_A = \mathrm{Tr}_B|\Psi_{AB}\rangle\langle\Psi_{AB}|$. For mixed states the situation is much more complicated and the simple formula is replaced by the convex roof construction, leading to the well-known entanglement of formation (EOF), defined by [3]

$$\mathrm{EOF}(\rho_{AB}) = \min_{p_k,\Psi_k} \sum_k p_k E(\Psi_k), \tag{2}$$

where the minimum is performed over all decompositions $\rho_{AB} = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$. For the two-qubit case EOF can be reduced to the celebrated Wootters concurrence,

$$C(\rho_{AB}) \equiv \max\{0, \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\}, \tag{3}$$

where $\{\alpha_i\}$ are the square roots of the four eigenvalues of the matrix $\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)$ taken in decreasing order [4]. (For an introduction to entanglement measures, see, for example, the review Ref. [5].)

A simple way to characterize mixed bipartite entanglement is based on the Peres-Horodecki criterion, also known as the *PPT condition* [6,7]: if a state ρ_{AB} is separable then its *partial transposition* $\rho_{AB}^\tau = (\mathbb{I} \otimes \tau)\rho_{AB}$ is necessarily positive semidefinite. Such a condition becomes necessary and sufficient only for the two-qubit and qubit-qutrit cases [8]. A measure of entanglement called *negativity* can be defined as follows:

$$N(\rho_{AB}) \equiv \tfrac{1}{2}\left(\left\|\rho_{AB}^\tau\right\|_1 - 1\right), \tag{4}$$

where $\|\rho_{AB}^\tau\|_1$ is the trace norm of ρ_{AB}^τ. Such a definition provides a convex function which is nonincreasing under local operation and classical communication [9,10].

A relevant feature of mixed bipartite states is the relation between entanglement and purity [11]. In particular, for a given purity $P = \mathrm{Tr}(\rho_{AB}^2)$, one may ask which state of the same purity displays maximal entanglement [12]. The concept of a maximally entangled mixed state (MEMS) for two qubits was introduced by Ishizaka and Hiroshima as states such that any entanglement measure cannot be increased by any global unitary [13]. They proposed a family of optimal states that was also supported by Munro *et al.* [14]. Such a family is recovered by means of the transformation maximizing the entanglement in the spectrum constrained analog problem, found by Verstraete *et al.* [15]. Despite recent numerical efforts, no analytical results are available for higher-dimensional cases [16,17].

In this paper we analyze a similar problem. We ask what is the maximal entanglement achievable by a bipartite system with fixed marginal states ρ_A and ρ_B. Such an assumption of fixed marginals is known to introduce constraints on the spectrum of the joint state ρ_{AB} in the form of linear inequalities, as shown by Klyachko [18,19]. However, such constraints do not directly tell about possible correlations among the subsystems. Therefore, focusing on bipartite entanglement in such scenarios, we investigate MEMS with respect to fixed marginals which provide the upper bound on entanglement stemming from local information only.

The paper is organized as follows. In Sec. II we review the known results for two-qubit states, including a

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characterization of the optimal states as extremal points of the convex set with fixed marginals, originally discussed in [20,21]. In Sec. III we present a family of candidate MEMS with respect to fixed marginals, supported by an insightful physical interpretation from the point of view of entanglement distillation. Finally, in Sec. IV, we present numerical studies comparing our candidate states with the results of numerical optimization for the case of two qutrits, which supports our conjecture.

II. KNOWN RESULTS

The problem of characterizing mixed bipartite entanglement of states with fixed marginal properties was first introduced in [22]. In particular, a special class of two-qubit states under scrutiny there was denoted as maximally entangled *marginally* mixed states (MEMMS), i.e., MEMS with respect to certain local purities. Clearly, only in the two-qubit case, a given value of both $P_A = \mathrm{Tr}(\rho_A^2)$ and $P_B = \mathrm{Tr}(\rho_B^2)$ uniquely determines the local spectra. Throughout the work, we assume instead complete knowledge of the marginal states.

A. Two-qubits case

Let us start our analysis in a pedagogical fashion and introduce a suitable representation of states with fixed marginals. This is described only in the two-qubit case, but its generalization to arbitrary high dimensions is straightforward. Let ρ_A and ρ_B be two-qubit states. We fix local bases such that the states of the two subsystems are given in diagonal form:

$$\rho_A = \mathrm{diag}\{1-\lambda_A, \lambda_A\}, \quad \rho_B = \mathrm{diag}\{1-\lambda_B, \lambda_B\}, \tag{5}$$

with the lowest eigenvalues such that $\lambda_A, \lambda_B \in [0, \frac{1}{2}]$. Assuming the ordering $\lambda_A \geqslant \lambda_B$, any joint state ρ_{AB} with marginals (5) can be represented as follows:

$$\rho_{AB} = \rho_A \otimes \rho_B + \Delta, \tag{6}$$

where Δ is such that $\mathrm{Tr}_A\Delta = \mathrm{Tr}_B\Delta = 0$, and it contains all possible correlations, quantum and classical, admitted by the two subsystems compatible with fixed marginals ρ_A and ρ_B. It is easy to see that the most general two-qubit matrix form of (6) is the following [23]:

$$\rho_{AB} = \rho_A \otimes \rho_B + \left(\begin{array}{cc|cc} \varepsilon & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ & -\varepsilon & \Delta_{23} & -\Delta_{13} \\ \hline & & -\varepsilon & -\Delta_{12} \\ (\mathrm{c.c}) & & & \varepsilon \end{array}\right), \tag{7}$$

where one has to choose the entries of Δ such that $\rho_{AB} \geqslant 0$. Let us observe that, in order to obtain a non-negative diagonal elements, one finds

$$-\lambda_A\lambda_B \leqslant \varepsilon \leqslant \lambda_B(1-\lambda_A). \tag{8}$$

Two-qubit MEMMS states are thus achieved by maximizing concurrence or negativity of states in the form (7). However, in this case it is sufficient to consider the subclass of X states only (nonzero diagonal and antidiagonal), since it includes also the two-qubit MEMS [13–15,32]. X states are common in quantum information theory because of their sparse structure, allowing for many analytic computations [24]. Important families of two-qubit states such as Bell, Werner, or isotropic states are within this class. Hence we consider

$$\rho_{AB} = \rho_A \otimes \rho_B + \left(\begin{array}{cc|cc} \varepsilon & \cdot & \cdot & \Delta_{14} \\ & -\varepsilon & \Delta_{23} & \cdot \\ \hline & & -\varepsilon & \cdot \\ (\mathrm{c.c}) & & & \varepsilon \end{array}\right). \tag{9}$$

Such a simple structure yields the following concurrence:

$$C(\rho_{AB}) = 2\max\{0, |\Delta_{23}| - \sqrt{\rho_{11}\rho_{44}}, |\Delta_{14}| - \sqrt{\rho_{22}\rho_{33}}\}, \tag{10}$$

where $\{\rho_{ij}\}_{i,j=1,\ldots,4}$ are matrix elements of ρ_{AB} [4]. It is useful, according to (8), to parametrize ε via $\varepsilon = s\lambda_B - \lambda_A\lambda_B$, where $s \in [0, 1]$. Finally, positivity of ρ_{AB} is simply controlled by the following inequalities for a given s:

$$\begin{aligned} |\Delta_{23}|^2 &\leqslant \lambda_B(1-s)(\lambda_A - \lambda_B s), \\ |\Delta_{14}|^2 &\leqslant s\lambda_B(1-\lambda_A-\lambda_B+\lambda_B s). \end{aligned} \tag{11}$$

