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Continuous k-to-1 functions between complete graphs of even order

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ABSTRACT

A function between graphs is k-to-1 if each point in the co-domain has precisely k pre-images in the domain. Given two graphs, G and H, and an integer $k \geq 1$, and considering G and H as subsets of $\mathbb{R}^3$, there may or may not be a k-to-1 continuous function (i.e. a k-to-1 map in the usual topological sense) from G onto H. In this paper we review and complete the determination of whether there are finitely discontinuous, or just infinitely discontinuous k-to-1 functions between two intervals, each of which is one of the following: $]0, 1[$, $[0, 1[$ and $[0, 1]$. We also show that for k even and $1 \leq r < 2s$, $(r, s) \neq (1, 1)$ and $(r, s) \neq (3, 2)$, there is a k-to-1 map from K_{2r} onto K_{2s} if and only if $k \geq 2s$.

1. Introduction

The study of k-to-1 continuous functions (or maps) between topological spaces dates back quite a while. In the particular case where the topological space is a graph (each edge being homeomorphic to $[0, 1]$), it dates back to 1939 when Harrold [3] showed that the graph consisting of two vertices and an edge between them, i.e. $[0, 1]$, could not be mapped (continuously) 2-to-1 onto any non-trivial topological space. Thus, if $G = K_2$ and H is any graph, then there does not exist a 2-to-1 map from G onto H. One question raised by Harrold's result is to wonder what happens with other intervals, not just closed ones. This has been investigated earlier, but not all possibilities appear to have been gone into so far; for example the case when the open unit interval $]0, 1[$ is mapped onto the closed unit interval $[0, 1]$. Here we complete this line of investigation for k-to-1 maps.

In [10], Hilton initiated a study of k-to-1 maps from a complete graph K_n onto a complete graph K_m. The ultimate objective is to determine all triples (n, k, m) for which there is a k-to-1 map from K_n onto K_m. In this and some subsequent papers we shall largely answer this question. In this paper our main task will be to show that if $1 \leq r < 2s$, k is even and $(r, s) \notin \{(1, 1), (3, 2)\}$, then there is a k-to-1 map from K_{2r} onto K_{2s} if and only if $k \geq 2s$.

1.1. Preliminary results and definitions

In this section, our main task is to state and prove some preliminary lemmas on continuous k-to-1 functions (henceforth called k-to-1 maps) acting on intervals, or between graphs. We also state some basic definitions. We should draw attention to the fact that we are denoting half-open intervals, or open intervals, by a reverse square bracket: $]a, b]$, $[a, b[$ or $]a, b[$. The notation (a, b) will be reserved for the ordered pair, a then b.

Definition 1.1. (i) If a function $f : A \to B$ is ***k* -to-1** for $k \in \mathbb{N}$, then for all $y \in \text{Rng}(A)$, the number of elements in $f^{-1}(y)$ is equal to k.

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(ii) A function $f : A \to B$ is **$(\leq k)$ -to-1** for $k \in \mathbb{N}$ if, for all $y \in \text{Rng}(A)$, the number of elements in $f^{-1}(y)$ is at most k.
(iii) A **map** is a continuous function $f : A \to B$.

Note that if f is a $(\leq k)$-to-1 function from A onto B, then each point of B has at least one pre-image.

A homeomorphic image in $\mathbb{R}^3$ of a closed bounded interval in $\mathbb{R}$ will be called an **arc**. A **topological graph *G*** is the union of a finite number of arcs which intersect only at their end points. The end points of the arcs will be called **vertices**. The pair of vertices (a, b) which have the property that a and b are the end points of an arc will be called an **edge** and may be denoted by ab. The vertex set $V(G)$ and the edge set $E(G)$ constitute an abstract graph, and (so long as it causes no confusion) we may take G to also denote the **abstract graph** with vertex set $V(G)$ and edge set $E(G)$. The normal definitions in abstract graph theory will be employed, including *inter alia* path, walk, connected, and Eulerian circuit (see any standard introductory book on graph theory, e.g. [14]).

Every simple abstract graph (simple meaning that there are no loops or multiple edges) can be represented by a topological graph in the 3-dimensional space $\mathbb{R}^3$ whose edges are straight line segments joining the two vertices. For simple graphs this can be done by labelling the vertices in the vertex-set $V(G)$ by vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ such that the pth vertex $\boldsymbol{v}_p$ (for $p = 1, 2, \ldots, n$) has coordinates $(p, p^2, p^3) \in \mathbb{R}^3$. At this point we remark that a parallelepiped formed by taking three vectors based on the same origin has volume equal to zero if and only if the vectors are coplanar [1]. It follows that if the three points were collinear, then the parallelepiped would have zero volume. Thus,

- no three distinct vertices $\boldsymbol{v}_j$, $\boldsymbol{v}_k$ and $\boldsymbol{v}_l$ are collinear since $\begin{vmatrix} j & j^2 & j^3 \\ k & k^2 & k^3 \\ l & l^2 & l^3 \end{vmatrix} \neq 0$; and
- no four distinct vertices $\boldsymbol{v}_i$, $\boldsymbol{v}_j$, $\boldsymbol{v}_k$ and $\boldsymbol{v}_l$ are co-planar since $\begin{vmatrix} (i-j) & (i^2-j^2) & (i^3-j^3) \\ (i-k) & (i^2-k^2) & (i^3-k^3) \\ (i-l) & (i^2-l^2) & (i^3-l^3) \end{vmatrix} \neq 0$.

Therefore, all the edges $e_q = [\boldsymbol{v}_i, \boldsymbol{v}_j]$ (for $q = 1, 2, \ldots, m$ and for $1 \leq i, j \leq n$) in the edge-set $E(G) = \{e_1, \ldots, e_m\}$ can be represented by straight-line segments joining the two end-vertices.

A well-known result about continuous images of intervals is stated in the following lemma.

Lemma 1.2. (i) *If I is a subinterval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval.*
(ii) *If the interval I in* (i) *is closed and bounded, then so is $f(I)$.*

A standard definition which is of importance in this work is given below.

Definition 1.3. The number of edges a point x_i is incident with is known as the **order** (or valency or degree) of the point, and is denoted by $O(x_i)$. The order of a point which is not a vertex is two. If X is a set of points, then $O(X) = \sum_{x \in X} O(x)$.

In the sequel, we need to refer to the following two results, which we prove below.

Lemma 1.4. *Let f be a map from $[a, b]$ onto $[c, d]$ such that $f(a) = c$ and $f(b) = d$. Let $\beta \in]c, d[$ be such that $f^{-1}(\beta) = \{\alpha_1, \ldots, \alpha_n\}$ for some n, and none of (α_i, β) is a maximum or a minimum. Then n is odd.*

Proof. We consider the map f from $[a, b]$ onto $[c, d]$ such that $f(a) = c$ and $f(b) = d$. We let $\beta \in]c, d[$ be such that $f^{-1}(\beta) = \{\alpha_1, \ldots, \alpha_n\}$ for some n, and none of (α_i, β) is a maximum or a minimum.

We start by assuming that n is even, i.e. $n = 2p$ for $p \in \mathbb{N}$, and we define the function $g(x) = f(x) - \beta$. Then,

(i) $g(a) = f(a) - \beta = c - \beta < 0$ (since $\beta > c$);
(ii) $g(b) = f(b) - \beta = d - \beta > 0$ (since $d > \beta$);
(iii) $g(\alpha_i) = f(\alpha_i) - \beta = \beta - \beta = 0$.

Thus, for $1 \leq i \leq n$, α_i are the roots of $g(x)$. We order these roots such that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$.

We consider the first root α_1. Since $g(a) < 0$, then for each $x \in [a, \alpha_1[$, $g(x) < 0$, otherwise if $g(x) > 0$, by the Intermediate Value Theorem there is a root smaller than α_1, a contradiction. Also, if $x \in]\alpha_1, \alpha_1 + \delta[$, where δ is sufficiently small, and $g(x) < 0$, then $(\alpha_1, 0)$ is a local maximum, a contradiction. Therefore, for $x \in]\alpha_1, \alpha_1 + \delta[$, then $g(x) > 0$.

Furthermore, if $x \in]\alpha_1, \alpha_2[$, then $g(x) > 0$, since otherwise, by the Intermediate Value Theorem, there exists another root between α_1 and α_2, a contradiction.

Similarly, since for $x \in]\alpha_1, \alpha_2[$, $g(x) > 0$ and α_2 is a root, then for each $x \in]\alpha_2, \alpha_3[$, $g(x) < 0$. In general, for $x \in]\alpha_{2t-1}, \alpha_{2t}[$, $g(x) > 0$, and for $x \in]\alpha_{2t}, a_{2t+1}[$, $g(x) < 0$; $1 \leq t \leq p$. In particular,

- for $x \in]a_{2p-1}, a_{2p}[$ then $g(x) > 0$, and
- for $x \in]a_{2p}, 1]$ then $g(x) < 0$.

But by (ii), $g(b) > 0$, hence we have a contradiction. Therefore, n is odd. □

Theorem 1.5. *Let m be odd, $m \geq 1$. Let the intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_m, b_m]$ be disjoint, except possibly for the end points. For $1 \leq i \leq m$, let f_i be a continuous function mapping the interval $[a_i, b_i]$ onto an interval $[c, d]$ such that $f_i(a_i) = c$ and $f_i(b_i) = d$ or $f_i(a_i) = d$ and $f_i(b_i) = c$. Then there is a $\beta \in]c, d[$ such that $\sum_{i=1}^{m} |f_i^{-1}(\beta)| = n$ for some odd n.*

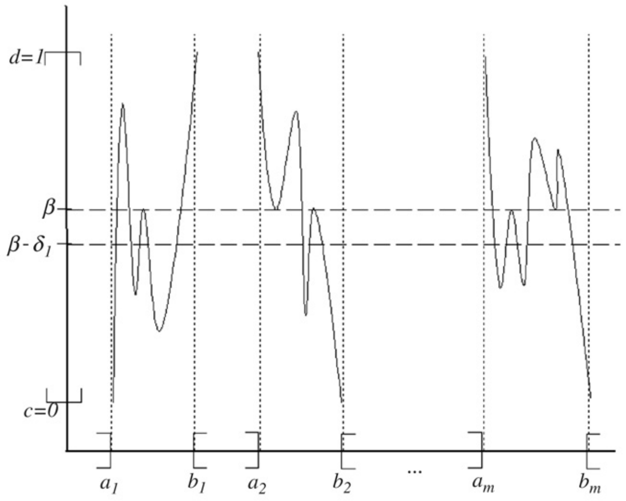

Fig. 1. Illustration of proof of Theorem 1.5.

