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Note

Multiple cross-intersecting families of signed sets

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ABSTRACT

A *k-signed r-set on* $[n] = \{1, \dots, n\}$ is an ordered pair (A, f), where A is an r-subset of $[n]$ and f is a function from A to $[k]$. Families $\mathcal{A}_1, \dots, \mathcal{A}_p$ are said to be *cross-intersecting* if any set in any family $\mathcal{A}_i$ intersects any set in any other family $\mathcal{A}_j$. Hilton proved a sharp bound for the sum of sizes of cross-intersecting families of r-subsets of $[n]$. Our aim is to generalise Hilton's bound to one for families of k-signed r-sets on $[n]$. The main tool developed is an extension of Katona's cyclic permutation argument.

1. Introduction

For an integer n, the n-set $\{1, \dots, n\}$ is denoted by $[n]$. The *power set* $\{A\colon A \subseteq X\}$ of a set X is denoted by 2^X, and the *uniform* sub-family $\{Y \subseteq X\colon |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$.

If $\mathcal{F}$ is a family of sets and x is an element of the union of all sets in $\mathcal{F}$, then we call the sub-family of $\mathcal{F}$ consisting of those sets that contain x a *star of* $\mathcal{F}$ with *centre* x.

A family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ intersect. Note that a star of a family is trivially intersecting.

The classical Erdős–Ko–Rado (EKR) Theorem [13] says that if $r \leqslant n/2$, then an intersecting sub-family $\mathcal{A}$ of $\binom{[n]}{r}$ has size at most $\binom{n-1}{r-1}$, i.e. the size of a star of $\binom{[n]}{r}$; if $r < n/2$, then $\mathcal{A}$ attains the bound if and only if $\mathcal{A}$ is a star of $\binom{[n]}{r}$ (see [13,20]). Two alternative short and beautiful proofs of the EKR Theorem were obtained by Katona [21] and Daykin [8]. In his proof, Katona introduced a very elegant averaging technique called the *cycle method*. Daykin's proof is based on a fundamental result known as the Kruskal–Katona Theorem [22,23]. The EKR Theorem inspired a wealth of results and continues to do so; the survey papers [9,15] are recommended.

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Families $\mathcal{A}_1, \dots, \mathcal{A}_p$ are said to be *cross-intersecting* if for any distinct i and j in $[p]$, any set in $\mathcal{A}_i$ intersects any set in $\mathcal{A}_j$.

Hilton [19] established the following best possible cross-intersection result.

Theorem 1.1. *(See Hilton [19].) Let $r \leqslant n/2$ and $p \geqslant 2$. Let $\mathcal{A}_1, \dots, \mathcal{A}_p$ be cross-intersecting sub-families of $\binom{[n]}{r}$. Then*

$$\sum_{i=1}^{p} |\mathcal{A}_i| \leqslant \begin{cases} \binom{n}{r} & \text{if } p \leqslant \frac{n}{r}; \\ p\binom{n-1}{r-1} & \text{if } p \geqslant \frac{n}{r}. \end{cases}$$

If equality holds and $\mathcal{A}_1 \neq \emptyset$, then, unless $p = 2 = n/r$, one of the following holds:

(i) $p < n/r$, $\mathcal{A}_1 = \binom{[n]}{r}$ *and* $\mathcal{A}_2 = \cdots = \mathcal{A}_p = \emptyset$;
(ii) $p > n/r$ *and* $|\mathcal{A}_1| = \cdots = |\mathcal{A}_p| = \binom{n-1}{r-1}$;
(iii) $p = n/r$ *and* $\mathcal{A}_1, \dots, \mathcal{A}_p$ *are as in* (i) *or* (ii).

The EKR Theorem follows from this result: set $p > n/r$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_p$. Note that if $r > n/2$, then it is trivial that the maximum sum of sizes is $p\binom{n}{r}$ because any two r-subsets of $[n]$ intersect.