Due to the simplicity of (10), one can independently maximize both the right-hand sides of (11) and observe that the maximum is reached when $s = 1$, $|\Delta_{23}| = 0$, $|\Delta_{14}| = \sqrt{(1-\lambda_A)\lambda_B}$, giving rise to the following state:

$$\tilde{\rho}_{AB} = \left(\begin{array}{cc|cc} 1-\lambda_A & \cdot & \cdot & \sqrt{(1-\lambda_A)\lambda_B} \\ \cdot & 0 & \cdot & \cdot \\ \hline \cdot & \cdot & \lambda_A - \lambda_B & \cdot \\ \sqrt{(1-\lambda_A)\lambda_B} & \cdot & \cdot & \lambda_B \end{array}\right), \tag{12}$$

with negativity given by

$$N(\tilde{\rho}_{AB}) = \tfrac{1}{2}\left(\lambda_A - \lambda_B - \sqrt{(\lambda_A-\lambda_B)^2 + 4\lambda_B(1-\lambda_A)}\right).$$

This represents the upper bound for a two-qubit system with arbitrarily fixed marginals, in accordance with [22]. Interestingly, $\tilde{\rho}_{AB}$ can be written as follows:

$$\tilde{\rho}_{AB} = (1-\eta)|\Psi_{\mathrm{mc}}\rangle\langle\Psi_{\mathrm{mc}}| + \eta|10\rangle\langle 10|, \tag{13}$$

where $\{|0\rangle, |1\rangle\}$ is the computational basis in $\mathbb{C}^2$, $\eta = \lambda_A - \lambda_B$, and $|\Psi_{\mathrm{mc}}\rangle\langle\Psi_{\mathrm{mc}}|$ is a *maximally correlated* rank-1 projector. Recall that a state σ_{mc} maximally correlated (or Schmidt-correlated) in the computational basis in $\mathbb{C}^d \otimes \mathbb{C}^d$ reads [25]

$$\sigma_{\mathrm{mc}} = \sum_{i,j=0}^{d-1} \alpha_{ij}|ii\rangle\langle jj|. \tag{14}$$

Moreover, the state (14) has all its (at most d) eigenvectors in the form $|\Psi_k\rangle = \frac{1}{\sqrt{d}}\sum_k \lambda_i^{(k)}|ii\rangle$. In order to provide a simple visual representation that (12) is the optimal state, we construct a negativity vs global purity plot (N-P), shown in Fig. 1. This allows us to compare the negativity of $\tilde{\rho}_{AB}$ with that of a set of randomly generated states from (7). In what follows we briefly recall a further characterization of the optimal state as an extremal point of a convex set. The motivation is simple: negativity is a convex function and the set of states with fixed marginals is also convex, hence the maximum must be attained by an extremal point [26].

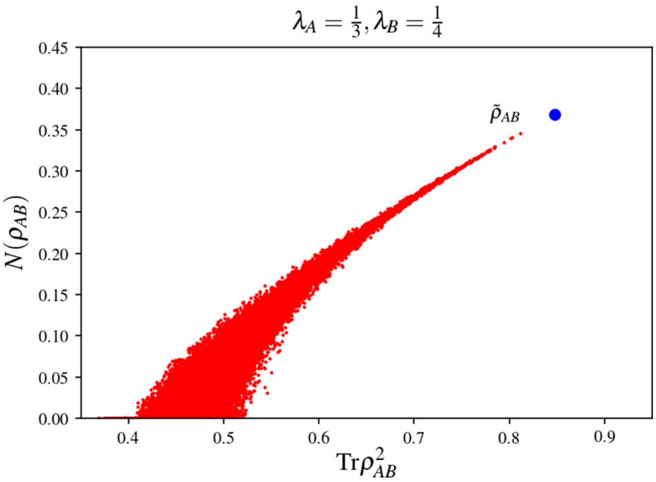

FIG. 1. *N-P* plot with 20 000 randomly generated two-qubit states with marginals $\lambda_A = \frac{1}{3}$ and $\lambda_B = \frac{1}{4}$. Note that both purity and negativity are maximized by the same state $\tilde{\rho}_{AB}$ (13).

B. Optimal states as extremal points

Let us denote with $C(\rho_A, \rho_B)$ the convex set of two-qudit states with fixed marginals ρ_A and ρ_B. The characterization of the extremal points of $C(\rho_A, \rho_B)$ was provided first by Parthasarathy in [20]. Here we follow instead the approach by Rudolph, based on the duality between positive operators and completely positive (CP) maps [21]. We recall that a map Λ is CP if and only if (iff) the map $\mathrm{id}_k \otimes \Lambda$ is positive $\forall\, k \in \mathbb{N}_+$, where id_k denotes the identity map. A powerful tool providing a duality between states and CP maps is given by the Choi-Jamiołkowski isomorphism [27]. For each CP map Λ, one can assign a legitimate density matrix ρ_Λ:

$$\rho_\Lambda = [\mathrm{id}_d \otimes \Lambda](P_d^+), \quad P_d^+ = \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle\langle jj|, \tag{15}$$

where P_d^+ is a maximally entangled projector and id_d is an identity map. Since the above duality is bijective, one has the following inverse CP map for any state ρ:

$$\Lambda_\rho[\sigma] = \mathrm{Tr}_2[\mathrm{id}_d \otimes \sigma^T \rho]. \tag{16}$$

This allows us to describe the convex structure of the set of states with fixed marginals at the level of the corresponding maps. In particular, one can exploit the known characterization of extremal maps in terms of their Kraus representation of $\Lambda_\rho[\sigma] = \sum_\alpha K_\alpha \sigma K_\alpha^\dagger$, namely, that ρ_Λ is extremal iff $\{K_\alpha^\dagger K_\beta\}_{\alpha,\beta=1,\dots,d^2}$ is a linearly independent set of matrices [28]. Moreover, the constraint of fixed marginals ρ_A and ρ_B is expressed via Eq. (16):

$$\begin{aligned} \frac{1}{d}\Lambda_\rho(\mathbb{I}_d) &= \frac{1}{d}\sum_\alpha K_\alpha K_\alpha^\dagger = \rho_A, \\ \frac{1}{d}\Lambda_\rho^*(\mathbb{I}_d) &= \frac{1}{d}\sum_\alpha K_\alpha^\dagger K_\alpha = \rho_B, \end{aligned} \tag{17}$$

where Λ^* denotes the canonical dual. Thus, the extremality condition amounts at proving that the set,

$$\{K_\alpha^\dagger K_\beta \oplus K_\beta K_\alpha^\dagger\}_{\alpha,\beta=1,\dots,d^2}, \tag{18}$$

is linearly independent, i.e., the two sets $\{K_\alpha^\dagger K_\beta\}_{\alpha,\beta=1,\dots,d^2}$ and $\{K_\beta K_\alpha^\dagger\}_{\alpha,\beta=1,\dots,d^2}$ are *jointly* linearly independent [29]. As an example, we have that the only extremal two-qubit state for $C(\frac{1}{2}\mathbb{I}_2, \frac{1}{2}\mathbb{I}_2)$ is the maximally entangled projector P_2^+ [20,21]. The criterion given by conditions (17) and (18) can be applied to our case in order to construct examples of extremal points in $C(\rho_A, \rho_B)$. Note that the optimal rank-2 state (12) is retrieved by means of the following Kraus operators:

$$\begin{aligned} K_1 &= \begin{pmatrix} 0 & 0 \\ \sqrt{\lambda_A - \lambda_B} & 0 \end{pmatrix}, \\ K_2 &= \begin{pmatrix} \sqrt{1-\lambda_A} & 0 \\ 0 & \sqrt{\lambda_B} \end{pmatrix}. \end{aligned} \tag{19}$$

