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The Gap Between Linear Elasticity and the Variational Limit of Finite Elasticity in Pure Traction Problems

Francesco Maddalena, Danilo Percivale
& Franco Tomarelli

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Abstract

A limit elastic energy for the pure traction problem is derived from re-scaled nonlinear energies of a hyperelastic material body subject to an equilibrated force field. We prove that the strains of minimizing sequences associated to re-scaled nonlinear energies weakly converge, up to subsequences, to the strains of minimizers of a limit energy, provided an additional compatibility condition is fulfilled by the force field. The limit energy is different from the classical energy of linear elasticity; nevertheless, the compatibility condition entails the coincidence of related minima and minimizers. A strong violation of this condition provides a limit energy which is unbounded from below, while a mild violation may produce unboundedness of strains and a limit energy which has infinitely many extra minimizers which are not minimizers of standard linear elastic energy. A consequence of this analysis is that a rigorous validation of linear elasticity fails for compressive force fields that infringe up on such a compatibility condition.

Contents

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1. Introduction

The Linear Theory of Elasticity ([19]) has a prominent role among Mathematical-Physics theories for its clarity, rigorous mathematical status and persistence. It was a great achievement of the previous centuries, and it inspired many other theories of Continuum Mechanics and led to the formulation of a more general theory named *Nonlinear Elasticity* ([22,32]), also known as *Finite Elasticity*, which underlines that no smallness assumptions are required.

There was always agreement amongst scholars that the relation between the linear and the nonlinear theory amounts to the linearization of strain measure under the assumption of small displacement gradients; this is the precondition advocated in almost all of the texts on elasticity. Nevertheless, only at the beginning of present century, when appropriate tools of mathematical analysis were suitably tuned, did the problem of a rigorous deduction of any particular theory based on some *approximation hypotheses* from a more general *exact* theory become a scientific issue related to the general problem of *validation* of a theory, as explained in [31].

In this conceptual framework, G. Dal Maso, M. Negri and D. Percivale in [13] proved that problems ruled by linear elastic energies can be rigorously deduced from problems ruled by non-linear energies in the case of *Dirichlet and mixed boundary conditions*; they did this by exploiting De Giorgi Γ-convergence theory ([12,14]). This result clarified the mathematical consistency of the linear boundary value problems, under displacements and forces prescribed on the boundary of a three-dimensional material body, via a rigorous deduction from the nonlinear elasticity theory. We mention several papers facing issues in elasticity which are connected with the context of our paper: [1–9,17,21,23–25,27–30].

The present paper tackles the same general question which was studied in [13], but here we deal with the *pure traction problem*, i.e. the case where the elastic body is subject to a system of equilibrated forces and no Dirichlet condition is imposed on the boundary.

We consider a bounded open set $\Omega \subset \mathbb{R}^N,\ N = 2, 3$ as the reference configuration of a hyperelastic material body, hence the stored energy due to a deformation $\mathbf{y}$ can be expressed as a functional of the deformation gradient $\nabla\mathbf{y}$ as follows:

$$\int_\Omega \mathcal{W}(\mathbf{x}, \nabla\mathbf{y})\, d\mathbf{x},$$

where $\mathcal{W} : \Omega \times \mathcal{M}^{N\times N} \to [0, +\infty]$ is a frame indifferent function, $\mathcal{M}^{N\times N}$ is the set of real $N \times N$ matrices and $\mathcal{W}(\mathbf{x}, \mathbf{F}) < +\infty$ if and only if $\det \mathbf{F} > 0$.

Then due to frame indifference there exists a function $\mathcal{V}$ such that

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{V}(\mathbf{x}, \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}))\,, \qquad \forall\, \mathbf{F} \in \mathcal{M}^{N\times N}, \ \text{ a.e. } \mathbf{x} \in \Omega.$$

We set $\mathbf{F} = \mathbf{I} + h\mathbf{B}$, where $h > 0$ is an adimensional small parameter and

$$\mathcal{V}_h(\mathbf{x}, \mathbf{B}) := h^{-2}\mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}).$$

We assume that the reference configuration has zero energy and is stress free, i.e.

$$\mathcal{W}(\mathbf{x}, \mathbf{I}) = 0, \text{ hence } \quad D\mathcal{W}(\mathbf{x}, \mathbf{I}) = \mathbf{0} \quad \text{for a.e. } \mathbf{x} \in \Omega\,,$$

and that $\mathcal{W}$ is regular enough in the second variable. Then Taylor's formula entails

$$\mathcal{V}_h(\mathbf{x}, \mathbf{B}) = \mathcal{V}_0(\mathbf{x}, \mathrm{sym}\mathbf{B}) + o(1) \qquad \text{as } h \to 0_+,$$

where $\mathrm{sym}\mathbf{B} := \frac{1}{2}(\mathbf{B}^T + \mathbf{B})$ and

$$\mathcal{V}_0(\mathbf{x}, \mathrm{sym}\mathbf{B}) := \frac{1}{2}\mathrm{sym}\mathbf{B}\, D^2\mathcal{V}(\mathbf{x}, \mathbf{0})\, \mathrm{sym}\mathbf{B}.$$

If the deformation $\mathbf{y}$ is close to the identity up to a small displacement, say $\mathbf{y}(\mathbf{x}) = \mathbf{x} + h\mathbf{v}(\mathbf{x})$ with bounded $\nabla\mathbf{v}$, then, by setting $\mathbb{E}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v})$, one plainly obtains that

$$\lim_{h\to 0} \int_\Omega \mathcal{V}_h(\mathbf{x}, \nabla\mathbf{v})\, d\mathbf{x} = \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}))\, d\mathbf{x}\,. \tag{1.1}$$

Historically, relationship (1.1) was considered as the main justification of linearized elasticity, but such pointwise convergence does not even entail that minimizers fulfilling a given fixed Dirichlet boundary condition actually converge to the minimizers of the corresponding limit boundary value problem; this phenomenon is made explicit by the Example 3.5 in [13] which exhibits a lack of compactness when $\mathcal{V}$ has several minima.

To set the Dirichlet problem in a variational perspective, referring to a prescribed vector field $\mathbf{v}_0 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ as the boundary condition on a given closed subset Σ of $\partial\Omega$ with $\mathcal{H}^{N-1}(\Sigma) > 0$ and to a given load $\mathbf{g} \in L^2(\Omega, \mathbb{R}^N)$, one has to study the asymptotic behavior of the sequence of functionals $\mathcal{I}_h$, which is defined as

$$\mathcal{I}_h(\mathbf{v}) = \begin{cases} \displaystyle\int_\Omega \mathcal{V}_h(\mathbf{x}, \nabla\mathbf{v})\, d\mathbf{x} - \int_\Omega \mathbf{g}\cdot\mathbf{v}\, d\mathbf{x} & \text{if } \mathbf{v} \in H^1_{\mathbf{v}_0,\Sigma} \\ +\infty & \text{else in } H^1(\Omega, \mathbb{R}^N), \end{cases}$$

where $H^1_{\mathbf{v}_0,\Sigma}$ denotes the closure in $H^1(\Omega, \mathbb{R}^N)$ of the space of displacements $\mathbf{v} \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ such that $\mathbf{v} = \mathbf{v}_0$ on Σ; it was proved in [13] that (under natural growth conditions and suitable regularity hypotheses on $\mathcal{W}$) every sequence $\mathbf{v}_h$ fulfilling

$$\mathcal{I}_h(\mathbf{v}_h) = \inf \mathcal{I}_h + o(1)$$

has a subsequence converging weakly in $H^1(\Omega, \mathbb{R}^N)$ to the (unique) minimizer $\mathbf{v}_*$ of the functional $\mathcal{I}$ representing the total energy in linear elasticity, that is

$$\mathcal{I}(\mathbf{v}) = \begin{cases} \displaystyle\int_\Omega \mathcal{V}_0\left(\mathbf{x}, \mathbb{E}(\mathbf{v})\right)\, d\mathbf{x} - \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}\, d\mathcal{H}^{n-1}(\mathbf{x}) & \text{if } \mathbf{v} \in H^1_{\mathbf{v}_0,\Sigma} \\ +\infty & \text{else in } H^1(\Omega; \mathbb{R}^N)\,, \end{cases}$$

and that the re-scaled energies converge, namely

$$\lim_{h\to 0} \mathcal{I}_h(\mathbf{v}_h) = \mathcal{I}(\mathbf{v}_*) = \int_\Omega \mathcal{V}_0\left(\mathbf{x}, \mathbb{E}(\mathbf{v}_*)\right)\, d\mathbf{x} - \int_\Omega \mathbf{g}\cdot\mathbf{v}_*\, d\mathbf{x}\,.$$

Such a result is a complete variational justification of linearized elasticity, at least as far as Dirichlet and mixed boundary value problems are concerned. Thus it is natural to ask whether a similar result holds true also for pure traction problems whose variational formulation is described in the sequel.

In the present paper we focus our analysis on Neumann boundary conditions, say the pure traction problem in elasticity. More precisely, we assume that $\mathbf{f} \in L^2(\partial\Omega;\mathbb{R}^N)$, $\mathbf{g} \in L^2(\Omega;\mathbb{R}^N)$ are, respectively, the prescribed boundary and body force fields such that the whole system of forces is equilibrated, namely, the condition of equilibrated load

$$\mathcal{L}(\mathbf{z}) = 0 \qquad \forall \mathbf{z} : \mathbb{E}(\mathbf{z}) \equiv \mathbf{0} \tag{1.2}$$

(which is a standard necessary condition for pure traction in linear elasticity), is assumed with

$$\mathcal{L}(\mathbf{v}) := \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{v}\, d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\,.$$

We consider the sequence of energy functionals

$$\mathcal{F}_h(\mathbf{v}) = \int_{\Omega} \mathcal{V}_h(\mathbf{x},\nabla\mathbf{v})\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v})\,, \tag{1.3}$$

and we inquire whether the asymptotic relationship $\mathcal{F}_h(\mathbf{v}_h) = \inf\mathcal{F}_h + o(1)$ as $h \to 0_+$ implies, up to subsequences, some kind of weak convergence of $\mathbf{v}_h$ to a minimizer $\mathbf{v}_0$ of a suitable limit functional in $H^1(\Omega;\mathbb{R}^N)$.

First we emphasize that in the case of Neumann condition on the whole boundary things are not so plain. Indeed even by choosing Ω Lipschitz and assuming the simplest dependance of stored energy density $\mathcal{W}$ on the deformation gradient $\mathbf{F}$, say (see [10])

$$\mathcal{W}(\mathbf{x},\mathbf{F}) = \begin{cases} |\mathbf{F}^T\mathbf{F}-\mathbf{I}|^2 & \text{if } \det\mathbf{F} > 0 \\ \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

if $\mathbf{g} \equiv 0$, $\mathbf{f} = f\mathbf{n}$, $f < 0$ and $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$ (so that the global condition (1.2) holds true), then by the same techniques as used in [13] one can exhibit the Gamma limit of $\mathcal{F}_h$ with respect to weak H^1 topology:

$$\Gamma(w\,H^1)\lim_{h\to 0}\ \mathcal{F}_h(\mathbf{v}) = \mathcal{E}(\mathbf{v})\,, \tag{1.5}$$

where

$$\mathcal{E}(\mathbf{v}) = 4\int_{\Omega} |\mathbb{E}(\mathbf{v})|^2\,\mathrm{d}\mathbf{x} - f\int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, d\mathcal{H}^{N-1}(\mathbf{x})\ , \tag{1.6}$$

i.e. the classical linear elasticity formulation which achieves a finite minimum over $H^1(\Omega,\mathbb{R}^N)$ since the condition of equilibrated loads is fulfilled. Nevertheless, with exactly the same choices, there is a sequence $\mathbf{w}_h$ in $H^1(\Omega,\mathbb{R}^N)$ such that $\mathcal{F}_h(\mathbf{w}_h) \to -\infty$ as $h \to 0^+$ (see Remark 2.7): although minimizers of $\mathcal{E}$ over $H^1(\Omega;\mathbb{R}^N)$ exist, functionals $\mathcal{F}_h$ are not uniformly bounded from below and these facts seem to suggest that, in presence of compressive forces acting on the boundary, minimizing sequences of $\mathcal{F}_h$ do not converge to minimizers of $\mathcal{E}$.

Moreover, it is worth noting that if $\mathcal{W}$ fulfils (1.4) and $\mathbf{g} \equiv \mathbf{f} \equiv \mathbf{0}$, and hence $\inf \mathcal{F}_h = 0$ for every $h > 0$, then by choosing a fixed nontrivial $N \times N$ skew-symmetric matrix $\mathbf{W}$, a real number $0 < 2\alpha < 1$ and setting

$$\mathbf{z}_h := h^{-\alpha}\,\mathbf{W}\,\mathbf{x}\,, \tag{1.7}$$

we get $\mathcal{F}_h(\mathbf{z}_h) = \inf \mathcal{F}_h + o(1)$; nevertheless $\mathbf{z}_h$ has no subsequence weakly converging in $H^1(\Omega; \mathbb{R}^N)$, see Remark 2.4.

Therefore here, in contrast to [13], we cannot expect weak $H^1(\Omega; \mathbb{R}^N)$ compactness of minimizing sequences, not even in the simplest case of null external forces. Although this fact is common to pure traction problems in linear elasticity, we emphasize that in general nonlinear elasticity setting this difficulty cannot be easily circumvented by plain translations since $\mathcal{F}_h(\mathbf{v}_h) \neq \mathcal{F}_h(\mathbf{v}_h - \mathbb{P}\mathbf{v}_h)$, with $\mathbb{P}$ projection on infinitesimal rigid displacements.

We deal with this issue in the paper [26], showing nonetheless that at least for some special $\mathcal{W}$, if $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$ then up to subsequences $\mathcal{F}_h(\mathbf{v}_h - \mathbb{P}\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$.

In order to achieve some kind of precompactness for the sequences $\mathbf{v}_h$ fulfilling $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$, we work with a very weak notion of convergence: the weak $L^2(\Omega; \mathbb{R}^N)$ convergence of linear strains. Therefore our approach requires the analysis of variational limit of $\mathcal{F}_h$ with respect to this convergence. Since weak L^2 convergence of linear strains does not imply an analogous convergence of the skew symmetric part of the displacement gradients, it can be expected that the Γ limit functional is different from the classical linearized elasticity functional which is the pointwise limit of $\mathcal{F}_h$ (for a reformulation of classical linearized elasticity with linear strain tensor as the "primary" unknown instead of displacement, see [11]).