Proof. We may suppose, without loss of generality, that $[c, d] = [0, 1]$, and that $]a_1, b_1[,]a_2, b_2[, \ldots,]a_m, b_m[$ are m disjoint open unit intervals on the real line. Let $\beta \in]c, d[$ and, for $1 \le i \le m$, suppose that $f_i(x) = \beta$ for n_i different points $x \in]a_i, b_i[$, or, in other words, suppose that the graph of the line $y = \beta$ intersects the curve f_i at n_i points. If, for each i, none of these points $(x, f_i(x))$ are local maxima or local minima, then, by Lemma 1.4, $n_1, n_2, \ldots, n_m$ are odd, and since m is odd by hypothesis, it follows that $\sum_{i=1}^{m} n_i$ is also odd, and the result is proved.

Thus, we may assume that, of all these points $(x, f_i(x))$, p are local maxima, q are local minima (where $p+q \ge 1$), and the remaining t are neither local maxima nor local minima; henceforth we refer to these are 'crossing points'. For $1 \le i \le m$, let t_i be the number of crossing points for the function f_i.

If $p \ge q$, we consider the line $y = \beta - \delta_1$, where $\delta_1 > 0$ is chosen sufficiently small so that, for $1 \le i \le m$, the line $y = \beta - \delta_1$ intersects the graph of f_i at n_{i1} points, where:

- for each of the maximum turning points touching the line $y = \beta$, two new crossing points are introduced;
- for each of the minimum turning points, no intersections are now present;
- for each crossing point, the number of intersections remains unchanged; and
- no new intersection points other than the ones mentioned above are introduced.

Fig. 1 illustrates this situation. Thus $\sum_{i=1}^{m} n_{i1} = 2p + t$.

If $p < q$, then we can apply a similar argument with the line $y = \beta + \delta_2$ (where $\delta_2 > 0$ is chosen sufficiently small), with the roles of the maximum and minimum turning points interchanged. Then, for each i, $1 \le i \le m$, we define n_{i1} to be the number of points in which the line $y = \beta + \delta_2$ intersects the graph of f_i such that $\sum_{i=1}^{m} n_{i1} = 2q + t$.

In either case, by Lemma 1.4, for each i, n_{i1} is odd, and so, since m is assumed to be odd as well, $\sum_{i=1}^{m} n_{i1}$ is odd. Thus $\sum_{i=1}^{m} |f_i^{-1}(\beta - \delta_1)|$ is odd if $p \ge q$, and $\sum_{i=1}^{m} |f_i^{-1}(\beta + \delta_2)|$ is odd if $p < q$, so Theorem 1.5 is true (possibly with $\beta - \delta_1$ or $\beta + \delta_2$ replacing β). □

1.2. Folds

Definition 1.6. A (p, q, r)-fold on $[a, b]$ is a map f from $[a, b]$ onto $[c, d]$ such that $F(a) = c$ and $F(b) = d$ and

(i) $\left|F^{-1}(c)\right| = p$
(ii) $\left|F^{-1}(d)\right| = r$
(iii) for all $y \in]c, d[$, $\left|F^{-1}(y)\right| = q$.

In slightly abnormal notation we denote the map F by $F(p, q, r)$. In the work that follows, we will require three special 'folds'. These are defined and constructed in terms of the functions shown below.

Definition 1.7. The map $\boldsymbol{F(m+1, 2m+1, m+1)}$ is the special $(m+1, 2m+1, m+1)$-fold from $[a, b]$ onto $[c, d]$ constructed as follows.

We consider the interval $[a, b]$ and split it into $2m + 1$ equal intervals $[a_i, a_{i+1}]$, for $0 \le i \le 2m$, such that $a_i = a + ih$ where $h = \frac{b-a}{2m+1}$. We take the first two intervals $[a, a_1]$ and $[a_1, a_2]$ and let the required function F restricted to $[a, a_2]$ be defined as follows.

(i) $F(a) = c$, $F(a_1) = d$ and $F(a_2) = c$;
(ii) $F(]d, d_1[)$ is the straight line segment joining c to d;
(iii) $F(]d_1, d_2[)$ is the straight line segment joining d to c.

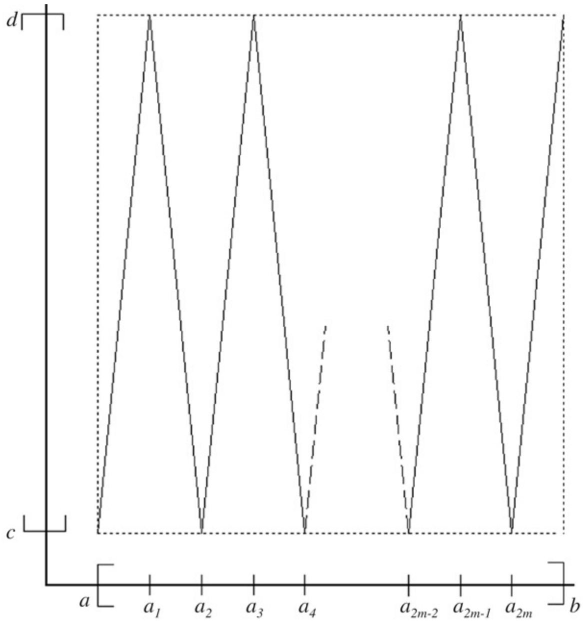

Fig. 2. $F(m+1, 2m+1, m+1)$.

Then the map F from $[a, b]$ onto $[c, d]$ is defined as the periodic function with period $2h$ obtained by extending the function f restricted to $[a, a_2]$ to the whole interval $[a, b]$.

Fig. 2 illustrates the map $F(m+1, 2m+1, m+1)$.

Definition 1.8. The map $\boldsymbol{F(m+1, 2m+1, 1)}$ is the special $(m+1, 2m+1, 1)$-fold from $[a, b]$ onto $[c, d]$ constructed as follows.

We consider the interval $[a, b]$ and the interval $[c, d]$ and let their respective mid-points be a_1 and c_1. Then we apply the function $F(m+1, 2m+1, m+1)$ from $[a, a_1]$ onto $[c, c_1]$. We consider the intervals $[a_1, b]$ and $[c_1, d]$ and let their respective midpoints be a_2 and c_2. We apply again the function $F(m+1, 2m+1, m+1)$ from $[a_1, a_2]$ onto $[c_1, c_2]$.

This procedure is repeated recursively by

(i) considering the intervals $[a_i, b]$ and $[c_i, d]$ and letting the respective mid-points be a_{i+1} and c_{i+1}, and
(ii) applying the function $F(m+1, 2m+1, m+1)$ from $[a_i, a_{i+1}]$ onto $[c_i, c_{i+1}]$.

The required function $F(m+1, 2m+1, 1)$ is thus obtained by taking the union of all the functions as described above from the interval $[a, b[$ onto the interval $[c, d[$ and mapping the point b to d.

Fig. 3 illustrates the map $F(m+1, 2m+1, 1)$.

Definition 1.9. The map $\boldsymbol{F(1, 2m+1, 1)}$ is the special $(1, 2m+1, 1)$-fold from $[a, b]$ onto $[c, d]$ constructed as follows.

We consider the interval $[a, b]$ and the interval $[c, d]$ and let their respective mid-points be a_1 and c_1. We take the interval $[a_1, b]$ and apply the function $f_1 = F(m+1, 2m+1, 1)$ from $[a_1, b]$ onto $[c_1, d]$. Then we take the interval $[a, a_1]$ and apply the function $f_2 = F(1, 2m+1, m+1)$ from $[a, a_1]$ onto $[c, c_1]$, obtained by rotating the function $F(m+1, 2m+1, 1)$ applied from $[a_1, b]$ onto $[c_1, d]$ through an angle of π radians about the point with coordinates (a_1, c_1).

Since $f_1(a_1) = f_2(a_1) = c_1$, then the union of the functions f_1 and f_2 gives the required function $F(1, 2m+1, 1)$ from $[a, b]$ onto $[c, d]$.

Fig. 4 illustrates the function $F(1, 2m+1, 1)$.

Note. The functions $F(m+1, 2m+1, m+1)$, $F(m+1, 2m+1, 1)$ and $F(1, 2m+1, 1)$ will be used to 'fold' edges in the domain so that these can be projected downwards onto edges in the co-domain. Thus, it will be useful to view these functions as the actual 'folding' of the edges as illustrated in Fig. 5.

For instance, Fig. 5(i) shows how an edge (a, b) in the domain graph can be $(m+1, 2m+1, m+1)$-folded such that it is projected downwards onto the edge (c, d) of the graph in the co-domain, in such a way that $|F^{-1}(c)| = m+1 = |F^{-1}(d)|$ and for all the points $y \in]c, d[$, $|F^{-1}(y)| = 2m+1$. Similarly Fig. 5(ii) and (iii) show respectively an $(m+1, 2m+1, 1)$-fold and a $(1, 2m+1, 1)$-fold projected downwards.

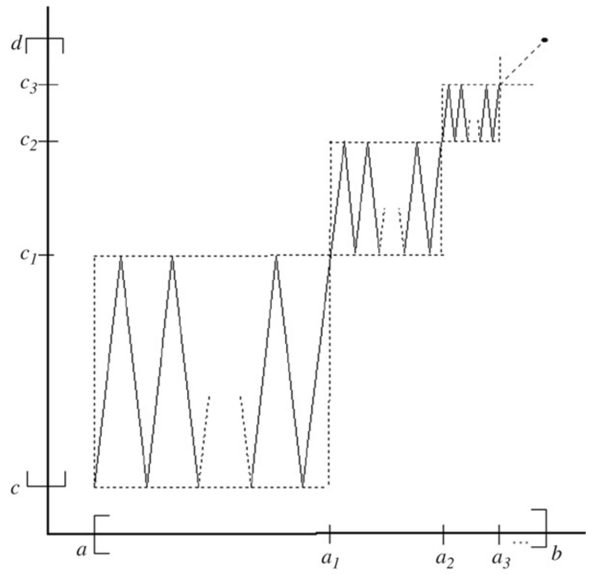

Fig. 3. $F(m+1, 2m+1, 1)$.

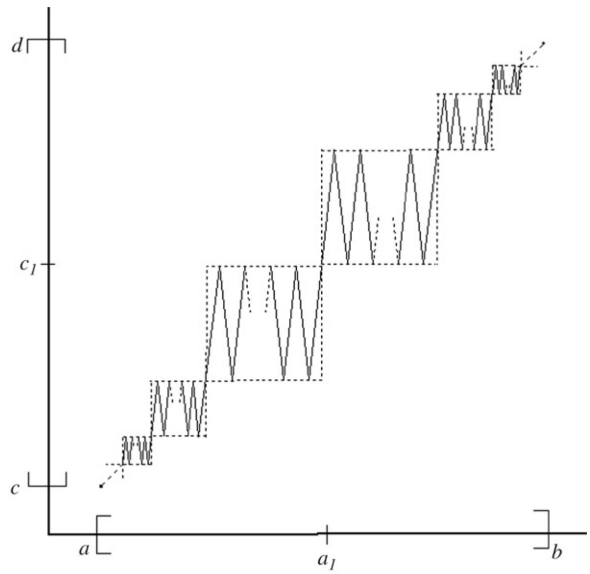

Fig. 4. $F(1, 2m+1, 1)$.