One can easily check that the fixed marginals and extremality conditions hold (cf. Appendix A). Moreover, defining $e_{ij} = |i\rangle\langle j|$, the corresponding rank-2 extremal, given by (15), reads

$$\rho_\Lambda = \frac{1}{2}\sum_{i,j=1}^{2}\sum_{\alpha=1}^{2} e_{ij} \otimes K_\alpha\, e_{ij}\, K_\alpha^\dagger \tag{20}$$

and coincides with the optimal state (12). The parametrization of the class of extremal states for arbitrarily given marginals in higher dimensions ($\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$) is out of the aim of this work and will not be discussed here. Nevertheless, we will adopt in the next section the extremality condition as a further check on the candidate MEMS with respect to fixed marginals. One can find the following necessary condition for extremal points in $C(\rho_A, \rho_B)$ [20]:

$$\mathrm{rank}(\rho) \leqslant \sqrt{2d^2 - 1}. \tag{21}$$

This observation turns out to be useful for the numerical studies discussed later in Sec. IV.

III. HIGHER DIMENSIONS

In this section we discuss the properties of a family of states within which we identify candidates for two-qutrit MEMS with respect to marginals. A crucial observation is that all candidate states are *quasidistillable*, i.e., states for which a singlet fraction arbitrarily close to unity can be obtained in the distillation process [30]. A connection between two-qubit MEMS and quasidistillable states was highlighted previously in [31,32].

A. A family of candidates

As an attempt to directly generalize the two-qubit results, we focus on the extension of the form (13) to higher dimensions $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, namely,

$$\tilde{\rho} = (1-\eta)\sigma_{\mathrm{mc}} + \sum_{i \neq j} p_{ij} |ij\rangle\langle ij|, \tag{22}$$

where $\eta = \sum_{i\neq j} p_{ij}$, and σ_{mc} indicates a maximally correlated state of the form (14). Note that by replacing σ_{mc} with P_d^+, one obtains a possible generalization of isotropic states [33]. Furthermore, the family defined by Eq. (22) belongs to a wider class known as *circulant* states, which reduces to X states in $\mathbb{C}^2 \otimes \mathbb{C}^2$ [34]. For the rest of the work, we examine the two-qutrit case for which the matrix structure of (22) reads

$$\tilde{\rho} = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \cdot & \cdot & \cdot & \Delta_{15} & \cdot & \cdot & \cdot & \Delta_{19} \\ & \rho_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \rho_{33} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline & & & \rho_{44} & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \rho_{55} & \cdot & \cdot & \cdot & \Delta_{59} \\ & & & & & \rho_{66} & \cdot & \cdot & \cdot \\ \hline & & & & & & \rho_{77} & \cdot & \cdot \\ & & & & & & & \rho_{88} & \cdot \\ \text{(c.c.)} & & & & & & & & \rho_{99} \end{array}\right) \tag{23}$$

and satisfies $\mathrm{Tr}\tilde{\rho} = 1$, the positivity condition, and compatibility with the following marginals:

$$\rho_A = \mathrm{diag}\{1-\lambda_1-\lambda_2, \lambda_1, \lambda_2\},$$
$$\rho_B = \mathrm{diag}\{1-\mu_1-\mu_2, \mu_1, \mu_2\}, \tag{24}$$

where $\lambda_1 \geqslant \lambda_2$, $\mu_1 \geqslant \mu_2$ correspond to the decreasingly ordered local eigenvalues, and thus $\lambda_2, \mu_2 \leqslant \frac{1}{3}$. Without loss of generality we also assume $\lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2$. The negativity of (23) is simply given by

$$N(\tilde{\rho}) = \tfrac{1}{2}[(|A| - A) + (|B| - B) + (|C| - C)], \tag{25}$$

where

$$A = \tfrac{1}{2}\left(\rho_{22} + \rho_{44} - \sqrt{4|\Delta_{15}|^2 + (\rho_{22} - \rho_{44})^2}\right),$$
$$B = \tfrac{1}{2}\left(\rho_{33} + \rho_{77} - \sqrt{4|\Delta_{19}|^2 + (\rho_{33} - \rho_{77})^2}\right),$$
$$C = \tfrac{1}{2}\left(\rho_{66} + \rho_{88} - \sqrt{4|\Delta_{59}|^2 + (\rho_{66} - \rho_{88})^2}\right). \tag{26}$$

Note that if at least one of the diagonal elements in each term of (26) is zero, we already reach the maximum number of negative eigenvalues of the partial transpose. Moreover, $N(\tilde{\rho})$ increases monotonically with $|\Delta_{ij}|$, and thus it is favorable to have the maximum number of zeros (four) in the diagonal, which can always be chosen independently in (26). Maximum negativity within our family is then attained by the following three states:

$$\tilde{\rho}^{(1)}_{AB} = (1 - p_{10} - p_{20})\,|\Psi^{(1)}_{\mathrm{mc}}\rangle\langle\Psi^{(1)}_{\mathrm{mc}}| + p_{10}\,|10\rangle\langle 10| + p_{20}\,|20\rangle\langle 20|,$$
$$p_{10} = \lambda_1 - \mu_1,\ \ p_{20} = \lambda_2 - \mu_2, \tag{27}$$

valid when $\lambda_1 > \mu_1$ and $\lambda_2 > \mu_2$,

$$\tilde{\rho}^{(2)}_{AB} = (1 - p_{10} - p_{12})\,|\Psi^{(2)}_{\mathrm{mc}}\rangle\langle\Psi^{(2)}_{\mathrm{mc}}| + p_{10}\,|10\rangle\langle 10| + p_{12}\,|12\rangle\langle 12|,$$
$$p_{10} = \lambda_1 + \lambda_1 - (\mu_1 + \mu_2),\ \ p_{12} = \mu_2 - \lambda_2 \tag{28}$$

when $\lambda_2 < \mu_2$, and finally,

$$\tilde{\rho}^{(3)}_{AB} = (1 - p_{20} - p_{21})\,|\Psi^{(3)}_{\mathrm{mc}}\rangle\langle\Psi^{(3)}_{\mathrm{mc}}| + p_{20}\,|20\rangle\langle 20| + p_{21}\,|21\rangle\langle 21|,$$
$$p_{20} = \lambda_1 + \lambda_1 - (\mu_1 + \mu_2),\quad p_{21} = \mu_1 - \lambda_1 \tag{29}$$

when $\lambda_1 < \mu_1$. As an example the matrix form of $\tilde{\rho}^{(1)}_{AB}$ reads as follows:

$$\tilde{\rho}^{(1)}_{AB} = \left(\begin{array}{ccc|ccc|ccc} 1-\lambda_1-\lambda_2 & \cdot & \cdot & \cdot & \sqrt{\mu_1(1-\lambda_1-\lambda_2)} & \cdot & \cdot & \cdot & \sqrt{\mu_2(1-\lambda_1-\lambda_2)} \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \lambda_1-\mu_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sqrt{\mu_1(1-\lambda_1-\lambda_2)} & \cdot & \cdot & \cdot & \mu_1 & \cdot & \cdot & \cdot & \sqrt{\mu_1\mu_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_2-\mu_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \sqrt{\mu_2(1-\lambda_1-\lambda_2)} & \cdot & \cdot & \cdot & \sqrt{\mu_1\mu_2} & \cdot & \cdot & \cdot & \mu_2 \end{array}\right). \tag{30}$$