We mention that other authors have considered the maximum product problem (see [25,28]); the main result in [25] implies that for any $r \leqslant n/2$ and $k \geqslant 2$, the product of sizes of k cross-intersecting sub-families of $\binom{[n]}{r}$ is a maximum if they are all the same star of $\binom{[n]}{r}$. In this paper, we are interested in the maximum sum problem.

For any family $\mathcal{A}$, we define $\mathcal{A}^*$ to be the sub-family of $\mathcal{A}$ consisting of those sets in $\mathcal{A}$ that intersect each set in $\mathcal{A}$, and we set $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}^*$. So $\mathcal{A}'$ consists of those sets in $\mathcal{A}$ that do not intersect all sets in $\mathcal{A}$.

In [5], the following extension of the EKR Theorem is proved and shown to immediately yield Theorem 1.1 (it is also shown that in case (ii) of Theorem 1.1, $\mathcal{A}_1 = \cdots = \mathcal{A}_p$ and $\mathcal{A}_1$ is a star of $\binom{[n]}{r}$).

Theorem 1.2. *(See Borg [5].) Let $r \leqslant n/2$, and let $\mathcal{A} \subseteq \binom{[n]}{r}$. Then*

$$|\mathcal{A}^*| + \frac{r}{n}|\mathcal{A}'| \leqslant \binom{n-1}{r-1},$$

and if $n > 2r$ then equality holds if and only if either $\mathcal{A}' = \binom{[n]}{r}$ and $\mathcal{A}^* = \emptyset$ or $\mathcal{A}' = \emptyset$ and $\mathcal{A}^*$ is a star of $\binom{[n]}{r}$.

The proof was obtained by extending Daykin's proof of the EKR Theorem. It will be easy to see from the proof of our main result (Theorem 1.4 below) how Theorem 1.1 follows from Theorem 1.2.

As explained below, in this paper we provide an analogue of Theorem 1.2 for *signed sets* and use it to obtain an analogue of Theorem 1.1 also for signed sets.

For $r \in [n]$ and a positive integer k, let $\mathcal{S}_{n,r,k}$ be the family of *k-signed r-sets on* $[n]$ given by

$$\mathcal{S}_{n,r,k} = \left\{ \{(x_1, s_1), \dots, (x_r, s_r)\} \colon \{x_1, \dots, x_r\} \in \binom{[n]}{r},\ s_1, \dots, s_r \in [k] \right\}.$$

A well-known analogue of the EKR Theorem for signed sets was first stated by Meyer [26] and proved in different ways by Deza and Frankl [9] and Bollobás and Leader [4].

Theorem 1.3. *(See Deza and Frankl [9], Bollobás and Leader [4].) Let $r \leqslant n$ and $k \geqslant 2$. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{S}_{n,r,k}$. Then $|\mathcal{A}| \leqslant \binom{n-1}{r-1}k^{r-1}$, and if $kn > 2r$ then equality holds if and only if $\mathcal{A}$ is a star of $\mathcal{S}_{n,r,k}$.*

The proof of Deza and Frankl is based on the well-known shifting technique (see [15]), whereas the proof of Bollobás and Leader is based on Katona's cycle method. There are several other papers in the general area, for example [1–3,7,10–12,14,16–18,24,27].

This brings us to our analogue of Theorem 1.1 for signed sets.