1.3. *k*-to-1 functions between intervals

In the very first work about k-to-1 mappings, Schweigert (as quoted by Harrold [3]) showed that an arc can be k-to-1 mapped onto a circle provided that $k \geq 3$. His result is stated in Lemma 1.10.

Lemma 1.10 (*[3]*). *There exists a k-to-1 map from an arc onto a circle for* $k \geq 3$.

The diagram in Fig. 6(a) shows the mapping Schweigert used for $k = 3$. This mapping can be clearly extended to all values of k by extending the arc and wrapping it round the circle for the required number of times. The circle can also be treated as the closed interval $[0, 1]$, with the point 1 identified with the point 0. Fig. 6(b) shows a 5-to-1 mapping of the closed interval $[0, 1]$ onto the interval $[0, 1]$ with the points 0 and 1 identified, or, equivalently, the circle.

Since those early days, many other researchers have published results on k-to-1 mappings between intervals, as listed below. Heath [4] considered the set of discontinuities needed for a 2-to-1 function from a closed interval (or an arc) onto any Hausdorff space (defined in Definition 1.11); the result is stated here without proof in Lemma 1.12. As a corollary, she also gave the result in Lemma 1.13.

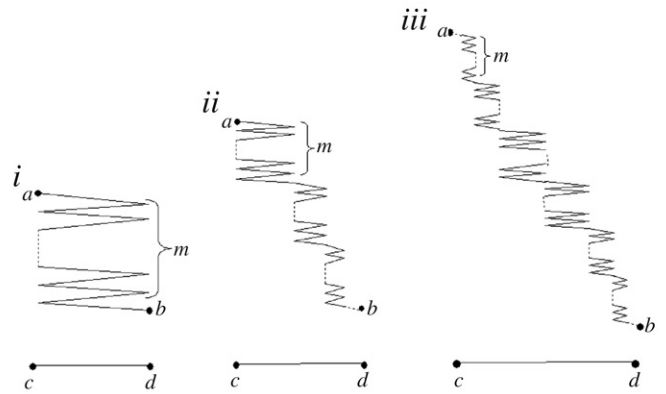

Fig. 5. Downward projection of folds.

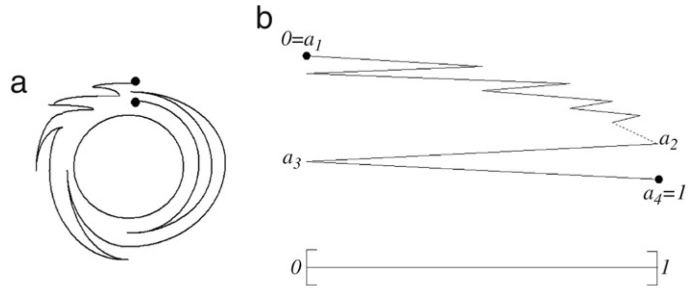

Fig. 6. 3-to-1 and 5-to-1 maps onto the circle.

Definition 1.11. Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be *separated by neighbourhoods* if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint, i.e. $U \cap V = \emptyset$.

X is a *Hausdorff Space* if any two distinct points of X can be separated by neighbourhoods.

Lemma 1.12 ([4]). *If f is a 2-to-1 function from* $[0, 1]$ *onto any Hausdorff space, then the set of discontinuities is infinite.*

Lemma 1.13 ([4]). *There is no 2-to-1 function with just finitely many discontinuities from* $]0, 1[$ *onto any Hausdorff space.*

Another result by Heath follows.

Lemma 1.14 ([5]). *For every* $k > 2$, *there is a finitely discontinuous k-to-1 function from* $[0, 1[$ *onto* $]0, 1[$.

Katsuura on his own and also with Kellum gave other results for k-to-1 mappings between intervals, as shown in the following five lemmas.

Lemma 1.15 ([13]). *If f is a k-to-1 function from* $[0, 1]$ *onto* $[0, 1]$ *and* $k \geq 2$, *then f has infinitely many discontinuities.*

Lemma 1.16 ([12]). *For every* $k > 2$ *there is a finitely discontinuous k-to-1 function from* $]0, 1[$ *onto* $]0, 1[$. *Moreover, if k is odd, this function can be taken to be continuous.*

Lemma 1.17 ([12]). *For every* $k > 2$ *there is a finitely discontinuous k-to-1 function from* $[0, 1]$ *onto* $]0, 1[$.

Lemma 1.18 ([12]). *For every* $k > 2$ *there is a finitely discontinuous k-to-1 function from* $[0, 1]$ *onto* $[0, 1[$.

Lemma 1.19 ([12]). *If f is a k-to-1* ($k \geq 2$) *function from* $]0, 1[$ *onto* $[0, 1[$, *then f must have infinitely many discontinuities.*

We here prove two other results, given in Lemmas 1.21 and 1.22. In the proof of Lemma 1.21, we need one result which was proved by Katsuura and Kellum in [13], stated below.

Lemma 1.20 ([13]). *If f is a k-to-1 map from an open set U of real numbers onto* $]0, 1[$, *then the number of components* $]a, b[$ *of U is not more than k.*

Lemma 1.21 extends Lemma 1.15.

Lemma 1.21. *There is no exactly k-to-1 function, for $k \geq 2$, from any of* $]0, 1[$, $[0, 1[$ *and* $[0, 1]$ *onto* $[0, 1]$ *that has finitely many discontinuities.*

The proof in the cases when the domain is $]0, 1[$ or $[0, 1[$ is very similar to the proof in the case when the domain is $[0, 1]$ due to Katsuura and Kellum [13], and we give a proof of all three at the same time.

Proof. We assume that f is a k-to-1 function from one of these intervals onto $[0, 1]$, and, for a contradiction, assume that f is finitely discontinuous. The argument is slightly different in each of these cases, and we point out the differences as we go along. Let Case 1 deal with the domain $]0, 1[$, Case 2 with $[0, 1[$, and Case 3 with $[0, 1]$. Let the non-zero points of discontinuity in the domain be $c_1, c_2, \ldots, c_p$ for some $p \geq 0$, and consider the set

$$\Gamma_1 = \{0, 1, f(c_1), \ldots, f(c_p)\} \text{ in Case 1,}$$
$$\Gamma_2 = \{0, 1, f(0), f(c_1), \ldots, f(c_p)\} \text{ in Case 2, and}$$
$$\Gamma_3 = \{0, 1, f(0), f(1), f(c_1), \ldots, f(c_p)\} \text{ in Case 3}$$

of points in the range. Denote the set Γ_j, for $j = 1, 2, 3$ according to the case we are dealing with, by $Y = \{y_1, y_2, \ldots, y_n\}$ for some $n > 0$, where $0 = y_1 < y_2 < \cdots < y_{n-1} < y_n = 1$. The points in the domain corresponding to $f^{-1}(y_i)$, for $1 \leq i \leq n$, are labelled $x_1, x_2, \ldots, x_{kn}$, where

$$0 < x_1 < x_2 < \cdots < x_{kn-1} < x_{kn} < 1 \text{ in Case 1,}$$
$$0 = x_1 < x_2 < \cdots < x_{kn-1} < x_{kn} < 1 \text{ in Case 2, and}$$
$$0 = x_1 < x_2 < \cdots < x_{kn-1} < x_{kn} = 1 \text{ in Case 3.}$$

In Case 1 we put $x_0 = 0$, and in Cases 1 and 2 we put $x_{kn+1} = 1$. By Lemma 1.2, for $2 \leq j \leq kn$, $f([x_{j-1}, x_j])$ is an interval, so, for $2 \leq i \leq n$, $f^{-1}(]y_{i-1}, y_i[)$ is the union of a finite number of intervals. For $2 \leq i \leq n$, we put $f^{-1}(]y_{i-1}, y_i[) = U_i$, and define the function g_i as f restricted to U_i such that U_i is mapped onto $]y_{i-1}, y_i[$. Then, for each i, $2 \leq i \leq n$, U_i is the union of finitely many open intervals of the form $]x_{j-1}, x_j[$, say m_i of them. Hence,

$$\sum_{i=2}^{n} m_i = \begin{cases} kn+1 & \text{in Case 1,} \\ kn & \text{in Case 2,} \\ kn-1 & \text{in Case 3.} \end{cases}$$

However for each i, $2 \leq i \leq n$, the function $g_i : U_i \to]y_{i-1}, y_i[$ is continuous and onto, and by Lemma 1.20, for each $i = 2, \ldots, n$, the number of components of U_i is not more than k.

This implies that

$$\sum_{i=2}^{n} m_i \leq \sum_{i=2}^{n} k = k(n-1) = kn - k,$$

so that

$$kn - k \geq \sum_{i=2}^{n} m_i = \begin{cases} kn+1 & \text{in Case 1,} \\ kn & \text{in Case 2,} \\ kn-1 & \text{in Case 3,} \end{cases}$$

giving us a contradiction for all values of $k \geq 1$ in Cases 1 and 2, and $k \geq 2$ in Case 3.

Hence the set of discontinuities is infinite. □

Lemma 1.22. *For every $k \geq 2$, there is a finitely discontinuous k-to-1 function from* $[0, 1[$ *onto* $[0, 1[$.

Proof. We divide the domain into k equal intervals $[a_i, a_{i+1}[$, for $i = 1, 2, \ldots, k$ such that $0 = a_1 < a_2 < \cdots < a_{k+1} = 1$, and on each interval let the function $f : [a_i, a_{i+1}[\to [0, 1[$ be one-to-one. □

The last result we prove in this section is that there does not exist a 2-to-1 finitely discontinuous function from $[0, 1[$ onto $]0, 1[$. The proof we use is an adaptation of the proof given by Heath [5].

Lemma 1.23. *There is no 2-to-1 function from* $[0, 1[$ *onto* $]0, 1[$ *that has finitely many discontinuities.*

Proof. For contradiction, we assume there exists such a function. We let $D \subseteq [0, 1[$ be the set of discontinuities $c_1, \ldots, c_{t-1}$ (for some $t \geq 0$) of f together with the point 0. We consider the open interval of the image less the p points (for some $p \geq 0$) of $f(D)$. This is the union of $(p+1)$ disjoint open intervals, $J_1, \ldots, J_{p+1}$. Now, since f is 2-to-1, $f^{-1}(f(D))$ has $2p$ points $0 = x_1 < \cdots < x_{2p}$. We let $x_{2p+1} = 1$. Since f restricted to each $]x_i, x_{i+1}[$ for $i = 1, 2, \ldots, 2p$ is continuous, by Lemma 1.2(i) it maps each $]x_i, x_{i+1}[$ into only one J_i. For each $j = 1, 2, \ldots, p+1$, we let m_j denote the number of open intervals $]x_i, x_{i+1}[$ that map into J_j. Then,

(i) $\sum_{j=1}^{p+1} m_j = 2p$, since there are $2p$ of the open intervals $]x_i, x_{i+1}[$, and

(ii) $\sum_{j=1}^{p+1} m_j \geq \sum_{j=1}^{p+1} 2 = 2(p+1)$, since $m_j \geq 2$ for each j by Lemma 1.13.