Indeed under some natural assumptions on $\mathcal{W}$, a careful application of the Rigidity Lemma of [18] shows that if $\mathbb{E}(\mathbf{v}_h)$ are bounded in L^2 then, up to subsequences, $\sqrt{h}\nabla \mathbf{v}_h$ converges strongly in L^2 to a constant skew symmetric matrix and the variational limit of the sequence $\mathcal{F}_h$, with respect to the w-L^2 convergence of linear strains, turns out to be

$$\mathcal{F}(\mathbf{v}) \;:=\; \min_{\mathbf{W}} \int_\Omega \mathcal{V}_0\left(\mathbf{x},\, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2\right) \mathrm{d}\mathbf{x} \;-\; \mathcal{L}(\mathbf{v})\,, \tag{1.8}$$

where the minimum is evaluated over skew symmetric $N \times N$ matrices $\mathbf{W}$ and

$$\mathcal{V}_0(\mathbf{x}, \mathbf{B}) \;:=\; \frac{1}{2}\,\mathbf{B}^T D^2\mathcal{V}(\mathbf{x}, \mathbf{0})\,\mathbf{B} \qquad \forall\, \mathbf{B} \in \mathcal{M}^{N\times N}_{sym}\,. \tag{1.9}$$

We emphasize that the functional $\mathcal{F}$ in (1.8) is different from the functional $\mathcal{E}$ of linearized elasticity defined as

$$\mathcal{E}(\mathbf{v}) := \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}))\,\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v}),$$

since if $\mathbf{v}(\mathbf{x}) = \frac{1}{2}\mathbf{W}^2\mathbf{x}$ with $\mathbf{W} \neq \mathbf{0}$ skew symmetric matrix, then $\mathcal{F}(\mathbf{v}) = -\mathcal{L}(\mathbf{v}) < \mathcal{E}(\mathbf{v})$.

Nevertheless if $N = 2$ then (see Remark 2.6)

$$\mathcal{F}(\mathbf{v}) \;=\; \mathcal{E}(\mathbf{v}) - \frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x},\mathbf{I})\mathrm{d}\mathbf{x}\right)^{-1}\left[\left(\int_\Omega D\mathcal{V}_0(\mathbf{x},\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,\mathrm{d}\mathbf{x}\right)^{-}\right]^2,$$

hence $\mathcal{F}(\mathbf{v}) = \mathcal{E}(\mathbf{v})$ if

$$N = 2\,, \qquad \text{and} \qquad \int_\Omega D\mathcal{V}_0(x,\mathbf{I})\cdot\mathbb{E}(\mathbf{v})\,dx \geq 0\,.$$

In particular, if $N = 2$ and $\mathcal{W}$ is the *Saint Venant–Kirchhoff energy density* (1.4), then the previous inequality reduces to

$$\int_\Omega \operatorname{div}\mathbf{v}\,\mathrm{d}\mathbf{x} \geq 0,$$

which means, roughly speaking, that the area of Ω is less than the area of the related deformed configuration $\mathbf{y}(\Omega)$, where $\mathbf{y}(\mathbf{x}) = \mathbf{x} + h\mathbf{v}(\mathbf{x})$ and $h > 0$.

The main results of present paper are stated in Theorems 2.2 and 4.1; they show that under a suitable compatibility condition on the forces (subsequent formula (1.10)) the pure traction problem in linear elasticity is deduced via Γ-convergence from pure traction problem in nonlinear elasticity, referring to weak L^2 convergence of the linear strains.

Precisely Theorem 2.2 states that, if the loads $\mathbf{f},\ \mathbf{g}$ fulfil (1.2) together with the next compatibility condition

$$\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{W}^2\mathbf{x}\,d\mathcal{H}^{N-1} + \int_\Omega \mathbf{g}\cdot\mathbf{W}^2\mathbf{x}\,\mathrm{d}\mathbf{x} \;<\; 0 \qquad \forall\,\text{skew symmetric matrix } \mathbf{W}\neq\mathbf{0}\,, \tag{1.10}$$

then every sequence $\mathbf{v}_h$ with $\mathcal{F}(\mathbf{v}_h) = \inf\mathcal{F}_h + o(1)$ has a subsequence such that the corresponding linear strains converge weakly in L^2 to the linear strain of a minimizer of $\mathcal{F}$, together with convergence (without relabeling) of energies $\mathcal{F}_h(\mathbf{v}_h)$ to $\min\mathcal{F}$. Under the same assumptions Theorem 4.1 states that minimizers of $\mathcal{F}$ coincide with the ones of linearized elasticity functional $\mathcal{E}$, thus providing a full justification of pure traction problems in linear elasticity at least if (1.10) is satisfied. In particular, as is shown in Remark 2.8, this is true when $\mathbf{g} \equiv 0,\ \mathbf{f} = f\mathbf{n}$ with $f > 0$ and $\mathbf{n}$ is the outer unit normal vector to $\partial\Omega$, that is when we are in presence of tension-like surface forces.

Regarding the physical interpretation and motivations of compatibility condition (1.10), we refer to subsequent Remarks 2.7 and 2.8.

Moreover, if there exists an $N\times N$ skew symmetric matrix such that the strict inequality is reversed in (1.10), then functional $\mathcal{F}$ is unbounded from below; see Remark 4.5 and Example 4.6. On the other hand, if inequality in (1.10) is satisfied in a weak sense by every skew symmetric matrix, then $\operatorname{argmin}\mathcal{F}$ contains $\operatorname{argmin}\mathcal{E}$, $\min\mathcal{F} = \min\mathcal{E}$ but $\mathcal{F}$ may have infinitely many minimizing critical points which are not minimizers of $\mathcal{E}$ (see Proposition 4.3).

Summarizing, only two cases are allowed: either $\min\mathcal{F} = \min\mathcal{E}$ or $\inf\mathcal{F} = -\infty$; actually the second case arises in the presence of compressive surface load.

By oversimplifying we could say that $\mathcal{F}$ somehow preserves memory of instabilities which are typical of finite elasticity, while they disappear in the linearized model described by $\mathcal{E}$.

In light of Theorem 2.2 and of the remarks and examples of Section 4, it seems reasonable that, as far as pure traction problems are concerned, the range of validity of linear elasticity should be restricted to a certain class of external loads, explicitly those verifying (1.10), a remarkable example in such class is a uniform normal tension load at the boundary as in Remark (2.8); while in the other cases equilibria of a linearly elastic body could be better described through critical points of $\mathcal{F}$, whose existence in general seems to be an interesting and open problem.

Concerning the structure of the new functional, we emphasize that actually $\mathcal{F}$ is different from the classical linear elasticity energy functional $\mathcal{E}$, though there are many relations between their minimizers (see Theorem 4.1). Further and more detailed information about functional $\mathcal{F}$ (a suitable property of weak lower semicontinuity, lack of subadditivity, convexity in 2D, nonconvexity in 3D) are described and proved in the paper [26].

2. Notation and Main Result

Assume that the reference configuration of an elastic body is a

$$\text{bounded, connected open set } \Omega \subset \mathbb{R}^N \text{ with Lipschitz boundary, } N = 2, 3. \tag{2.1}$$

The generic point $\mathbf{x} \in \Omega$ has components x_j referring to the standard basis vectors $\mathbf{e}_j$ in $\mathbb{R}^N$; $\mathcal{L}^N$ and $\mathcal{B}^N$ denote respectively the σ-algebras of Lebesgue measurable and Borel measurable subsets of $\mathbb{R}^N$. For every $\alpha \in \mathbb{R}$ we set $\alpha^+ = \alpha \vee 0$, $\alpha^- = -\alpha \vee 0$.

The notation for vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^N$ and $N \times N$ real matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$ are as follows: $\mathbf{a} \cdot \mathbf{b} = \sum_j \mathbf{a}_j \mathbf{b}_j$; $\mathbf{A} \cdot \mathbf{B} = \sum_{i,j} \mathbf{A}_{i,j} \mathbf{B}_{i,j}$; $[\mathbf{AB}]_{i,j} = \sum_k \mathbf{A}_{i,k} \mathbf{B}_{k,j}$; $|\mathbf{F}|^2 = \mathrm{Tr}(\mathbf{F}^T \mathbf{F}) = \sum_{i,j} F_{i,j}^2$ denotes the squared Euclidean norm of $\mathbf{F}$ in the space $\mathcal{M}^{N \times N}$ of $N \times N$ real matrices; $\mathbf{I} \in \mathcal{M}^{N \times N}$ denotes the identity matrix, $SO(N)$ denotes the group of rotation matrices, $\mathcal{M}^{N\times N}_{sym}$ and $\mathcal{M}^{N\times N}_{skew}$ denote respectively the sets of symmetric and skew-symmetric matrices. For every $\mathbf{B} \in \mathcal{M}^{N\times N}$ we define $\mathrm{sym}\, \mathbf{B} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$ and $\mathrm{skew}\, \mathbf{B} := \frac{1}{2}(\mathbf{B} - \mathbf{B}^T)$.

It is well known that matrix exponential maps $\mathcal{M}^{N\times N}_{skew}$ to $SO(N)$ and is surjective on $SO(N)$ (see [20]). Therefore for every $\mathbf{R} \in SO(N)$ there exist $\vartheta \in \mathbb{R}$ and $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$, $|\mathbf{W}|^2 = 2$ such that $\exp(\vartheta\, \mathbf{W}) = \mathbf{R}$. By taking into account that $\mathbf{W}^3 = -\mathbf{W}$ if $N = 2, 3$, the Taylor's series expansion of $\vartheta \to \exp(\vartheta\, \mathbf{W}) = \sum_{k=0}^{\infty} \vartheta^k \mathbf{W}^k / k!$ yields the *Euler-Rodrigues formula*

$$\exp(\vartheta\, \mathbf{W}) \;=\; \mathbf{R} \;=\; \mathbf{I} \,+\, \sin\vartheta\, \mathbf{W} \,+\, (1 - \cos\vartheta)\, \mathbf{W}^2 \qquad N = 2, 3\,. \tag{2.2}$$

We recall an elementary issue which proves useful in our analysis:

$$\text{if } \mathbf{W} \in \mathcal{M}^{N\times N}_{skew}\,, \;\; |\mathbf{W}|^2 = 2\,, \;\; N = 2, 3\,, \quad \text{then} \;\; |\mathbf{W}^2|^2 = 2\,, \tag{2.3}$$

and we set

$$\mathbb{K} := \{\tau(\mathbf{R} - \mathbf{I}) : \tau \geq 0,\; \mathbf{R} \in SO(N)\}\,. \tag{2.4}$$

For every $\mathcal{U} : \Omega \times \mathcal{M}^{N\times N} \to \mathbb{R}$, with $\mathcal{U}(\mathbf{x}, \cdot) \in C^2$ a.e. $\mathbf{x} \in \Omega$, we denote by $D\mathcal{U}(\mathbf{x}, \cdot)$ and $D^2\mathcal{U}(\mathbf{x}, \cdot)$ respectively the gradient and the hessian of $\mathcal{U}$ with respect to the second variable.

For every displacements field $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$, $\mathbb{E}(\mathbf{v}) := \operatorname{sym}\nabla\mathbf{v}$ denotes the infinitesimal strain tensor field, $\mathcal{R} := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N) : \mathbb{E}(\mathbf{v}) = \mathbf{0}\}$ denotes the space spanned by the set of the infinitesimal rigid displacements and $\mathbb{P}\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto $\mathcal{R}$.

We set $⨍_\Omega \mathbf{v}d\mathbf{x} = |\Omega|^{-1} \int_\Omega \mathbf{v}d\mathbf{x}$.

We consider a body made of a hyperelastic material, say there exists a $\mathcal{L}^N{\times}\mathcal{B}^{N^2}$ measurable $\mathcal{W} : \Omega \times \mathcal{M}^{N\times N} \to [0, +\infty]$ such that, for a.e. $\mathbf{x} \in \Omega$, $\mathcal{W}(\mathbf{x}, \nabla\mathbf{y}(\mathbf{x}))$ represents the stored energy density, when $\mathbf{y}(x)$ is the deformation and $\nabla\mathbf{y}(\mathbf{x})$ is the deformation gradient.

Moreover we assume that, for a.e. $\mathbf{x} \in \Omega$,

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = +\infty \quad \text{if } \det\mathbf{F} \le 0 \quad \text{(orientation preserving condition)}, \tag{2.5}$$

$$\mathcal{W}(\mathbf{x}, \mathbf{RF}) = \mathcal{W}(\mathbf{x}, \mathbf{F}) \quad \forall\,\mathbf{R}\in SO(N) \quad \forall\,\mathbf{F} \in \mathcal{M}^{N\times N} \quad \text{(frame indifference)}, \tag{2.6}$$

$$\exists \text{ a neighborhood } \mathcal{A} \text{ of } SO(N) \text{ s.t. } \quad \mathcal{W}(\mathbf{x}, \cdot) \in C^2(\mathcal{A}), \tag{2.7}$$

$$\exists\, C > 0 \text{ independent of } \mathbf{x} :\ \mathcal{W}(\mathbf{x}, \mathbf{F}) \ge C|\mathbf{F}^T\mathbf{F} - \mathbf{I}|^2 \ \ \forall\,\mathbf{F}\in\mathcal{M}^{N\times N} \text{ (coerciveness)}, \tag{2.8}$$

$$\mathcal{W}(\mathbf{x}, \mathbf{I}) = 0\,, \text{ hence} \qquad D\mathcal{W}(\mathbf{x}, \mathbf{I}) = 0\,, \qquad \text{for a.e. } \mathbf{x} \in \Omega\,, \tag{2.9}$$

that is the reference configuration has zero energy and is stress free, so by (2.6) we also get

$$\mathcal{W}(\mathbf{x}, \mathbf{R}) = 0, \ \ D\mathcal{W}(\mathbf{x}, \mathbf{R}) = 0 \qquad \forall\,\mathbf{R} \in SO(N)\,.$$

By frame indifference there exists a $\mathcal{L}^N \times \mathcal{B}^N$-measurable $\mathcal{V} : \Omega \times \mathcal{M}^{N\times N} \to [0, +\infty]$ such that for every $\mathbf{F} \in \mathcal{M}^{N\times N}$,

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{V}(\mathbf{x}, \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})), \tag{2.10}$$

and by (2.7),

$$\exists \text{ a neighborhood } \mathcal{O} \text{ of } \mathbf{0} \text{ such that } \mathcal{V}(\mathbf{x}, \cdot) \in C^2(\mathcal{O}), \ \text{ a.e. } x \in \Omega\,. \tag{2.11}$$

In addition we assume that there exists $\gamma > 0$ independent of $\mathbf{x}$ such that

$$\left|\mathbf{B}^T\, D^2\mathcal{V}(\mathbf{x}, \mathbf{D})\, \mathbf{B}\right| \le 2\gamma\, |\mathbf{B}|^2 \quad \forall\,\mathbf{D}\in\mathcal{O}, \ \forall\,\mathbf{B}\in\mathcal{M}^{N\times N}. \tag{2.12}$$

By (2.9) and Taylor expansion with the Lagrange reminder we get, for a.e. $\mathbf{x} \in \Omega$ and suitable $t \in (0, 1)$ depending on $\mathbf{x}$ and on $\mathbf{B}$,

$$\mathcal{V}(\mathbf{x}, \mathbf{B}) = \frac{1}{2}\, \mathbf{B}^T D^2\mathcal{V}(\mathbf{x}, t\mathbf{B})\, \mathbf{B}\,. \tag{2.13}$$

Hence, by (2.12),

$$\mathcal{V}(\mathbf{x}, \mathbf{B}) \le \gamma\, |\mathbf{B}|^2 \qquad \forall\,\mathbf{B} \in \mathcal{M}^{N\times N} \cap \mathcal{O}\,. \tag{2.14}$$

According to (2.10) for a.e. $\mathbf{x}\in\Omega,\ h>0$ and every $\mathbf{B}\in\mathcal{M}^{N\times N}$, we set

$$\mathcal{V}_h(\mathbf{x},\mathbf{B}) := \frac{1}{h^2}\,\mathcal{W}(\mathbf{x},\mathbf{I}+h\mathbf{B}) = \frac{1}{h^2}\,\mathcal{V}(\mathbf{x}, h\mathrm{sym}\,\mathbf{B}+\tfrac{1}{2}h^2\mathbf{B}^T\mathbf{B})\,. \tag{2.15}$$

Taylor's formula with (2.9) and (2.15) entails $\mathcal{V}_h(\mathbf{x},\mathbf{B})=\frac{1}{2}(\mathrm{sym}\mathbf{B})D^2\mathcal{V}(\mathbf{x},\mathbf{0})(\mathrm{sym}\,\mathbf{B})+o(1)$, so

$$\mathcal{V}_h(\mathbf{x},\mathbf{B}) \ \to\ \mathcal{V}_0(\mathbf{x},\mathrm{sym}\,\mathbf{B}) \text{ as } h\to 0_+\,, \tag{2.16}$$

where the pointwise limit of integrands is the quadratic form $\mathcal{V}_0$ defined by

$$\mathcal{V}_0(\mathbf{x},\mathbf{B}) \ :=\ \frac{1}{2}\,\mathbf{B}^T D^2\mathcal{V}(\mathbf{x},\mathbf{0})\,\mathbf{B}\quad \text{a.e. } \mathbf{x}\in\Omega,\ \mathbf{B}\in\mathcal{M}^{N\times N}\,. \tag{2.17}$$

The symmetric fourth order tensor $D^2\mathcal{V}(\mathbf{x},\mathbf{0})$ in (2.17) plays the role of classical linear elasticity tensor.