Theorem 1.4. *Let $r \leqslant n$, $k \geqslant 2$, $p \geqslant 2$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_p$ be cross-intersecting sub-families of $\mathcal{S}_{n,r,k}$. Then*

$$\sum_{i=1}^{p} |\mathcal{A}_i| \leqslant \begin{cases} \binom{n}{r} k^r & \text{if } p \leqslant \frac{kn}{r}; \\ p\binom{n-1}{r-1} k^{r-1} & \text{if } p \geqslant \frac{kn}{r}. \end{cases}$$

Suppose equality holds and $\mathcal{A}_1 \neq \emptyset$:

(i) *if $p < \frac{kn}{r}$ then $\mathcal{A}_1 = \mathcal{S}_{n,r,k}$ and $\mathcal{A}_2 = \cdots = \mathcal{A}_p = \emptyset$;*
(ii) *if $p > \frac{kn}{r}$ then $\mathcal{A}_1 = \cdots = \mathcal{A}_p$ and $\mathcal{A}_1$ is a star of $\mathcal{S}_{n,r,k}$;*
(iii) *if $p = \frac{kn}{r} > 2$ then $\mathcal{A}_1, \ldots, \mathcal{A}_p$ are as in* (i) *or* (ii).

Theorem 1.3 follows from this result: set $p > n/r$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_p$. Theorem 1.1 can be viewed as the cross-intersection result for 1-signed r-sets on $[n]$. We remark that the case $r = n$ in Theorem 1.4 is a special case of [6, Theorem 1.5], which employs a method that is different from the one used here.

We will show that Theorem 1.4 is a consequence of the following analogue of Theorem 1.2.

Theorem 1.5. *Let $r \leqslant n$ and $k \geqslant 2$. Let $\mathcal{A} \subseteq \mathcal{S}_{n,r,k}$. Then*

$$|\mathcal{A}^*| + \frac{r}{kn}|\mathcal{A}'| \leqslant \binom{n-1}{r-1} k^{r-1},$$

and if $kn > 2r$ then equality holds if and only if either $\mathcal{A}' = \mathcal{S}_{n,r,k}$ and $\mathcal{A}^* = \emptyset$ or $\mathcal{A}' = \emptyset$ and $\mathcal{A}^*$ is a star of $\mathcal{S}_{n,r,k}$.

We prove this extension of Theorem 1.3 by refining Katona's cycle method and employing the idea of *good cyclic orderings* (see the proof of Theorem 1.5) in the proof of Theorem 1.3 by Bollobás and Leader [4]. The refinement of the cycle method is given by Lemma 2.1 in the next section, and it is inspired by Theorem 1.2. It is important to point out that, as is clear from the proof of Theorem 1.5, Lemma 2.1 also leads to a Katona-type proof of Theorem 1.2.

2. Proofs

The key idea in our work is to extend Katona's method [21] to give a result for *any* family, whether intersecting or not; this is achieved in Lemma 2.1 below. The other key point is to decide what kind of object should play the role of the cyclic orderings; this will become clear at the very beginning of the proof of Theorem 1.5.

If σ is a cyclic ordering of the elements of a set X and the elements of a subset A of X are consecutive in σ, then we say that A *meets* σ.

Lemma 2.1. *Let $m \geqslant 2r$, and let X be a set of size m. Let σ be a cyclic ordering of X. Let $\mathcal{C} = \{C \in \binom{X}{r}\colon C$ meets $\sigma\}$. For any $\mathcal{B} \subseteq \mathcal{C}$ we have*

$$|\mathcal{B}^*| + \frac{r}{m}|\mathcal{B}'| \leqslant r,$$

and if $m > 2r$ then equality holds if and only if either $\mathcal{B}' = \mathcal{C}$ and $\mathcal{B}^* = \emptyset$ or $\mathcal{B}' = \emptyset$ and $|\mathcal{B}^*| = r$.

Proof. Clearly there are m r-subsets of X that meet σ, i.e. $|\mathcal{C}| = m$. So the result is straightforward if $\mathcal{B}^* = \emptyset$. Suppose $\mathcal{B}^* \neq \emptyset$. Let $B^* \in \mathcal{B}^*$, and let $x_1, \ldots, x_r$ be the consecutive points in σ such that $B^* = \{x_1, \ldots, x_r\}$. For $i \in [r]$, let C_i be the r-set in $\mathcal{C}$ beginning with x_i in σ, and let C_i' be the r-set in $\mathcal{C}$ ending with x_i in σ. Let $\mathcal{D} = \{C_1, \ldots, C_r\} \cup \{C_1', \ldots, C_r'\}$. Note that $B^* = C_1 = C_r'$ and hence $\mathcal{D} = \{B^*\} \cup \{C_2, \ldots, C_r\} \cup \{C_1', \ldots, C_{r-1}'\}$. By the definitions of $\mathcal{B}^*$ and $\mathcal{B}'$, we have $\mathcal{B}^* \cup \mathcal{B}' \subseteq \mathcal{D}$ (because