Hence $2p = \sum_{j=1}^{p} m_j \geq 2(p+1)$, which is a contradiction.

Thus, the set of discontinuities is infinite. □

All the above results give a complete characterization of k-to-1 surjective functions between intervals. These are summarised in the following theorem.

Theorem 1.24. *For $k \geq 2$, the following chart shows if there is a finitely discontinuous k-to-1 function between the specified domain and range; if there is no finitely discontinuous function, then any k-to-1 function is infinitely discontinuous. In one case (indicated) there is a continuous k-to-1 map.*

<table>
<tr><th>Range \ Domain</th><th>[0,1]</th><th>]0,1[</th><th>[0,1[</th></tr>
<tr><td>[0,1]</td><td>$k \geq 2$
Inf. Dcts.
(Lemma 1.15 [13])</td><td>$k \geq 2$
Inf. Dcts.
(Lemma 1.21)</td><td>$k \geq 2$
Inf. Dcts.
(Lemma 1.21)</td></tr>
<tr><td rowspan="3">]0,1[</td><td>$k = 2$
Inf. Dcts.
(Lemma 1.12 [4])</td><td>$k = 2$
Inf. Dcts.
(Lemma 1.13 [4])</td><td>$k = 2$
Inf. Dcts.
(Lemma 1.23)</td></tr>
<tr><td rowspan="2">$k \geq 3$
Fin. Dcts.
(Lemma 1.17 [12])</td><td>$k \geq 3$, k odd
Continuous
(Lemma 1.16 [12])</td><td rowspan="2">$k \geq 3$
Fin. Dcts.
(Lemma 1.14 [5])</td></tr>
<tr><td>$k \geq 3$, k even
Fin. Dcts.
(Lemma 1.16 [12])</td></tr>
<tr><td rowspan="2">[0,1[</td><td>$k = 2$
Inf. Dcts.
(Lemma 1.12 [4])</td><td rowspan="2">$k \geq 2$
Inf. Dcts.
(Lemma 1.19 [12])</td><td rowspan="2">$k \geq 2$
Fin. Dcts.
(Lemma 1.22)</td></tr>
<tr><td>$k \geq 3$
Fin. Dcts.
(Lemma 1.18 [12])</td></tr>
</table>

2. Conditions for k-to-1 functions between graphs

In one of the early papers about k-to-1 functions between graphs, Gottschalk [2] established the following important and intuitive result.

Lemma 2.1 ([2]). *If f is an $(\leq k)$-to-1 map from a graph G onto a graph H, and if for $y \in H$, $f^{-1}(y) = \{x_1, \ldots, x_n\} \in G$, then $\sum_{i=1}^{n} O(x_i) \leq k\,O(y)$.*

An important corollary of Lemma 2.1 for k-to-1 maps between graphs is that vertices of G must be mapped onto vertices of H. More precisely we have

Lemma 2.2. *Let f be a k-to-1 map from a graph G onto a graph H, let w be a point of H of order at most 2, and let $f^{-1}(w) = \{x_1, \ldots, x_k\}$ where $O(x_i) \geq 2$. Then $O(w) = O(x_i) = 2$ (for $1 \leq i \leq k$).*

Proof. By Lemma 2.1,

$$2k \leq \sum_{i=1}^{k} O(x_i) = O\left(f^{-1}(w)\right) \leq kO(w) \leq 2k,$$

from which the assertion follows. □

Another quick way to check for the existence of a k-to-1 finitely discontinuous function between two graphs is to check their Euler numbers (defined in Definition 2.3). This was established by Heath [5] and is reproduced here in Theorem 2.4.

Definition 2.3. The **Euler Number** of a connected graph G, denoted by $\mathcal{E}(G)$, is defined to be the number of edges less the number of vertices, or in symbols $\mathcal{E}(G) = |E(G)| - |V(G)|$.

Theorem 2.4 ([5]). *There is a k-to-1 finitely discontinuous function from a graph G onto a graph H if and only if:*

(i) *$\mathcal{E}(G) \leq k\mathcal{E}(H)$ for $k > 2$ and*
(ii) *$\mathcal{E}(G) = k\mathcal{E}(H)$ for $k = 2$.*

In [7], Heath and Hilton gave a necessary and sufficient condition for extending a k-to-1 function from a vertex set N of G onto a vertex set M of H to a k-to-1 (continuous) map from G onto H. We state their result as Theorem 2.6 below. They make use of the adjacency matrix of H together with another associated matrix (the Inverse Adjacency Matrix defined in Definition 2.5 below).

Definition 2.5 (*[7]*). We consider two simple graphs G and H with vertex sets N and M respectively, and suppose that f is a k-to-1 correspondence from N onto M.

- The ***Adjacency Matrix*** A for H is the matrix indexed by $M \times M$ such that the entry $A(p, q)$, for vertices p and q in M, is defined to be the number of edges in H with end-points p and q.
- The ***Inverse Adjacency Matrix*** B for G, H and f, also indexed by $M \times M$, is the matrix such that the entry $B(p, q)$, for vertices p and q in M, is defined to be the number of edges in G with one end-point in $f^{-1}(p)$ and the other end-point in $f^{-1}(q)$.

Here we note that an Adjacency Matrix can be defined for every non-empty subset of the vertex set. Also

(a) $A(p, p)$ is zero since H is not allowed to have loops;
(b) $B(p, p)$ is equal to the number of edges in G with both endpoints in $f^{-1}(p)$;
(c) both A and B are symmetric.

The result proved by Heath and Hilton is the following.

Theorem 2.6 (*[7]*). *Suppose G and H are graphs and f is a k-to-1 correspondence from a vertex set of G onto a vertex set of H. Then f extends to a k-to-1 map from G onto H if and only if, the adjacency matrix A and the inverse adjacency matrix B satisfy:*

(1) $k\,O(p) \geq O\left(f^{-1}(p)\right)$ *for each vertex p in H,*
(2) *each off-diagonal entry of $kA - B$ is even and non-negative, and*
(3) *if k is odd then each entry of $B - A$ is non-negative; and if k is even then, for each vertex p of H,*

$$B(p, p) \geq \sum_{q \neq p} \max\left\{A(p, q) - \frac{1}{2}B(p, q), 0\right\}.$$

Hilton [10] developed the Heath–Hilton result [7] further. Before describing it, let us remark that, if it helps to construct a k-to-1 map from G onto H, we have the option of enlarging the vertex sets by the introduction of additional vertices of degree 2 into edges; this process is known as subdividing the edges. Thus we may speak of "a vertex set" of G or H, rather than "the vertex set". In the sequel, if we describe loosely an edge of G being mapped into H as "passing through" a vertex v of H, it is to be understood that really an extra vertex is introduced into the edge of G, and that this extra vertex is mapped onto v.

When we try to use Theorem 2.6 to show that there is a k-to-1 map from G onto H, we have to choose some k-to-1 correspondence from some vertex set of G onto the vertex set of H such that (1), (2) and (3) of Theorem 2.6 are satisfied. After doing this, we quickly find that much of the argument is repetitious, and that it can be incorporated into a result which we give here as Theorem 2.7. Theorem 2.7 catches us after we have started the definition of f and done the "hard part", and we have got to the point where finishing the construction of f is "just routine". In our description we shall employ an intermediate (or metaphorical) term — a *loop*. This is purely a useful aid to thought and description, and we would prefer not to incorporate it into any formal structure, or to try to bypass it. The idea is that, in the course of trying to describe our mapping f, we may metaphorically map several vertices of G and the edges between them onto the same vertex of H — we do not wish these edges to disappear, but it is not convenient yet to say what we shall do with them. They are just there – edges to be mapped – and we call them "loops". Thus if the two vertices at the ends of the edge e of G are mapped to the same vertex v of H, then the edge will form a "loop" on v. The final destination of the "loop" in the map f is taken care of in the "just routine" part of the map, and is incorporated in Theorem 2.7. How this is actually coped with can be read in the proof of Theorem 2.7 in [10]. Meanwhile we need to describe some terminology concerning the interim stage from which it is just routine to finish the construction of f. We have a $(\leq k)$-to-1 function f_0 from G onto H, but f_0 is embellished by the presence of "loops" at various vertices of H, that is, edges of G whose final destination we do not specify. Thus f_0 is, more accurately, a "partial $(\leq k)$-to-1 function" in the computer science sense from G onto H. The function f_0 restricted to a vertex set N_0 of G maps N_0 onto M_0 (the vertex set of H), has adjacency matrix A_0 and inverse adjacency matrix B_0. However, the inverse adjacency matrix takes account of the loops on each vertex of H. The entry $B(p, p)$ on the diagonal of B is the number of loops on the vertex p of H, or in other words, it is the number of edges of G both of the end-vertices of which are mapped onto p by the partial function f_0.

Note that Theorem 2.7 is a slightly corrected version of a theorem in [10].

Theorem 2.7 (*[10]*). *Let G and H be simple graphs with no isolated vertices. Let G_0 be obtained from G by introducing some extra vertices (of order 2), let $H_0 = H$, and let N_0 and M_0 be the vertex sets of G_0 and H_0 respectively (so $M_0 = M$). Let f_0 be a partial $(\leq k)$-to-1 function (embellished with some loops as described above) from the vertex set N_0 of G onto the vertex set M_0 of H, with the property that, if k is odd the entries of $B_0 - A_0$ are non-negative, and if k is even then for all $p \in M_0$, $\sum_q B_0(p, q) \geq 1$*

(where A_0 is the adjacency matrix of H, indexed by $M_0 \times M_0$, and B_0 is the inverse adjacency matrix for G_0, H_0 and f_0, also indexed by $M_0 \times M_0$). Then the function f_0 can be extended to an exactly k-to-1 map f from G (with vertex set $N = f^{-1}(M_0) \supseteq N_0$) onto H (with vertex set $M = M_0$) in such a way that, if B denotes the inverse adjacency matrix for G, H and f, still indexed by $M_0 \times M_0$, then $B(p, q) - B_0(p, q)$ is non-negative and even for all $p, q \in M_0$, $p \neq q$, if and only if:

(1) $O\left(f_0^{-1}(p)\right) + 2\left(k - |f_0^{-1}(p)|\right) \le k.\ O(p)$ *for all* $p \in M_0$;
(2) $kA_0 - B_0$ *has even and non-negative off-diagonal elements; and*
(3) *if k is even, then for each* $p \in M_0$

$$B_0(p, p) + \left(k - |f_0^{-1}(p)|\right) \ge \sum_{q \neq p} \max\left\{A_0(p, q) - \frac{1}{2}B_0(p, q), 0\right\}.$$

Theorem 2.6 of Heath and Hilton is considerably more elegant than Theorem 2.7, but for the kind of application we make in this paper, it is less useful. The deduction of Theorem 2.7 from the Heath–Hilton Theorem makes great use of the various folds described in Section 1.2. For a more "hands on" illustration of the use of folds, the reader could at this point read the proof of Lemma 3.10 about the construction of a 6-to-1 map from K_6 onto K_4.