The specific form of $|\Psi^{(i)}_{\mathrm{mc}}\rangle\langle\Psi^{(i)}_{\mathrm{mc}}|$, $i = 1, 2, 3$ in Eqs. (27), (28), and (29) is easily found by properly adjusting partial traces. Maximal negativity within the family (22) is thus attained when σ_{mc} is rank 1, that is, when the state has the following structure, similar to the two-qubit MEMMS (12):

$$\tilde{\rho}_{AB} = (1 - \eta)|\Psi_{\mathrm{mc}}\rangle\langle\Psi_{\mathrm{mc}}| + \sum_{i\neq j} p_{ij}|ij\rangle\langle ij|, \tag{31}$$

where $\eta = \sum_{i\neq j} p_{ij}$, namely, a convex combination of a rank-1 projector and a classical state with at most two nonzero entries. To conclude this section we state the following proposition (proven in Appendix A):

Proposition. All candidate states, Eqs. (27), (28), and (29), are extremal points in the convex set $C(\rho_A, \rho_B)$ of states with fixed marginals ρ_A and ρ_B.

In what follows we show that the same states can be found from an entanglement distillation perspective, i.e., imposing that states of the form (22) are *quasidistillable*.

B. Quasidistillable states

As introduced before, quasidistillable states are mixed entangled states for which a singlet fraction arbitrarily close to unity can be distilled with nonzero probability. In this section we recall the main feature of such states in the general two-qudit case and provide a criterion to identify them within the class (22). The main motivation is that the two-qubit MEMMS (12) is also a quasidistillable state. Interestingly, we will show that all candidate states in Eqs. (27), (28), and (29) are again quasidistillable. As usual, we denote the computational basis in $\mathbb{C}^d \otimes \mathbb{C}^d$ with $\{|ij\rangle\}_{i,j=1,\ldots,d}$. Let us start from the following [30]:

Definition. A state ρ is said **quasidistillable** iff there exist two sequences of filtering operators $\{A_n\}$ and $\{B_n\}$ such that

$$\frac{\Lambda^{(n)}(\rho)}{\mathrm{Tr}[\Lambda^{(n)}(\rho)]} = \frac{(A_n \otimes B_n)\rho(A_n^\dagger \otimes B_n^\dagger)}{\mathrm{Tr}[(A_n \otimes B_n)\rho(A_n^\dagger \otimes B_n^\dagger)]} \xrightarrow[n\to\infty]{} P_d^+ \quad (32)$$

and the probabilities $p_n = \mathrm{Tr}[\Lambda^{(n)}(\rho)] \to 0$.

Note that the filtering operators can be taken Hermitian so we can simply restrict to A_n and B_n. In order to characterize quasidistillable states within (22), we state our two main results concerning first maximally correlated states only and the structure of our candidate MEMS with respect to fixed marginals (31), proven in Appendixes B and C:

Theorem 1. A maximally correlated state σ_{mc} is quasidistillable iff it is of rank 1, i.e., $\sigma_{\mathrm{mc}} = |\Psi_{\mathrm{mc}}\rangle\langle\Psi_{\mathrm{mc}}|$, and $\mathrm{s-rank}(|\Psi_k\rangle) = d$ (Schmidt rank).

Theorem 2. Let ρ be a state of the form (31), i.e., a convex mixture of a maximally correlated rank-1 projector and a classical state. ρ is quasidistillable iff among the set of $p_{ij} \neq 0$ there are no looping indices, i.e., $p_{ij}p_{jk}\dots p_{li} = 0$.

In [30], the authors proved that the following two-qutrit state,

$$\rho = \eta P_3^+ + \frac{(1-\eta)}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|), \quad (33)$$

with $0 < \eta < 1$ is not quasidistillable. Indeed, one has $p_{01}p_{12}p_{20} \neq 0$, that is, p_{ij} meets the loop condition. However, the following state,

$$\rho = \eta P_3^+ + \frac{(1-\eta)}{3}(|10\rangle\langle 10| + |12\rangle\langle 12| + |20\rangle\langle 20|), \quad (34)$$

is quasidistillable according to the sequence of filtering operators $\{A_n\}$ and $\{B_n\}$ provided in the proof of Theorem 2 in Appendix B. Furthermore, structures similar to our candidate states Eqs. (27)–(29) can be recovered by means of the following:

Corollary. If ρ of the form (31) is quasidistillable, it has at most $\binom{d}{2}$ nonzero diagonal elements.

Proof. Let ρ be of the form (7) with $p_{ij} > 0\, \forall i > j$. It easy to see from Theorem 2 that ρ is quasidistillable and has exactly $\binom{d}{2}$ nonzero elements. If we consider a further nonzero element from the remaining set $(i < j)$ we would have $p_{i_0,j_0}p_{j_0,i_0} \neq 0$ for at least one couple of indexes (i_0, j_0), meaning that such a ρ is no more quasidistillable.

Therefore, only one element is allowed in the two-qubit case and at most three for two-qutrit cases. Some special cases of (31) are the following:

$$\rho = (1-\eta)|\Psi_{\mathrm{mc}}\rangle\langle\Psi_{\mathrm{mc}}| + |i_0\rangle\langle i_0| \otimes \sum_j^{d-1} p_{i_0,j}|j\rangle\langle j|,$$

$$\rho = (1-\eta)|\Psi_{\mathrm{mc}}\rangle\langle\Psi_{\mathrm{mc}}| + \sum_i^{d-1} p_{i,j_0}|i\rangle\langle i| \otimes |j_0\rangle\langle j_0|, \quad (35)$$

that is, with some fixed index i_0 or j_0 in one of the two marginal subspaces. As a final remark, we have observed that the maximization of negativity within the family (22) with fixed marginals yields candidate states satisfying Theorem 2. In particular, the requirement of having the maximal number (three) of negative eigenvalues of ρ^τ yields at most three nonzero elements in the classical term. Moreover, the two-qutrit candidates display *only* two nonzero $p_{i,j}$ such that the indices do not *loop*, in the above sense. This leads us to the conjecture that all MEMS with respect to fixed marginals are quasidistillable in arbitrary dimensions.