$B^* \in \mathcal{B}^*$) and, since $r \leqslant m/2$, $C'_{j-1} \notin \mathcal{B}^* \cup \mathcal{B}'$ for any $j \in [2, r]$ such that $C_j \in \mathcal{B}^*$. It follows that there are at least $|\mathcal{B}^*| - 1$ sets in $\mathcal{D} \setminus (\mathcal{B}^* \cup \mathcal{B}')$, and hence $|\mathcal{B}'| \leqslant |\mathcal{D}| - |\mathcal{B}^*| - (|\mathcal{B}^*| - 1) = 2r - 2|\mathcal{B}^*|$. So

$$\left|\mathcal{B}^*\right| + \frac{r}{m}\left|\mathcal{B}'\right| \leqslant \left|\mathcal{B}^*\right| + \frac{1}{2}\left|\mathcal{B}'\right| \leqslant \left|\mathcal{B}^*\right| + \frac{1}{2}\left(2r - 2\left|\mathcal{B}^*\right|\right) = r,$$

and it is immediate from this expression that if $\frac{r}{m} < \frac{1}{2}$ then equality holds throughout if and only if $|\mathcal{B}^*| = r$ and $\mathcal{B}' = \emptyset$. Hence the result. $\square$

Katona [21] proved the above result for intersecting sub-families of $\mathcal{C}$. Our result applies to *any* sub-family.

Lemma 2.2. *Let $r \leqslant n$ and $k \geqslant 2$ such that $kn > 2r$. Suppose $\emptyset \neq \mathcal{F} \subseteq \mathcal{S}_{n,r,k}$ such that for any $A \in \mathcal{F}$ and $B \in \{S \in \mathcal{S}_{n,r,k}\colon\ A \cap S = \emptyset\}$, $B \in \mathcal{F}$. Then $\mathcal{F} = \mathcal{S}_{n,r,k}$.*

Proof. We are given that $\mathcal{F}$ contains some set F. For simplicity, we may assume that $F = \{(1,1), (2,1), \ldots, (r,1)\}$. Let A be an arbitrary set in $\mathcal{S}_{n,r,k}$ other than F. We are required to show that $A \in \mathcal{F}$.

Suppose $k \geqslant 3$. Then there exist integers $s_1, \ldots, s_r \in [k]$ such that the set $B = \{(1, s_1), \ldots, (r, s_r)\}$ is disjoint from both F and A. By the conditions of the lemma, we get $B \in \mathcal{F}$, which in turn implies $A \in \mathcal{F}$.