By (2.8) we get

$$\mathcal{V}_h(x,\mathbf{B}) \ =\ \frac{1}{h^2}\,\mathcal{W}(x,\mathbf{I}+h\mathbf{B}) \ \geq\ C\,|\,2\,\mathrm{sym}\mathbf{B}\ +\ h\,\mathbf{B}^T\mathbf{B}\,|^2, \tag{2.18}$$

so that (2.17) and (2.18) imply the ellipticity of $\mathcal{V}_0$ as follows:

$$\mathcal{V}_0(\mathbf{x},\ \mathrm{sym}\,\mathbf{B}) \ \geq\ 4\,C\,|\,\mathrm{sym}\,\mathbf{B}|^2 \qquad \text{a.e. } \mathbf{x}\in\Omega,\ \mathbf{B}\in\mathcal{M}^{N\times N}\,. \tag{2.19}$$

Let $\mathbf{f}\in L^2(\partial\Omega;\mathbb{R}^N)$ and $\mathbf{g}\in L^2(\Omega;\mathbb{R}^N)$ be, respectively, the surface and body force field.

For a suitable choice of the adimensional parameter $h>0$, the functional representing the total energy is labeled by $\mathcal{F}_h : H^1(\Omega;\mathbb{R}^N)\to\mathbb{R}\cup\{+\infty\}$ and defined as follows:

$$\mathcal{F}_h(\mathbf{v}) \ :=\ \int_\Omega \mathcal{V}_h(\mathbf{x},\nabla\mathbf{v})\,\mathrm{d}\mathbf{x}\ -\ \mathcal{L}(\mathbf{v})\,, \tag{2.20}$$

where

$$\mathcal{L}(\mathbf{v}) := \int_{\partial\Omega}\mathbf{f}\cdot\mathbf{v}\,d\mathcal{H}^{n-1}\ +\ \int_\Omega \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}. \tag{2.21}$$

In this paper we are interested in the asymptotic behavior as $h\downarrow 0_+$ of functionals $\mathcal{F}_h$ and to this aim we introduce the limit energy functional $\mathcal{F}: H^1(\Omega;\mathbb{R}^N)\to\mathbb{R}$ defined by

$$\mathcal{F}(\mathbf{v}) \ =\ \min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}}\int_\Omega \mathcal{V}_0\left(\mathbf{x},\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2\right)\mathrm{d}\mathbf{x}\ -\ \mathcal{L}(\mathbf{v})\,. \tag{2.22}$$

We emphasize that the minimum in right-hand side of definition (2.22) exists; more precisely, the finite dimensional minimization problem has exactly two solutions which differ only by a sign, since, by (2.19),

$$\lim_{|\mathbf{W}|\to+\infty,\ \mathbf{W}\in\mathcal{M}^{N\times N}_{skew}}\int_\Omega \mathcal{V}_0\left(\mathbf{x},\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2\right)\,\mathrm{d}\mathbf{x}\ =\ +\infty, \tag{2.23}$$

and $\mathcal{V}_0(\mathbf{x},\cdot)$ is strictly convex by (2.17), and (2.19).

All along this paper we assume (2.1) together with the *standard structural conditions* (2.5)–(2.9), (2.12) as it is usually done in scientific literature concerning elasticity theory and we refer to the notations (2.10), (2.15), (2.17), (2.20)–(2.22).

The pair $\mathbf{f},\ \mathbf{g}$ describing the load is said to be *equilibrated* if

$$\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{z}\, d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g}\cdot\mathbf{z}\, \mathrm{d}\mathbf{x} \;=\; 0 \quad \forall\, \mathbf{z}\in\mathcal{R}\,, \tag{2.24}$$

and it is said to be *compatible* if

$$\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{W}^2\,\mathbf{x}\, d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g}\cdot\mathbf{W}^2\,\mathbf{x}\, \mathrm{d}\mathbf{x} \;<\; 0 \qquad \forall\,\mathbf{W}\in\mathcal{M}^{N\times N}_{skew} \text{ s.t. } \mathbf{W}\neq\mathbf{0}. \tag{2.25}$$

Definition 2.1. We say that $\mathbf{v}_j \in H^1(\Omega;\mathbb{R}^N)$ is a *minimizing sequence* of the sequence of functionals $\mathcal{F}_{h_j}$, if $(\mathcal{F}_{h_j}(\mathbf{v}_j) - \inf \mathcal{F}_{h_j}) \to 0$ as $h_j \to 0_+$.

We will show (see Lemma 3.1) that, if compatibility (2.25) holds true, then $\inf_h \inf \mathcal{F}_h > -\infty$, hence for every infinitesimal sequence h_j a minimizing sequences of the sequence of functionals $\mathcal{F}_{h_j}$ do exist.

Now we can state the main result, whose proof is postponed.

Theorem 2.2. *Assume that the standard structural conditions and* (2.24), (2.25) *hold true. Then for every sequence of strictly positive real numbers* $h_j \to 0_+$ *there exist minimizing sequences of the sequence of functionals* $\mathcal{F}_{h_j}$.

Moreover, for every minimizing sequence $\mathbf{v}_j \in H^1(\Omega;\mathbb{R}^N)$ *of* $\mathcal{F}_{h_j}$ *there exists a subsequence, a displacement* $\mathbf{v}_0 \in H^1(\Omega;\mathbb{R}^N)$, *and a constant matrix* $\mathbf{W}_0 \in \mathcal{M}^{N\times N}_{skew}$ *such that, without relabeling,*

$$\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v}_0) \quad \text{weakly in } L^2(\Omega;\mathcal{M}^{N\times N})\,, \tag{2.26}$$

$$\sqrt{h_j}\,\nabla\mathbf{v}_j \to \mathbf{W}_0 \quad \text{strongly in } L^2(\Omega;\mathcal{M}^{N\times N})\,. \tag{2.27}$$

$$\lim_{j\to+\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \;=\; \mathcal{F}(\mathbf{v}_0) \;=\; \min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v})\,, \tag{2.28}$$

$$\mathcal{F}(\mathbf{v}_0) = \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0) - \tfrac{1}{2}\mathbf{W}_0^2)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}_0). \tag{2.29}$$

Remark 2.3. In the sequel (see Corollary 4.2) we prove that structural assumptions together with (2.24) and (2.25) entail even further refinement in previous statement: explicitly $\mathbf{W}_0 = \mathbf{0}$.

Remark 2.4. It is worth underlining that in contrast to the case of the Dirichlet problem faced in [13], here, in the pure traction problem, we cannot expect even weak $H^1(\Omega;\mathbb{R}^N)$ convergence of minimizing sequences.

Indeed choose $\mathbf{f} = \mathbf{g} \equiv 0$ and

$$\mathcal{W}(\mathbf{x},\mathbf{F}) \;=\; \begin{cases} |\mathbf{F}^T\mathbf{F} - \mathbf{I}|^2 \text{ if } \det\mathbf{F} > 0\,, \\ +\infty \;\text{ otherwise}\,, \end{cases} \tag{2.30}$$

$$\mathbf{v}_j := h_j^{-\alpha}\,\mathbf{W}\,\mathbf{x} \qquad \text{with } \mathbf{W}\in\mathcal{M}^{N\times N}_{skew}\,,\; 0 < 2\alpha < 1\,,\; h_j \to 0_+ \tag{2.31}$$

Then $\mathcal{F}_{h_j}(\mathbf{v}_j) = o(1)$ and, due to $\inf \mathcal{F}_{h_j} = 0$, the sequence $\mathbf{v}_j$ is a minimizing sequence which has no subsequence weakly converging in $H^1(\Omega; \mathbb{R}^N)$. It is well known that such a phenomenon takes place for pure traction problems in linear elasticity too, but in nonlinear elasticity this difficulty cannot be easily circumvented in general, since the fact that $\mathbf{v}_j$ is a minimizing sequence does not also entail that $\mathbf{v}_j - \mathbb{P}\mathbf{v}_j$ is minimizing sequence. In [26] we show that for some special integrand $\mathcal{W}$, as in the case of *Saint Venant–Kirchhoff energy density*, if $\mathbf{v}_j$ is a minimizing sequence, then $\mathbf{w}_j := \mathbf{v}_j - \mathbb{P}\mathbf{v}_j$ is a minimizing sequence too, and there exists a (not relabeled) subsequence of functionals $\mathcal{F}_{h_j}$ such that the related *minimizing subsequence* $\mathbf{w}_j$ converges weakly in $H^1(\Omega; \mathbb{R}^N)$ to a minimizer $\mathbf{v}_0$ of $\mathcal{F}$, provided (2.24) and (2.25) hold true.

Remark 2.5. A careful inspection of the proof (see also [13]) shows that Theorem 2.2 remains true if coercivity condition (2.8) on $\mathcal{W}$ is weakened by assuming either

$$\exists\, C > 0 \text{ independent of } \mathbf{x}: \quad \mathcal{W}(\mathbf{x}, \mathbf{F}) \geq C \operatorname{dist}^2\big(\mathbf{F}, SO(N)\big) \quad \forall\, \mathbf{F} \in \mathcal{M}^{N\times N}, \tag{2.32}$$

or by assuming these three conditions on $\mathcal{V}$:

$$\inf_{|\mathbf{B}|\geq\rho}\ \inf_{x\in\Omega} \mathcal{V}(x, \mathbf{B}) > 0 \ \ \forall\, \rho > 0, \tag{2.33}$$

$$\exists\, \alpha > 0,\ \rho > 0 \ \text{ such that } \inf_{x\in\Omega} \mathcal{V}(x, \mathbf{B}) \geq \alpha |\mathbf{B}|^2 \ \ \forall\ |\mathbf{B}| \leq \rho, \tag{2.34}$$

$$\liminf_{|\mathbf{B}|\to+\infty} \frac{1}{|\mathbf{B}|} \inf_{x\in\Omega} \mathcal{V}(x, \mathbf{B}) > 0. \tag{2.35}$$

This can be shown by exploiting Lemma 3.1 in [13]; notice that under such weaker assumption the constant appearing on the right-hand side of inequality (3.4) must be modified accordingly in Lemma 3.1.

It is worth noting that (2.8) implies (2.32), (2.33), (2.34) and (2.35).

In this Section we show some properties of the limit functional $\mathcal{F}$ and several preliminary results to be used in the proof of Theorem 2.2.

Remark 2.6. If $N = 2$, then for every $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$ there is $a \in \mathbb{R}$ such that $\mathbf{W}^2 = -a^2\mathbf{I}$, hence (2.22) reads

$$\mathcal{F}(\mathbf{v}) = \min_{a\in\mathbb{R}} \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) + \tfrac{a^2}{2}\mathbf{I})\, d\mathbf{x} - \mathcal{L}(\mathbf{v}), \tag{2.36}$$

therefore, a minimizer $a_*(\mathbf{v})$ of functional (2.36) (for $a \in \mathbb{R}$ with fixed $\mathbf{v}$) fulfils

$$a_*^3(\mathbf{v}) \int_\Omega \mathcal{V}_0(x, \mathbf{I})\, d\mathbf{x} + a_*(\mathbf{v}) \int_\Omega D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v})\, d\mathbf{x} = 0,$$

that is,

$$a_*^2(\mathbf{v}) = \left(\int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbf{I})\, dx\right)^{-1} \left(\int_\Omega D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v})\, d\mathbf{x}\right)^{-} \tag{2.37}$$

and

$$\mathcal{F}(\mathbf{v}) \;=\; \int_\Omega \mathcal{V}_0\left(\mathbf{x}\,,\, \mathbb{E}(\mathbf{v}) + \tfrac{a_*^2(\mathbf{v})}{2}\mathbf{I}\right)\, \mathrm{d}\mathbf{x} \;-\; \mathcal{L}(\mathbf{v}). \tag{2.38}$$

Hence, by taking into account that $\mathcal{V}_0$ is a quadratic form, it is readily seen that for $N = 2$,

$$\begin{aligned}
&\mathcal{F}(\mathbf{v}) \;= \\
&= \int_\Omega \mathcal{V}_0\left(\mathbf{x}, \mathbb{E}(\mathbf{v})\right)\, \mathrm{d}\mathbf{x} - \frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbf{I})\mathrm{d}\mathbf{x}\right)^{-1}\left[\left(\int_\Omega D\mathcal{V}_0(\mathbf{x}, \mathbf{I})\cdot\mathbb{E}(\mathbf{v})\, \mathrm{d}\mathbf{x}\right)^{-}\right]^2 - \mathcal{L}(\mathbf{v}) \\
&= \mathcal{E}(\mathbf{v}) - \frac{1}{4}\left(\int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbf{I})\mathrm{d}\mathbf{x}\right)^{-1}\left[\left(\int_\Omega D\mathcal{V}_0(\mathbf{x}, \mathbf{I})\cdot\mathbb{E}(\mathbf{v})\, \mathrm{d}\mathbf{x}\right)^{-}\right]^2 .
\end{aligned} \tag{2.39}$$

Even more explicitly, if $N = 2,\ \lambda,\ \mu > 0$ and

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \begin{cases} \mu|\mathbf{F}^T\mathbf{F} - I|^2 + \frac{\lambda}{2}|\ \mathrm{Tr}\,(\mathbf{F}^T\mathbf{F} - I)|^2 & \text{if } \det \mathbf{F} > 0 \\ +\infty \quad \text{otherwise,} \end{cases} \tag{2.40}$$

then $\mathcal{V}_0(\mathbf{x}, \mathbf{B}) \;=\; 4\mu|\mathbf{B}|^2 + 2\lambda|\mathrm{Tr}\mathbf{B}|^2$, and we get

$$a_*^2(\mathbf{v}) \;=\; |\Omega|^{-1}\left(\int_\Omega \mathrm{div}\, \mathbf{v}\, \mathrm{d}\mathbf{x}\right)^{-} . \tag{2.41}$$

This conclusion could be approximately rephrased as follows: in 2D the global energy $\mathcal{F}(\mathbf{v})$ of a displacement $\mathbf{v}$ is the same of linearized elasticity if the area of the associated deformed configuration $\mathbf{y}(\Omega) = (\mathbf{I} + \mathbf{v})(\Omega)$ is not less than the area of Ω.