IV. NUMERICAL RESULTS

The aim of this section is to provide a set of numerical observations in order to legitimate our states [Eqs. (27)–(29)] as good candidates for two-qutrit MEMS with respect to fixed marginals. To begin with, we observe that a key ingredient is generation of random states with fixed marginals ρ_A and ρ_B, i.e., an element of $C(\rho_A, \rho_B)$. To this aim, we have adopted two procedures. First, for the two-qubit case we algorithmically generated random correlation elements of Eq. (7) and check the positivity of the resulting ρ_{AB}. This procedure was used to generate points in the N-P plot in Fig. 1. A more efficient method is to choose a state randomly and to minimize numerically[1] a distance function from the set $C(\rho_A, \rho_B)$. Such a distance is simply defined as

$$\rho \mapsto \left|\left|\mathrm{Tr}_B\rho - \rho_A\right|\right|_2^2 + \left|\left|\mathrm{Tr}_A\rho - \rho_B\right|\right|_2^2. \quad (36)$$

Having a random initial state from the set $C(\rho_A, \rho_B)$, we proceed by maximizing the negativity function. We stay in the set $C(\rho_A, \rho_B)$ during the minimization, adding the mentioned function (36) to the (negated) negativity as a penalty function, with a factor controlling the accuracy.

In the minimization procedure, we represent states as $\rho = AA^\dagger$, where A is a square complex matrix (9×9 for two qutrits), if ρ has an unrestricted rank. Note, however, that according to [20] we have that $\mathrm{rank}(\rho) \leqslant \sqrt{2d^2 - 1}$ for extremal states. For the two-qutrit case, the latter is $\sqrt{17} \approx 4.12$ and we can limit our search to rank-4 states only, represented by complex matrices A of size 9×4, which reduces the (real) dimension of the problem from 162 to 72.

A restricted, one-dimensional set of examples is shown in Fig. 2, where one can see a satisfactory agreement between the negativity of the candidate states (blue line) and the results of numerical optimization (red crosses) for a particular set of marginals. A second set of examples is obtained spanning over the two lowest marginal eigenvalues independently, thus keeping λ_1 and μ_1 fixed. For the set of points in Fig. 3 we choose $\lambda_1 = \mu_1 = \frac{1}{3}$ and span over uniformly distributed values of the allowed domain for λ_2 and μ_2 and compare the negativity surface from the candidate states with numerical optimums. Note that for such a choice we have one candidate only, since all candidate states collapse in one. As a last series of examples we choose $\lambda_1 = 0.25$, $\mu_1 = 0.3$ so that the candidate is given by $\tilde{\rho}_{AB}^{(3)}$ in (29) and the range for λ_2, μ_2 is restricted by the assumption $\lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2$ (see Fig. 4). To summarize, all the above results strongly support our conjecture that our quasidistillable states [Eqs. (27)–(29)] are legitimate candidates for two-qutrit MEMS with respect to fixed marginals and motivate the search for analytical proofs in further studies.

[1] For this we use the SCIPY function *minimize*.

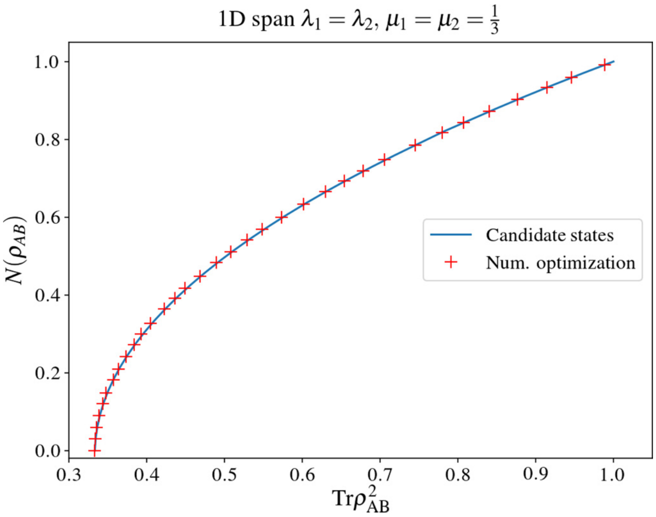

FIG. 2. Negativity vs global purity (N-P) plot. We analyze a particular configuration with one marginal maximally mixed and another spanning over one eigenvalue only, namely, $\lambda_1 = \lambda_2$. The blue line describes the negativity of the candidate states (27) for $P \in [\frac{1}{3}, 1]$. Red crosses represent the negativity values obtained from numerical optimization.

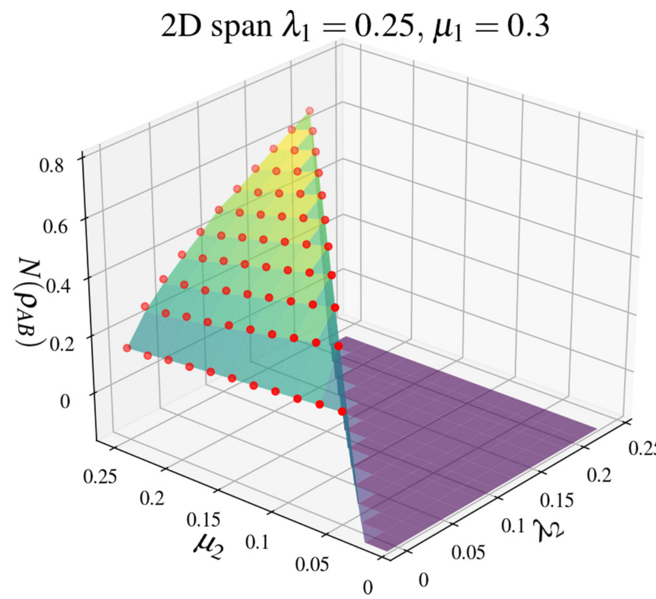

FIG. 4. Second three-dimensional plot of negativity as a function of the lowest marginal eigenvalues λ_2, μ_2. Here λ_1, μ_1 are chosen such that the domain of interest is restricted.

V. CONCLUSIVE REMARKS

In this work we have observed strong numerical evidence that the two-qutrit states [see Eqs. (27), (28), and (29)] are good candidates as MEMS with respect to fixed marginals. The main feature of our reasoning is the generalization of two special properties of the two-qubit state (12), i.e., its simple structure and the property of being quasidistillable. It is shown that these states are always quasidistillable, and hence we provide another interesting application of quasidistillable states in quantum information. Such a strong link between the two concepts deserves to be investigated in further studies. Moreover, a possible obvious generalization of our problem can be thought for multipartite entanglement in the presence of many fixed marginal states. Other similar versions can be considered such as the bounds of mutual information, coherence, or the study of the such bounds in the presence of fixed marginal purities, as the original problem in [22]. The difference is the corresponding set of states is not convex, and we cannot rely on the extremality property. Finally, concerning our problem, it is worth remarking that both the maximization of negativity and purity lead to the same optimal state. This is true for the two-qubit case and for the two-qutrit family defined by (22), and there is numerical evidence for general two-qutrit states. This observation will be also object of further investigations. We hope that further characterizations of extremal points in $C(\rho_A, \rho_B)$ in future studies could lead to other observations strengthening our conjecture and pave the way to analytical proofs.