Now suppose $k = 2$. Then $n \geqslant r + 1$ as $kn > 2r$. We first show that $\mathcal{S}_{r,r,2} \subset \mathcal{F}$. Let A_1 be an arbitrary set in $\mathcal{S}_{r,r,2}$ that intersects F on exactly $r-1$ elements, say $A_1 \cap F = \{(1,1), \ldots, (r-1,1)\}$. Then the set $A'_1 = \{(1,2), \ldots, (r-1,2), (r+1,2)\} \in \mathcal{S}_{n,r,2}$ is disjoint from both F and A_1. So $A'_1 \in \mathcal{F}$, which in turn implies $A_1 \in \mathcal{F}$. Therefore $\mathcal{F}$ contains $\mathcal{A}_1 = \{S \in \mathcal{S}_{r,r,2}\colon\ |S \cap F| = r-1\}$. Suppose $r \geqslant 2$. Clearly, for any set $A_2 \in \mathcal{A}_2 = \{S \in \mathcal{S}_{r,r,2}\colon\ |S \cap F| = r-2\}$, there exist $S \in \mathcal{A}_1$ and $A'_2 \in \mathcal{S}_{n,r,k}$ such that $|S \cap A_2| = r-1$ and A'_2 is disjoint from both S and A_2. So $A'_2 \in \mathcal{F}$ and hence $A_2 \in \mathcal{F}$. Thus $\mathcal{A}_2 \subset \mathcal{F}$. We can keep on repeating this step until we obtain $\mathcal{S}_{r,r,2} \subset \mathcal{F}$. Finally, if $A \notin \mathcal{S}_{r,r,2}$, then there exists a set in $\mathcal{S}_{r,r,2}$ that is disjoint from A, and hence $A \in \mathcal{F}$. $\square$

Proof of Theorem 1.5. Let $X = [n] \times [k]$. Let $\mathcal{S} = \mathcal{S}_{n,r,k}$. For a cyclic ordering σ of X, a family $\mathcal{F} \subseteq \mathcal{S}$ and a set $S \in \mathcal{S}$, let $\mathcal{F}_\sigma = \{F \in \mathcal{F}\colon\ F \text{ meets } \sigma\}$ and

$$\Phi(\sigma, S) = \begin{cases} 1 & \text{if } S \text{ meets } \sigma; \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\left(\mathcal{A}^*\right)_\sigma \cup \left(\mathcal{A}'\right)_\sigma = (\mathcal{A}_\sigma)^* \cup (\mathcal{A}_\sigma)' \quad \text{and} \quad \left(\mathcal{A}^*\right)_\sigma \subseteq (\mathcal{A}_\sigma)^*. \tag{1}$$

We call a cyclic ordering σ of X *good* if any n elements $(x_1, s_1), \ldots, (x_n, s_n)$ of X appearing consecutively in σ are such that $x_1, \ldots, x_n$ are distinct and hence $\{x_1, \ldots, x_n\} = [n]$. This means that if the elements of X are listed in the order they appear in a good cyclic ordering starting from an arbitrary element, then the list takes the form $(x_1, s_{11}), \ldots, (x_n, s_{1n})$, $(x_1, s_{21}), \ldots, (x_n, s_{2n})$, $\ldots, (x_1, s_{k1}), \ldots, (x_n, s_{kn})$; note that for any $i \in [n]$, $\{s_{1i}, \ldots, s_{ki}\} = [k]$. Let N be the set of all good cyclic orderings of X. The size of N is $h = n!(k!)^n/|X| = (n-1)!(k-1)!(k!)^{n-1}$ (note that the division by $|X|$ comes from the fact that we are regarding any cyclic ordering and any rotation of it as the same). Any set in $\mathcal{S}$ meets $l = r!(n-r)!((k-1)!)^r(k!)^{n-r}$ cyclic orderings in N. Thus we have

$$\begin{aligned} l\left(\left|\mathcal{A}^*\right| + \frac{r}{kn}\left|\mathcal{A}'\right|\right) &= \left(\sum_{A^* \in \mathcal{A}^*} l\right) + \frac{r}{kn}\left(\sum_{A' \in \mathcal{A}'} l\right) \\ &= \sum_{A^* \in \mathcal{A}^*} \sum_{\sigma \in N} \Phi\left(\sigma, A^*\right) + \frac{r}{kn} \sum_{A' \in \mathcal{A}'} \sum_{\sigma \in N} \Phi\left(\sigma, A'\right) \\ &= \sum_{\sigma \in N}\left(\sum_{A^* \in \mathcal{A}^*} \Phi\left(\sigma, A^*\right) + \frac{r}{kn} \sum_{A' \in \mathcal{A}'} \Phi\left(\sigma, A'\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\sigma \in N} \left(\left|(\mathcal{A}^*)_\sigma\right| + \frac{r}{kn}\left|(\mathcal{A}')_\sigma\right| \right) \\
&\leqslant \sum_{\sigma \in N} \left(\left|(\mathcal{A}_\sigma)^*\right| + \frac{r}{kn}\left|(\mathcal{A}_\sigma)'\right| \right) \quad \text{(by (1))} && (2) \\
&\leqslant \sum_{\sigma \in N} r \quad \text{(by Lemma 2.1)} && (3) \\
&= rh,
\end{aligned}
$$