Remark 2.7. The compatibility condition (2.25) cannot be dropped in Theorem 2.2 even if the (necessary) condition (2.24) holds true. Moreover plain substitution of strong with weak inequality in (2.25) leads to a lack of compactness for minimizing sequences.

Indeed, if $\mathbf{n}$ denotes the outer unit normal vector to $\partial\Omega$ and we choose $\mathbf{f} = f\mathbf{n}$ with $f < 0$, $\mathbf{g} \equiv 0$ then, by the Divergence Theorem,

$$\int_{\partial\Omega} \mathbf{f}\cdot \mathbf{W}^2\mathbf{x}\, d\mathcal{H}^{N-1} \;=\; f\,(\mathrm{Tr}\mathbf{W}^2)\,|\Omega| \;>\; 0 \qquad \forall\, \mathbf{W}\in\mathcal{M}^{N\times N}_{skew}\setminus\{\mathbf{0}\}\,, \tag{2.42}$$

so that the strict inequality in (2.25) is reversed in a strong sense by any $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew} \setminus \{\mathbf{0}\}$; fix a sequence of positive real numbers such that $h_j \to 0_+$ $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$, $\mathbf{W} \not\equiv \mathbf{0}$, and set $\mathbf{v}_j = h_j{}^{-1}(\frac{1}{2}\mathbf{W}^2 + \frac{\sqrt{3}}{2}\mathbf{W})\,\mathbf{x}$, then $\mathbf{I} + h\mathbf{v}_j = \mathbf{I} + \left(\frac{1}{2}\mathbf{W}^2 + \frac{\sqrt{3}}{2}\mathbf{W}\right) \in SO(N)$, due to representation (2.2) with $\vartheta = \pi/3$. Hence, by frame indifference,

$$\mathcal{F}_{h_j}(\mathbf{v}_j) = \mathcal{L}(\mathbf{v}_j) = -\frac{f}{2h_j}\int_{\partial\Omega} \mathbf{W}^2\mathbf{x}\cdot\mathbf{n}\, d\mathcal{H}^{n-1} = -\frac{f}{2h_j}(\mathrm{Tr}\mathbf{W}^2)|\Omega| \to -\infty. \tag{2.43}$$

On the other hand, assume that $N = 2, 3$, $\Omega \subset \mathbb{R}^N$ a bounded open set with Lipschitz boundary and $\mathcal{W}$ as in (2.30) and $\mathbf{f} = \mathbf{g} \equiv \mathbf{0}$, so that the compatibility inequality is susbstituted by the weak inequality; if $\mathbf{v}_j$ are still defined as above then, hence by frame indifference,

$$\mathcal{F}_{h_j}(\mathbf{v}_j) = 0 = \inf \mathcal{F}_{h_j} \ , \tag{2.44}$$

namely, $\mathbf{v}_j$ is a minimizing sequence for $\mathcal{F}_{h_j}$, but $\mathbb{E}(\mathbf{v}_j)$ has no weakly convergent subsequences in $L^2(\Omega; \mathcal{M}^{N\times N})$.

Remark 2.8. It is worth noticing that the compatibility condition (2.25) holds true when $\mathbf{g} \equiv 0$, $\mathbf{f} = f\mathbf{n}$ with $f > 0$ and $\mathbf{n}$ the outer unit normal vector to $\partial\Omega$.

Indeed let $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$, $\mathbf{W} \not\equiv \mathbf{0}$: hence by (2.24) and the Divergence Theorem we get

$$\int_{\partial\Omega} \mathbf{f}\cdot \mathbf{W}^2\,\mathbf{x}\, d\mathcal{H}^{N-1} \ = \ f\,(\mathrm{Tr}\mathbf{W}^2)\,|\Omega| \ < \ 0, \tag{2.45}$$

thus proving (2.25) in this case. This means that in case of uniform tension-like force field at the boundary and null body force field the compatibility condition holds true.

Remark 2.9. It is possible to observe some analogy between the energy functional (2.22) and the results in [15,16], where the approximate theory of small strain together with *moderate rotations* is discussed under suitable kinematical assumptions. More precisely, if $\mathbf{F} = \mathbf{I} + h\nabla\mathbf{v}$ is the deformation gradient and $\mathbf{F} = \mathbf{R}\mathbf{U}$ is the polar decomposition, [15] shows that the assumptions

$$\mathbf{R} = \mathbf{I} + O(\sqrt{h}), \quad \mathbf{U} = \mathbf{I} + O(h) \qquad \text{as } h \to 0_+, \tag{2.46}$$

in the sense of pointwise convergence, are equivalent to

$$\mathbb{E}(\mathbf{v}) = O(1), \quad h(\mathrm{skew}\nabla\mathbf{v}) = O(\sqrt{h}) \qquad \text{as } h \to 0_+, \tag{2.47}$$

still in the sense of pointwise convergence. Therefore,

$$\mathbf{U} = \mathbf{I} + h\big(\mathbb{E}(\mathbf{v}) - \tfrac{1}{2}(\mathrm{skew}\nabla\mathbf{v})^2\big) + o(h), \tag{2.48}$$

and the pointwise limit of $\mathcal{F}_h$ (not the Γ-limit !) becomes

$$\int_\Omega \mathcal{V}_0\big(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}(\mathrm{skew}\nabla\mathbf{v})^2\big)\,\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v})\,,$$

which is quite similar to (2.22).

We highlight the fact that (2.46) cannot be understood in the sense of $L^2(\Omega, \mathcal{M}^{N\times N})$ whenever $\mathbf{v} \equiv \mathbf{v}_*$ on a closed subset Σ of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Sigma) > 0$, since by Korn and Poincarè inequalities we get

$$\int_\Omega |\nabla\mathbf{v}|^2\,\mathrm{d}\mathbf{x} \le C\Big(\int_\Omega |\mathbb{E}(\mathbf{v})|^2\,\mathrm{d}\mathbf{x} + \int_\Sigma |\mathbf{v}_*|^2\, d\mathcal{H}^{N-1}\Big)\,,$$

therefore if $\mathbb{E}(\mathbf{v}) = O(1)$ then $h\nabla\mathbf{v} = O(h)$, thus contradicting the second of (2.46). On the other hand a careful application of the rigidity Lemma of [18] show that if $\mathbb{E}(\mathbf{v}) = O(1)$ and $\mathbf{U} = \mathbf{I} + O(h)$ in the sense of $L^2(\Omega, \mathcal{M}^{N\times N})$, then there exists a constant skew symmetric matrix $\mathbf{W}$ such that $h\nabla\mathbf{v}^T\nabla\mathbf{v} = -\mathbf{W}^2 + o(1)$ in the sense of $L^1(\Omega, \mathcal{M}^{N\times N})$ (see the proof of Lemma 3.4 below). Therefore

$$\mathbf{U} = \mathbf{I} + h\big(\mathbb{E}(\mathbf{v}) - \mathbf{W}^2/2\big) + o(h), \tag{2.49}$$

where equality is understood in the sense of $L^1(\Omega, \mathcal{M}^{N\times N})$ and $\mathbf{W}$ a constant skew symmetric matrix.

3. Proofs

We recall some basic inequalities exploited in the sequel, for both reader's convenience and labeling the related constants.

Poincaré Inequality. There exists a constant $C_P = C_P(\Omega)$ such that

$$\| \mathbf{v} - ⨍_\Omega \mathbf{v} \|_{L^2(\Omega;\mathbb{R}^N)} + \| \mathbf{v} - ⨍_\Omega \mathbf{v} \|_{L^2(\partial\Omega;\mathbb{R}^N)} \le C_P \, \|\nabla\mathbf{v}\|_{L^2(\Omega;\mathcal{M}^{N\times N})} \quad \forall\, \mathbf{v} \in H^1(\Omega;\mathbb{R}^N)\,. \tag{3.1}$$

Korn Inequality. There exists a constant $C_K = C_K(\Omega)$ such that

$$\|\mathbf{v}-\mathbb{P}\mathbf{v}\|_{L^2(\Omega;\mathbb{R}^N)} + \|\mathbf{v}-\mathbb{P}\mathbf{v}\|_{L^2(\partial\Omega;\mathbb{R}^N)} \le C_K \, \|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega;\mathcal{M}^{N\times N})} \quad \forall\, \mathbf{v} \in H^1(\Omega;\mathbb{R}^N)\,. \tag{3.2}$$

Geometric Rigidity Inequality ([18]). There exists a constant $C_G = C_G(\Omega)$ such that for every $\mathbf{y} \in H^1(\Omega;\mathbb{R}^N)$ there is an associated rotation $\mathbf{R} \in SO(N)$ such that we have

$$\int_\Omega |\nabla\mathbf{y} - \mathbf{R}|^2 \,\mathrm{d}\mathbf{x} \le C_G \int_\Omega \mathrm{dist}^2(\nabla\mathbf{y}; SO(N)) \,\mathrm{d}\mathbf{x}. \tag{3.3}$$

The first step in our analysis is the next lemma showing that if (2.24), (2.25) hold true then the functionals $\mathcal{F}_h$ are bounded from below uniformly with respect to $h > 0$; this implies the existence of minimizing sequences of the sequence of functionals $\mathcal{F}_{h_j}$ (see Definition 2.1).

Lemma 3.1. *Assume* (2.24) *and* (2.25). *Then*

$$\inf_{h>0} \inf_{\mathbf{v}\in H^1} \mathcal{F}_h(\mathbf{v}) \ \ge\ -\frac{C_P^2\, C_G}{C}\left(\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2\right), \tag{3.4}$$

where C is the coercivity constant in (2.18) *and C_P, C_G are the constants related to the basic inequalities above.*

Actually the claim holds true even if strict inequality is replaced by weak inequality in (2.25).

Proof. Let $\mathbf{v} \in H^1(\Omega;\mathbb{R}^N)$ and $\mathbf{y} = \mathbf{x} + h\mathbf{v}$. Since $\det\nabla\mathbf{y} > 0$ a.e., by polar decomposition for a.e. $\mathbf{x}$ there exist a rotation $\mathbf{R}_h(\mathbf{x})$ and a symmetric positive definite matrix $\mathbf{U}_h(\mathbf{x})$ such that $\nabla\mathbf{y}(\mathbf{x}) = \mathbf{R}_h(\mathbf{x})\mathbf{U}_h(\mathbf{x})$, hence $\nabla\mathbf{y}^T\nabla\mathbf{y} = \mathbf{U}_h^2$, so that for a.e. $\mathbf{x}$,

$$\begin{aligned} |\nabla\mathbf{y}^T\nabla\mathbf{y} - \mathbf{I}|^2 &= |\mathbf{U}_h^2 - \mathbf{I}|^2 = |(\mathbf{U}_h - \mathbf{I})(\mathbf{U}_h + \mathbf{I})|^2 \ge |\mathbf{U}_h - \mathbf{I}|^2 = \\ &= |(\nabla\mathbf{y} - \mathbf{R}_h)|^2 \ \ge\ \mathrm{dist}^2(\nabla\mathbf{y}, SO(N))\,. \end{aligned} \tag{3.5}$$

By (2.8), (3.5) and the Geometric Rigidity Inequality (3.3) there exists a constant rotation $\mathbf{R}$ such that

$$\begin{aligned}\mathcal{F}_h(\mathbf{v}) &\geq C h^{-2} \int_\Omega |\nabla \mathbf{y}^T \nabla \mathbf{y} - \mathbf{I}|^2 \, d\mathbf{x} - h^{-1} \mathcal{L}(\mathbf{y} - \mathbf{x}) \geq \\ &\geq \frac{C}{C_G} h^{-2} \int_\Omega |\nabla \mathbf{y} - \mathbf{R}|^2 \, d\mathbf{x} - h^{-1} \mathcal{L}(\mathbf{y} - \mathbf{x}).\end{aligned} \tag{3.6}$$

If

$$\mathbf{c} := |\Omega|^{-1} \int_\Omega (\mathbf{y} - \mathbf{R}\mathbf{x}) \, d\mathbf{x} \,,$$

then by Poincaré inequality (3.1), we have that

$$\|\mathbf{y}-\mathbf{R}\mathbf{x}-\mathbf{c}\|_{L^2(\Omega)} + \|\mathbf{y}-\mathbf{R}\mathbf{x}-\mathbf{c}\|_{L^2(\partial\Omega)} \leq C_P \, \|\nabla(\mathbf{y}-\mathbf{R}\mathbf{x})\|_{L^2} = C_P \, \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2} \,,$$

and by (2.24) and Young inequality we get, for every $\alpha > 0$,

$$\begin{aligned}\mathcal{L}(\mathbf{y} - \mathbf{R}\mathbf{x} - \mathbf{c}) &\leq C_P \, \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2} \left(\|\mathbf{f}\|_{L^2(\partial\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)} \right) \leq \\ &\leq \alpha^{-1} \frac{C_P}{2} \, \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \alpha \frac{C_P}{2} \left(\|\mathbf{f}\|_{L^2} + \|\mathbf{g}\|_{L^2} \right)^2 \\ &\leq \alpha^{-1} \frac{C_P}{2} \, \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \alpha \, C_P \left(\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2 \right).\end{aligned}$$

By choosing $\alpha = h \, C_P \, C_G / C$,

$$\begin{aligned}\mathcal{L}(\mathbf{y} - \mathbf{x}) &= \mathcal{L}(\mathbf{y} - \mathbf{R}\mathbf{x} - \mathbf{c}) + \mathcal{L}(\mathbf{R}\mathbf{x} - \mathbf{x}) \leq \\ &\leq \alpha^{-1} \frac{C_P}{2} \, \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \alpha \, C_P \left(\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2 \right) + \mathcal{L}(\mathbf{R}\mathbf{x} - \mathbf{x}) = \\ &= h^{-1} \frac{C/C_G}{2} \, \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \frac{C_P^2}{C/C_G} h \left(\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2 \right) + \mathcal{L}(\mathbf{R}\mathbf{x} - \mathbf{x}).\end{aligned} \tag{3.7}$$

Exploiting the standard representation (2.2) of the rotation $\mathbf{R} = \mathbf{I} + \mathbf{W} \sin\vartheta + (1 - \cos\vartheta)\mathbf{W}^2$ for suitable $\vartheta \in \mathbb{R}$ and $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$ with $|\mathbf{W}|^2 = 2$, by (2.24) and (2.25) we get

$$\mathcal{L}\big((\mathbf{R} - \mathbf{I})\mathbf{x}\big) = (\sin\vartheta) \, \mathcal{L}(\mathbf{W}\mathbf{x}) + (1 - \cos\vartheta) \, \mathcal{L}(\mathbf{W}^2) < 0 \,, \tag{3.8}$$

hence, by (3.6), (3.7) and (3.8), we conclude that

$$\begin{aligned}\mathcal{F}_h(\mathbf{v}) &\geq \frac{C/C_G}{2} h^{-2} \int_\Omega |\nabla \mathbf{y} - \mathbf{R}|^2 \, d\mathbf{x} - \frac{C_P^2}{C/C_G} \left(\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2 \right) - h^{-1} \mathcal{L}\big((\mathbf{R} - \mathbf{I})\mathbf{x}\big) > \\ &> -\frac{C_P^2 \, C_G}{C} \left(\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2 \right) \qquad \forall \mathbf{v} \in H^1(\Omega; \mathbb{R}^N), \ \forall h > 0 \,.\end{aligned} \tag{3.9}$$

□

Lemma 3.2. *Let* $\mathbf{v}_n \in H^1(\Omega; \mathbb{R}^N)$ *be a sequence such that* $\mathbb{E}(\mathbf{v}_n) \rightharpoonup \mathbf{T}$ *in* $L^2(\Omega; \mathcal{M}^{N\times N})$*. Then there exists* $\mathbf{w} \in H^1(\Omega; \mathbb{R}^N)$ *such that* $\mathbf{T} = \mathbb{E}(\mathbf{w})$*. If, in addition,* $\nabla \mathbf{v}_n \rightharpoonup \mathbf{G}$ *in* $L^2(\Omega; \mathbb{R}^N)$*, then there exists a constant matrix* $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$ *such that* $\nabla \mathbf{w} = \mathbf{G} - \mathbf{W}$.