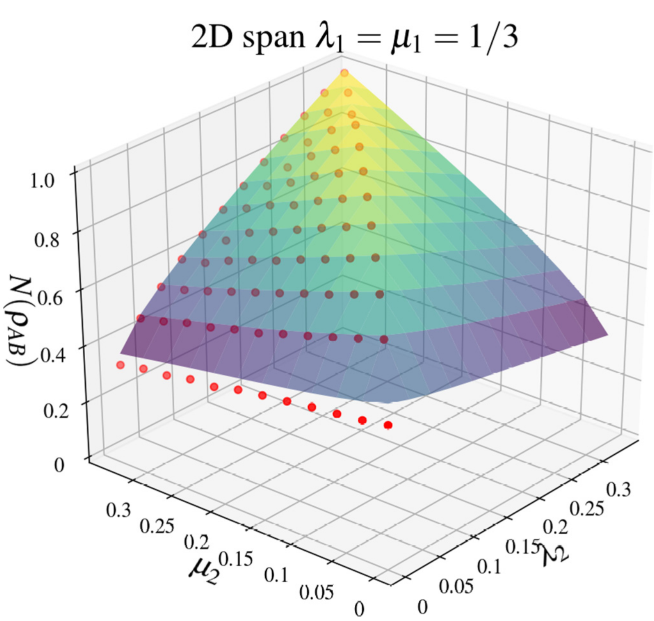

FIG. 3. Three-dimensional plot of negativity as a function of λ_2, μ_2. Red points obtained from numerical optimization are compared with the negativity surface obtained from our candidate.

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APPENDIX A: (EXTREMAL STATES)

In this Appendix we show that the two-qubit MEMMS (12) and all the candidate states, Eqs. (27), (28), and (29), are extremal points in the convex set of states with fixed marginals $C(\rho_A, \rho_B)$. According to the conditions (17) and (18), a generic state in $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, defined as

$$\rho_\Lambda = [\mathrm{id}_d \otimes \Lambda](P_d^+) = \rho_\Lambda = \frac{1}{d} \sum_{i,j=1}^{d} \sum_{\alpha=1}^{d^2} e_{ij} \otimes K_\alpha \, e_{ij} \, K_\alpha^\dagger, \tag{A1}$$

is extremal in $C(\rho_A, \rho_B)$ iff $\Lambda(\mathbb{I}_d) = d\rho_A$, $\Lambda^*(\mathbb{I}_d) = d\rho_B$, and the set $\{K_\alpha^\dagger K_\beta \oplus K_\beta K_\alpha^\dagger\}_{\alpha,\beta=1,\dots,d^2}$ is linearly independent. For the rank-2 two-qubit MEMMS state (12), we have the following suitable family of Kraus operators:

$$K_1 = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$
$$K_1 K_1^\dagger + K_2 K_2^\dagger = \begin{pmatrix} |a|^2 + |x|^2 & 0 \\ 0 & |b|^2 + |y|^2 \end{pmatrix}$$
$$K_1^\dagger K_1 + K_2^\dagger K_2 = \begin{pmatrix} |b|^2 + |x|^2 & 0 \\ 0 & |a|^2 + |y|^2 \end{pmatrix}. \tag{A2}$$

Choosing $a = 0$ implies $|x| = \sqrt{1 - \lambda_A}$, $|y| = \sqrt{\lambda_B}$, and $|b| = \sqrt{\lambda_A - \lambda_B}$. Moreover, the Kraus operators satisfy

$$K_1^\dagger K_2 = \sqrt{\lambda_B(\lambda_A - \lambda_B)}|0\rangle\langle 1| = (K_2^\dagger K_1)^\dagger,$$
$$K_1 K_2^\dagger = \sqrt{(1 - \lambda_B)(\lambda_A - \lambda_B)}|1\rangle\langle 0| = (K_2 K_1^\dagger)^\dagger. \tag{A3}$$

Thus, the two sets $\{K_\alpha^\dagger K_\beta\}_{\alpha,\beta=1,2}$ and $\{K_\beta K_\alpha^\dagger\}_{\alpha,\beta=1,2}$ are jointly linear independent, and we have

$$\rho_\Lambda = \tilde{\rho}_{AB} = \begin{pmatrix} 1 - \lambda_A & \cdot & \cdot & \sqrt{(1-\lambda_A)\lambda_B} \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \lambda_A - \lambda_B & \cdot \\ \sqrt{(1-\lambda_A)\lambda_B} & \cdot & \cdot & \lambda_B \end{pmatrix}.$$

By means of a similar argument one finds the corresponding Kraus operators for the candidate states $\tilde{\rho}_{AB}^{(i)}$, $i = 1, 2, 3$. We have for $\tilde{\rho}_{AB}^{(1)}$,

$$K_1 = \sqrt{1 - \lambda_1 - \lambda_2}\, |0\rangle\langle 0| + \sqrt{\mu_1}\, |1\rangle\langle 1| + \sqrt{\mu_2}\, |2\rangle\langle 2|$$
$$K_2 = \sqrt{\lambda_1 - \mu_1}\, |1\rangle\langle 0|, \quad K_3 = \sqrt{\lambda_2 - \mu_2}\, |2\rangle\langle 0|,$$

which produce the following state via (A2):

$$\rho_\Lambda = \left(\begin{array}{ccc|ccc|ccc} \alpha_{00}^2 & \cdot & \cdot & \cdot & \alpha_{00}\alpha_{11} & \cdot & \cdot & \cdot & \alpha_{00}\alpha_{22} \\ & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline & & & \lambda_1 - \mu_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \alpha_{11}^2 & \cdot & \cdot & \cdot & \alpha_{11}\alpha_{22} \\ & & & & & 0 & \cdot & \cdot & \cdot \\ \hline & & & & & & \lambda_2 - \mu_2 & \cdot & \cdot \\ & & & & & & & 0 & \cdot \\ & & & & & & & & \mu_2 \end{array}\right),$$

where $\alpha_{00} = \sqrt{1 - \lambda_1 - \lambda_2}$, $\alpha_{11} = \sqrt{\mu_1}$, $\alpha_{22} = \sqrt{\mu_2}$. One sees that such a state coincides with $\tilde{\rho}_{AB}^{(1)}$ (30). Other possible Kraus operators for the states (28) and (29) are found as

$$K_1 = \sqrt{1 - \lambda_1 - \lambda_2}|0\rangle\langle 0| + \sqrt{\mu_1}\, |1\rangle\langle 1| + \sqrt{\lambda_2}\, |2\rangle\langle 2|,$$
$$K_2 = \sqrt{\lambda_1 + \lambda_2 - (\mu_1 + \mu_2)}|1\rangle\langle 0|, \quad K_3 = \sqrt{\mu_2 - \lambda_2}\, |1\rangle\langle 2|,$$

$$K_1 = \sqrt{1 - \lambda_1 - \lambda_2}\, |0\rangle\langle 0| + \sqrt{\lambda_1}\, |1\rangle\langle 1| + \sqrt{\mu_2}\, |2\rangle\langle 2|,$$
$$K_2 = \sqrt{\mu_2 - \lambda_2}|2\rangle\langle 1|, \quad K_3 = \sqrt{\lambda_1 + \lambda_2 - (\mu_1 + \mu_2)}\, |2\rangle\langle 0|,$$

valid for $\tilde{\rho}_{AB}^{(2)}$ and $\tilde{\rho}_{AB}^{(3)}$, respectively.

APPENDIX B: (THEOREM 1)

Before proving Theorem 1, let us state the following lemma concerning filtering operators A_n^T and B_n.