which yields the inequality in the theorem.

Suppose $\mathcal{A}' = \emptyset$. Then the above immediately gives us $|\mathcal{A}^*| \leqslant \binom{n-1}{r-1}k^{r-1}$. This is in fact Theorem 1.3, which also tells us that the bound is attained only by stars of $\mathcal{S}$ if $kn > 2r$.

Now suppose $kn > 2r$, $\mathcal{A}' \neq \emptyset$ and we have equality in the theorem. So we have equality in (2) and (3). By (1) and the equality in (2), we clearly have

$$(\mathcal{A}^*)_\sigma = (\mathcal{A}_\sigma)^* \quad \text{and} \quad (\mathcal{A}')_\sigma = (\mathcal{A}_\sigma)'. \tag{4}$$

The equality in (3) and Lemma 2.1 give us that for any $\sigma \in N$, if $(\mathcal{A}_\sigma)' \neq \emptyset$ then $(\mathcal{A}_\sigma)' = \mathcal{S}_\sigma$ (and $(\mathcal{A}_\sigma)^* = \emptyset$). Thus, by (4),

$$\text{for any } \sigma \in N, \quad \text{if } (\mathcal{A}')_\sigma \neq \emptyset \quad \text{then } (\mathcal{A}')_\sigma = \mathcal{S}_\sigma. \tag{5}$$

Let A be an arbitrary set $\{(x_1, p_1), \ldots, (x_r, p_r)\}$ in $\mathcal{A}'$. Let B be an arbitrary set $\{(y_1, q_1), \ldots, (y_r, q_r)\}$ in $\{S \in \mathcal{S}\colon\ A \cap S = \emptyset\}$. Let $X = \{x_1, \ldots, x_r\}$, $Y = \{y_1, \ldots, y_r\}$, $m = |X \cap Y|$.

As we now show, there exists $\sigma_{A,B} \in N$ such that both A and B meet $\sigma_{A,B}$. If $m = r$ (i.e. $X = Y$), then this is straightforward since $A \cap B = \emptyset$. If $m = 0$ (i.e. $X \cap Y = \emptyset$, and so $2r \leqslant n$), then clearly there exist members of N in which the elements $(x_1, p_1), \ldots, (x_r, p_r), (y_1, q_1), \ldots, (y_r, q_r)$ of $A \cup B$ appear consecutively in the given order. Now suppose $1 \leqslant m \leqslant r-1$. Let $z_1, \ldots, z_m$ be the elements of $X \cap Y$. We may re-label the elements of A as $(u_1, p'_1), \ldots, (u_{r-m}, p'_{r-m}), (z_1, p'_{r-m+1}), \ldots, (z_m, p'_r)$ and the elements of B as $(z_1, q'_1), \ldots, (z_m, q'_m), (v_1, q'_{m+1}), \ldots, (v_{r-m}, q'_r)$, and it is clear from the order in which we listed the elements that there exist members σ of N such that both A and B meet σ.