Proof. Since $\mathbb{E}(\mathbf{v}_n) \rightharpoonup \mathbf{T}$ in $L^2(\Omega; M^{N\times N})$, we have $\mathbb{E}\big(\mathbf{v}_n - \mathbb{P}\mathbf{v}_n\big) = \mathbb{E}\big(\mathbf{v}_n\big)$ are equibounded in $L^2(\Omega; \mathcal{M}^{N\times N})$, then by Korn inequality, $\mathbf{v}_n - \mathbb{P}\mathbf{v}_n$ are equibounded in $H^1(\Omega; \mathbb{R}^N)$, where $\mathbb{P}$ the projection on the set $\mathcal{R}$ of infinitesimal rigid displacements. Therefore, up to subsequences, we can assume that $\mathbf{w}_n := \mathbf{v}_n - \mathbb{P}\mathbf{v}_n \rightharpoonup \mathbf{w}$ in $H^1(\Omega; \mathbb{R}^N)$ and we get

$$\nabla \mathbf{w}_n = \mathbb{E}(\mathbf{w}_n) + \mathrm{skew}\nabla \mathbf{w}_n = \mathbb{E}(\mathbf{v}_n) + \mathrm{skew}\nabla \mathbf{w}_n\,. \tag{3.10}$$

Hence there exists $\mathbf{S} \in L^2(\Omega; \mathcal{M}^{N\times N}_{skew})$ such that $\mathrm{skew}\nabla \mathbf{w}_n \rightharpoonup \mathbf{S}$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ and by letting $n \to +\infty$ into (3.10) we have $\mathbf{T}+\mathbf{S} = \nabla \mathbf{w}$. Since $\mathbf{S} \in L^2(\Omega; \mathcal{M}^{N\times N}_{skew})$, it is readily seen that $\mathbb{E}(\mathbf{w}) = \mathbf{T}$, and if, in addition, $\nabla \mathbf{v}_n \rightharpoonup \mathbf{G}$ in $L^2(\Omega; \mathcal{M}^{N\times N})$, then there exists a constant $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$ such that $\nabla \mathbb{P}\mathbf{v}_n \rightharpoonup \mathbf{W}$ in $L^2(\Omega; \mathcal{M}^{N\times N})$, actually converging in the finite dimensional space of constant skew symmetric matrices, thus proving the Lemma. □

Remark 3.3. It is worth noting that if $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbf{T}$ in $L^2(\Omega; \mathcal{M}^{N\times N})$, then by Lemma 3.2 there exists $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ such that $\mathbf{T} = \mathbb{E}(\mathbf{v})$, and if $\mathbf{T} = \mathbb{E}(\mathbf{w})$ for some $\mathbf{w} \in H^1(\Omega; \mathbb{R}^N)$, then $\mathbf{v} - \mathbf{w}$ is an infinitesimal rigid displacement in Ω, *i*.e. $\mathbb{E}(\mathbf{v} - \mathbf{w}) = \mathbf{0}$.

Next we show a preliminary convergence property; we compute a kind of Gamma limit of the sequence of functional $\mathcal{F}_h$ with respect to weak L^2 convergence of linearized strains.

Lemma 3.4. *(**energy convergence**) Assume that* (2.24) *holds true and let* $h_j \to 0$ *be a decreasing sequence. Then we have that:*

i) *For every* $\mathbf{v}_j, \mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ *such that* $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v})$ *in* $L^2(\Omega; \mathcal{M}^{N\times N})$ *we have*

$$\liminf_{j\to+\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \mathcal{F}(\mathbf{v}).$$

ii) *For every* $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ *there exists a sequence* $\mathbf{v}_j \in H^1(\Omega; \mathbb{R}^N)$ *such that* $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v})$ *in* $L^2(\Omega; \mathcal{M}^{N\times N})$ *and*

$$\limsup_{j\to+\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \leq \mathcal{F}(\mathbf{v}).$$

Proof. First we prove i). We set $\mathbf{y}_j = \mathbf{x} + h_j \mathbf{v}_j$ and denote various positive constants by C', C'', ..., L', L''. We may assume without restriction that $\mathcal{F}_{h_j}(\mathbf{v}_j) \leq C'$; by taking into account (2.8), we get

$$C h_j^{-2} \int_\Omega |\nabla \mathbf{y}_j^T \nabla \mathbf{y}_j - \mathbf{I}|^2 \,d\mathbf{x} - \mathcal{L}(\mathbf{v}_j) \;\leq\; \mathcal{F}_{h_j}(\mathbf{v}_j),$$

and by (2.24),

$$h_j^{-2}\int_\Omega |\nabla \mathbf{y}_j^T\nabla \mathbf{y}_j-\mathbf{I}|^2\,d\mathbf{x}\ \le\ C'\,+\,\mathcal{L}(\mathbf{v}_j)\ =\ C'+\mathcal{L}(\mathbf{v}_j-\mathbb{P}\mathbf{v}_j),$$

where $\mathbb{P}\,\mathbf{v}_j$ is the projection of $\mathbf{v}_j$ onto the set of infinitesimal rigid displacements.

Hence, by Korn inequality, we have

$$h_j^{-2}\int_\Omega |\nabla \mathbf{y}_j^T\nabla \mathbf{y}_j-\mathbf{I}|^2\,d\mathbf{x}\ \le\ C'\,+\,C''\left(\int_\Omega |\mathbb{E}(\mathbf{v}_j)|^2\,d\mathbf{x}\right)^{\frac12}\ \le\ C'''. \tag{3.11}$$

Inequality (3.11), together with the Rigidity Lemma of [18] and (3.5), implies that for every h_j there exists a constant rotation $\mathbf{R}_j\in SO(N)$, and a constant C''', dependent only on Ω, such that

$$\int_\Omega |\nabla \mathbf{y}_j-\mathbf{R}_j|^2\,d\mathbf{x}\le C''''h_j^2;$$

that is,

$$\int_\Omega |\mathbf{I}+h_j\nabla\mathbf{v}_j-\mathbf{R}_j|^2\,d\mathbf{x}\le C''''h_j^2. \tag{3.12}$$

Due to the representation (2.2) of rotations, for every $j\in\mathbb{N}$ there exist $\vartheta_j\in(-\pi,\pi]$ and $\mathbf{W}_j\in\mathcal{M}^{N\times N}_{skew}$, $|\mathbf{W}_j|^2=2$ such that $\mathbf{R}_j=\exp(\vartheta_j\mathbf{W}_j)$ and

$$\mathbf{R}_j\ =\ \exp(\vartheta_j\mathbf{W}_j)\ =\ \mathbf{I}\ +\ \sin\vartheta_j\,\mathbf{W}_j\ +\ (1-\cos\vartheta_j)\,\mathbf{W}_j^2, \tag{3.13}$$

hence, by (3.12),

$$\int_\Omega |h_j\nabla\mathbf{v}_j-\sin\vartheta_j\mathbf{W}_j-(1-\cos\vartheta_j)\mathbf{W}_j^2|^2\,d\mathbf{x}\le C''''h_j^2\,. \tag{3.14}$$

Since

$$\mathrm{sym}\Big(h_j\nabla\mathbf{v}_j-\sin\vartheta_j\mathbf{W}_j-(1-\cos\vartheta_j)\mathbf{W}_j^2\Big)=h_j\mathbb{E}(\mathbf{v}_j)-(1-\cos\vartheta_j)\mathbf{W}_j^2,$$

we get

$$\int_\Omega |\mathbb{E}(\mathbf{v}_j)-(1-\cos\vartheta_j)h_j^{-1}\mathbf{W}_j^2|^2\,d\mathbf{x}\le C''''.$$

By recalling that $\mathbb{E}(\mathbf{v}_j)\rightharpoonup\mathbb{E}(\mathbf{v})$ in $L^2(\Omega;\mathcal{M}^{N\times N})$ and $|\mathbf{W}_j^2|^2=2$ due to (2.3), we deduce, for suitable $L>0$,

$$\left|1-\cos\vartheta_j\right|=\frac{1}{\sqrt2}\left|(1-\cos\vartheta_j)\mathbf{W}_j^2\right|\le Lh_j, \tag{3.15}$$

hence

$$|\sin\vartheta_j|\le\sqrt{2Lh_j}. \tag{3.16}$$

By (3.14) and (3.15) we have

$$\int_\Omega |\sqrt{h_j}\nabla\mathbf{v}_j-h_j^{-1/2}\sin\vartheta_j\mathbf{W}_j|^2\,d\mathbf{x}\ \le\ (C'''+2L|\Omega|)\,h_j\,, \tag{3.17}$$

hence, by compactness of the sequence $h^{-1/2}\sin\vartheta_j \mathbf{W}_j$ in $\mathcal{M}^{N\times N}_{skew}$, there exists a constant matrix $\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}$ such that, up to subsequences,

$$\sqrt{h_j}\nabla\mathbf{v}_j \;\to\; \mathbf{W} \qquad \text{strongly in } L^2(\Omega;\mathcal{M}^{N\times N}), \tag{3.18}$$

and therefore

$$h_j\nabla\mathbf{v}_j^T\nabla\mathbf{v}_j \;\to\; \mathbf{W}^T\mathbf{W} \;=\; -\mathbf{W}^2 \qquad \text{strongly in } L^1(\Omega;\mathcal{M}^{N\times N})\,. \tag{3.19}$$

By Lemma 4.2. of [13], for every $k\in\mathbb{N}$, there exists an increasing sequence of Caratheodory functions $\mathcal{V}^k_j:\Omega\times\mathcal{M}^{N\times N}_{sym}\to[0,+\infty)$ and a measurable function $\mu^k:\Omega\to(0,+\infty)$ such that $\mathcal{V}^k_j(\mathbf{x},\cdot)$ is convex for a.e. $\mathbf{x}\in\Omega$ and satisfies

$$\mathcal{V}^k_j(\mathbf{x},\mathbf{D}) \le \mathcal{V}(x,h_j\mathbf{D})/h_j^2 \qquad \forall\,\mathbf{D}\in\mathcal{M}^{N\times N}_{sym}\,, \tag{3.20}$$

$$\mathcal{V}^k_j(\mathbf{x},\mathbf{D}) = \left(1-\frac{1}{k}\right)\mathcal{V}_0(\mathbf{x},\mathbf{D}) \qquad \text{for } \mathcal{V}_0(\mathbf{x},\mathbf{D})\le\mu^k(x)/h_j^2\,. \tag{3.21}$$

By setting $\mathbf{D}_j := \mathbb{E}(\mathbf{v}_j)+\frac{1}{2}h_j\nabla\mathbf{v}_j^T\nabla\mathbf{v}_j$, properties (3.19), (3.20) and (3.21) entail

$$\int_\Omega \mathcal{V}_{h_j}(x,\nabla\mathbf{v}_j)\,\mathrm{d}\mathbf{x} \ge \int_\Omega \mathcal{V}^k_j(x,\mathbf{D}_j)\,\mathrm{d}\mathbf{x} \tag{3.22}$$

and

$$\lim_{j\to+\infty}\mathcal{V}^k_j(\mathbf{x},\mathbf{D}) = \left(1-\frac{1}{k}\right)\mathcal{V}_0(\mathbf{x},\mathbf{D}) \qquad \text{a.e. } \mathbf{x}\in\Omega,\ \forall\,\mathbf{D}\in\mathcal{M}^{N\times N}_{sym}. \tag{3.23}$$

Then by taking into account that

$$\mathbf{D}_j \;\rightharpoonup\; \mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2 \;\text{ in } L^1(\Omega;\mathcal{M}^{N\times N}),$$

(3.22) and Lemma 4.3 of [13] yield

$$\begin{aligned}\liminf_{j\to+\infty}\int_\Omega \mathcal{V}_{h_j}(x,\nabla\mathbf{v}_j)\,\mathrm{d}\mathbf{x} &\ge \liminf_{j\to+\infty}\int_\Omega \mathcal{V}^k_j(x,\mathbf{D}_j)\,\mathrm{d}\mathbf{x}\\ &\ge \int_\Omega (1-1/k)\mathcal{V}_0\Big(\mathbf{x},\,\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2\Big)\,\mathrm{d}\mathbf{x} \quad \forall k\in\mathbb{N}\,.\end{aligned}$$

Up to subsequences, $\mathbf{v}_j-\mathbb{P}\mathbf{v}_j\rightharpoonup\mathbf{w}$ in $H^1(\Omega;\mathbb{R}^N)$, moreover, $\mathbb{E}(\mathbf{v})=\mathbb{E}(\mathbf{w})$. Then, by (2.24) for every $k\in\mathbb{N}$, we obtain

$$\begin{aligned}\liminf_{j\to+\infty}\mathcal{F}_{h_j}(\mathbf{v}_j) \ge & \int_\Omega \Big(1-\frac{1}{k}\Big)\mathcal{V}_0(\mathbf{x},\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2)\,\mathrm{d}\mathbf{x}-\mathcal{L}(\mathbf{w}) =\\ = & \int_\Omega \Big(1-\frac{1}{k}\Big)\mathcal{V}_0(\mathbf{x},\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2)\,\mathrm{d}\mathbf{x}-\mathcal{L}(\mathbf{v}).\end{aligned}$$

Taking the supremum as $k\to\infty$, we deduce

$$\liminf_{j\to+\infty}\mathcal{F}_{h_j}(\mathbf{v}_j) \ge \int_\Omega \mathcal{V}_0(\mathbf{x},\mathbb{E}(\mathbf{v})-\tfrac{1}{2}\mathbf{W}^2)\,\mathrm{d}\mathbf{x}-\mathcal{L}(\mathbf{v}) \ge \mathcal{F}(\mathbf{v}), \tag{3.24}$$

which proves i).