Lemma 1. Let $\{A_n^T\}$ and $\{B_n\}$ be filtering operators for some state ρ in a quasidistillation process and $0 \leqslant a_1^{(n)} \leqslant \cdots \leqslant a_d^{(n)}$ and $0 \leqslant b_1^{(n)} \leqslant \cdots \leqslant b_d^{(n)}$ their singular eigenvalues. Then, at least one among $a_1^{(n)}, b_1^{(n)}$ must tend to zero as $n \to \infty$.

Proof. Suppose that both $a_1^{(n)}, b_1^{(n)} \geqslant \gamma > 0 \quad \forall n$, and let us consider $|\Psi\rangle = \sum_i d_{ii}|ii\rangle$ satisfying Eq. (32). Then $\mathrm{s-rank}(|\Psi_k\rangle) = d$ and the matrix $D = \{d_{ii}\}$ has full rank, i.e., $d_{ii} \geqslant \delta > 0$. Equation (32) is then equivalent to

$$\frac{B_n D A_n}{\|B_n D A_n\|_{\text{H-S}}} \xrightarrow[n\to\infty]{} \frac{\mathbb{I}}{\sqrt{d}}, \tag{B1}$$

where $\|\omega\|_{\text{H-S}} = \sqrt{\mathrm{Tr}\omega\omega^\dagger}$ is the Hilbert-Schmidt norm. Note that $\|B_n D A_n\|_{\text{H-S}} = \mathrm{Tr}[\Lambda^{(n)}(|\Psi\rangle\langle\Psi|)]^{\frac{1}{2}}$ so that it must tend to zero as $n \to \infty$. However, we have the following:

$$\mathrm{Tr}[\Lambda^{(n)}(|\Psi\rangle\langle\Psi|)] = \mathrm{Tr}[B_n D A_n A_n D B_n] \geqslant$$
$$\mathrm{Tr}[B_n D^2 B_n]\gamma^2 \geqslant \mathrm{Tr}[A^2]\gamma^2\delta \geqslant \gamma^4\delta > 0.$$

Therefore, at least one among $a_1^{(n)}, b_1^{(n)}$ must tend to zero. Let us now prove Theorem 1.

Proof (Theorem 1). Consider the quasidistillation of σ_{mc} (22) which has many eigenvectors $|\Psi_k\rangle$. As already mentioned, they all have diagonal coefficient matrices D^k with elements $D_{ij}^k = \lambda_{ij}^{(k)}$. Because of quasidistillation process, at least one of the eigenvectors satisfies (in terms of $|\Psi_k\rangle\langle\Psi_k|$) Eq. (32), which we shall drop a particular index denoting that vector and its coefficients matrix as $|\Psi\rangle$ and D accordingly. We shall show that if the mixture σ_{mc} is to satisfy (32) then it cannot admit any more eigenvectors but $|\Psi\rangle$.

Let $|\Psi'\rangle = \sum_i (d')_{ii}|ii\rangle$ be another arbitrary eigenvector with its corresponding coefficients matrix D'. We shall show that either it vanishes or is proportional to $|\Psi\rangle$. There are three alternatives: the ratio $\frac{\Lambda^{(n)}(|\Psi'\rangle\langle\Psi'|)}{|\Psi\rangle\langle\Psi|}$ can (i) converge to a strictly positive constant, (ii) diverge to infinity, or (iii) converge to zero.[2] This corresponds to situations where a weight at the transformed eigenvector $|\Psi\rangle$ is comparable, dominates, or is dominated in the limit of large n, respectively.

Consider first the case (i). Here we have

$$\frac{\sqrt{\mathrm{Tr}[\Lambda^{(n)}(|\Psi\rangle\langle\Psi|)]}}{\sqrt{\mathrm{Tr}[\Lambda^{(n)}(|\Psi'\rangle\langle\Psi'|)]}} = \frac{\|B_nDA_n\|_{\text{H-S}}}{\|B_nD'A_n\|_{\text{H-S}}} \xrightarrow[n\to\infty]{} c > 0,$$

where, of course, $D' = \{d'_{ii}\}$. If we call X_n the left-hand side of Eq. (B1), we have by assumption the following:

$$X'_n = \frac{B_nD'A_n}{\|B_nD'A_n\|_{\text{H-S}}} \xrightarrow[n\to\infty]{} \frac{\mathbb{I}}{\sqrt{d}}. \tag{B2}$$

Both X_n and X'_n have bounded inversion, so we have

$$X_n(X'_n)^{-1} = \frac{\|B_nD'A_n\|_{\text{H-S}}}{\|B_nDA_n\|_{\text{H-S}}} B_nDA_n(A_n)^{-1}(D')^{-1}(B_n)^{-1} \xrightarrow[n\to\infty]{} \mathbb{I}, \tag{B3}$$

or, equivalently,

$$B_nD(D')^{-1}(B_n)^{-1} \xrightarrow[n\to\infty]{} c\mathbb{I}. \tag{B4}$$

Let us now transpose Eq. (B4) into its matrix representation in the basis $\{|b_i^{(n)}\rangle\}$ of the eigenvectors of B_n corresponding to the increasingly ordered eigenvalues $b_i^{(n)}$:

$$\langle b_i^{(n)}|B_nD(D')^{-1}(B_n)^{-1}|b_j^{(n)}\rangle = b_i^{(n)}(b_j^{(n)})^{-1}\langle b_i^{(n)}|D(D')^{-1}|b_j^{(n)}\rangle \xrightarrow[n\to\infty]{} c\delta_{ij}. \tag{B5}$$

The products $b_i^{(n)}(b_j^{(n)})^{-1}$ define a set of coefficients which can be represented in the following matrix form:

$$\begin{pmatrix} b_1^{(n)} \\ \vdots \\ b_d^{(n)} \end{pmatrix} \cdot \left((b_1^{(n)})^{-1} \quad \dots \quad (b_n^{(n)})^{-1} \right) = \begin{pmatrix} 1 & b_1^{(n)}(b_2^{(n)})^{-1} & b_1^{(n)}(b_3^{(n)})^{-1} & \dots \\ b_2^{(n)}(b_1^{(n)})^{-1} & 1 & b_2^{(n)}(b_3^{(n)})^{-1} & \dots \\ b_3^{(n)}(b_1^{(n)})^{-1} & b_3^{(n)}(b_2^{(n)})^{-1} & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix}, \tag{B6}$$

in which we can easily see that each element $b_i^{(n)}(b_j^{(n)})^{-1} \geqslant 1$ in the lower triangle. Therefore, in order to have Eq. (B5) satisfied, $D(D')^{-1}$ must be upper triangular in the basis of the eigenvectors of B_n. However, since D and D' commute, its product is Hermitian, so it must be such in the basis $|b_i\rangle$ being the limit of the eigenbases $|b_i^{(n)}\rangle$ (again, in the sense of a compactness argument). This means eventually that it must be diagonal in that limit, which leads to the conclusion that $\langle b_i|D(D')^{-1}|b_j\rangle = c\delta_{ij}$. In other words, $D \propto D'$, so effectively, $|\Psi'\rangle$ is proportional to $|\Psi\rangle$ and in this sense is removed form the eigenrepresentation of σ_{mc}.