In the same paper, Hilton [10] examined the *initial* and *threshold* values of k for which there exists a k-to-1 map between two graphs. These terms are defined below.

Definition 2.8 ([10]).

(i) The ***initial value*** $\boldsymbol{j(G, H)}$ is the least integer k such that there is a k-to-1 map from G onto H. If there is no such least integer, then we put $j(G, H) = \infty$. The ***initial even value*** $\boldsymbol{j_e(G, H)}$ and the ***initial odd value*** $\boldsymbol{j_o(G, H)}$ are defined similarly, except that k is restricted to being even or odd, respectively.
(ii) The ***threshold value*** $\boldsymbol{t(G, H)}$ is the least positive integer k_0 such that, for all $k \ge k_0$ there is a k-to-1 map from G onto H. If there is no such least value, then we put $t(G, H) = \infty$. The ***threshold even value*** $\boldsymbol{t_e(G, H)}$ and the ***threshold odd value*** $\boldsymbol{t_o(G, H)}$ are similarly defined, except that k_0 is restricted to being even or odd, respectively.

Note. Clearly, $j(G, H) \le t(G, H)$; $j_e(G, H) \le t_e(G, H)$; and $j_o(G, H) \le t_o(G, H)$.

A very useful result which was proved by Hilton [10] (but based on an earlier result of Heath and Hilton [8]) concerns the relationship between the initial even value and the threshold even value of k, and the initial odd value and the threshold odd value of k.

Theorem 2.9 ([10]). *Let $|E(G)| \ge 1$ and let H be connected such that $|E(H)| \ge |V(H)|$. If H contains no vertices of degree 1, then $j_e(G, H) = t_e(G, H)$ and $j_o(G, H) = t_o(G, H)$.*

3. *k*-to-1 maps between complete graphs

The main result of this paper is presented in the theorem below.

Theorem 3.1. *Let $2s > r \ge 1$. The initial even value of k for which there exists a k-to-1 map from $G = K_{2r}$ onto $H = K_{2s}$ is*

$$j_e(K_{2r}, K_{2s}) = \begin{cases} \infty & \text{if } (r, s) = (1, 1), \\ 6 & \text{if } (r, s) = (3, 2), \\ 2s & \text{otherwise}. \end{cases}$$

In view of Theorem 2.9, this implies that there is a k-to-1 map from K_{2r} onto K_{2s} for all even values of $k \ge 2s$ whenever $r < 2s$, $(r, s) \neq (1, 1)$, $(r, s) \neq (3, 2)$.

Corollary 3.2. *Let $1 \le r < 2s$, $(r, s) \neq (1, 1)$, $(r, s) \neq (3, 2)$. Then there is a k-to-1 map from K_{2r} onto K_{2s} if and only if $k \ge 2s$.*

Various parts of the proof of Theorem 3.1 are separated out in the various lemmas of this section (Lemmas 3.3–3.10). But before the proof of Theorem 3.1, we would like to make one further definition, that of an ***Eulerian double circuit*** of a connected graph. Given a connected simple graph G, we can form another graph $2G$ by doubling each edge, that is, by replacing each edge ab by two edges joining a and b. The graph $2G$ is connected and each vertex of $2G$ has even degree, so $2G$ has an Eulerian circuit. Therefore G itself has a walk in which the initial and final vertices are the same, and in which each edge is contained exactly twice. We call this walk an Eulerian double circuit E of G. If, for some reason, we need to specify the Eulerian double circuit as well as an initial and final vertex v_0, say, we shall write (v_0, E, v_0). Note that the number of times that a vertex $v \neq v_0$ is encountered going round (v_0, E, v_0) is $d_G(v)$, and, if $v = v_0$, it is $d_G(v_0) + 1$ (counting the initial and final encounters as being distinct).

Proof of Theorem 3.1. We first note that if $(r, s) = (1, 1)$ then, by Lemma 1.15 (Katsuura and Kellum's result of 1987), $j_e(K_{2r}, K_{2s}) = \infty$. So from now on we assume that $1 \le r < 2s$ and $(r, s) \neq (1, 1)$, and that k is even.

Let us deal with the case $r = 1$. Then $s \ge 2$. Let the vertices of $H = K_{2s}$ be $v_1, v_2, \ldots, v_{2s}$ and map the two vertices of $G = K_2$ onto v_1, and let the edge of G be mapped onto (v_1, E, v_1), where E is an Eulerian double circuit of H, and v_1 is the initial and final vertex. Let f_0 be this $(\le 2s)$-to-1 map from G onto H.

To help the reader get more familiar with the main ideas in all the cases, we describe this case with more detail than in the subsequent cases.

Each time that the Eulerian double circuit encounters one of the vertices of K_{2s}, then, for the purposes of interpreting Theorem 2.7 in this case, we implicitly insert a vertex into the edge of K_2. Thus, when all the implicitly inserted vertices are accounted for, we find that we have inserted $(2s-1)(2s-1)+2s-2$ vertices, so that the K_2 has become a path P with $(2s-1)^2+2s$ vertices. The graph G with these extra vertices is denoted by G_0.

Between any two vertices of K_{2s} there are now mapped two edges (of the path P), so that $B_0(i,j)=2$ if $i\neq j$, $1\leq i\leq 2s$, $1\leq j\leq 2s$, while between any two vertices of P mapped onto v_i there are no edges, so that $B_0(i,i)=0$, $1\leq i\leq 2s$. Thus the inverse adjacency matrix is given by

$$B_0(i,j)=\begin{cases}1 & \text{if } i\neq j\\ 0 & \text{if } i=j.\end{cases}$$

Since $\sum_j B_0(i,j)\geq 1$ for all $v_i\in M_0$, if we verify that Properties 1–3 of Theorem 2.7 are satisfied, then the $(\leq 2s)$-to-1 map f_0 from G onto H can be extended to an exactly $2s$-to-1 map from G onto H.

If $p=v_i$, $2\leq i\leq 2s$, then $O\left({f_0}^{-1}(p)\right)$ is twice the number of times the Eulerian double circuit (v_1,E_2,v_1) encounters the vertex v_i; in other words, $O\left({f_0}^{-1}(p)\right)=2(2s-1)$. Also $|{f_0}^{-1}(p)|$ is the number of vertices of the path P that are mapped onto v_i, namely $2s-1$. Thus, for $p=v_i$, $2\leq i\leq 2s$

$$O\left({f_0}^{-1}(p)\right)+2\left(k-\left|{f_0}^{-1}(p)\right|\right)=2(2s-1)+2k-2(2s-1)=2k<k(2s-1)=kO(p),$$

since $s\geq 2$. The only difference in the case when $p=v_1$ is that $|{f_0}^{-1}(p)|=2s$, so again

$$O\left({f_0}^{-1}(p)\right)+2\left(k-\left|{f_0}^{-1}(p)\right|\right)\leq k\,O(p).$$

Thus, Property (1) holds.

The off-diagonal elements of kA_0-B_0 are all equal to $k-2=2s-2$, which is even and non-negative since $s\geq 2$. Therefore Property (2) is satisfied.

Finally, for $p=v_i$, $2\leq i\leq 2s$,

$$B_0(p,p)+\left(k-\left|{f_0}^{-1}(p)\right|\right)=0+(k-(2s-1))=1, \text{ and}$$

$$\sum_{q\neq p}\max\left(A_0(p,q)-\frac{1}{2}B(p,q),0\right)=\sum_{q\neq p}\max\left(1-\frac{1}{2}\times 2,0\right)=0, \text{ so}$$

$$B_0(p,p)+\left(k-\left|{f_0}^{-1}(p)\right|\right)\geq\sum_{q\neq p}\max\left(A_0(p,q)-\frac{1}{2}B(p,q),0\right).$$

If $p=v_1$, then $B_0(p,p)=0$ and $|{f_0}^{-1}(p)|=2s$, so

$$B_0(p,p)+\left(k-\left|{f_0}^{-1}(p)\right|\right)=0=\sum_{q\neq p}\max\left(A_0(p,q)-\frac{1}{2}B(p,q),0\right),$$

so Property (3) is also satisfied.

Hence it follows that f_0 can be extended to a $2s$-to-1 map from K_2 onto K_{2s} for $s\geq 2$.

To show that $j_e(K_2,K_{2s})=2s$, we still need to show that there is no k-to-1 map from K_2 onto K_{2s} for $k\leq(2s-2)$. However, this is proved under Case I (Lemma 3.3) below.

From now we assume that $r<2s$ and that k is even. We consider the three cases:

(I) $r<s$
(II) $r=s$
(III) $s<r<2s$

separately. In each case, we first prove that we cannot find a (continuous) k-to-1 map from G onto H for $k\leq(2s-2)$, and then show that it is possible to construct a $2s$-to-1 mapping between the two graphs. At this point we note that all the vertices of G must map to vertices of H, for, if one vertex of G is mapped onto a point of valency 2 in H, then by Lemma 2.1

$$2k\geq(2r-1)+2(k-1)=2r-1+2k-2,$$

implying that $2r\leq 3$, a contradiction. Lemmas 3.3–3.10 complete the proof. □

Case (I) $r<s$.

Lemma 3.3. *If $1\leq r<s$, then there does not exist a k-to-1 map f from $G=K_{2r}$ onto $H=K_{2s}$ for k even and $k\leq(2s-2)$.*

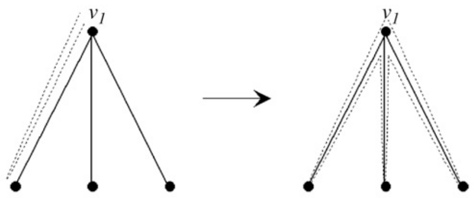

Fig. 7.

Proof. We assume, for contradiction, that there exists a $(2s-2)$-to-1 map from G onto H. Let G_0 denote the graph G with the extra vertices of valency 2 added, so that, for each vertex $v \in V(H), f^{-1}(v)$ is a set of k vertices of G_0. Consider a vertex x in H. Then x has degree $(2s-1)$.

Since G has $2r \le 2s-2$ vertices, at least two vertices in H do not have any of the vertices of G mapped onto them, but only vertices of G_0 of valency two. We consider one such vertex x of H. Since the map is $(2s-2)$-to-1, there are $(2s-2)$ vertices a_i in G_0, $1 \le i \le 2s-2$, having valency 2 mapped onto x. Thus there are $2(2s-2) = 4(s-1)$ intervals of edges of G_0 with end-vertices a_i (two for each a_i) mapped onto the intervals $[x, y_j]$ of edges incident to x in H, for $1 \le j \le 2s-1$.