We are left to prove claim ii). To this end, we set, for every $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$,

$$\mathbf{W_v} \in \operatorname{argmin} \left\{ \int_\Omega \mathcal{V}_0\Big(\mathbf{x},\ \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2\Big)\, d\mathbf{x} :\ \mathbf{W} \in \mathcal{M}^{N\times N}_{skew} \right\}. \tag{3.25}$$

Without relabeling, $\mathbf{v}$ denotes also a fixed compactly supported extension in $H^1(\mathbb{R}^N; \mathbb{R}^N)$ of the given $\mathbf{v}$ (such extension exists since Ω is Lipschitz due to (2.1)).

We may define a recovery sequence $\mathbf{w}_j \in C^1(\overline{\Omega}; \mathbb{R}^N)$ for every j, as follows: set

$$\mathbf{w}_j = h_j^{-1/2}\, \mathbf{W_v}\, \mathbf{x} + \mathbf{v} \star \varphi_j, \tag{3.26}$$

where $\varphi_j(\mathbf{x}) = \varepsilon_j^{-N} \varphi(\mathbf{x}/\varepsilon_j)$ is a mollifier supported in $B_{\varepsilon_j}(\mathbf{0})$, and the sequence ε_j is chosen in such a way that $h_j \varepsilon_j^{-3} \to 0$ holds true. Sobolev embedding entails $\mathbf{v} \in L^6(\mathbb{R}^N; \mathbb{R}^N)$, since $\mathbf{v} \in H^1(\mathbb{R}^N; \mathbb{R}^N)$ and $N = 2, 3$; then, by the Young Theorem, and since $0 < \varepsilon \le 1$, we have

$$\|\nabla(\mathbf{v}\star\varphi_j)\|_{L^\infty} \le \|v\|_{L^6} \|\mathbf{v}\varphi_j\|_{L^{6/5}} \le \varepsilon_j^{-N/6-1} \|\nabla\varphi\|_{L^{6/5}} \|\mathbf{v}\|_{L^6} \le \varepsilon_j^{-3/2} \|\nabla\varphi\|_{L^{6/5}} \|\mathbf{v}\|_{L^6}. \tag{3.27}$$

By $\nabla\mathbf{w}_j = h_j^{-1/2}\mathbf{W_v} + \nabla(\mathbf{v} \star \varphi_j)$ and $\mathbf{W_v}^T = -\mathbf{W_v}$ we get

$$\begin{aligned} \mathbb{E}(\mathbf{w}_j) &= \mathbb{E}(\mathbf{v}) \star \varphi_j, \\ h_j \nabla\mathbf{w}_j^T \nabla\mathbf{w}_j &= -\mathbf{W_v}^2 + h_j \nabla(\mathbf{v}\star\varphi_j)^T\, \nabla(\mathbf{v}\star\varphi_j) + h_j^{1/2}\big(\nabla(\mathbf{v}\star\varphi_j)^T\, \mathbf{W_v} - \mathbf{W_v}\nabla(\mathbf{v}\star\varphi_j)\big), \end{aligned}$$

hence, by taking into account (3.27) and $h_j\varepsilon_j^{-3} \to 0$, we get

$$\mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2}h_j \nabla\mathbf{w}_j^T \nabla\mathbf{w}_j \to \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W_v}^2 \quad \text{in } L^2(\Omega, \mathbb{R}^N), \tag{3.28}$$

$$h_j\Big(\mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2}h_j \nabla\mathbf{w}_j^T \nabla\mathbf{w}_j\Big) \to \mathbf{0} \quad \text{in } L^\infty(\Omega, \mathbb{R}^N). \tag{3.29}$$

Therefore Taylor's expansion of $\mathcal{V}$ entails

$$\lim_{j\to+\infty} \mathcal{V}_{h_j}(\mathbf{x}, \mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2}h_j \nabla\mathbf{w}_j^T \nabla\mathbf{w}_j) = \mathcal{V}_0\left(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W_v}^2\right) \qquad \text{for a.e. } \mathbf{x} \in \Omega, \tag{3.30}$$

and taking into account (2.14), (2.15), (2.16) and (3.29), we have

$$\mathcal{V}_{h_j}(\mathbf{x}, \mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2}h_j \nabla\mathbf{w}_j^T \nabla\mathbf{w}_j) \le \gamma\, |\mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2}h_j \nabla\mathbf{w}_j^T \nabla\mathbf{w}_j|^2, \tag{3.31}$$

hence the Lebesgue dominated convergence theorem yields

$$\mathcal{F}_{h_j}(\mathbf{w}_j) \to \min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \int_\Omega \mathcal{V}_0\left(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W_v}^2\right) d\mathbf{x} - \mathcal{L}(\mathbf{v}),$$

thus proving ii). □

Remark 3.5. If $\mathcal{W}$ is a convex function of $\mathbf{F}^T\mathbf{F}-\mathbf{I}$, then (3.24) is a straightforward consequence of the weak $L^1(\Omega;\mathcal{M}^{N\times N})$ convergence of $\mathbf{B}_j$ and the construction of $\mathcal{V}_j^k$ can be avoided in the proof. Hence the restriction to decreasing sequences h_j (needed in order to apply Lemmas 4.2 and 4.3 of [13]) can be removed in the assumptions of Lemma 3.4 if $\mathcal{W}$ is convex.

Lemma 3.1 entails the existence of minimizing sequences for the sequence of functionals $\mathcal{F}_h$. Next, tecnical lemma 3.6 shows a (very weak) relative compactness property of these sequences: if $\mathbf{v}_j$ is a minimizing sequence then $E(\mathbf{v}_j)$ is equibounded in L^2. The key idea of the proof consists in showing that $||E(\mathbf{v}_j)||_{L^2}\to+\infty$ entails the contradiction $E(\mathbf{v}_j)/||E(\mathbf{v}_j)||_{L^2}\to 0$ strongly in L^2, and by a careful analysis of all possible cases related to different balance of involved parameters we get

Lemma 3.6. *(**Compactness of minimizing sequences**) Assume that (2.24) and (2.25) hold true, $h_j\to 0_+$ is a sequence of strictly positive real numbers and the sequence of displacements $\mathbf{v}_j\in H^1(\Omega;\mathbb{R}^N)$ fulfil $(\mathcal{F}_{h_j}(\mathbf{v}_j)-\inf\mathcal{F}_{h_j})\to 0$, namely $\mathbf{v}_j$ is a minimizing sequence for F_{h_j}.*

Then there exists $M>0$ such that $\|\mathbb{E}(\mathbf{v}_j)\|_{L^2}\le M$.

Proof. By Lemma 3.1 there exists c such that

$$-\infty\;<\;c\;\le\;\inf\mathcal{F}_{h_j}\;\le\;\mathcal{F}_{h_j}(\mathbf{0})\;=\;0. \tag{3.32}$$

Assume, by contradiction, that $t_j:=\|\;\mathbb{E}(\mathbf{v}_j)\;\|_{L^2}\to+\infty$ and set $\mathbf{w}_j=t_j^{-1}\mathbf{v}_j$. By Lemma 3.2 there exist $\mathbf{w}\in H^1(\Omega;\mathbb{R}^N)$ and a subsequence such that without relabeling $\mathbb{E}(\mathbf{w}_j)\rightharpoonup\mathbb{E}(\mathbf{w})$ in $L^2(\Omega;\mathcal{M}^{N\times N})$. By (3.32) we can assume up to subsequences that $\mathcal{F}_{h_j}(\mathbf{v}_j)\le 1$.

By setting $\mathbf{y}_j=\mathbf{x}+h_j\mathbf{v}_j=\mathbf{x}+h_jt_j\mathbf{w}_j$, arguing as at the beginning of the Lemma 3.4 proof and exploiting Korn inequality (3.2), we obtain that for every $j\in\mathbb{N}$ there exists a constant rotation $\mathbf{R}_j\in SO(N)$ such that

$$\int_\Omega|\nabla\mathbf{y}_j-\mathbf{R}_j|^2\,d\mathbf{x}\;\le\;h_j^2\;+\;C_K(\|f\|_{L^2(\partial\Omega)}+\|g\|_{L^2(\Omega)})\,t_j\,h_j^2,$$

that is, by setting $C'=C_K(\|f\|_{L^2(\partial\Omega)}+\|g\|_{L^2(\Omega)})$,

$$\int_\Omega|\mathbf{I}+h_jt_j\nabla\mathbf{w}_j-\mathbf{R}_j|^2\,d\mathbf{x}\;\le\;h_j^2\,(1+C'\,t_j). \tag{3.33}$$

Possibly up to further subsequence extraction, one among these three alternatives takes place:

$$a)\;h_jt_j\to\lambda>0\,,\qquad b)\;h_jt_j\to 0\,,\qquad c)\;h_jt_j\to+\infty\,.$$

If condition $a)$ holds true, we have

$$\int_\Omega\left|\nabla\mathbf{w}_j-\frac{\mathbf{R}_j-\mathbf{I}}{h_jt_j}\right|^2\,d\mathbf{x}\;\le\;\frac{1}{t_j^2}\;+\;\frac{C'}{t_j}\,,$$

hence, up to subsequences,

$$\nabla \mathbf{w}_j \to \frac{\mathbf{R}-\mathbf{I}}{\lambda} \tag{3.34}$$

strongly in $L^2(\Omega; \mathcal{M}^{N\times N})$ for a suitable constant matrix $\mathbf{R} \in SO(N)$, and by Lemma 3.2 we get $\nabla \mathbf{w} \in \mathbb{K} + \mathcal{M}^{N\times N}_{skew}$.

If condition b) holds true, then, by using formulæ (3.13) and (3.33), there exist $\vartheta_{h_j} \in (-\pi, \pi]$ and a constant matrix $\mathbf{W}_j \in \mathcal{M}^{N\times N}_{skew}$ with $|\mathbf{W}_j|^2 = |\mathbf{W}_j^2|^2 = 2$ such that

$$\int_\Omega |h_j t_j \nabla \mathbf{w}_j - \sin \vartheta_{h_j} \mathbf{W}_j - (1-\cos\vartheta_j)\mathbf{W}_j^2|^2 \,d\mathbf{x} \le h_j^2 (1 + C' t_j)\,. \tag{3.35}$$

Since $\mathbf{W}_j$ and $\mathbf{W}_j^2$ are respectively skew-symmetric and symmetric, (3.35) yields

$$\int_\Omega \left| \mathbb{E}(\mathbf{w}_j) - \frac{(1-\cos\vartheta_j)}{h_j t_j}\mathbf{W}_j^2 \right|^2 d\mathbf{x} \le t_j^{-2} + C' t_j^{-1}, \tag{3.36}$$

and, bearing in mind that $\int_\Omega |\mathbb{E}(\mathbf{w}_j)|^2 \,d\mathbf{x} = 1$, we get

$$\left|\frac{(1-\cos\vartheta_j)}{h_j t_j}\right| = \frac{1}{\sqrt{2}} \left|\frac{(1-\cos\vartheta_j)}{h_j t_j}\mathbf{W}_j^2\right| \le C'', \tag{3.37}$$

hence

$$\left|\sin\vartheta_{h_j}\right| \le \sqrt{2(1-\cos\vartheta_{h_j})} \le \sqrt{2\,C''\,h_j\,t_j}. \tag{3.38}$$

Estimate (3.35), together with $t_j \to +\infty$, yields

$$\int_\Omega \left| \sqrt{h_j t_j}\nabla \mathbf{w}_j - \frac{\sin\vartheta_j}{\sqrt{h_j t_j}}\mathbf{W}_j - \frac{(1-\cos\vartheta_j)}{\sqrt{h_j t_j}}\mathbf{W}_j^2 \right|^2 d\mathbf{x} \le C'''\,h_j\,. \tag{3.39}$$

By (3.37) we know that $1-\cos\vartheta_j = o(\sqrt{h_j t_j})$, hence (3.38) and (3.39) entail the existence of a constant matrix $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$ such that, up to subsequences, $\sqrt{h_j t_j}\nabla \mathbf{w}_j \to \mathbf{W}$ strongly in $L^2(\Omega; \mathcal{M}^{N\times N})$. Moreover, by (2.8) and Korn inequality,

$$\begin{aligned} t_j^2 \int_\Omega |\mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2} h_j t_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j|^2 \,d\mathbf{x} &\le C^{IV} + \mathcal{L}(\mathbf{w}_j) \\ &= C^{IV} + \mathcal{L}(\mathbf{w}_j - \mathbb{P}\mathbf{w}_j) \\ &\le C^{V} \left(\int_\Omega |\mathbb{E}(\mathbf{w}_j)|^2 \,d\mathbf{x} \right)^{\frac{1}{2}}, \end{aligned}$$

hence

$$\int_\Omega |\mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2} h_j t_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j|^2 \,d\mathbf{x} \to 0.$$

On the other hand, by the Fatou Lemma,

$$\liminf_{j\to+\infty} \int_\Omega |2\mathbb{E}(\mathbf{w}_j) + h_j t_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j|^2 \,d\mathbf{x} \ge \int_\Omega |2\mathbb{E}(\mathbf{w}) - \mathbf{W}^2|\,d\mathbf{x},$$

and we get $2\mathbb{E}(\mathbf{w}) = \mathbf{W}^2$, which implies $\nabla\mathbf{w} = \text{skew}\nabla\mathbf{w} + \frac{1}{2}\mathbf{W}^2$, hence $\text{skew}\big(\nabla\mathbf{w}\big)$ is a gradient field, and there is a constant skew-symmetric matrix $\mathbf{Z}$ such that $\text{skew}\big(\nabla\mathbf{w}\big) = \mathbf{Z}$. By setting

$$\mathbf{R} := \mathbf{I} + \frac{1}{2}\mathbf{W}^2 + \frac{\sqrt{3}}{2}\mathbf{W},$$

and by applying formula (3.13), we get a $\mathbf{R} \in SO(N)$ that is $\nabla\mathbf{w} - (\mathbf{R} - \mathbf{I}) \in \mathcal{M}^{N\times N}_{skew}$, which implies $\nabla\mathbf{w} \in \mathbb{K} + \mathcal{M}^{N\times N}_{skew}$ whenever condition b) holds true.

Eventually, if condition c) holds true, by (3.33) we get

$$\int_\Omega \left| \nabla\mathbf{w}_j - \frac{\mathbf{R}_j - \mathbf{I}}{h_j t_j} \right|^2 \mathrm{d}\mathbf{x} \le \frac{A}{t_j^2} + \frac{C'}{t_j}, \tag{3.40}$$

and by taking into account that $h_j t_j \to +\infty$ and $|\mathbf{R}_j - \mathbf{I}|$ is bounded, by (3.33) we get $\nabla\mathbf{w}_{h_j} \to \mathbf{0}$ strongly in $L^2(\Omega; M^{N\times N})$, hence $\nabla\mathbf{w} \in \overline{\mathbb{K}} + \mathcal{M}^{N\times N}_{skew}$ still by Lemma 3.2.