Finally, since $A \in (\mathcal{A}')_{\sigma_{A,B}}$ and $B \in \mathcal{S}_{\sigma_{A,B}}$, we have $B \in (\mathcal{A}')_{\sigma_{A,B}}$ by (5). So $B \in \mathcal{A}'$. Therefore $\mathcal{A}' = \mathcal{S}$ by Lemma 2.2. Hence the result. $\square$

Proof of Theorem 1.4. Let $\mathcal{A} = \bigcup_{i=1}^p \mathcal{A}_i$. Clearly $\mathcal{A}^* = \bigcup_{i=1}^p \mathcal{A}_i^*$ and $\mathcal{A}' = \bigcup_{i=1}^p \mathcal{A}'_i$. Suppose $\mathcal{A}'_i \cap \mathcal{A}'_j \neq \emptyset$, $i \neq j$. Let $A \in \mathcal{A}'_i \cap \mathcal{A}'_j$. Then there exists $A_i \in \mathcal{A}'_i$ such that $A \cap A_i = \emptyset$, which is a contradiction because $A \in \mathcal{A}_j$. So $\mathcal{A}'_i \cap \mathcal{A}'_j = \emptyset$ for $i \neq j$, and hence $|\mathcal{A}'| = \sum_{i=1}^p |\mathcal{A}'_i|$. Applying Theorem 1.5, we therefore get

$$\sum_{i=1}^p |\mathcal{A}_i| = \sum_{i=1}^p |\mathcal{A}'_i| + \sum_{i=1}^p |\mathcal{A}_i^*| \leqslant |\mathcal{A}'| + p|\mathcal{A}^*| \leqslant \binom{n}{r}k^r + \left(p - \frac{kn}{r}\right)|\mathcal{A}^*|. \tag{6}$$

Suppose $p < \frac{kn}{r}$. Then $\sum_{i=1}^p |\mathcal{A}_i| \leqslant \binom{n}{r}k^r$, and equality holds if and only if $\mathcal{A}^* = \emptyset$ and $\mathcal{A} = \mathcal{A}' = \mathcal{S}_{n,r,k}$. If $A \in \mathcal{A}_1$ and B is a set in $\mathcal{S}_{n,r,k} \setminus \mathcal{A}_1$ that does not intersect A, then $B \notin \mathcal{A}_i$, $i = 2, \ldots, p$, and hence $B \in \mathcal{S}_{n,r,k} \setminus \mathcal{A}$. Thus, if $\mathcal{A} = \mathcal{S}_{n,r,k}$ then the conditions of Lemma 2.2 hold for $\mathcal{A}_1$ (recall that $\mathcal{A}_1 \neq \emptyset$), and therefore $\mathcal{A}_1 = \mathcal{A} = \mathcal{S}_{n,r,k}$. Hence (i).

Next, suppose $p > \frac{kn}{r}$. Then, by (6) and Theorem 1.5,

$$\sum_{i=1}^p |\mathcal{A}_i| \leqslant \binom{n}{r}k^r + \left(p - \frac{kn}{r}\right)\binom{n-1}{r-1}k^{r-1} = p\binom{n-1}{r-1}k^{r-1},$$

and equality holds if and only if $\mathcal{A}_1^* = \cdots = \mathcal{A}_k^* = \mathcal{A}^*$ and $|\mathcal{A}^*| = \binom{n-1}{r-1}k^{r-1} = |\mathcal{A}|$, in which case $\mathcal{A}$ is a star of $\mathcal{S}_{n,r,k}$ by Theorem 1.3. Hence (ii).

Finally, suppose $p = \frac{kn}{r}$. Then, by (6), $\sum_{i=1}^{p} |\mathcal{A}_i| \leqslant |\mathcal{A}'| + \frac{kn}{r}|\mathcal{A}^*| \leqslant \binom{n}{r}k^r$. Suppose $p > 2$, i.e. $\frac{kn}{r} > 2$. If $\mathcal{A}^* = \emptyset$ then $\mathcal{A}$ is as in the case $p < \frac{kn}{r}$, and it is immediate from Theorem 1.5 that if $\mathcal{A}^* \neq \emptyset$ then $\mathcal{A}^*$ is as in the case $p > \frac{kn}{r}$. Hence (iii). $\square$

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