Each $[x, y_j]$ must have an interval from at least one of the edges of G_0 with end-vertex a_i mapped onto it, and thus, after distributing $(2s-1)$ of the $4(s-1)$ intervals, we have $(2s-3)$ intervals left. Thus one of the intervals $[x, y_j]$, without loss of generality $[x, y_1]$, has only one edge of G_0 with end-vertex a_i mapped onto it. Letting $f^{-1}([x, y_1]) = [a_h, w_h]$, where $f(a_h) = x$ and $f(w_h) = y_1$, by Theorem 1.5 there exists a $z \in]x, y_1[$ such that $f^{-1}(z) = \{\alpha_1, \dots, \alpha_t\}$ where t is odd. But t is equal to $(2s-2)$, and hence is even, a contradiction. Thus there does not exist a $(2s-2)$-to-1 map from G onto H, and by Theorem 2.9 there is no k-to-1 map from K_{2r} onto K_{2s} for $1 \le r < s$ if k is even and $k \le (2s-2)$. □

Lemma 3.4. *Let $2 \le r < s$. Then there exists a $2s$-to-1 map f from the graph $G = K_{2r}$ onto the graph $H = K_{2s}$.*

Proof. To show the existence of a (continuous) k-to-1 map from K_{2r} onto K_{2s} such that $k = 2s$, we will consider two separate cases, as follows:

Case (i) when the valency of any vertex in the range is greater than the number of edges of the graph in the domain, that is $2s-1 > r\ (2r-1)$;

Case (ii) when the valency of any vertex in the range is less than or equal to the number of edges of the graph in the domain, that is $(2s-1) \le r(2r-1)$.

Case (i) $2s-1 > r(2r-1)$.

Here the valency of any vertex in the co-domain graph H is greater than the number of edges in the domain graph G.

We consider all the $2r$ vertices x_i, $1 \le i \le 2r$, in G and map them to one vertex v_1 of H. Then, since in G there are $\binom{2r}{2} = r(2r-1)$ edges, we obtain $r(2r-1)$ "loops" on v_1. Mapping one loop on each of $r(2r-1)$ edges of H incident to v_1, we have $(2s-1)-r(2r-1)$ edges left. We "invent" this number of further loops by taking one of the existing loops which is already mapped to an edge $[v_1, z]$ of H, and extending it to cover the edges that still have no pre-images as shown in Fig. 7. By doing this, the number of pre-images of v_1 is increased by $(2s-1-r(2r-1))$, to become $|f^{-1}(v_1)| = 3r+2s-1-2r^2$. Since $0 \le (2r-1)(r-1)$, it follows quickly that $3r+2s-1-2r^2 \le 2s$. Note also that each vertex $v_i \ne v_1$ now has one "new" vertex mapped onto it (the "new" vertices being points of valency 2), and that each edge of H incident with v_1 has two edges mapped onto it. We let G_0 be the graph G with these new extra vertices of valency 2 in it, and let $H_0 = H$.

We denote the present $(\le 2s)$-to-1 function from G_0 to H_0 by f_0 and note that the inverse adjacency matrix B_0 is given by

$$B_0(1,1) = 0;$$
$$B_0(1,i) = B_0(i,1) = 2, \quad \text{for } i = 2, \dots, 2s;$$
$$B_0(i,j) = B_0(j,i) = 0, \quad \text{for } 2 \le i \le 2s, 2 \le j \le 2s.$$

Now, since $\sum_q B_0(p,q) \ge 1$ for all $p \in M_0$, if we verify that Properties 1–3 of Theorem 2.7 are satisfied, then it follows that the present $(\le 2s)$-to-1 function f_0 from G to H can be extended to an exactly $2s$-to-1 map f from G onto H.

In fact, for each $p \in M_0$,
if $p = v_1$, then

$$\begin{aligned} O\left(f_0^{-1}(p)\right) + 2\left(k - \left|f_0^{-1}(p)\right|\right) &= 2(2s-1) + 2\left[2s - \left(3r+2s-1-2r^2\right)\right] \\ &= 4r^2 - 6r + 4s < 4s^2 - 6s + 4s, \text{ since } r < s, \\ &= 2s(2s-1) = k\,O(p); \end{aligned}$$

if $p = v_i$, $i = 2, \dots, 2s$, then

$$O\left(f_0^{-1}(p)\right) + 2\left(k - \left|f_0^{-1}(p)\right|\right) = 1(2) + 2(2s-1) = 4s < 2s(2s-1) = k\,O(p), \text{ since } s \ge r \ge 2,$$

and thus Property (1) holds.

Also, the off-diagonal elements of $kA_0 - B_0$ are equal to either $(k-2)$ or k, which, in either case, is even and non-negative since $k = 2s$. Thus, Property (2) is satisfied.

Finally, for each $p \in M_0$,

if $p = v_1$, then

$$\begin{aligned} B_0(p,p) + \left(k - \left|{f_0}^{-1}(p)\right|\right) &= 0 + \left[2s - \left(3r + 2s - 1 - 2r^2\right)\right] \\ &= (2r-1)(r-1) \geq 0 \\ &= \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\}, \quad \text{since } r \geq 1; \end{aligned}$$

if $p = v_i$, $i = 2, \ldots, 2s$, then

$$B_0(p,p) + \left(k - \left|{f_0}^{-1}(p)\right|\right) = 0 + (2s-1) > 2s - 2 = \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\},$$

so Property (3) is also satisfied.

Hence it follows that, in this case, the function f_0 can be extended to a $2s$-to-1 map from K_{2r} onto K_{2s}.

Case (ii) $2s - 1 \leq r(2r-1)$

In this case $r \geq 2$. Also, the valency of any vertex in the co-domain graph H is less than or equal to the number of edges in the domain graph G.

We consider all the $2r$ vertices x_i, $1 \leq i \leq 2r$, in G and again map them to one vertex v_1 of H. Then we again obtain $r(2r-1)$ "loops" on v_1. We map one loop onto each of the $2s-1$ edges of H incident to v_1, leaving us with $(r(2r-1) - (2s-1))$ loops which are not yet mapped. On each of the mapped loops there is now (defined implicitly) a vertex of valency 2. We let G_0 denote G with these extra vertices inserted, and let $H_0 = H$.

We let f_0 be this partial ($\leq 2s$)-to-1 function from G_0 into H_0 (with additional loops). The inverse adjacency matrix B_0 is given by

$$\begin{aligned} &B_0(1,1) = r(2r-1) - (2s-1); \\ &B_0(1,i) = B_0(i,1) = 2, \quad \text{for } i = 2, \ldots, 2s; \\ &B_0(i,j) = B_0(j,i) = 0, \quad \text{for } 2 \leq i \leq 2s,\ 2 \leq j \leq 2s. \end{aligned}$$

Now again, since $\sum_q B_0(p,q) \geq 1$ for all $p \in M_0$, if we verify that Properties (1)–(3) of Theorem 2.7 are satisfied, then it follows that the ($\leq 2s$)-to-1 function f_0 from G_0 to H_0 can be extended to an exactly $2s$-to-1 map f from G onto H.

In fact, for each $p \in M_0$,

if $p = v_1$, then

$$\begin{aligned} O\left({f_0}^{-1}(p)\right) + 2\left(k - \left|{f_0}^{-1}(p)\right|\right) &= [2(2s-1) + 2(r(2r-1) - (2s-1))] + 2(2s-2r) \\ &= 2r(2r-3) + 4s < 2s(2s-3) + 4s, \quad \text{since } r < s, \\ &= 2s(2s-1) = k\,O(p); \end{aligned}$$

if $p = v_i$, $i = 2, \ldots, 2s$, then

$$O\left({f_0}^{-1}(p)\right) + 2\left(k - \left|{f_0}^{-1}(p)\right|\right) = 1(2) + 2(2s-1) = 4s < 2s(2s-1) = k\,O(p), \quad \text{since } s \geq 2,$$

and thus Property (1) holds.

Also the off-diagonal elements of $kA_0 - B_0$ are equal to either $(k-2)$ or k, which, in either case, is even and non-negative since $k = 2s$. Thus Property 2 is satisfied.

Finally, for each $p \in M_0$,

if $p = v_1$, then

$$\begin{aligned} B_0(p,p) + \left(k - \left|{f_0}^{-1}(p)\right|\right) &= [r(2r-1) - (2s-1)] + (2s-2r) \\ &= (2r-1)(r-1) \geq 0 \\ &= \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\}, \quad \text{since } r \geq 2; \end{aligned}$$

if $p = v_i$, $i = 2, \ldots, 2s$, then

$$B_0(p,p) + \left(k - \left|{f_0}^{-1}(p)\right|\right) = 0 + (2s-1) > 2s - 2 = \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\},$$

so Property (3) is also satisfied.

Hence it follows that the function f_0 can be extended to a $2s$-to-1 map from K_{2r} onto K_{2s} for $r < s$. □

Case (II) $r = s$.

Lemma 3.5. *If $r = s > 1$, then there does not exist a k-to-1 map f from $G = K_{2r}$ onto $H = K_{2s}$ for k even and $k \leq (2s-2)$.*

Proof. We assume that a $(2s-2)$-to-1 map f exists and consider a vertex x in H having degree $(2r-1)$.

Let G_0 denote the graph G with extra vertices of valency 2 added so that, for each vertex $v \in V(H)$, $f^{-1}(v)$ is a set of k vertices of G_0. Suppose first that all the vertices mapped to x have valency 2, that is $f^{-1}(x) \in V(G_0) \setminus V(G)$. Since the map is $(2s-2)$-to-1, there are $(2s-2) = (2r-2)$ vertices $a_i (1 \leq i \leq 2r-2)$ having valency 2 mapped onto x. Thus there are $2(2r-2) = 4(r-1)$ intervals of edges of G with end-vertices a_i (two for each a_i) mapped to the intervals of edges on x, say $[x, y_j]$, where $1 \leq j \leq 2r-1$, in H.

Each $[x, y_j]$ in H must have an interval with end-vertex a_i from at least one of the edges of G_0 mapped onto it, and thus after distributing $(2r-1)$ of the $4(r-1)$ intervals of G_0, we have only $(2r-3)$ intervals left. Thus at least two of the intervals $[x, y_j]$ of H, without loss of generality $[x, y_1]$ and $[x, y_2]$, each have exactly one pre-image. We consider one of them, $[x, y_1]$ say, and let $f^{-1}([x, y_1]) = [a_h, w_h]$, where $f(a_h) = x$ and $f(w_h) = y_1$. By Theorem 1.5, there exists a $z \in]x, y_1[$ such that $f^{-1}(z) = \{\alpha_1, \ldots, \alpha_n\}$ where n is odd. But n is equal to $(2r-2)$, and hence is even, a contradiction.