By summarizing, in all three cases if $t_j := \| \mathbb{E}(\mathbf{v}_j) \|_{L^2} \to +\infty$ and $\mathcal{F}_{h_j}(t_j\mathbf{w}_j) \le C$ then $\nabla(t_j^{-1}\mathbf{v}_j) = \nabla\mathbf{w}_j \to \nabla\mathbf{w}$ strongly in $L^2(\Omega; \mathcal{M}^{N\times N})$ and $\nabla\mathbf{w} \in \overline{\mathbb{K}} + \mathcal{M}^{N\times N}_{skew} = \mathbb{K} + \mathcal{M}^{N\times N}_{skew}$.

Therefore, $\mathbb{E}(\mathbf{w}_j) \to \mathbb{E}(\mathbf{w})$ strongly in $L^2(\Omega; \mathcal{M}^{N\times N})$.

Since $\widetilde{\mathbf{w}}_j := \mathbf{w}_j - \mathbb{P}\mathbf{w}_j$ are equibounded in $H^1(\Omega; \mathbb{R}^N)$, every subsequence of $\widetilde{\mathbf{w}}_j$ has a weakly convergent subsequence and if $\widetilde{\mathbf{w}}$ is one of the limits we get $\mathbb{E}(\widetilde{\mathbf{w}}) = \mathbb{E}(\mathbf{w})$ hence by (2.24) $\mathcal{L}(\widetilde{\mathbf{w}}) = \mathcal{L}(\mathbf{w})$. Therefore every subsequence of $\mathcal{L}(\widetilde{\mathbf{w}}_j)$ has a subsequence which converges to $\mathcal{L}(\mathbf{w})$ that is the whole sequence $\mathcal{L}(\widetilde{\mathbf{w}}_j)$ converges to $\mathcal{L}(\mathbf{w})$, hence

$$-\mathcal{L}(\mathbf{w}) = -\limsup_{j\to+\infty} \mathcal{L}(\widetilde{\mathbf{w}}_j) = -\limsup_{j\to+\infty} \mathcal{L}(\mathbf{w}_j) \le \liminf_{j\to+\infty} t_j^{-1}\mathcal{F}_{h_j}(\mathbf{v}_j). \tag{3.41}$$

Since (3.32) entails $\limsup t_j^{-1}\mathcal{F}_{h_j}(\mathbf{v}_j) \le 0$, by (3.41) we get $\mathcal{L}(\mathbf{w}) \ge 0$.

By taking into account that $\nabla\mathbf{w} \in \mathbb{K} + \mathcal{M}^{N\times N}_{skew}$, then we have either $\nabla\mathbf{w} \in \mathcal{M}^{N\times N}_{skew}$ or

$$\begin{gathered}\mathbf{w}(\mathbf{x}) = \tau(\mathbf{R} - \mathbf{I})\mathbf{x} + \mathbf{A}\mathbf{x} + \mathbf{c}, \text{ for some } \tau > 0, \ \mathbf{R} \in SO(N), \\ \mathbf{R} \ne \mathbf{I}, \ \mathbf{A} \in \mathcal{M}^{N\times N}_{skew}, \ \mathbf{c} \in \mathbb{R}^N.\end{gathered}$$

The second case cannot occur, since in such a case, by (2.2), there would exist $\vartheta \in \mathbb{R}$ with $\cos\vartheta < 1$ and $\mathbf{W} \in \mathcal{M}^{N\times N}_{sym}$, $\mathbf{W} \not\equiv \mathbf{0}$ such that $\mathbf{R} = \mathbf{I} + (1 - \cos\vartheta)\mathbf{W}^2 + (\sin\vartheta)\mathbf{W} \in SO(N)$, hence (2.24) and (2.25) would entail the contradiction

$$\begin{aligned}\mathcal{L}(\mathbf{w}) &= \tau\int_{\partial\Omega} \mathbf{f}\cdot(\mathbf{R} - \mathbf{I})\mathbf{x}\, d\mathcal{H}^{N-1} + \tau\int_\Omega \mathbf{g}\cdot(\mathbf{R} - \mathbf{I})\mathbf{x} = \\ &= \tau(1 - \cos\vartheta)\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{W}^2\mathbf{x}\, d\mathcal{H}^{N-1} + \tau(1 - \cos\vartheta)\int_\Omega \mathbf{g}\cdot\mathbf{W}^2\mathbf{x} < 0.\end{aligned} \tag{3.42}$$

Hence $\nabla\mathbf{w} \in \mathcal{M}^{N\times N}_{skew}$, that is $\mathbb{E}(\mathbf{w}) = \mathbf{0}$, which is again a contradiction, since $\|\mathbb{E}(\mathbf{w}_j)\|_{L^2} = 1$ and $\mathbb{E}(\mathbf{w}_j) \to \mathbb{E}(\mathbf{w})$ in $L^2(\Omega; \mathcal{M}^{N\times N})$. □

Proof of Theorem 2.2 -. First we notice that minimizing sequences for $\mathcal{F}_{h_j}$ do exist for every sequence of positive real numbers h_j converging to 0, thanks to Lemma 3.1.

Fix a sequence of real numbers $h_j > 0$ converging to 0 and a minimizing sequence $\mathbf{v}_j$ for $\mathcal{F}_{h_j}$.

Up to a preliminary extraction of a subsequence we can assume that h_j is decreasing.

Since $-\infty < \inf \mathcal{F}_{h_j} \leq 0$ there exists $C > 0$ such that $\mathcal{F}_{h_j}(\mathbf{v}_j) \leq C$, hence, by Lemma 3.6,

$$\|\mathbb{E}(\mathbf{v}_j)\|_{L^2} \leq C,$$

and by Lemma 3.2, there exists $\mathbf{v}_0 \in H^1(\Omega; \mathbb{R}^N)$ such that, up to subsequences, $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v}_0)$ in $L^2(\Omega; \mathcal{M}^{N\times N})$, thus proving (2.26). By Lemma 3.4 we get that

$$\liminf_{j\to+\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \mathcal{F}(\mathbf{v}_0)\,.$$

Still by Lemma 3.4, for every $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ there exists $\widetilde{\mathbf{v}}_j \in H^1(\Omega; \mathbb{R}^N)$ such that $\mathbb{E}(\widetilde{\mathbf{v}}_j) \rightharpoonup \mathbb{E}(\mathbf{v})$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ and

$$\limsup_{j\to+\infty} \mathcal{F}_j(\widetilde{\mathbf{v}}_{h_j}) \leq \mathcal{F}(\mathbf{v})\,,$$

hence

$$\mathcal{F}(\mathbf{v}_0) \leq \liminf_{n\to+\infty} \mathcal{F}_j(\mathbf{v}_{h_j}) \leq \liminf_{n\to+\infty} (\inf \mathcal{F}_{h_j} + o(1)) \leq \limsup_{j\to+\infty} \mathcal{F}_{h_j}(\widetilde{\mathbf{v}}_j) \leq \mathcal{F}(\mathbf{v}), \tag{3.43}$$

and (2.28) is proven.

Eventually we notice that (2.27) follows by labeling with $\mathbf{W}_0$ the skew symmetric matrix $\mathbf{W}$ in the convergence relationship (3.18) obtained in the proof of claim i) in Lemma 3.4.

Thus we have only to prove (2.29): to this end, by arguing as in the proof of claim i) in Lemma 3.4, we have (3.24), hence we get

$$\mathcal{F}(\mathbf{v}_0) = \liminf_{j\to+\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0) - \tfrac{1}{2}\mathbf{W}_0^2)\,\mathrm{d}\mathbf{x} \geq \mathcal{F}(\mathbf{v}_0),$$

thus proving (2.29). □

4. Limit Problem and Linear Elasticity

We denote by $\mathcal{E} : H^1(\Omega; \mathbb{R}^N) \to \mathbb{R}$ the energy functional of classical linear elasticity

$$\mathcal{E}(\mathbf{v}) := \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}))\,\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v})\,. \tag{4.1}$$

Notice that (1.6) is just a particular model case of (4.1) corresponding to (1.4).

As was already emphasized, the inequality $\mathcal{F} \leq \mathcal{E}$ always holds true. Moreover the two functionals cannot coincide: indeed $\mathcal{F}(\mathbf{v}) < \mathcal{E}(\mathbf{v})$ whenever $\mathbf{v}(\mathbf{x}) = \frac{1}{2}\mathbf{W}^2\mathbf{x}$ with $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$. However we can show that the two functionals $\mathcal{F}$ and $\mathcal{E}$, notwithstanding their differences, have the same minimum and same set of minimizers when the loads are equilibrated and compatible, that is they fulfil both (2.24) and (2.25).

The next results clarify the relationship between the minimizers of classical linear elasticity functional $\mathcal{E}$ and the minimizers of functional $\mathcal{F}$ defined in (1.8) , and coincident with the Gamma limit of nonlinear energies $\mathcal{F}_h$, in the sense stated in Theorem 2.2.

Theorem 4.1. *Assume that* (2.24) *and* (2.25) *hold true. Then*

$$\min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) \;=\; \min_{\mathbf{w}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{E}(\mathbf{w}) \tag{4.2}$$

and

$$\operatorname{argmin}_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)}\mathcal{F} \;=\; \operatorname{argmin}_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)}\mathcal{E}. \tag{4.3}$$

Proof. Both functionals $\mathcal{F}$, $\mathcal{E}$ do have minimizers under conditions (2.24) and (2.25), thanks to Theorem 2.2. Taking into account that $\mathcal{F}(\mathbf{v}) \leq \mathcal{E}(\mathbf{v})$ for every $\mathbf{v} \in H^1(\Omega;\mathbb{R}^N)$, and setting $\mathbf{z_W}(\mathbf{x}) := \frac{1}{2}\mathbf{W}^2\mathbf{x}$ for every $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$, we get $\mathbb{E}(\mathbf{z_W}) = \frac{1}{2}\mathbf{W}^2$ and

$$\begin{aligned}
&\min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{E}(\mathbf{v}) \;\geq\; \min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) = \\
&\min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \left\{ \min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \left\{ \int_\Omega \mathcal{V}_0(\mathbf{x},\, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}) \right\} \right\} = \\
&\min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \left\{ \min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{V}_0(\mathbf{x},\, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}) \right\} \right\} = \\
&\min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \left\{ \min_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{V}_0(\mathbf{x},\, \mathbb{E}(\mathbf{v}-\mathbf{z_W}))\, d\mathbf{x} - \mathcal{L}(\mathbf{v}-\mathbf{z_W}) - \mathcal{L}(\mathbf{z_W}) \right\} \right\} = \\
&\min_{\mathbf{z}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{E}(\mathbf{z}) \;-\; \max_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \mathcal{L}(\mathbf{z_W}) \;=\; \min_{H^1(\Omega;\mathbb{R}^N)} \mathcal{E}\,,
\end{aligned} \tag{4.4}$$

where the last inequality follows by $\mathcal{L}(z_\mathbf{W}) \leq 0$, due to (2.25) and $\mathcal{L}(\mathbf{0})$. Therefore (4.2) is proved and we are left to show (4.3).

First, assume $\mathbf{v} \in \operatorname{argmin}_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)}\mathcal{F}$ and let

$$\mathbf{W_v} \in \operatorname{argmin}\left\{ \int_\Omega \mathcal{V}_0\Big(\mathbf{x},\, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2\Big)\, d\mathbf{x} :\; \mathbf{W} \in \mathcal{M}^{N\times N}_{skew} \right\}. \tag{4.5}$$

If $\mathbf{W_v} \neq \mathbf{0}$ then, by setting $\mathbf{z}_{\mathbf{W_v}} = \frac{1}{2}\mathbf{W}_\mathbf{v}^2\,\mathbf{x}$, we get $\mathbb{E}(\mathbf{z}_{\mathbf{W_v}}) = \nabla \mathbf{z}_{\mathbf{W_v}} = \frac{1}{2}\mathbf{W}_\mathbf{v}^2$ and, by compatibility (2.25), we obtain

$$\min \mathcal{F} = \mathcal{F}(\mathbf{v}) = \int_\Omega \mathcal{V}_0\Big(\mathbf{x},\ \mathbb{E}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}})\Big)\,d\mathbf{x} - \mathcal{L}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W_v}}) = \\ \mathcal{E}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W_v}}) \geq \min \mathcal{E} - \mathcal{L}(\mathbf{z}_{\mathbf{W_v}}) > \min \mathcal{E}\,, \tag{4.6}$$

which is a contradiction. Therefore $\mathbf{W_v} = \mathbf{0}$, $\mathbf{z}_{\mathbf{W_v}} = \mathbf{0}$, and all the inequalities in (4.6) turn out to be equalities, hence we get $\mathcal{F}(\mathbf{v}) = \mathcal{E}(\mathbf{v}) = \min \mathcal{E} = \min \mathcal{F}$, say, $\mathbf{v} \in \operatorname{argmin}_{H^1(\Omega;\mathbb{R}^N)}\mathcal{E}$ and $\operatorname{argmin}_{H^1(\Omega;\mathbb{R}^N)}\mathcal{F} \subset \operatorname{argmin}_{H^1(\Omega;\mathbb{R}^N)}\mathcal{E}$.

In order to show the opposite inclusion, we assume $\mathbf{v} \in \operatorname{argmin}_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)}\mathcal{E}$ and, still referring to the choice (4.5), we set $\mathbf{z}_{\mathbf{W_v}} = \frac{1}{2}\mathbf{W}_\mathbf{v}^2\,\mathbf{x}$. Then

$$\mathcal{F}(\mathbf{v}) = \int_\Omega \mathcal{V}_0\Big(\mathbf{x},\ \mathbb{E}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}})\Big)\,d\mathbf{x} - \mathcal{L}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W_v}}) = \\ = \mathcal{E}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W_v}}) \geq \mathcal{F}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W_v}})\,. \tag{4.7}$$

This leads to the contradiction $\mathcal{F}(\mathbf{v}) > \mathcal{F}(\mathbf{v}-\mathbf{z}_{\mathbf{W_v}})$ if $\mathbf{z}_{\mathbf{W_v}} \neq \mathbf{0}$, due to (2.25); therefore $\mathbf{z}_{\mathbf{W_v}} = \mathbf{0}$, and we have equalities in place of inequalities in (4.7). Therefore $\mathcal{E}(\mathbf{v}) = \mathcal{F}(\mathbf{v})$ and $\mathbf{v} \in \operatorname{argmin}_{H^1(\Omega;\mathbb{R}^N)}\mathcal{F}$. □

Corollary 4.2. *Assume that the standard structural assumptions hold true along with* (2.24) *and* (2.25)*, and that* $\mathbf{W}_0 \in \mathcal{M}^{N\times N}_{skew}$ *is the matrix whose existence is warranted by in Theorem* 2.2*. Then* $\mathbf{W}_0 = \mathbf{0}$*.*

Proof. Let $\mathbf{v}_0$ be in $\operatorname{argmin}\mathcal{F}$, let $\mathbf{W}_0$ be the skew symmetric matrix in the claim of Theorem 2.2, and assume by contradiction that $\mathbf{W}_0 \neq \mathbf{0}$. By (4.2) and (4.3) we get

$$\int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0) - \tfrac{1}{2}\mathbf{W}_0^2)\,d\mathbf{x} = \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0))\,d\mathbf{x},$$

and by taking into account that $\mathcal{V}_0$ is a quadratic form,

$$\int_\Omega \mathcal{V}_0(\mathbf{x}, \tfrac{1}{2}\mathbf{W}_0^2)\,d\mathbf{x} - \frac{1}{2}\int_\Omega D\mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0))\cdot \mathbf{W}_0^2\,d\mathbf{x} = 0.$$

Since, by (4.3), $\mathbf{v}_0 \in \operatorname{argmin}\mathcal{E}$, the Euler-Lagrange equation yields

$$\int_\Omega D\mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0))\cdot \mathbf{W}_0^2\,d\mathbf{x} = \mathcal{L}(\mathbf{W}_0^2\mathbf{x})\,,$$

hence, by (2.25),

$$0 \leq \int_\Omega \mathcal{V}_0(\mathbf{x}, \tfrac{1}{2}\mathbf{W}_0^2)\,d\mathbf{x} = \frac{1}{2}\mathcal{L}(\mathbf{W}_0^2\mathbf{x}) < 0,$$

which is a contradiction.