Consider now the case (ii) from the alternative options (i–iii). Here we have, by assumption,

$$\frac{\mathrm{Tr}[\Lambda^{(n)}(|\Psi'\rangle\langle\Psi'|)]}{\mathrm{Tr}[\Lambda^{(n)}(|\Psi\rangle\langle\Psi|)]} = \frac{\|B_nD'A_n\|_{\text{H-S}}}{\|B_nDA_n\|_{\text{H-S}}} \xrightarrow[n\to\infty]{} 0. \tag{B7}$$

Therefore, Eq. (B2) becomes

$$X'_n = \frac{B_nD'A_n}{\|B_nD'A_n\|_{\text{H-S}}} \xrightarrow[n\to\infty]{} \mathbf{0}.$$

Let us consider this time the product $X'_n(X_n)^{-1}$:

$$X'_n(X_n)^{-1} = \frac{\|B_nDA_n\|_{\text{H-S}}}{\|B_nD'A_n\|_{\text{H-S}}} B_nD'A_n(A_n)^{-1}D^{-1}(B_n)^{-1} \xrightarrow[n\to\infty]{} \mathbf{0}, \tag{B8}$$

which, applying the same above reasoning, becomes

$$\frac{\|B_nDA_n\|_{\text{H-S}}}{\|B_nD'A_n\|_{\text{H-S}}} \cdot b_i^{(n)}(b_j^{(n)})^{-1} \cdot \langle b_i^{(n)}|D'D^{-1}|b_j^{(n)}\rangle \xrightarrow[n\to\infty]{} 0.$$

Assumption (B7) implies that the fraction of norms diverges in the above formula. Thus, again by the property of the matrix (B6), we have that the matrix $D(D')^{-1}$ must be *strictly* upper triangular (i.e., with vanishing diagonal) in the limit basis, which, by its hermiticity, implies that $\langle b_i|D'D^{-1}|b_j\rangle = 0$. Thus, since D is invertible, $D' = \mathbf{0}$, which means that $|\Psi'\rangle$ compatible with (ii) cannot exist. The last case (iii) can be immediately resolved by permuting the roles of $|\Psi\rangle$ and $|\Psi'\rangle$ and concluding that $|\Psi\rangle$ cannot vanish by assumption, which leads to the expected contradiction.

APPENDIX C: (THEOREM 2)

Proof. Since the sum in Eq. (31) is separable, it must tend to zero when applying filtering, namely,

$$\frac{1}{\mathrm{Tr}[\Lambda^{(n)}(\rho)]} \sum_{i\neq j} p_{ij}\Lambda^{(n)}(|ij\rangle\langle ij|) \xrightarrow[n\to\infty]{} 0.$$

Moreover, due to Theorem 1, we also have that quasidistillability implies that each eigenvector must vanish in the limit when applying filtering:

$$\frac{\Lambda^{(n)}(|ij\rangle\langle ij|)}{\mathrm{Tr}[\Lambda^{(n)}(|ij\rangle\langle ij|)]} \xrightarrow[n\to\infty]{} 0 \quad i \neq j. \tag{C1}$$

[2] If there are some oscillations in those sequences, then we can always find subsequences of filters that realize quasidistillation, satisfying the classification (i-iii).

Let us then apply $\Lambda^{(n)}$ to a generic state $|\Psi\rangle = \sum_{i,j=1} \alpha_{ij}|ij\rangle$. It is easy to see that the state

$$\sum_{i,j} \alpha_{ij}\left(\frac{A_n|i\rangle}{\mathrm{Tr}[\Lambda^{(n)}(|\tilde{\Psi}\rangle\langle\tilde{\Psi}|)^{1/4}]}\right)\otimes\left(\frac{B_n|j\rangle}{\mathrm{Tr}[\Lambda^{(n)}(|\tilde{\Psi}\rangle\langle\tilde{\Psi}|)^{1/4}]}\right)$$
$$= \sum_{i,j} \alpha_{ij}\left|a_n^{(i)}, b_n^{(j)}\right\rangle$$

is normalized and that Eq. (C1) is then equivalent to $\|a_n^{(i)}\| \cdot \|b_n^{(j)}\| \to 0 \;\; \forall i \neq j$. Thus

$$\prod_{i\neq j} \|a_n^{(i)}\| \cdot \|b_n^{(j)}\| \xrightarrow[n\to\infty]{} 0.$$

Therefore, if there is a loop in the set of indexes (i.e., $p_{ij}p_{jk}\dots p_{li} \neq 0$) we have

$$\|a_n^{(i)}\| \cdot \|b_n^{(j)}\| \cdot \|a_n^{(j)}\| \cdot \|b_n^{(k)}\| \cdot \dots \cdot \|a_n^{(l)}\| \cdot \|b_n^{(i)}\| \xrightarrow[n\to\infty]{} 0,$$

which after suitable reordering gives

$$\left(\|a_n^{(i)}\| \cdot \|b_n^{(i)}\|\right)\left(\|a_n^{(j)}\| \cdot \|b_n^{(j)}\|\right)\cdots \left(\|a_n^{(l)}\| \cdot \|b_n^{(l)}\|\right) \xrightarrow[n\to\infty]{} 0. \tag{C2}$$

Equation (C2) implies that at least one among $(\|a_n^{(i)}\| \cdot \|b_n^{(i)}\|)$ would vanish in the limit and thus the maximally correlated part σ_{MC} cannot have maximal Schmidt rank. This argument proves that if ρ has the form (31) and it is quasidistillable, then necessarily, $p_{ij}p_{jk}\dots p_{li} = 0$. In what follows, we show that this condition is also sufficient for quasidistillability.

Let A_n and B_n be operators with the following representation in the computational basis:

$$A_n = \begin{pmatrix} n^{\alpha_1-1} & 0 & 0 & \cdots & 0 \\ 0 & n^{\alpha_2-1} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & n^{\alpha_d-1} \end{pmatrix},$$

$$B_n = \begin{pmatrix} n^{-\alpha_1} & 0 & 0 & \cdots & 0 \\ 0 & n^{-\alpha_2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & n^{-\alpha_d} \end{pmatrix},$$

where $\alpha_i \in [0, \frac{1}{2}]$ is a set of real numbers. Note that the structure of A_n and B_n is in accordance with the result proved in Lemma 1, since, in particular, all eigenvalues vanish in the limit. One can easily see that the filtering map $\Lambda^{(n)}$ defined by these two operators yields the following:

$$\Lambda^{(n)}[|i\rangle\langle i| \otimes |i\rangle\langle i|] = \frac{1}{n^2}|i\rangle\langle i| \otimes |i\rangle\langle i|,$$
$$\Lambda^{(n)}[|i\rangle\langle j| \otimes |i\rangle\langle j|] = \frac{1}{n^2}|i\rangle\langle j| \otimes |i\rangle\langle j|,$$
$$\Lambda^{(n)}[|i\rangle\langle i| \otimes |j\rangle\langle j|] = \frac{1}{n^2}(|i\rangle\langle i| \otimes |j\rangle\langle j|)n^{2(\alpha_i-\alpha_j)}.$$

In other words, A_n and B_n are constructed in such a way to distill a state ρ of the form (7) iff all the inequalities $\alpha_i < \alpha_j$ hold for every $i \neq j$. We can also see that if there are no loops of indexes the inequalities $\alpha_i < \alpha_j \; \forall i \neq j$ amount to a certain number p of order relations between at least $p+1$ real numbers. Such a set is always compatible, and, therefore, it is always possible to choose $\{\alpha_i\}$ in such a way that A_n and B_n filter any ρ of the form (31).

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