Thus at least one of the vertices mapped to x must have valency greater than 2, and so we may suppose that, for each $x \in V(H), f^{-1}(x) \cap V(G) \neq \emptyset$. Hence there exists a vertex $v \in V(G)$ such that $f(v) = x$, and the same holds for each vertex x of H, and so each $x \in V(H)$ has one vertex of degree $(2r-1)$ and $(2r-3)$ points of valency 2 mapped to it. Therefore there are $(2r-1) + 2(2r-3) = (6r-7)$ intervals of G mapped to the $(2r-1)$ intervals of H having end-point x.

Claim. *At least one interval $[x, y_j]$ in H has an odd number of pre-images, i.e. intervals of G_0 with end-vertices a_i mapped onto it.*

Proof. Assume not. Then, since each interval $[x, y_j]$ of H has at least one pre-image (because f is onto), each such interval has at least two pre-images. Thus $2(2r-1)$ of the $(6r-7)$ intervals of G_0 are accounted for, leaving $(2r-5)$ intervals. These remaining intervals of G_0 must be disposed of in such a way that if an interval of H is assigned one pre-image, then it must be assigned two intervals. But $(2r-5)$ is odd, implying that at least one of the intervals $[x, y_j]$ of H must be assigned an odd number of these remaining intervals, and so, must be assigned an odd number of intervals of G altogether.

Suppose that f maps an odd number, m, of intervals of G onto an interval $[x, y_j]$ of H. Suppose that the intervals mapped onto $[x, y_j]$ are $[v_1, w_1], [v_2, w_2], \ldots, [v_m, w_m]$, where $v_1, v_2, \ldots, v_m$ are not necessarily distinct. For $1 \leq i \leq m$, let the restriction of f to $[v_i, w_i]$ be denoted by f_i. Then, by Theorem 1.5, there is a point $\beta \in]x, y_j[$ such that $\sum_{i=1}^{m} |f_i^{-1}(\beta)|$ is odd, contradicting the fact that f is a k-to-1 map with k even.

Therefore, there does not exist a $(2s-2)$-to-1 map from G onto H, and by Theorem 2.9 there is no k-to-1 map from K_{2r} onto K_{2s} for $r = s > 1$ if k is even and $k \leq (2s-2)$. □

Lemma 3.6. *Let $r = s \geq 2$. Then there exists a 2s-to-1 map f mapping the graph $G = K_{2r}$ onto the graph $H = K_{2s}$.*

Proof. We denote the vertex-set of G by $V(G) = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2r}\}$ and the vertex-set of H by $V(H) = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{2r}\}$. For each edge $[x_i, x_j] \in E(G)$, we let the midpoint be $\boldsymbol{y}_{i,j} = \frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}$.

First, we consider all the $2r$ vertices x_i, $1 \leq i \leq 2r$, in the domain graph G and map them to a single vertex v_1 of the co-domain graph H, so that $f(\boldsymbol{x}_i) = \boldsymbol{v}_1$ for $1 \leq i \leq 2r$. The $\binom{2r}{2} = r(2r-1)$ edges of G then form $r(2r-1)$ loops on v_1 in H. Next take $2r-1$ of these loops and map them, one each, onto the $2r-1$ edges of H incident with v_1, so that, if $[v_1, v_q]$ is such an edge, then the vertex v_q has a point of valency 2 mapped onto it.

Let G_0 denote the graph G with an extra vertex in each edge $[x_1, x_i]$ of G, for $2 \leq i \leq 2r$, (so $2r-1$ extra vertices altogether). Let $H_0 = H$. Once again, at this stage, let f_0 be this partial $(\leq k)$-to-1 function from G_0 to H_0 (with additional loops). The inverse adjacency matrix B_0 is given by:

$$B_0(1,1) = r(2r-1) - (2r-1) = (2r-1)(r-1);$$
$$B_0(1,i) = B_0(i,1) = 2, \quad \text{for } i = 2, \ldots, 2r;$$
$$B_0(i,j) = B_0(j,i) = 0, \quad \text{for } 2 \leq i \leq 2r, 2 \leq j \leq 2r.$$

Now, as before, since $\sum_q B_0(p,q) \geq 1$ for all $p \in M_0$, we only need to verify that Properties (1)–(3) of Theorem 2.6 are satisfied.

For each $p \in M_0$,
if $p = v_1$, then

$$O\left(f_0^{-1}(p)\right) + 2\left(k - \left|f_0^{-1}(p)\right|\right) = (2r(2r-1)) + 2(2r-2r) = 2r(2r-1) = kO(p);$$

if $p = v_i$, $i = 2, \ldots, 2r$, then

$$O\left(f_0^{-1}(p)\right) + 2\left(k - \left|f_0^{-1}(p)\right|\right) = 1(2) + 2(2r-1) = 4r < 2r(2r-1) = kO(p),$$

since $r \geq 2$, and thus Property (1) holds.

Also the off-diagonal elements of $kA_0 - B_0$ are equal to either $(k-2)$ or k, which, in either case, is even and non-negative since $k = 2r$. Thus Property (2) is satisfied.

Finally, for each $p \in M_0$,

if $p = v_1$, then

$$B_0(p,p) + \left(k - \left|f_0^{-1}(p)\right|\right) = (2r-1)(r-1) + (2r-2r) > 0 = \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\};$$

if $p = v_i$, $i = 2, \ldots, 2r$, then

$$B_0(p,p) + \left(k - \left|f_0^{-1}(p)\right|\right) = 0 + (2r-1) > 2r-2 = \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\},$$

satisfying also Property (3).

Hence it follows from Theorem 2.7 that the function f_0 can be extended to a $2s$-to-1 map from K_{2r} onto K_{2s} for $r = s$. □

Case (III) $s < r < 2s$.

The last situation we need to consider to conclude the proof of Theorem 3.1 is when $s < r < 2s$.

Lemma 3.7. *Let $2 \leq s < r < 2s$. Then there does not exist a k-to-1 map f from $G = K_{2r}$ onto $H = K_{2s}$ for k even and $k \leq (2s-2)$.*

Proof. We assume, for contradiction, that a k-to-1 map exists for $k = 2s - 2$.

By Lemma 2.2 all the vertices of G must map to vertices of H. Let G_0 denote the graph G with extra vertices of valency 2 inserted so that, for each vertex $v \in V(H)$, $f^{-1}(v)$ is a set of k vertices of G_0. Therefore we can have the following two cases.

Case (i): At least two vertices of H have an odd number of vertices of G mapped onto them.

We let v_1 in H be one of the vertices having an odd number α_1 of vertices of G mapped onto it. We let γ be the number of intervals of edges of G which are mapped onto the intervals of edges on v_1 in H. Each of the α_1 vertices of G mapped onto v_1 is adjacent to the remaining $2r - \alpha_1$ vertices of G mapped to the other vertices of H. This accounts for $\alpha_1(2r - \alpha_1)$ intervals of edges of G which are mapped onto the intervals of edges of H which are incident with v_1. There are $\binom{\alpha_1}{2}$ edges of G both end-vertices of which map to v_1 in H, and with each of these there are two intervals of edges of G which are mapped to intervals of edges of H incident with v_1. So these account altogether for a further $2\binom{\alpha_1}{2}$ intervals of edges of G which are mapped to intervals of edges of H incident with v_1. There are $k - \alpha_1$ vertices of G_0 of valency 2 which are mapped to v_1, and so there are a final $2(k - \alpha_1)$ intervals of edges of G_0 which are mapped to intervals of edges of H incident with v_1. Thus $\gamma = \alpha_1(2r - \alpha_1) + 2\binom{\alpha_1}{2} + 2(k - \alpha_1)$. Since α_1 is odd, γ is also odd. Also the number of edges of H incident with v_1 is $(2s-1)$, and thus is odd. Mapping the γ intervals of G_0 onto the $(2s-1)$ intervals of H incident to v_1, we get that at least one interval incident to v_1, say $[v_1, y_j]$, is the image of an odd number of intervals $[x_1, w_1], [x_2, w_2], \ldots, [x_m, w_m]$ of G_0 where $x_1, x_2, \ldots, x_m \in f^{-1}(v)$ ($x_1, x_2, \ldots, x_m$ are not all distinct). For $1 \leq i \leq m$, let the restriction of f to $[x_i, w_i]$ be denoted by f_i. Then, by Theorem 1.5, there is a point $\beta \in]v_1, y_j[$ such that $\sum_{i=1}^{m} |f_i^{-1}(\beta)|$ is odd, contradicting the fact that f is a k-to-1 mapping with k even. Therefore Case (i) does not occur.

Case (ii): all the vertices of H have an even number of vertices of G mapped onto them.

If all the vertices of H have an even number of vertices of G mapped onto them, then since $r < 2s$, there is at least one vertex of H, say v_α, such that all its k pre-images are vertices of G_0 of valency 2. Denote these by $x_1, x_2, \ldots, x_k$. Therefore in all there are $2k$ intervals incident with the points x_i which are mapped to intervals of edges incident to v_α. Each edge must have at least one interval mapped onto it, and so, since k is even, it follows by Theorem 1.5that each edge must have at least two intervals mapped onto it. Since v_α has valency $2s-1$, mapping the $2k$ intervals onto the edges incident with v_α, it follows that $2k \geq 2(2s-1)$, so that $k \geq 2s-1$. But since k is even, $k \geq 2s$, contradicting the assumption that $k = 2s-2$. Thus Case (ii) does not occur either.

Thus there does not exist a $(2s-2)$-to-1 map from G onto H, and by Theorem 2.9 there is no k-to-1 map from K_{2r} onto K_{2s} for $2 \leq s < r < 2s$ if k is even and $k \leq (2s-2)$. □

Lemma 3.8. *Let $3 \leq s < r < 2s$. Then there exists a 2s-to-1 map f mapping the graph $G = K_{2r}$ onto the graph $H = K_{2s}$.*

Proof. Let $s \geq 3$. We consider the $2r$ vertices $x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{r1}, x_{r2}$ of G and map the two vertices x_{i1}, x_{i2} onto the vertex v_i of H, for $1 \leq i \leq r$. The four edges of G joining the pair x_{i1}, x_{i2} of vertices of G to the pair x_{j1}, x_{j2} when $1 \leq j < i \leq r$ are mapped onto the edge $v_i v_j$ of H. Then there are $(2s-r)$ vertices of H, $v_{r+1}, \ldots, v_{2s}$, which have no vertices of G mapped onto them, and, thus, $\sum_q B_0(p,q)$ is not at least one for all $p \in M_0$ and we cannot apply Theorem 2.7 at this stage. However, we note that each pair of vertices x_{i2} and x_{i2} mapped onto v_i gives rise to a loop. Thus, there are a total of r loops, of which we consider, without loss of generality, the loop $[x_{r1}, x_{r2}]$.