Hence $\mathbf{W}_0 = \mathbf{0}$. □

If the strong inequality in (2.25) is replaced by a weak inequality, then Theorem 4.1 cannot hold true, as is shown by the following general result:

Proposition 4.3. *If the structural assumptions together with* (2.24) *are fulfilled, but* (2.25) *is replaced by*

$$\mathcal{L}(\mathbf{W}^2\mathbf{x}) \le 0 \qquad \forall\, \mathbf{W} \in \mathcal{M}^{N\times N}_{skew}\,, \tag{4.8}$$

then argmin$\mathcal{F}$ *is still nonempty and*

$$\min \mathcal{F} = \min \mathcal{E}\,, \tag{4.9}$$

but the coincidence of minimizer sets is replaced by the inclusion

$$\mathrm{argmin}\mathcal{E} \subset \mathrm{argmin}\mathcal{F}\,. \tag{4.10}$$

If (4.8) *holds true and there exists* $\mathbf{U} \in \mathcal{M}^{N\times N}_{skew}$, $\mathbf{U} \neq \mathbf{0}$ *such that* $\mathcal{L}(\mathbf{U}^2\mathbf{x}) = 0$*, then* $\mathcal{F}$ *admits infinitely many minimizers which are not minimizers of* $\mathcal{E}$*; more precisely,*

$$\mathrm{argmin}\mathcal{E} \underset{\neq}{\subset} \mathrm{argmin}\mathcal{E} + \left\{\mathbf{U}^2\mathbf{x} \,:\, \mathbf{U} \in \mathcal{M}^{N\times N}_{skew},\ \mathcal{L}(\mathbf{U}^2\mathbf{x}) = 0\right\} \subset \mathrm{argmin}\mathcal{F}\,, \tag{4.11}$$

where the last inclusion is an equality in 2D:

$$\mathrm{argmin}\mathcal{E} \underset{\neq}{\subset} \mathrm{argmin}\mathcal{E} + \{-t\,\mathbf{x} \,:\, t \ge 0\} = \mathrm{argmin}\mathcal{F}\,, \qquad \textit{if } N = 2\,. \tag{4.12}$$

Proof. The set argmin$\mathcal{E}$ is nonempty by classical arguments. Fix $\mathbf{v}_* \in \mathrm{argmin}\mathcal{E}$. Then, for every $\mathbf{v} \in H^1(\Omega;\mathbb{R}^N)$, and for every $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$, by setting $\mathbf{z_W} = \frac{1}{2}\mathbf{W}^2\mathbf{x}$, we get

$$\begin{aligned}\mathcal{F}(\mathbf{v}_*) \le \mathcal{E}(\mathbf{v}_*) \le \mathcal{E}(\mathbf{v} - \mathbf{z_W}) &= \int_\Omega \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2)\,\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v} - \mathbf{z_W})\\ &\le \int_\Omega \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2)\,\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v}),\end{aligned} \tag{4.13}$$

hence, for every $\mathbf{v} \in H^1(\Omega;\mathbb{R}^N)$,

$$\mathcal{F}(\mathbf{v}_*) \le \min_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \int_\Omega \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2)\,\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v}) = \mathcal{F}(\mathbf{v}), \tag{4.14}$$

thus proving that argmin$\mathcal{F}$ is nonempty as well as (4.10).

Moreover, by setting

$$\mathbf{W_v} \in \mathrm{argmin}\left\{\int_\Omega \mathcal{V}_0\big(x, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2\big)\,\mathrm{d}\mathbf{x} :\ \mathbf{W} \in \mathcal{M}^{N\times N}_{skew}\right\}, \qquad \forall\,\mathbf{v} \in H^1(\Omega;\mathbb{R}^N), \tag{4.15}$$

condition (4.8) entails that for every $\mathbf{v}_* \in \mathrm{argmin}\,\mathcal{E}$,

$$\mathcal{F}(\mathbf{v}_*) = \int_\Omega \mathcal{V}_0\big(x, \mathbb{E}(\mathbf{v}_* - \mathbf{z}_{\mathbf{W}_{\mathbf{v}_*}})\big)\,\mathrm{d}\mathbf{x} - \mathcal{L}(\mathbf{v}_*) = \mathcal{E}(\mathbf{v}_* - \mathbf{z}_{\mathbf{W}_{\mathbf{v}_*}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}_*}}) \ge \mathcal{E}(\mathbf{v}_*), \tag{4.16}$$

hence (4.9) follows by $\mathcal{F} \le \mathcal{E}$.

If (4.8) holds true, $\mathcal{L}(\mathbf{z_U}) = 0$ for some $\mathbf{0} \neq \mathbf{U} \in \mathcal{M}^{N\times N}_{skew}$ and $\mathbf{v}^* \in \operatorname{argmin} \mathcal{E}$, then, by comparing the finite dimensional minimization over $\mathbf{W}$ with evaluation at $\mathbf{W} = \mathbf{U}$ and exploiting (4.9), we get

$$\mathcal{F}(\mathbf{v}_* + \mathbf{z_U}) = \min_{\mathbf{W}} \int_\Omega \mathcal{V}_0\big(x, \mathbb{E}(\mathbf{v}_*) + \tfrac{1}{2}\mathbf{U}^2 - \tfrac{1}{2}\mathbf{W}^2\big)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}_* + \mathbf{z_U}) \leq$$
$$\leq \int_\Omega \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}_*))\, dx - \mathcal{L}(\mathbf{v}_*) = \mathcal{E}(\mathbf{v}_*) = \min \mathcal{E} = \min \mathcal{F}\,, \tag{4.17}$$

that is, $\mathbf{v}_* + \mathbf{z_U} \in \operatorname{argmin}\mathcal{F}$. Since $\mathcal{V}_0$ is strictly convex we get $\operatorname{argmin}\mathcal{E} = \{\mathbf{v}_* + \mathbf{z} : \mathbf{z} \in \mathcal{R}\}$, hence $\mathcal{E}(\mathbf{v}_* + \mathbf{z_U}) > \mathcal{E}(\mathbf{v}_*)$, thus proving the strict inclusion in (4.11).

Concerning the last claim, if (4.8) holds true, $\mathcal{L}(\mathbf{z_U}) = 0$ for some $\mathbf{0} \neq \mathbf{U} \in \mathcal{M}^{N\times N}_{skew}$, $\mathbf{v}^* \in \operatorname{argmin}\mathcal{F}$ and $N = 2$, then $\mathcal{M}^{2\times 2}_{skew}$ is a 1D space, therefore we can assume $\mathbf{U} = (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1)$, $\mathbf{U}^2 = -\mathbf{I}$, $\mathcal{M}^{2\times 2}_{skew} = \operatorname{span}\mathbf{U}$ and $\mathbf{W}_{\mathbf{v}_*} = \lambda\mathbf{U}$ for some $\lambda \in \mathbb{R}$, and by (4.9),

$$\min \mathcal{E} \;=\; \min \mathcal{F} \;=\; \mathcal{F}(\mathbf{v}_*) \;=\; \int_\Omega \mathcal{V}_0\Big(\mathbb{E}(\mathbf{v}_*) - \frac{1}{2}\mathbf{W}^2_{\mathbf{v}_*}\Big)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}_*) \;=$$
$$=\; \int_\Omega \mathcal{V}_0\Big(\mathbb{E}(\mathbf{v}_*) - \frac{\lambda^2}{2}\mathbf{U}^2\Big)\, d\mathbf{x} - \mathcal{L}(\mathbf{v}_* - \mathbf{z}_{\lambda U}) \;=\; \mathcal{E}(\mathbf{v}_* - \mathbf{z}_{\lambda U})\,,$$

that is $(\mathbf{v}_* - \mathbf{z}_{\lambda U}) \in \operatorname{argmin}\mathcal{E}$ for every $\mathbf{v}_* \in \operatorname{argmin}\mathcal{F}$, therefore we get

$$\operatorname{argmin}\mathcal{F} - \{\,\mathbf{z}_{\lambda\mathbf{U}} \,:\, \lambda \in \mathbb{R}\} \subset \operatorname{argmin}\mathcal{E}\,, \quad \operatorname{argmin}\mathcal{F} \subset \operatorname{argmin}\mathcal{E} + \{\,\mathbf{z}_{\lambda\mathbf{U}} \,:\, \lambda \in \mathbb{R}\}\,,$$

hence by $z_{\lambda\mathbf{U}} = \frac{\lambda^2}{2}\mathbf{U}^2\mathbf{x} = -\frac{\lambda^2}{2}\mathbf{x}$ we obtain the equality in place of the last inclusion in (4.11), hence (4.12). □

The next example depicts the above Proposition in a simple explicit case.

Example 4.4. Let $\Omega = (-1/2, 1/2)^2$, $\mathbf{g} \equiv \mathbf{0}$, $\mathbf{f} = (\mathbf{1}_{S_+} - \mathbf{1}_{S_-})\mathbf{e}_2 + (\mathbf{1}_{T_+} - \mathbf{1}_{T_-})\mathbf{e}_1$ where $S_\pm$ denotes, respectively, the right and the left sides, and $T_\pm$ the upper and the lower sides of the square. A straightforward computation gives, for suitable $\lambda \in \mathbb{R}$,

$$\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{W}^2\,\mathbf{x}\, d\mathcal{H}^{N-1} = -\lambda^2 \int_{\partial\Omega} \mathbf{f}\cdot\mathbf{x}\, d\mathcal{H}^{N-1} = 0 \qquad \forall\, \mathbf{W} \in \mathcal{M}^{2\times 2}_{skew}\,. \tag{4.18}$$

Then, since (2.24) and (4.8) are fulfilled, by (4.12) in Proposition 4.3 we have the whole description of the set $\operatorname{argmin}\mathcal{F}$: for every choice of $\mathcal{V}_0$ satisfying the standard structural hypotheses, $\mathcal{F}$ has infinitely many minimizers $\mathbf{v}$ which are not minimizers of $\mathcal{E}$; explicitly, they are of the kind

$$\mathbf{v} = (\mathbf{v}_* - t\mathbf{x}) \in \operatorname{argmin}\mathcal{F} \setminus \operatorname{argmin}\mathcal{E} \qquad \text{if } \mathbf{v}_* \in \operatorname{argmin}\mathcal{E},\; t > 0\,.$$

It is quite natural to ask whether condition (2.25), which is essential in the proof of Theorem 2.2, may be dropped in order to obtain at least the existence of $\min \mathcal{F}$: the answer is negative.

Actually the next remark shows that, when the inequality in compatibility condition (2.25) is reversed for at least one choice of the skew-symmetric matrix $\mathbf{W}$, then $\mathcal{F}$ is unbounded from below.

Remark 4.5. If

$$\exists\, \mathbf{W}_* \in \mathcal{M}^{N\times N}_{skew} \ : \qquad \mathcal{L}(\mathbf{z}_{\mathbf{W}_*}) > 0\,, \quad \text{where } \mathbf{z}_{\mathbf{W}_*} = \frac{1}{2}\mathbf{W}_*^2\mathbf{x}\,, \tag{4.19}$$

then

$$\inf_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) \ = \ -\infty. \tag{4.20}$$

Indeed, by arguing as in (4.4) and replacing $\min_{H^1}$ with $\inf_{H^1}$, we get

$$\inf_{H^1(\Omega;\mathbb{R}^N)} \mathcal{F} \ = \ \min_{H^1(\Omega;\mathbb{R}^N)} \mathcal{E} \ - \ \sup_{\mathbf{W}\in\mathcal{M}^{N\times N}_{skew}} \mathcal{L}(\mathbf{z}_{\mathbf{W}}) \qquad \text{where } \ \mathbf{z}_{\mathbf{W}} = \frac{1}{2}\mathbf{W}^2\mathbf{x}\,. \tag{4.21}$$

Hence

$$\inf_{H^1(\Omega;\mathbb{R}^N)} \mathcal{F} \ \le \ \min_{H^1(\Omega;\mathbb{R}^N)} \mathcal{E} \ - \ \tau\mathcal{L}(\mathbf{z}_{\mathbf{W}_*}) \qquad \forall\, \tau > 0\,,$$

which entails (4.20).

The next example shows that in the case of uniform compression along the whole boundary functional, $\mathcal{F}$ is unbounded from below.

Example 4.6. Assume $\Omega \subset \mathbb{R}^N$ is a Lipschitz, connected open set, $N = 2, 3$, $\mathbf{g} \equiv \mathbf{0}$, $\mathbf{f} = -\mathbf{n}$, where $\mathbf{n}$ denotes the outer unit normal vector to $\partial\Omega$.

Then (4.19) holds true, hence, by Remark 4.5, $\inf_{\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)} \mathcal{F}(\mathbf{v}) \ = \ -\infty$.

Indeed, for every $\mathbf{W} \in \mathcal{M}^{N\times N}_{skew}$ such that $|\mathbf{W}|^2 = 2$, we obtain

$$\begin{aligned}\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{W}^2\mathbf{x}\, d\mathcal{H}^{N-1} \ &= -\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{W}^2\mathbf{x}\, d\mathcal{H}^{N-1} \\ &= -\int_{\Omega} \operatorname{div}(\mathbf{W}^2\mathbf{x})\, \mathrm{d}\mathbf{x} \ = \ -\,|\Omega|\,\mathrm{Tr}\,\mathbf{W}^2 \ = \ 2\,|\Omega| > 0\,.\end{aligned}$$

This means that any Lipschitz open set turns out to be always unstable when uniformly compressed in the direction of the inward normal vector along its boundary. Therefore the linearized model proves inadequate for such a case, even for a small load.

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Francesco Maddalena
Dipartimento di Meccanica, Matematica, Management,
Politecnico di Bari,
Via Re David 200,
70125 Bari
Italy.
e-mail: francesco.maddalena@poliba.it

and

Danilo Percivale
Dipartimento di Ingegneria Meccanica,
Università di Genova,
Piazzale Kennedy, Fiera del Mare, Padiglione D,
16129 Genova
Italy.
e-mail: percivale@diptem.unige.it

and

Franco Tomarelli
Dipartimento di Matematica,
Politecnico di Milano,
Piazza Leonardo da Vinci 32,
20133 Milano
Italy.
e-mail: franco.tomarelli@polimi.it

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