We now consider the graph $2K_{2s-r+1}$, where K_{2s-r+1} is the complete subgraph of H induced by the vertices $v_r, v_{r+1}, \ldots, v_{2s}$. Recall that $2K_{2s-r+1}$ indicates that each edge of K_{2s-r+1} is replaced by a double edge. The graph $2K_{2s-r+1}$ is Eulerian; let E denote an Eulerian double circuit, and let (v_r, E, v_r) denote the Eulerian double circuit starting and ending at v_r. We map the loop $[x_{r1}, x_{r2}]$ onto (v_r, E, v_r); we also map each loop $[x_{i1}, x_{i2}]$, for $1 \leq i \leq r-1$ onto the walk $(v_i, v_{2s}, v_i, v_{2s-1}, v_i, \dots, v_{r+1}, v_i)$. In this way, all the edges of H of type $[v_\alpha, v_\beta]$, where either $\{1 \leq \alpha \leq r-1$ and $r+1 \leq \beta \leq 2s\}$ or $\{r \leq \alpha < \beta \leq 2s\}$ have two intervals mapped onto them. Also, the vertices v_i of H for $i = 1, \dots, r$ have $2 + (2s - r - 1) = 2s - r + 1$ points of G mapped onto them, and the vertices v_i for $i = r+1, \dots, 2s$ have $(2s - r) + (r - 1) = 2s - 1$ points of G mapped onto them.

Let G_0 be the graph G with all the points of G mapped onto vertices of H now being called vertices of G_0. Let $H_0 = H$. Let f_0 be this ($\leq 2s$)-to-1 function from G_0 to H_0. The inverse adjacency matrix B_0 is given by:

$$B_0(i,i) = 0 \quad \text{for } 1 \leq i \leq 2s;$$
$$B_0(i,j) = B_0(j,i) = 4, \quad \text{for } 1 \leq i \leq r, 1 \leq j \leq r, i \neq j;$$
$$B_0(i,j) = B_0(j,i) = 2, \quad \text{for } 1 \leq i \leq r, r+1 \leq j \leq 2s;$$
$$B_0(i,j) = B_0(j,i) = 2, \quad \text{for } r+1 \leq i \leq 2s, r+1 \leq j \leq 2s, i \neq j.$$

Now, since $\sum_q B_0(p,q) \geq 1$ for all $p \in M_0$, if we verify that Properties (1)–(3) of Theorem 2.7 are satisfied, then the ($\leq 2s$)-to-1 function f_0 from G_0 to H_0 can be extended to an exactly $2s$-to-1 map f from G onto H.

For each $p \in M_0$,

if $p = v_i, i = 1, \dots r$, then

$$\begin{aligned} O\left(f_0^{-1}(p)\right) + 2\left(k - \left|f_0^{-1}(p)\right|\right) &= [4(r-1) + 2(2s-r-1)] + 2[2s - (2s-r+1)] \\ &= 4r + 4s - 8 \leq 4(2s-1) + 4s - 8, \text{ since } r \leq 2s-1, \\ &= 12s - 12 < 2s(2s-1) = k\,O(p), \quad \text{for } s \geq 3; \end{aligned}$$

if $p = v_i, i = r+1, \dots, 2s$, then

$$\begin{aligned} O\left(f_0^{-1}(p)\right) + 2\left(k - \left|f_0^{-1}(p)\right|\right) &= 2(2s-1) + 2[2s - (2s-1)] \\ &= 4s < 2s(2s-1) = k\,O(p), \quad \text{for } s \geq 2, \end{aligned}$$

and thus Property (1) holds.

Also, the off-diagonal elements of $kA_0 - B_0$ are equal to either $(k-4)$ or $(k-2)$, which, in either case, is even and non-negative since $k = 2s \geq 4$. Thus Property (2) is satisfied.

Finally, for each $p \in M_0$,

if $p = v_i, i = 1, \dots, r$, then, since $r > s$

$$\begin{aligned} &B_0(p,p) + \left(k - \left|f_0^{-1}(p)\right|\right) = 0 + (2s - (2s - r + 1)) = r - 1 \geq s, \text{ since } r \geq s+1, \\ &> 0 = \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\} \end{aligned}$$

if $p = v_i, i = r+1, \dots, 2s$, then

$$B_0(p,p) + \left(k - \left|f_0^{-1}(p)\right|\right) = 0 + (2s - (2s-1)) = 1 > 0 = \sum_{q \neq p} \max\left\{A_0(p,q) - \frac{1}{2}B_0(p,q), 0\right\};$$

so Property (3) is also satisfied.

Hence it follows from Theorem 2.7 that the function f_0 can be extended to a $2s$-to-1 map from K_{2r} onto K_{2s} for $3 \leq s < r < 2s$. □

Lemma 3.9. *There is no 4-to-1 map from K_6 onto K_4.*

Proof. Suppose there is a 4-to-1 map from $G = K_6$ onto $H = K_4$. Let the vertices of G be $x_1, x_2, \dots, x_6$ and the vertices of H be v_1, v_2, v_3, v_4.

By Lemma 2.2 all the vertices of G must map to vertices of H. Then some vertex of H, say v_1, has p vertices of G mapped onto it, where $p \in \{2, 3, 4\}$ so that $f^{-1}(v_1)$ consists of p points of order 5 and $4-p$ points of order 2. But then, by Gottschalk's inequality (Lemma 2.1), $5p + 2(4-p) \leq 4 \times 3$, so $p \leq 1$, a contradiction. □

Lemma 3.10. $j_e(K_6, K_4) = 6$.

Proof. In view of Lemma 3.9 and Theorem 2.9, we only need to show that a 6-to-1 map from $G = K_6$ onto $H = K_4$ exists.

For the sake of variety, and also to illustrate the use of folds, we provide a different kind of proof to show that there is a 6-to-1 map from $G = K_6$ onto $H = K_4$. The techniques we use here are the ones used to derive Theorem 2.7 from the Heath–Hilton characterization given in Theorem 2.6.

Let the vertices of K_6 be $x_1, x_2, \dots, x_6$ and of K_4 be v_1, v_2, v_3, v_4. We map x_1, x_2 to v_1; x_3, x_4 to v_2; and x_5, x_6 to v_3. We map also the four edges joining x_1 and x_2 to x_3 and x_4 in G 1-to-1 onto the edge v_1v_2 in H. Then the edge v_1v_2 in H has four edges mapped onto it. We replace one of the 1-to-1 maps by a (2, 3, 2)-fold. We treat similarly the maps of the edges joining

x_3, x_4 to x_5, x_6 onto v_2v_3, and the maps of the edges joining x_1, x_2 to x_5, x_6 onto v_1v_3. At this stage, the map is 6-to-1 onto the edges of the K_3 in H induced by v_1, v_2 and v_3, but only 4-to-1 on v_1, v_2, v_3.

We next map the "loop" on x_1x_2 onto the edge v_1v_4, the "loop" on x_3x_4 onto the edge v_2v_4, and the "loop" on x_5x_6 onto the edge v_3v_4. Then the map is 2-to-1 on the edges v_1v_4, v_2v_4 and v_3v_4 and is 3-to-1 on v_4.

We replace one of the 1-to-1 maps onto v_1v_4 by a (3, 5, 1)-fold, one of the 1-to-1 maps onto v_2v_4 by a (3, 5, 3)-fold, and one of the 1-to-1 maps onto v_3v_4 by a (2, 3, 2)-fold and the other by a (2, 3, 1)-fold. We now have the desired 6-to-1 map from K_6 onto K_4. □

4. Further developments

(a) We have very strong, but unfortunately not quite complete, results in our programme of determining all values of m, n and k for which there is a k-to-1 map from K_n onto K_m. We hope to publish these in later papers. The cases about which we are still uncertain all occur when $n > m$.

(b) If a fold $F(x + 1, 2x + 1, 1)$ is used in a k-to-1 map from G onto H then the construction leads to the occurrence of a limit point. It is natural to wonder if there has to be such a limit point, or if it could be avoided somehow. We call the fold $F(x+1, 2x+1, 1)$ an ***x-wiggle***. If the value of x is not material, then we shall refer to it as a ***wiggle***. Given graphs G and H and a finitely discontinuous k-to-1 function from G onto H, let $W(x)$ be the number of x-wiggles. We shall show in a forthcoming paper that if there is a finitely discontinuous k-to-1 function from G onto H, then

$$\sum_{x=1}^{\infty} xW(x) = k\mathcal{E}(H) - \mathcal{E}(G),$$

where, as earlier, $\mathcal{E}(G) = |E(G)| - |V(G)|$ is the Euler number of G. In particular, the number of wiggles in any 3-to-1 function from G onto H is $\max(0, 3\mathcal{E}(H) - \mathcal{E}(G))$.

(c) So far, abstract graph theory has not had a great impact on the study of k-to-1 maps between graphs. The exceptions which come to mind are the Euler number (see (b) just above and Jo Heath's theorem, Theorem 2.4), the adjacency matrix of a graph (see Theorem 2.6 due to Heath and Hilton), the Max-flow Min-cut theorem (which was used in the study of ($\leq k$)-to-1 maps by Heath and Hilton (see [6,9]), and Eulerian graphs. In [11] it is shown that if H is connected and H is not a cycle, then there is a number $\mu_e = \mu_e(G, H, k)$ such that if $k \geq \mu_e$ and k is even, then there is a k-to-1 map from G onto H. There is also a similar result with a number μ_o for odd values of k. The estimates for μ_e and μ_o are quite rough and it is not impossible that a more sophisticated argument using deeper results from graph theory might yield much better bounds for μ_e and μ_o.

(d) Finally we bring up the possibility of generalizing the Heath–Hilton characterization theorem for k-to-1 maps between graphs, Theorem 2.6. Consider more general objects than abstract graphs/topological graphs. Instead of an abstract graph, consider an hereditary 3-hypergraph H. In this we have 3-sets $\{a, b, c\}$, some 2-sets $\{a, b\}$ and some 1-sets $\{a\}$. If a 3-set $\{a, b, c\}$ lies in H, then so do all the 2- and 1-subsets of $\{a, b, c\}$. Similarly if a 2-set $\{a, b\}$ lies in H, then so do its 1-subsets. This has a natural topological representation. Each 3-set $\{a, b, c\}$ can be thought of as a piece of surface which is homeomorphic to the plane triangle with vertices (0, 0), (0, 1), (1, 0). The edges $\{a, b\}$, $\{b, c\}$, $\{c, a\}$ are then homeomorphic to the straight line segments joining (0, 0), (0, 1); (0, 1), (1, 0); and (0, 0), (1, 0). In this case, the sets $\{a\}$, $\{b\}$ and $\{c\}$ correspond to the vertices (0, 0), (0, 1) and (1, 0). Such a topological representation of an hereditary 3-hypergraph is sometimes called a ***3-complex***. It would be very interesting to find a way of characterizing k-to-1 maps between 3-complexes.

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